$\mathbb{D}$ynamics of $\mathbb{C}$ohomological $\mathbb{E}$xpanding $\mathbb{M}$appings II : Third and Fourth Main Results

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Abstract

Let $h : \mathcal{V} \longrightarrow \mathcal{V}$ be a Cohomological Expanding Mapping[1] of a smooth complex compact homogeneous manifold with $dim_{\mathbb{C}}(\mathcal{V}) = k \geq 1$ and Kodaira Dimension ≤ 0. We study the dynamics of such mapping from a probabilistic point of view, that is, we describe the asymptotic behavior of the orbit $O_h(x) = \{h^n(x), n \in \mathbb{N} \quad \text{or} \quad \mathbb{Z}\}$ of a generic point. Using pluripotential methods, we have constructed in our previous paper [1] a natural invariant canonical probability measure of maximal Cohomological Entropy ν_h such that $\chi_{2l}^{-m}(h^m)^*\Omega \to \nu_h \qquad \text{as} \quad m \to \infty$ for each smooth probability measure Ω in $\mathcal{V}$. We have also studied the main stochastic properties of ν_h and have shown that ν_h is a smooth equilibrium measure , ergodic, mixing, K-mixing, exponential-mixing. In this paper we are interested on equidistribution problems and we show in particular that ν_h reflects a property of equidistribution of periodic points by setting out the Third and Fourth Main Results in our study. Finally we conjecture that

$$\nu_h := T_l^+ \wedge T_{k-l}^-,$$

$$\dim_{\mathcal{H}}(\nu_h) = \Psi h_\chi(h),$$

$$\dim_{\mathcal{H}}(\mathrm{Supp} T_l^+) \geq 2(k-l) + \frac{\log \chi_{2l}}{\psi_l},$$

$$|\langle \nu_m^x - \nu_h, \zeta\rangle| \leq M\Big[1 + \log^+ \frac{1}{D(x,\mathfrak{T})}\Big]^{\beta/2} \|\zeta\|_{\mathcal{C}^\beta}\gamma^{-\beta m/2}$$

and

$$|\langle \nu_m^x - \nu_h, \zeta\rangle| \leq M\Big[1 + \log^+ \frac{1}{D(x,E_\gamma)}\Big]^{\beta/2} \|\zeta\|_{\mathcal{C}^\beta}\gamma^{-\beta m/2}.$$

***K*eywords** Complex Dynamics · Cohomological Expanding Mapping · Cohomological Degree · Cohomological Entropy · Cohomological Quotient .

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[1]cf Definition 1.6.

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Contents

1 Introduction

Note that in this paper we will generalize some results on Complex Dynamics due to Nessim Sibony, Tien-Cuong Dinh, Duc-Viet Vu and other authors. For this, we will use a new Method called the Cohomological Method introduced in our preview paper [1]. This new Method is based on the Concepts of Cohomological Degree, Cohomological Entropy and Cohomological Quocient which were introduced in ours preview papers [1], [42]. Let ν be the equilibrium measure of an endomorphism h. If ζ is an observable, $(\zeta \circ h^n)_{n\geq 0}$ can be seen as a sequence of dependent random variables. As the measure is invariant, these variables are distributed in an identical way, that is, the Borel sets $\{\zeta \circ h^n < b\}$ have the same measure ν for any fixed constant b. We recall some general facts of ergodic theory and probability theory. We refer to [38, 40] for the general theory. Consider a dynamic system of a map $h : \mathcal{V} \to \mathcal{V}$, measurable against a σ-algebra $\mathcal{F}$ on $\mathcal{V}$. The direct image of a probability measure ν by h is the measure of probability $h_*(\nu)$ defined by

$$h_*(\nu)(E) := \nu(h^{-1}(E))$$

for each measurable set E. Likewise, for any positive measurable function ζ, we have

$$\langle h_*(\nu), \zeta\rangle := \langle \nu, \zeta \circ h\rangle.$$

The measure ν is ***invariant*** if $h_*(\nu) = \nu$. When $\mathcal{V}$ is a compact metric space and h is continuous, the set $\mathcal{M}(h)$ of invariant probability measures is convex, compact and not empty: for any sequence of probability measures ν_N, the limit values of

$$\frac{1}{N}\sum_{j=0}^{N-1}(h^n)_*(\nu_N)$$

are invariant probability measures. A measurable set E is ***totally invariant*** if $\nu(E\setminus h^{-1}(E)) = \nu(h^{-1}(E)\setminus E) = 0$. An invariant probability measure ν is ***ergodic*** if any totally invariant set is of measure ν zero or complete . It is easy to show that ν is ergodic if and only if $\zeta \circ h = \zeta$, for $\zeta \in L^1(\nu)$, then ζ is constant. Here, we can replace $L^1(\nu)$ by $L^p(\nu)$ with $1 \leq p \leq +\infty$. The **ergodicity** of ν is also equivalent to the fact that it is extremal on $\mathcal{M}(h)$. We remember Birkhoff's ergodic theorem, which is the analogue of the law of large numbers for independent random variables [40].

Theorem 1.1 (Birkhoff). *Let $h : \mathcal{V} \to \mathcal{V}$ be a measurable map as above. Suppose that ν is an invariant ergodic probability measure . Let ζ be a function on $L^1(\nu)$. Then*

$$\frac{1}{N}\sum_{n=0}^{N-1}\zeta(h^n(x)) \to \langle \nu, \zeta\rangle$$

almost everywhere in relation to ν.

When $\mathcal{V}$ is a compact metric space, we can apply Birkhoff's theorem to continuous functions ζ and deduce that for ν almost all x

$$\frac{1}{N}\sum_{n=0}^{N-1}\delta_{h^n(x)} \to \nu,$$

where δ_x indicates the mass of Dirac at x. The sum

$$\mathrm{St}_N(\zeta) := \sum_{n=0}^{N-1}\zeta \circ h^n$$

is called ***Birkhoff sum***. Therefore, Birkhoff's theorem describes the behavior of $\frac{1}{N}\mathrm{St}_N(\zeta)$ for an observable ζ. A stronger notion than **ergodicity** is the notion of **mixing**. An invariant probability measure ν is ***mixing*** if for each measurable set E, F

$$\lim_{n\to\infty}\nu(h^{-n}(E)\cap F) = \nu(E)\nu(F).$$

Clearly, **mixing** implies **ergodicity**. It is not difficult to see that ν is mixing if, and only if, for any test functions ζ, η on $L^\infty(\nu)$ or on $L^2(\nu)$, we have

$$\lim_{n\to\infty}\langle \nu, (\zeta \circ h^n)\eta\rangle = \langle \nu, \zeta\rangle\langle \nu, \eta\rangle.$$

The Quantity

$$W_n(\zeta, \eta) := |\langle \nu, (\zeta \circ h^n)\eta\rangle - \langle \nu, \zeta\rangle\langle \nu, \eta\rangle|$$

is called ***the correlation on time*** n of ζ and η. Thus, **mixing** is equivalent to the convergence of $W_n(\zeta, \eta)$ to 0. We say that ν is ***K-mixing*** if for each $\eta \in L^2(\nu)$

$$\sup_{\|\zeta\|_{L^2(\nu)} \leq 1} W_n(\zeta, \eta) \to 0.$$

Note that **K-mixing** is equivalent to the fact that the σ -algebra $\mathcal{F}_\infty := \cap h^{-n}(\mathcal{F})$ contains only sets zero and complete measures. This is the strongest form of mixing for observables on $L^2(\nu)$. However, it is of interest to obtain quantitative information about the mixing speed for more regular observables, such as smooth functions or Hölder continuous. Now consider an endomorphism h of degree $\tau \geq 2$ of $\mathbb{P}^k$ as above and its equilibrium measure ν. We know that ν is totally invariant: $h^*(\nu) = \tau^k \nu$. If ζ is a continuous function, so

$$\langle \nu, \zeta \circ h \rangle = \langle \tau^{-k} h^*(\nu), \zeta \circ h \rangle = \langle \nu, \tau^{-k} h_*(\zeta \circ h) \rangle = \langle \nu, \zeta \rangle.$$

We use the obvious fact that $h_*(\zeta \circ h) = \tau^k \zeta$. Thus, ν is invariant. Mixing for measure ν was proved in [27].

Theorem 1.2. *Let h be an endomorphism of degree $\tau \geq 2$ of $\mathbb{P}^k$. So its measure of Green ν is K-mixing.*

The equilibrium measure ν satisfies remarkable stochastic properties that are quite difficult to obtain in the real dynamic systems scenario. Pluripotential methods replace the delicate estimates used in some real dynamic systems. Consider a dynamic system $h : (\mathcal{V}, \mathcal{F}, \nu) \to (\mathcal{V}, \mathcal{F}, \nu)$ as above, where ν is an invariant probability measure. Therefore, h^* defines a linear operator of norm 1 on $L^2(\nu)$. We say that h has *the Jacobian limited* if there is a constant $\kappa > 0$ such that $\nu(h(E)) \leq \kappa\nu(E)$ for each $E \in \mathcal{F}$. When $\mathcal{V}$ is a complex mamifold, it is necessarily orientable . Let $\mathcal{V}$ be a smooth complex compact homogeneous manifold with $dim_{\mathbb{C}}(\mathcal{V}) = k \geq 1$ and Kodaira dimension ≤ 0 and $h : \mathcal{V} \longrightarrow \mathcal{V}$ be a dominant surjective meromorphic endomorphism, that is, whose Jacobian is not identically null in any local chart. Let ω be a $(1,1)$-strictly positive Hermitian form on $\mathcal{V}$. Let ℓ be a prime number.

Definition 1.3. The i*-th* ***Cohomological Degree*** $\chi_i(h)$ of h is defined as the spectral radius of the pullback action h^* in the cohomology group ℓ -adic étale $H^i_{\text{ét}}(\mathcal{V}, \mathbf{Q}_\ell)$ independent of ℓ by: (cf [42] [6] [43] [2] [30] for more details)

$$\chi_i(h) = \rho\big(h^*|_{H^i_{\text{ét}}(\mathcal{V}, \mathbf{Q}_\ell)}\big).$$

Definition 1.4. We define the (l,n)*-th* ***Cohomological Quotient*** $\xi^n_l(h)$ of h as follows:

$$\xi^n_l(h) = \left[\frac{\chi_{2l-1}(h)}{\chi_{2l}(h)}\right]^n$$

Definition 1.5. The **Cohomological Entropy** of h is defined by

$$h_\chi(h) = \max_i \log \chi_{2i}(h).$$

Definition 1.6. We say that h is a **Cohomological Expanding Mapping** when h is dynamically compatible (that is $(h^n)^* = (h^*)^n$) and there is $l \in \{1, ..., k\}$ such that :

$$\xi^{-1}_l(h) > 1.$$

We will write χ_i for $\chi_i(h)$ and ξ^n_l for $\xi^n_l(h)$ if there is no confusion.
Let $(\mathcal{V}, \mathcal{F}, v)$ be a probability space and $h : \mathcal{V} \to \mathcal{V}$ be a measurable map that preserves v, that is, v is h_* -invariant: $h_* v = v$. The measure v is ***ergodic*** if for any measurable set E such that $h^{-1}(E) = E$, we have $v(E) = 0$ or $v(E) = 1$. This is equivalent to the property that v is extremal on the convex set of invariant probability measures (if v is mixing, so it is ergodic). When v is ergodic, Birkhoff's theorem implies that if η is an observable on $L^1(v)$ then

$$\lim_{n\to\infty} \frac{1}{n}\Big[\eta(x) + \eta(h(x)) + \cdots + \eta(h^{n-1}(x))\Big] = \langle v, \eta \rangle$$

for v - almost all x. Suppose now that $\langle v, \eta \rangle = 0$. Then, the previous limit is equal to 0. The theorem of limit central (TLC), when it occurs, provides the speed of this convergence. We say that η *satisfies the TLC* if there is a constant $\sigma > 0$ such that

$$\frac{1}{\sqrt{n}}\Big[\eta(x) + \eta(h(x)) + \cdots + \eta(h^{n-1}(x))\Big]$$

converges in distribution for the Gaussian random variable $\mathcal{N}(0,\sigma)$ of mean 0 and variance σ. Remember that η is a *coboundary* whether there is a function η' on $L^2(\nu)$ such that $\eta = \eta' - \eta' \circ h$. In that case, it is easy to see that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\Big[\eta(x) + \eta(h(x)) + \cdots + \eta(h^{n-1}(x))\Big] = \lim_{n\to\infty} \frac{1}{\sqrt{n}}\Big[\eta'(x) - \eta'(h^n(x))\Big] = 0$$

in distribution. Therefore, η does not satisfy the TLC (sometimes it is said that η satisfies the TLC by $\sigma = 0$). The TLC can be deduced from strong mixing, see [31,35,37,39]. In the following result, $Et(\eta|\mathcal{F}_n)$ indicates *the expectation* of η in relation to $\mathcal{F}_n$, that is, $\eta \mapsto Et(\eta|\mathcal{F}_n)$ is the orthogonal projection of $L^2(v)$ in the subspace generated by the measurable functions $\mathcal{F}_n$.

Theorem 1.7 (Gordin)**.** *Consider the decreasing sequence $\mathcal{F}_n := h^{-n}(\mathcal{F})$, $n \geq 0$, of algebras. Let η be a function with real value on $L^2(v)$ such that $\langle v, \eta\rangle = 0$. Suppose that*

$$\sum_{n\geq 0} \|Et(\eta|\mathcal{F}_n)\|_{L^2(v)} < \infty.$$

So, the positive number σ defined by

$$\sigma^2 := \langle v, \eta^2\rangle + 2\sum_{n\geq 1}\langle v, \eta(\eta\circ h^n)\rangle$$

is finite. It vanishes if and only if η is a coboundary. Furthermore, when $\sigma \neq 0$, then η satisfies the TLC with variance σ.

Note that σ is equal to the limit of $n^{-1/2}\|\eta + \cdots + \eta \circ h^{n-1}\|_{L^2(v)}$. The last expression is equal to $\|\eta\|_{L^2(v)}$ if the family $(\eta \circ h^n)_{n\geq 0}$ is orthogonal on $L^2(v)$. We refer to [38,40] for the notion of Lyapunov exponent.

Definition 1.8. An invariant positive measure is ***hyperbolic*** if its Lyapunov exponents are non-zero.

A function *quasi-p.s.h.* on $\mathcal{V}$ is a function of $\mathcal{V}$ on $[-\infty,\infty)$, which is locally the sum of a plurisubharmonic function and a smooth function. For a given $(1,1)$ -continuous form η, denote by $\mathrm{PSH}^0(\eta)$ the set of quasi-p.s.h. functions ζ such that $dd^c\zeta + \eta \geq 0$ and $\sup_{\mathcal{V}}\zeta = 0$. Endow $\mathrm{PSH}^0(\eta)$ with induced distance of $L^1(\mathcal{V})$ using natural inclusion $\mathrm{PSH}^0(\eta) \subset L^1(\mathcal{V})$. Remember from [33] that a complex measure ν on $\mathcal{V}$ is considered *PC* if each quasi-p.s.h. function is ν-integrable and for each sequence $(\zeta_n)_{n\in\mathbb{N}}$ of quasi-p.s.h. functions converging to ζ on L^1 , so that $dd^c\zeta_n + \eta \geq 0$ for some smooth form η independent of n, we have $\langle\nu,\zeta_n\rangle \to \langle\nu,\zeta\rangle$. A *pluripolar set* on $\mathcal{V}$ is a subset of $\mathcal{V}$ contained on $\{\zeta = -\infty\}$ for some quasi-p.s.h. function ζ. Any locally pluripolar set on $\mathcal{V}$ is pluripolar, cf [34]. In particular, this implies that there are profusely quasi-p.s.h singular functions on $\mathcal{V}$. Note that every PC measure has no mass on pluripolar sets. Next, we will consider the dynamics of h with $\xi_l^{-1}(h) > 1$. The Third Main Result is as follows.

Theorem 1.9. *Let $h : \mathcal{V} \to \mathcal{V}$ be a* ***Cohomological Expanding Mapping*** *with $l = k$. Let ν_h be the equilibrium measure of h. Let E_n (resp. RE_n) be the set of isolated periodic points (resp. periodic repellent) of period n. Let Λ_n be the set E_n , RE_n or their intersections with the support of ν_h. Then Λ_n is equidistributed asymptotically with respect to ν_h: we have*

$$\frac{1}{\chi_{2k}^n}\sum_{x\in\Lambda_n}\Gamma_x \to \nu_h \qquad \text{when} \quad n\to\infty,$$

where Γ_x indicates the mass of Dirac on x. In particular, we have $\#\Lambda_n = \chi_{2k}^n + o(\chi_{2k}^n)$ when $n \to \infty$.

In fact, when h admits positive dimensional analytical sets of periodic points, the classic Lefschetz formula does not allow estimate the number of isolated periodic points. The upper limit $\#\Lambda_n \leq \chi_{2k}^n + o(\chi_{2k}^n)$ is, in fact, obtained using a recent theory of the density of positive closed currents developed by Sibony-Dinh in [14]. See too [24], [26], [46], [23], [17]. Let J' be *the second set of indeterminacy* of h, that is, the set of points z such that $h^{-1}(z)$ have a positive dimension. It is an analytical set of codimension at least equal to 2. The open set of Zariski $\mathcal{V}\setminus J'$ is the set of points x such that the fiber $h^{-1}(x)$ contains exactly χ_{2k} contains exactly h^{-1} on subsets of $\mathcal{V}$. Define $J'_0 := J'$, $J'_{n+1} := J'_0 \cup h(J'_n)$ for $n \geq 0$ and $J'_\infty := \cup_{n\geq 0}J'_n$. Note that the set J'_∞ is characterized by the following property: the sequence of probability measures

$$\nu_0^x := \Gamma_x, \quad \nu_{n+1}^x := \chi_{2k}^{-1}h^*(\nu_n^x) \quad \text{ for } n \geq 0$$

is well defined if and only if $x \notin J'_\infty$. We have $\nu_n^x = \chi_{2k}^{-n}(h^n)^*(\Gamma_x)$. Thus , ν_n^x is the probability measure equidistributed on the fiber $h^{-n}(x)$ where the points on that fiber are counted with multiplicity. It is necessary to distinguish J'_∞ with the set $\cup_{n\geq 0}h^n(J')$, which is a priori minor. Let J be *the (first) set of indeterminacy* of h. Also define $J_0 := J$, $J_{n+1} := J_0 \cup h(J_n)$ for $n \geq 0$ and $J_\infty := \cup_{n\geq 0}J_n$. The set $J_\infty \setminus J'_\infty$ consists of points $x \notin J'_\infty$, so that the support of ν_n^x intersects J for some $n \geq 0$. Let's consider $x \notin J_\infty \cup J'_\infty$. The Fourth Main Result is as follows.

Theorem 1.10. *Let $h : \mathcal{V} \to \mathcal{V}$ and ν_h be as in Theorem 1.9. So there is a proper analytic set (possibly empty) $\mathcal{T}$ of $\mathcal{V}$, so that for $x \notin J_\infty \cup J'_\infty$ we have*

$$\frac{1}{\chi_{2k}^m}(h^m)^*\Gamma_x \to \nu_h \qquad \textit{when} \quad m \to \infty$$

if and only if $x \notin \mathcal{T}$.

Many meromorphic maps with $J' = \varnothing$ are not holomorphic. For example, if $f : \widehat{\mathbb{P}}^k \to \mathbb{P}^k$ is an explosion of $\mathbb{P}^k$ and $\pi : \widehat{\mathbb{P}}^k \to \mathbb{P}^k$ is a finite holomorphic map, so $\pi \circ f^{-1}$ is not holomorphic, but its second set of indeterminacy is empty. For holomorphic maps on $\mathbb{P}^k$, we have $J = J' = \varnothing$.

2 Third and Fourth Main Results

Recall that in this paper we generalize some results on Complex Dynamics due to Nessim Sibony, Tien-Cuong Dinh, Duc-Viet Vu and other authors. For this, we use a new Method called the Cohomological Method introduced in our previous paper [1]. This new Method is based on the Concepts of Cohomological Degree, Cohomological Entropy and Cohomological Quocient which were introduced in ours previous papers [5], [1], [42] . Let $\mathcal{V}$ be a compact Kähler manifold with dimension k and ω be a Kähler form on $\mathcal{V}$ normalized such that ω^k define a probability measure on $\mathcal{V}$. Let $h : \mathcal{V} \to \mathcal{V}$ be a **Cohomological Expanding Mapping**. *The iterated of order* n of h is defined by $h^n := h \circ \cdots \circ h$, n times, on a dense open set of Zariski and extends to a dominant meromorphic map on $\mathcal{V}$. We refer the reader to [12, 13, 24] for details. The first main result of this paper is as follows.

Theorem 2.1 (Theorem 1.9 "**Third Main Result**"). *Let $h : \mathcal{V} \to \mathcal{V}$ be a* ***Cohomological Expanding Mapping*** *with $l = k$. Let ν_h be the equilibrium measure of h. Let E_n (resp. RE_n) be the set of isolated periodic points (resp. repellent points) of period n. Let Λ_n be any of the sets E_n , RE_n or their intersections with the support of μ_h. Then Λ_n is equidistributed asymptotically with respect to ν_h: we have*

$$\frac{1}{\chi_{2k}^n}\sum_{x\in\Lambda_n}\Gamma_x \to \nu_h \qquad \textit{when} \quad n \to \infty,$$

where Γ_x denotes the mass of Dirac on x. In particular, we have $\#\Lambda_n = \chi_{2k}^n + o(\chi_{2k}^n)$ when $n \to \infty$.

In Theorem 2.1 , the last statement is an important point in its proof. Therefore the classical Lefschetz formula does not allow to estimate the number of isolated periodic points when h admits positive dimensional analytical sets of periodic points. So the upper limit $\#\Lambda_n \leq \chi_{2k}^n + o(\chi_{2k}^n)$ is obtained using the theory of the density of positive closed currents developed by Sibony and Dinh in [14]. One may need to construct enough periodic repellent points on the support of ν_h. To this end, we will construct on Section 4 good enough inverse branches of balls for h^n with controlled size, cf Proposition 4.1 below. This construction will be a generalization of the construction in [30]. The construction of inverse branches for holomorphic discs on projective varieties can be achieved using a method developed by Briend-Duval in [41]. Here we follow the approach developed by Dinh-Sibony in [46] which also allows the construction of discs and spheres on varieties of Kähler. An idea from Buff allows to get periodic repellent points [44]. Mas the presence of indeterminacy sets for **Cohomological Expanding Mappings** is the source of several delicate technical points . For example, the obstruction to the existence of inverse branches for balls, at least in this approach, may be greater than the orbits of critical values and locus of indeterminacy. One may construct and use a $(1, 1)$-current U closed and positive that allows to control this obstruction. Note that when $\mathcal{V}$ is a projective variety, a weaker version of Theorem 2.1 was stated in [24]. When h is a holomorphic endomorphism of $\mathbb{P}^k$, a similar version of the above theorem was obtained by Briend-Duval in [26]. His proof heavily uses the continuity of Hölder of the dynamical function of Green. For **Cohomological Expanding Mappings**, Green's dynamical function, even when it exists, is generally not continuous. A similar result for polynomial maps of dominant topological degree, in particular for a large family of rational maps on $\mathbb{P}^k$, was obtained by Sibony and Dinh [46]. For the case of dimension 1, see Brolin [23], Freire-Lopes-Mañé [47], Lyubich [47] and Tortrat [50]. This construction of inverse branches of balls also allows to study the equidistribution of pre-images of points by h^n. Let J' be *the second set of indeterminacy* of h, that is, the set of points z such that $h^{-1}(z)$ has a positive dimension. It is an analytical set of codimension at least equal to 2. The open set of Zariski $\mathcal{V} \setminus J'$ is the set of points x such that the fiber $h^{-1}(x)$ contain exactly χ_{2k} points counted with multiplicity, see Section 3 for the definition of the action of h and h^{-1} on subsets of $\mathcal{V}$. Define $J'_0 := J'$, $J'_{n+1} := J'_0 \cup h(J'_n)$ for $n \geq 0$ and $J'_\infty := \cup_{n\geq 0}J'_n$. Note that the set J'_∞ is characterized by the following property: the sequence of probability measures

$$\nu_0^x := \Gamma_x, \quad \nu_{n+1}^x := \chi_{2k}^{-1}h^*(\nu_n^x) \quad \text{for } n \geq 0$$

is well defined if and only if $x \notin J'_\infty$. We have $\nu_n^x = \chi_{2k}^{-n}(h^n)^*(\Gamma_x)$. Then ν_n^x is the measure of probability equidistributed on fiber $h^{-n}(x)$ where the points on this fiber are counted with multiplicity. It is necessary to distinguish J'_∞ with the set $\cup_{n\geq 0} h^n(J')$ which is a priori minor. Let J be *the (first) set of indeterminacy* of h. Also define $J_0 := J$, $J_{n+1} := J_0 \cup h(J_n)$ for $n \geq 0$ and $J_\infty := \cup_{n\geq 0} J_n$.The set $J_\infty \setminus J'_\infty$ consists of points $x \notin J'_\infty$ such that the support of ν_n^x intercepts J for some $n \geq 0$. Let's consider $x \notin J_\infty \cup J'_\infty$.

Theorem 2.2 (Theorem 1.10 "**Fourth Main Result**"). *Let $h : \mathcal{V} \to \mathcal{V}$ and ν_h be as in Theorem 1.9. So there is a proper analytic set (possibly empty) $\mathcal{T}$ of $\mathcal{V}$ such that for $x \notin J_\infty \cup J'_\infty$ we have*

$$\frac{1}{\chi_{2k}^m}(h^m)^*\Gamma_x \to \nu_h \qquad when \quad m \to \infty$$

if and only if $x \notin \mathcal{T}$.

When $\mathcal{V}$ is projective, was shown by Guedj in [24] that $\mathcal{T}$ is a finite or enumerable union of analytical sets, see also [13] in the case of compact varieties of Kähler . A similar result to the theorem above was obtained for holomorphic endomorphisms of $\mathbb{P}^k$ in [29, 41, 46]. It also applies to maps of the polynomial type with topological dominant degree. For the case of dimension 1, see [17, 23, 47, 50].

3 Current density

Suppose $l \leq k$. In this section, we define several operations for **Cohomological Expanding Mappings** and positive closed currents on compact varieties of Kähler. We also recall some elements of the theory of closed positive current density and establish a preparatory result. We refer the reader to Armand [3], [30], [8] and also to Demailly [11], Dinh-Sibony [15, 20] and Voisin [49] for basics of positive closed currents and quasi-plurisubharmonic functions (quasi-p.s.h.) and basic facts about the geometry of Kähler. Let $\mathcal{V}$ be a compact Kähler manifold with dimension k and ω be a Kähler form on $\mathcal{V}$ as above. If Q is a current on $\mathcal{V}$ and ζ is a test form of the right degree, the pairing $\langle Q, \zeta\rangle$ denotes the value of Q on ζ. If Q is a (p, p)-positive current on $\mathcal{V}$, its mass is given by the formula

$$\|Q\| := \langle Q, \omega^{k-p}\rangle.$$

Note that when Q is, moreover, closed, its mass depends only on its class of cohomology $\{Q\}$ on $H^{p,p}(\mathcal{V}, \mathbb{R})$. Here $H^{p,q}(\mathcal{V}, \mathbb{C})$ denotes the Hodge cohomology group of bidegree (p, q) of $\mathcal{V}$ and $H^{p,p}(\mathcal{V}, \mathbb{R}) := H^{p,p}(\mathcal{V}, \mathbb{C}) \cap H^{2p}(\mathcal{V}, \mathbb{R})$. Let's write $Q \leq Q'$ and $Q' \geq Q$ for two (p, p)-real currents Q, Q' if $Q' - Q$ is a positive current. We also write $c \leq c'$ and $c' \geq c$ for $c, c' \in H^{p,p}(\mathcal{V}, \mathbb{R})$ when $c' - c$ is the class of a (p, p)-closed positive current. If V is an analytic subset of pure dimension $k - p$ on $\mathcal{V}$, denote by $[V]$ the positive closed current of integration on V and $\{V\}$ its cohomology class on $H^{p,p}(\mathcal{V}, \mathbb{R})$. The cup product on $H^*(\mathcal{V}, \mathbb{C})$ is denoted by $\smile$. Now consider a **Cohomological Expanding Mapping** $h : \mathcal{V} \to \mathcal{V}$. Remember that h is holomorphic on an open Zariski set and the closing Γ of its graph on $\mathcal{V} \times \mathcal{V}$ is an irreducible analytical subset of dimension k. Let π_1 and π_2 be the canonical projections of $\mathcal{V} \times \mathcal{V}$ on its factors. If Z is a subset of $\mathcal{V}$, define $h(Z) := \pi_2(\pi_1^{-1}(Z) \cap \Gamma)$ and $h^{-1}(Z) := \pi_1(\pi_2^{-1}(Z) \cap \Gamma)$. *The (first) set of indeterminacy* J of h is the complement of the set of all points $z \in \mathcal{V}$ such that $h(z)$ is of dimension 0, or equivalently, that $h(z)$ contains only one point. *The second set of indeterminacy* J' of h is the complement of the set of all points z such that $h^{-1}(z)$ is of dimension 0, or equivalently, that $h^{-1}(z)$ contains exactly χ_{2k} points counted with multiplicity. Both J and J' are analytical subsets of $\mathcal{V}$ of codimension at least equal to 2. The **Cohomological Expanding Mapping** h induces linear operators on forms and currents. The presence of locus of indeterminacy makes these operators more delicate to handle. If ζ is a (p, q)-smooth form on $\mathcal{V}$, then $h^*(\zeta)$ is the (p, q)-current defined by

$$h^*(\zeta) := (\pi_1)_*(\pi_2^*(\zeta) \wedge [\Gamma]).$$

One can see that $h^*(\zeta)$ is a L^1-smooth form out of J. Its singularities throughout of J do not allow iterate the operation in the same way. However, the operation commutes with ∂ and $\overline{\partial}$. In particular, when ζ is closed or exact, so is $h^*(\zeta)$. Therefore, h^* induces a linear operator on $H^{p,q}(\mathcal{V}, \mathbb{C})$. One can iterate the previous operator and since h is a **Cohomological Expanding Mapping**, we have $(h^*)^n = (h^n)^*$. Similarly, the (p, q)-current $h_*(\zeta)$ is defined by

$$h_*(\zeta) := (\pi_2)_*(\pi_1^*(\zeta) \wedge [\Gamma]).$$

This is a L^1-smooth form out of the critical values of $\pi_{2|\Gamma}$. The operator h_* also commutes with $\partial, \overline{\partial}$ and induces a linear operator h_* on $H^{p,q}(\mathcal{V}, \mathbb{C})$. Let us now consider two particular cases of the pull-back operator h^* on currents that will be used later. If ζ is a continuous function on $\mathcal{V}$ then $h_*(\zeta)$ is a limited function on $\mathcal{V}$, that is continuous outside J'. So if ν is a positive measure without mass on J' we can define

$$\langle h^*(\nu), \zeta\rangle := \langle \nu, h_*(\zeta)\rangle.$$

One can see that $h^*(\nu)$ is a positive measure whose mass is equal to χ_{2k} times the mass of ν since π_2 restricted to Γ defines a branched covering of degree χ_{2k} on $\mathcal{V} \setminus J'$. If ν is the mass of Dirac at one point $x \notin J'$, then $h^*(\nu)$ is the sum of the Dirac masses on the fiber $h^{-1}(x)$ counted with multiplicity. If ν has no mass on J, the positive measure $h_*(\nu)$ given by

$$\langle h_*(\nu), \zeta \rangle := \langle \nu, h^*(\zeta) \rangle$$

for each continuous function ζ on $\mathcal{V}$, is well defined and has the same mass as ν. If ν is the mass of Dirac on $x \notin J$, then $h_*(\nu)$ is the mass of Dirac on $h(x)$. The second situation concerns the $(1, 1)$-closed positive currents . If Q is such a current on $\mathcal{V}$, we can write $Q = \alpha + dd^c u$ where α is a $(1, 1)$-closed smooth form on class $\{Q\}$ and u is a function quasi-p.s.h. . Then $u \circ \pi_2$ is a function quasi-p.s.h. on Γ and we define

$$h^*(Q) := h^*(\alpha) + (\pi_1)_*(dd^c(u \circ \pi_{2|\Gamma})).$$

Using local regularization of Q, one can see that $h^*(Q)$ is a $(1, 1)$-closed positive current. The operator is linear and continuous on Q. So, using Demailly's regularization of $(1, 1)$-currents on $\mathcal{V}$ [10], we can verify that the operator is compatible with the cohomology, that is, we have $\{h^*(Q)\} = h^*\{Q\}$. The operator h_* is defined in the same way on currents $(1, 1)$ closed positive and is also compatible with the cohomology. We will recall basic facts about positive closed currents density theory and provide an abstract result that will allow us to ignore Lefschetz's fixed point formula to limit the number of periodic points. We will restrict ourselves to the simplest situation necessary for the present paper. The reader is invited to consult [14] for more details. Let V be an irreducible subvariety of $\mathcal{V}$ of dimension l. Let $\pi : X \to V$ be the normal vector bundle of V on $\mathcal{V}$. For a point $x \in V$, if $\mathrm{Tan}_x\mathcal{V}$ and $\mathrm{Tan}_x V$ denote, respectively, the tangent spaces of $\mathcal{V}$ and of V on x, the fiber $X_x := \pi^{-1}(x)$ of X over x is canonically identified with the quotient space $\mathrm{Tan}_x\mathcal{V}/\mathrm{Tan}_x V$. The zero section of X is naturally identified with V. Denote by $\overline{X}$ the natural compactification of X, that is, the projectivization $\mathbb{P}(X \oplus \mathbb{C})$ of the vector bundle $X \oplus \mathbb{C}$, where $\mathbb{C}$ is the trivial line bundle over V. We still denote by π the natural projection of $\overline{X}$ to V. Denote by Z_λ the multiplication by λ on the fibers of X where $\lambda \in \mathbb{C}^*$, i.e. $Z_\lambda(u) := \lambda u$, $u \in X_x$, $x \in V$. This map extends to a holomorphic automorphism of $\overline{X}$. Let V_0 an open subset of V which is naturally identified with an open subset of section 0 on X. A diffeomorphism τ of a neighborhood of V_0 on $\mathcal{V}$ to a neighborhood of V_0 on X is called *allowable* if it satisfies the following essential three conditions: the restriction of τ to V_0 is the identity, the differential of τ at each point $x \in V_0$ is $\mathbb{C}$-linear and the composition of

$$X_x \hookrightarrow \mathrm{Tan}_x(X) \to \mathrm{Tan}_x(\mathcal{V}) \to X_x$$

is the identity, where the morphism $\mathrm{Tan}_x(X) \to \mathrm{Tan}_x(\mathcal{V})$ is given by the differential of τ^{-1} on x and the other maps are canonical, see [14] for details. Note that an allowable map is not necessarily holomorphic. When V_0 is small enough, there are local holomorphic coordinates in a small neighborhood U of V_0 on $\mathcal{V}$ so that over V_0 we naturally identify X with $V_0 \times \mathbb{C}^{k-l}$ and U with an open neighborhood of $V_0 \times \{0\}$ on $V_0 \times \mathbb{C}^{k-l}$ (we reduce U, if necessary). In this image, identity is an allowable holomorphic map. There are always allowable maps for $V_0 := V$. However, such an allowable global map is rarely holomorphic. Consider an allowable map τ as above. Let Q be a (p, p)-positive current closed on $\mathcal{V}$ without mass on V for simplicity. Define $Q_\lambda := (Z_\lambda)_*\tau_*(Q)$. The family (Q_λ) is relatively compact on $\pi^{-1}(V_0)$ when $\lambda \to \infty$: we can extract convergent subsequences $\lambda \to \infty$. The limit currents U are (p, p)-closed positive currents without mass on V that are V-conical , that is, $(Z_\lambda)_*U = U$ for any $\lambda \in \mathbb{C}^*$, in other words, U is invariant by Z_λ. Such a current U depends on the choice of $\lambda \to \infty$ but it is independent of the choice of τ. This property gives us great flexibility to work with allowable maps. In particular, using allowable global maps, we obtain (p, p)-closed positive currents U on $\overline{X}$. It is also known that the class of cohomology of U depends on Q, but it does not depend on the choice of U. This class is denoted by $\kappa^V(Q)$ and is called *the total tangent class* of Q in relation to V. The currents U are called *tangent currents* of Q along V. The mass of U and the norm of $\kappa^V(Q)$ are limited by a constant times the mass of Q. Let $-h$ be the $(1, 1)$-tautological class on $\overline{X}$. Remember that $H^*(\overline{X}, \mathbb{C})$ is a free $H^*(V, \mathbb{C})$-module generated by $1, h, \ldots, h^{k-l}$ (the fibers of $\overline{X}$ are of dimension $k - l$). So, we can write

$$\kappa^V(Q) = \sum_{j=\max(0,l-p)}^{\min(l,k-p)} \pi^*(\kappa_j^V(Q)) \smile h^{j-l+p}$$

where $\kappa_j^V(Q)$ is a class on $H^{l-j,l-j}(V, \mathbb{C})$ with the convention that $\kappa_j^V(Q) = 0$ out of range $\max(0, l-p) \leq j \leq \min(l, k-p)$. Let s be the maximum integer such that $\kappa_s^V(Q) \neq 0$. We call it *tangential h dimension* of Q along V. The class $\kappa_s^V(Q)$ is pseudo-effective, that is, it contains a positive closed current on V. The tangential h dimension of Q is also equal to the maximum integer $s \geq 0$ such that $U \wedge \pi^*(\omega_V^s) \neq 0$, where ω_V is any form of Kähler on V. In addition, if Q_n and Q are (p, p)-positive currents closed on $\mathcal{V}$ such that

$Q_n \to Q$, then $\kappa_j^V(Q_n) \to 0$ for $j > s$ and any limit class of $\kappa_s^V(Q_n)$ is pseudo-effective and is less than or equal to $\kappa_s^V(Q)$. The following result will allow us to limit the number of periodic points isolated from a meromorphic map. We identified here the cohomology group $H^{2k}(\mathcal{V}, \mathbb{C})$ with $\mathbb{C}$ using the integrals differential forms of high degree on $\mathcal{V}$.

Proposition 3.1. *Let Γ_n be complex submanifolds of pure dimension $k-l$ on $\mathcal{V}$. Suppose there is a sequence of positive numbers K_n such that $K_n \to \infty$ and $K_n^{-1}[\Gamma_n]$ converges to a (l,l)-current Q closed positive on $\mathcal{V}$. Suppose also that the h-tangent dimension of Q in relation to V is 0 and that $\{Q\} \smile \{V\} = 1$. So the number δ_n of isolated points at the intersection $\Gamma_n \cap V$, counted with multiplicity, satisfies $\delta_n \leq K_n + o(K_n)$ when $n \to \infty$.*

Proof. We use the Lemma 3.2 in this proof . Define $Q_n := K_n^{-1}[\Gamma_n]$. Extracting a subsequence allows us to assume that $\kappa^V(Q_n)$ converges to a class κ. As the h-tangent dimension of Q is zero, the above discussion implies that $\kappa = \lambda L$, where L is a fiber class of $\overline{E}$ and λ is a positive number. We also have $\kappa^V(Q) = \pi^*(\kappa_0^V(Q))$. In the above construction of $\kappa^V(Q)$ with an allowable global map, we see that Rham's class of cohomology of Q_λ on a neighborhood of $V_0 \times \{0\}$ does not depend on λ when $\lambda \to \infty$. It follows that $\{Q\} \smile \{V\} = \kappa^V(Q) \smile \{V\}$. This together with the hypothesis $\{Q\} \smile \{V\} = 1$ implies that $\kappa^V(Q) = L$. The above discussion about the superior semi-continuity of $\kappa_s^V(Q_n)$ implies that $\lambda \leq 1$. By Lemma 3.2, we can write $\kappa^V(Q_n) = \delta_n K_n^{-1} L + L_n$ where L_n is a pseudo-effective class. As $\kappa^V(Q_n)$ converges to $\kappa = \lambda L$, we deduce that the limit values of L_n are also equal to positive constants times L and then $\limsup \delta_n K_n^{-1} \leq \lambda$. The proposition follows. ■

We have used the following lemma.

Lemma 3.2 ([30]). *Let Γ be a submanifold of pure dimension $k-l$ on $\mathcal{V}$. Let $x_1, \ldots, x_N$ be the isolated points on $\Gamma \cap V$ and m_i be the multiplicity of the intersection of $\Gamma \cap V$ on x_i. So, any tangent current of $[\Gamma]$ along V is greater than or equal to $\sum m_i[\pi^{-1}(x_i)]$.*

Proof. Consider a small open set Z_0 on V that contains only one point x_i. As above, we identified E (resp. $\overline{E}$) on Z_0 with $Z_0 \times \mathbb{C}^{k-l}$ (resp. $Z_0 \times \mathbb{P}^{k-l}$), and a small neighborhood of Z_0 on $\mathcal{V}$ with an open neighborhood of $Z_0 \times \{0\}$ on $Z_0 \times \mathbb{C}^{k-l}$, and π with the canonical projection of $Z_0 \times \mathbb{P}^{k-l}$ on its first factor. Identity is then an allowable map. It is clear in this context that any tangent current of $[\Gamma]$ along V constructed as above is greater than or equal to $m_i[\pi^*(x_i)]$. The lemma follows. ■

4 Branches Constructing

Note that this section is a generalization of the construction in [2] , [30] .
Let $h : \mathcal{V} \to \mathcal{V}$ and ν_h be as in Theorem 1.9. The purpose of this section is to construct for generic small balls an almost maximum number of inverse branches in relation to h^n and that we control the size. Remember that $J, J', d_p, \chi_{2k}, \Gamma$ denote the sets of indeterminacy, the dynamic degree of order p, the cohomological degree and the closing of the graph of h on $\mathcal{V} \times \mathcal{V}$. By definition, the dynamic degree of order p and the cohomological degree of h^n are equal to d_p^n and χ_{2k}^n respectively. Denote by $J(h^n), J'(h^n), \Gamma_n$ the indeterminacy sets and the closing of the graph of h^n. Natural projections of $\mathcal{V} \times \mathcal{V}$ in their factors are denoted by π_1 and π_2. Also remember that $J'_0 := J'$, $J'_{n+1} := J'_0 \cup h(J'_n)$ for $n \geq 0$ and $J'_\infty := \cup_{n\geq 0} J'_n$. One must distinguish J'_∞ with the set $\cup_{n\geq 0} h^n(J')$ and the union of $J'(h^n)$ which are a priori minor. Choose an analytical subset Σ_0 of $\mathcal{V}$ containing $J, J', h(J), h^{-1}(J')$ such that π_2 restricted to $\Gamma \setminus \pi_2^{-1}(\Sigma_0)$ defines an unrestricted covering over $\mathcal{V} \setminus \Sigma_0$. Let Z be a connected subset of $\mathcal{V}$, for example, a holomorphic ball, a holomorphic disc or a family of discs through a point on $\mathcal{V}$. We call *an inverse branch of order n* of Z any continuous bijective map $f : Z \to Z_{-n}$ with $Z_{-n} \subset \mathcal{V}$ such that if we define $Z_{-i} := h(Z_{-i-1})$ with $0 \leq i \leq n-1$, then $Z_{-i} \cap \Sigma_0 = \varnothing$ for $0 \leq i \leq n$, $h : Z_{-i} \to Z_{-i+1}$ is a bijective map for $1 \leq i \leq n$, $Z_0 = Z$ and $h^n \circ f = \mathrm{id}$ on Z. Note that $h^{n-i} \circ g : Z \to Z_{-i}$ is an inverse branch of order i of Z and Z admits at most χ_{2k}^n inverse branches of order n. The condition $B_{-i} \cap \Sigma_0 = \varnothing$ implies that the inverse branch can be extended to any open set small enough containing Z using local inverses of the map h^n. We say that the inverse branch above is *of size smaller than* λ if the diameter of Z_{-n} is smaller than λ. We also call Z_{-n} *the image* of the inverse branch $f : Z \to Z_{-n}$.

Proposition 4.1 ([2], [30]). *There is a $(1,1)$-closed positive current U on $\mathcal{V}$ satisfying the following property. Let ε, ν be strictly positive numbers with $\nu \leq 1$ and let x be a point on $\mathcal{V}$ so that Lelong's number $\nu(U, x)$ of U on x is smaller than ν. So there is a constant $r > 0$ such that the ball $B(x, r)$ with center x and radius r admit at least $(1-\nu)\chi_{2k}^n$ inverse branches of order n and smaller in size than $(\chi_{2k-1}/\chi_{2k} + \varepsilon)^{n/2}$ for each $n \geq 0$.*

Proof. A Siu theorem says that $\{\nu(U,x)\geq c\}$ is a proper analytical subset of $\mathcal{V}$ for each $c>0$ [45]. Thus, the above proposition applies to generic points x on $\mathcal{V}$. In the construction of U, we will see that the set $\{\nu(U,x)>0\}$ contains the orbits of critical values and points of indeterminacy that are obviously an obstruction to obtain inverse branches of balls. However, $\{\nu(U,x)>0\}$ contains a priori other analytic sets which are a less obvious obstruction to the existence of inverse branches. It can be seen as a local of accumulation of the orbits of the points of indeterminacy. We now give the proof of the above proposition. The first step is to define the current U. Remember that operators $(h^n)_*$ act continuously on currents $(1,1)$ positive closed and these actions are compatible with the actions of $(h^n)_*$ on $H^{1,1}(\mathcal{V},\mathbb{R})$. If Q is a $(1,1)$-positive current closed on $\mathcal{V}$, its mass depends only on the class of cohomology $\{Q\}$. Therefore, for a fixed norm in cohomology, the mass of Q is comparable to the norm of $\{Q\}$. We then deduce the existence of a constant $L_0>0$ independent of Q, h and n so that

$$\|(h^n)_*(Q)\|\leq L_0\|(h^n)_*\|_{H^{1,1}(\mathcal{V},\mathbb{R})}\|Q\|.$$

By definition of χ_{2k-1}, one can fix an integer $N\geq 1$ large enough such that $L_0\|(h^N)_*\|_{H^{1,1}(\mathcal{V},\mathbb{R})}<(\Upsilon\chi_{2k})^N$, where $\chi_{2k-1}/\chi_{2k}<\Upsilon<\chi_{2k-1}/\chi_{2k}+\varepsilon$ is any fixed constant strictly less than 1. Define $\Sigma_{i+1}:=h(\Sigma_i)$ for $0\leq i\leq N-1$ and $\Sigma:=\cup_{0\leq i\leq N}\Sigma_i$. Therefore, any connected and simply connected set outside Σ admits the maximum number χ_{2k}^N of inverse branches of order N with images outside Σ_0. Choose a desingularization $\pi:\widehat{\Gamma}\to\Gamma$ which is a composition of explosions of Γ along smooth centers on or over $\pi_1^{-1}(\Sigma)\cap\Gamma$ and $\pi_2^{-1}(\Sigma)\cap\Gamma$. Define $\tau_i:=\pi_i\circ\pi$. One can choose π so that $\tau_1^{-1}(\Sigma)$ and $\tau_2^{-1}(\Sigma)$ are pure codimension 1 on $\widehat{\Gamma}$. By Blanchard's theorem [9], $\widehat{\Gamma}$ is a compact variety of Kähler . Fix a form of Kähler $\widehat{\omega}$ on $\widehat{\Gamma}$ that is bigger than $\tau_i^*(\omega)$. We also assume that $\widehat{\omega}$ is large enough that each ball of radius 1 on $\widehat{\Gamma}$ in relation to the metric $\widehat{\omega}$ biholomorph to $\mathbb{C}^k$. Denote by Σ' and Σ'' respectively, the union of codimension components 1 and the union of the components of codimension ≥ 2 of Σ. Define

$$P_0:=L_1\Upsilon^{-N}\chi_{2k}^N\big([\Sigma']+(\tau_2)_*(\widehat{\omega})\big),\qquad P:=\sum_{n=0}^{N}(h_*)^n(P_0)$$

and

$$U:=8\sum_{m\geq 0}(\Upsilon\chi_{2k})^{-mN}(h^N)_*^m(P).$$

Here $L_1\geq\delta_1^{-1}$ is a constant satisfying Lemma 4.2 below and δ_1 is a constant whose exact value will be determined after the Lemma 4.4 below. By definition of Υ, the last current is well defined. Note that one must distinguish operators $(h^N)_*^m$ and $(h^{Nm})_*$. The orbit of Σ is the obstruction to build inverse branches of balls. Now we apply Lemma 4.6 and the Proposition 4.3 for $\delta_0=\nu/4$. Let $x_{-n}^{(1)},\ldots,x_{-n}^{(s)}$ with $0\leq s\leq\chi_{2k}^n$ the distinct points on $h^{-n}(x)$ such that $h^i(x_{-n}^{(j)})\notin\Sigma$ for all $0\leq i\leq n$ and $1\leq j\leq s$. If $f:\Delta_{r_0}\to\Delta_{r_0,-n}$ is an inverse branch as in the Lemma 4.6, then $\Delta_{r_0,-n}$ contains exactly one of the points $x_{-n}^{(j)}$. We say that $x_{-n}^{(j)}$ is the center of $\Delta_{r_0,-n}$. Denote by $\mathcal{F}^{(j)}$ the set of $\Delta\in\mathcal{F}$ such that Δ_{r_0} admits an inverse branch of order n as in the Lemma 4.6 with center $x_{-n}^{(j)}$. Let P_n be the set of all j such that $\mathcal{L}(\mathcal{F}^{(j)})\geq\nu/4$. Let j be an element of P_n. We show that $B(x,r)$ admits an inverse branch of order n of size $\leq\Upsilon^{n/2}$ with center $x_{-n}^{(j)}$ for a suitable constant $r>0$ independent of n. Let $T^{(j)}$ be the intersection of $\mathcal{F}^{(j)}$ with $B(x,r_0)$. The inverse branches of Δ_{r_0} with $\Delta\in\mathcal{F}^{(j)}$ with images centered on $x_{-n}^{(j)}$ agree at the common intersection point x and form a map $f:T^{(j)}\to T_{-n}^{(j)}$ where $T_{-n}^{(j)}$ is the union of $\Delta_{r_0,-n}$ centered on $x_{-n}^{(j)}$ with $\Delta\in\mathcal{F}^{(j)}$. Define as above $x_{-i}^{(j)}:=h^{n-i}(x_{-n}^{(j)})$, $T_{-i}^{(j)}:=h^{n-i}(T_{-n}^{(j)})$ for $0\leq i\leq n$ and $\widehat{x}_{-i}^{(j)}:=\tau_1^{-1}(x_{-i-1}^{(j)})$, $\widehat{T}_{-i}^{(j)}:=\tau_1^{-1}(T_{-i-1}^{(j)})$ for $0\leq i\leq n-1$. By Lemma 4.6, we have $T_{-i}^{(j)}\cap\Sigma=\varnothing$ for $0\leq i\leq n$. So the map $\tau_1^{-1}\circ h^{n-i-1}\circ g$ extends holomorphically to a neighborhood of $T^{(j)}$. In addition, the image of $T^{(j)}$ is equal to $\widehat{T}_{-i}^{(j)}$ that is contained in the ball of radius $\frac{1}{2}\Upsilon^{i/2}\leq 1$ centered on $\widehat{x}_{-i}^{(j)}$ and does not intercept the hypersurface $\tau_1^{-1}(\Sigma)\cup\tau_2^{-1}(\Sigma)$. Remember that the metric $\widehat{\omega}$ on $\widehat{\Gamma}$ is chosen so that any ball of radius 1 is contained on an open set bihomorphic to a ball on $\mathbb{C}^k$. So Proposition 4.3 can be applied to this map. According to this proposition, for r small enough (equal to r_0 times a constant independent of n,i,j), all maps $h^{n-i}\circ f$ and $\tau_1^{-1}\circ h^{n-i-1}\circ f$ extend holomorphically to $B(x,r)$. Furthermore, his images, denoted by $B(x,r)_{-i}^{(j)}$ and $\widehat{B}(x,r)_{-i}^{(j)}$ respectively, have diameters less than or equal to $\Upsilon^{i/2}$. We also have $\widehat{B}(x,r)_{-i}^{(j)}\cap\tau_1^{-1}(\Sigma)=\varnothing$ and $\widehat{B}(x,r)_{-i}^{(j)}\cap\tau_2^{-1}(\Sigma)=\varnothing$ for $0\leq i\leq n-1$. It follows that $B(x,r)_{-i}^{(j)}\cap\Sigma=\varnothing$ for $0\leq i\leq n$. Then the extension of g defines an inverse branch of order n and size $\leq\Upsilon^{n/2}$ on $B(x,r)$. It now remains to show that P_n contains at least $(1-\nu)\chi_{2k}^n$ elements. By Lemma

4.6, we have $\sum_j \mathcal{L}(\mathcal{F}^{(j)}) \geq \chi^n_{2k}(1-\nu/2)^2$. Since $\mathcal{L}(\mathcal{F}^{(j)}) \leq \mathcal{L}(\mathcal{F}) = 1$, we deduce that the last sum is limited by $\#P_n + (\chi^n_{2k} - \#P_n)\nu/4$. It follows that $\#P_n + (\chi^n_{2k} - \#P_n)\nu/4 \geq \chi^n_{2k}(1-\nu/2)^2$. Thus, $\#P_n \geq (1-\nu)\chi^n_{2k}$. This completes the proof of the proposition. ■

The following Lemma shows that it is visible using the current U.

Lemma 4.2. *If L_1 is big enough, so for each $x \in \Sigma$ Lelong's number $\nu(P_0, x)$ of P_0 on x is bigger than* 1.

Proof. The Lemma is clear for $x \in \Sigma'$. Now consider a point $x \in \Sigma'' \setminus \Sigma'$. As the function $\nu(P_0, x)$ is semi-continuous superior on x regarding Zariski's topology, just check that $\nu(P_0, x)$ is positive on generic points $x \in \Sigma'' \setminus \Sigma'$. We then choose L_1 large enough to obtain Lelong numbers greater than 1. Therefore, we can assume that x is a regular point of $\Sigma'' \setminus \Sigma'$ and there is a point $\widehat{x} \in \tau_2^{-1}(x)$ such that $\tau_2^{-1}(\Sigma'')$ is a smooth hypersurface on $\widehat{x}$. Choose local coordinates $\widehat{z} = (\widehat{z}_1, \ldots, \widehat{z}_k)$ on a neighborhood of $\widehat{x}$ such that $\widehat{z} = 0$ on $\widehat{x}$ and $\tau_2^{-1}(\Sigma'')$ is given by $\widehat{z}_1 = 0$. As Σ'' has codimension ≥ 2, we can choose $\widehat{z}$ so that the hyperplanes $\widehat{H}_\xi := \{\widehat{z}_k = \xi\}$ parallel to $\{\widehat{z}_k = 0\}$ are sent to hypersurfaces, denoted by H_ξ, that contain Σ'' on a neighborhood of x. The average of $[\widehat{H}_\xi]$ with respect to the Lebesgue measure on ξ is a smooth form $\widehat{\Theta}$. So it is limited by a constant times $\widehat{\omega}$. On the other hand, as $[H_\xi]$ has a positive Lelong number on x, $(\tau_2)_*(\widehat{\Theta})$ has a positive Lelong number on x. We concluded that $(\tau_2)_*(\widehat{\omega})$ has a positive Lelong number on x. This completes the proof of the lemma. ■

We show that U satisfies the Proposition 4.1. Fix a point x on $\mathcal{V}$ such that $\nu(U, x) \leq \nu$. Also fix local holomorphic coordinates close to x. First, construct inverse branches for flat holomorphic disks through x and then we will extend these inverse branches to a small ball centered on x. We will identify a neighborhood of x for the unit ball on $\mathbb{C}^k$ for simplicity. The following version of the Sibony-Wong theorem is the tool for this extension. Let B_r be the ball of center 0 and radius r on $\mathbb{C}^k$. The family $\mathcal{F}$ of complex lines through 0 is parameterized by the projective space $\mathbb{P}^{k-1}$, which is endowed with the natural Fubini-Study metric. This metric induces a natural probability measure on $\mathcal{F}$ that we denote by $\mathcal{L}$. If Δ is an element of $\mathcal{F}$, denote by Δ_r its intersection with B_r.

Proposition 4.3 ([2], [30]). *Let $0 < \delta_0 \leq 1$ be a constant. Let $\mathcal{F}' \subset \mathcal{F}$ be such that $\mathcal{L}(\mathcal{F}') \geq \delta_0$, and A the intersection of $\mathcal{F}'$ with B_r. Let $h : A \to \mathbb{C}^l$ be a map that is holomorphic on each Δ_r for $\Delta \in \mathcal{F}'$ and that can be extended holomorphically to a neighborhood of* 0*. Then h can be extended to a holomorphic map of $B_{\lambda r}$ for $\mathbb{C}^l$, where $0 < \lambda \leq 1$ is a constant that depends on δ_0 but independent of l, $\mathcal{F}'$ e r. Furthermore, if the extension is still denoted by h then*

$$\sup_{B_{\lambda r}} \|h - h(0)\| \leq \sup_A \|h - h(0)\|.$$

In particular, if $\|h - h(0)\| < \rho$ and if $h(A)$ does not intercept a complex hypersurface Z the ball of center $h(0)$ and radius ρ, then $h(B_{\lambda r})$ satisfies the same property.

Proof. For $l = 1$ a similar result is due to Sibony-Wong [48]. One can deduce from their result the holomorphic extension of h for any dimension l. To obtain inequality in the proposition, assume $h(0) = 0$ for simplicity. Let z be a point on $B(0, \lambda r)$ we have to show that $\|h(z)\| \leq \sup_A \|h\|$. Composing h with a rotation on $\mathbb{C}^l$ lets assume that $h(z) = (\|h(z)\|, 0, \ldots, 0)$. One obtains the desired inequality using the Sibony-Wong theorem for the first function of coordinates of h. For the last statement, one can write $Z = \{f = 0\}$ where f is a holomorphic function on the ball of center $h(0)$ and radius ρ. The Sibony-Wong theorem applied to $1/f \circ h$ implies that $1/f \circ h$ is holomorphic on $B_{\lambda r}$. Thus $h(B_{\lambda r})$ does not intercept Z. The proposition follows. ■

To control the size of holomorphic discs, one needs the following lemma, see [15] for the case of any compact complex manifold C.

Lemma 4.4. *Let C be a compact complex manifold with a fixed Hermitian metric. Let $\delta_1 > 0$ be a small enough constant, depending on C. Let $f : \Delta_r \to C$ be a holomorphic map of the center disc* 0 *and radius r on $\mathbb{C}$ for C. Suppose the area of $f(\Delta_r)$, counted with multiplicity, is less than δ_1. So, for any $\varepsilon > 0$, there is a constant $0 < \lambda < 1$ independent of f and r such that the diameter of $f(\Delta_{\lambda r})$ is at most equal to $\varepsilon\sqrt{\text{area}(f(\Delta_r))}$.*

We will apply it to $C = \widehat{\Gamma}$. So from now on, we fix a constant δ_1 satisfying the last lemma for $\widehat{\Gamma}$. Note that the current U constructed above can be seen as obstructing the existence of good inverse branches for balls in the spirit of the Proposition 4.1. To measure the obstruction of the inverse branches of the disks through a point x, we must divide this current using complex lines through x. We will need the following known technical result, cf [15].

Lemma 4.5. *Let Q be a $(1,1)$-positive current closed on $\mathcal{V}$. So, for any constant $0 < \delta_2 < 1$ there is a constant $r > 0$ and a family $\mathcal{F}' \subset \mathcal{F}$, such that $\mathcal{L}(\mathcal{F}') \geq 1 - \delta_2$, and for each $\Delta \in \mathcal{F}'$ the measure $Q \wedge [\Delta_r]$ is well defined and with a mass less than or equal to $\nu(Q, x) + \delta_2$. Here, $\nu(Q, x)$ indicates the Lelong number of Q on x.*

Remember that we can write locally $Q = dd^c q$ with q a p.s.h. function . The measure $Q \wedge [\Delta_r]$ is well defined if q is not identically $-\infty$ on Δ_r. This property is valid for $\mathcal{L}$-almost every Δ and we have $Q \wedge [\Delta_r] := dd^c(q[\Delta_r])$. Now we are ready to construct inverse branches for disks through the point x under the assumptions of the Proposition 4.1. Fix a constant value $c_1 \geq \delta_1^{-1}$ in the definition of P satisfying the lemma 4.2. We have the following lemma.

Lemma 4.6 ([2], [30])**.** *There are a number $r_0 > 0$ and a family $\mathcal{F}_0 \subset \mathcal{F}$ with $\mathcal{L}(\mathcal{F}_0) \geq 1 - \nu/2$ satisfying the following property. For each complex line $\Delta \in \mathcal{F}_0$ and for each $n \geq 0$, the disk Δ_{r_0} admits at least $(1-\nu/2)\chi_{2k}^n$ inverse branches $f : \Delta_{r_0} \to \Delta_{r_0,-n}$ of order n so that if we define $\Delta_{r_0,-i} := h^{n-i}(\Delta_{r_0,-n})$ for $0 \leq i \leq n$ and $\widehat{\Delta}_{r_0,-i} := \tau_1^{-1}(\Delta_{r_0,-i-1})$ for $0 \leq i \leq n-1$, then $\Delta_{r_0,-i} \cap \Sigma = \varnothing$ for $0 \leq i \leq n$ and the diameters of $\widehat{\Delta}_{r_0,-i}$ for $0 \leq i \leq n-1$ are smaller than $\frac{1}{2}\Upsilon^{i/2}$.*

Note that as $\Delta_{r_0,-i} \cap \Sigma = \varnothing$ for $0 \leq i \leq n$, τ_1 defines a biholomorphic map between $\widehat{\Delta}_{r_0,-i}$ and $\Delta_{r_0,-i-1}$ and τ_2 defines a biholomorphic map between $\widehat{\Delta}_{r_0,-i}$ and $\Delta_{r_0,-i}$. Furthermore, as $\widehat{\omega} \geq \tau_1^*(\omega)$ and $\widehat{\omega} \geq \tau_2^*(\omega)$, the diameter of $\widehat{\Delta}_{r_0,-i}$ is greater than or equal to the diameters of $\Delta_{r_0,-i-1}$ and of $\Delta_{r_0,-i}$. So the diameter of $\Delta_{r_0,-i}$ is smaller than $\frac{1}{2}\Upsilon^{i/2}$ for $0 \leq i \leq n$.

Proof. Note that if f is an inverse branch of order n that satisfies the properties of the lemma, so $h \circ f$ is an inverse branch of order $n-1$ that satisfies the same properties. So, we just have to prove the lemma to $n = mN$ where m is an integer. By Lemma 4.4, we just need to limit the area $\widehat{\Delta}_{r_0,-i}$ by $\Upsilon^i/c_1 \leq \delta_1$ and then reduce the radius r_0 to get the control of the diameter . The rest of the lemma is obtained by induction on m. We will only consider disks Δ_r through x that are not contained in the orbit of Σ. This property applies to almost all discs. By Lemma 4.5 applied to U and for $\delta_2 := \nu/2$, we can choose a number r and a family $\mathcal{F}_0 \subset \mathcal{F}$ with $\mathcal{L}(\mathcal{F}_0) \geq 1 - \nu/2$ such that for $\Delta \in \mathcal{F}_0$ the measure $U \wedge [\Delta_r]$ is well defined and less than 2ν. Let ν_m be the mass of $\chi_{2k}^{-Nm}(h^N)_*^m(P) \wedge [\Delta_r]$. By definition of U, we have $\sum_{m\geq 0} \Upsilon^{-Nm}\nu_m \leq \nu/4$. By definition of m, for each $\Delta \in \mathcal{F}_0$ the disk Δ_r admits at least $\gamma_m := (1 - 2\sum_{0\leq i<m} \Upsilon^{-Ni}\nu_i)\chi_{2k}^{Nm}$ inverse branches $f_{-n}^{(s)} : \Delta_r \to \Delta_{r,-Nm}^{(s)}$ of order Nm so that the area of $\widehat{\Delta}_{r,-i}^{(s)}$ is smaller than Υ^i/c_1 and $\Delta_{r,-i}^{(s)} \cap \Sigma = \varnothing$ for $0 \leq i \leq Nm$. Here we use notation similar to that introduced in the lemma of the statement. The index s satisfies $1 \leq s \leq s_m$ for some integer s_m with $\gamma_m \leq s_m \leq \chi_{2k}^{Nm}$. So the above discussion implies the result. Assume this property for m. The case $m = 0$ is clear since the choice of $\mathcal{F}_0$ implies that Δ_r is out of Σ. We construct inverse branches of order $N(m+1)$. The property $\Delta_{r,-Nm}^{(s)} \cap \Sigma = \varnothing$ and the definition of Σ allows us to define the maximal number χ_{2k}^N inverse branches of order N for each $\Delta_{r,-Nm}^{(s)}$. Composing them with inverse branches of order Nm of Δ gives $\gamma_m\chi_{2k}^N$ inverse branches of order $N(m+1)$ for Δ. We will count and remove those which do not satisfy the area control. We call them *large size inverse branches*. We also have to remove posterior inverse branches whose images intersect Σ. First, we count the number of large size inverse branches of order N of $\Delta_{r,-Nm}^{(s)}$ for each s. For the matter of simplicity, we drop the letter s at the moment. Consider all inverse branches $g : \Delta_{r,-Nm} \to \Delta_{r,-Nm-n}$ of order $1 \leq n \leq N$ of $\Delta_{r,-Nm}$ so that the area of $\widehat{\Delta}_{r,-Nm-i}$ is limited by Υ^{Nm+i}/c_1 for $i \leq n-1$ but not for $i = n$. They are completely disjoint in the sense that these two different branches are not extensions of the same lower-order branch of $\Delta_{r,-Nm}$. The extensions of order $N(m+1)$ of these branches are exactly those of large size. Therefore, the number of large branches of order $N(m+1)$ extending g is χ_{2k}^{N-n}. Note that the area $\widehat{\Delta}_{r,-Nm-n}$ is the mass of $[\Delta_{r,-Nm-n}] \wedge (\tau_2)_*(\widehat{\omega})$. This mass is less than or equal to the mass of $[\Delta_{r,-Nm}] \wedge (h_*)^n(\tau_2)_*(\widehat{\omega})$ because h^n defines a biholomorphic map of a neighborhood of $\Delta_{r,-Nm-n}$ for a neighborhood of $\Delta_{r,-Nm}$. Therefore, the sum of these areas in all these branches f (exist at most χ_{2k}^N of such maps) is limited by $c_1^{-1}\Upsilon^N$ times the mass of $P \wedge [\Delta_{r,-Nm}]$. As the area of $\widehat{\Delta}_{r,-Nm-n}$ is bigger than Υ^{Nm+n}/c_1, the number of maps considered f is at most equal to Υ^{-Nm} times the mass of

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$P \wedge [\Delta_{r,-Nm}]$. The number of large inverse branches of order N of $\Delta_{r,-Nm}$ to remove is at most equal to $\Upsilon^{-Nm}\chi_{2k}^N$ times the mass of $P \wedge [\Delta_{r,-Nm}]$. Now the number M of all large size inverse branches of order $N(m+1)$ of Δ_r to remove is limited by $\Upsilon^{-Nm}\chi_{2k}^N$ times the mass of $\sum_s P \wedge [\Delta^{(s)}_{r,-Nm}]$. By the definition of inverse branches, the last mass is limited by one of the $(h^N)^m_*(P) \wedge [\Delta_r]$ which is equal to $\chi_{2k}^{Nm}\nu_m$. We concluded that $M \leq \Upsilon^{-Nm}\chi_{2k}^{N(m+1)}\nu_m$. Therefore, the number of inverse branches of order $N(m+1)$ satisfying the control of the area is greater than or equal to $\gamma_m\chi_{2k}^N - M \geq \gamma_{m+1} + \nu_m\chi_{2k}^{N(m+1)}$. It remains to remove the inverse branches whose images intercept Σ. Denote by t_{m+1} the number of inverse branches $f : \Delta_r \to \Delta_{r,-N(m+1)}$ of order $N(m+1)$ such that $\Delta_{r,-Nm-i} \cap \Sigma \neq \varnothing$ for some $1 \leq i \leq N$. By Lemma 4.2, the intersection of $[\Delta_{r,-Nm-i}]$ with the current S_0 is a positive measure of mass at least equal to 1. By definition of inverse branches, the map h^n is holomorphic and injector on a neighborhood of $\Delta_{r,-n}$ with image on a neighborhood of Δ_r for each $n \leq N(m+1)$. We deduce then that the mass of $(h^N)^m_*(P) \wedge [\Delta_r]$ is at least equal to t_{m+1}. It follows that $t_{m+1} \leq \nu_m\chi_{2k}^{Nm}$. We conclude that the number of inverse branches of order $N(m+1)$ satisfying the properties of the Lemma is at least equal to γ_{m+1}. This completes the proof of the Lemma. ■

5 Proof of Theorem 2.2

In this section, we give the proof of Theorem 2.2. In what follows, we only consider points x outside of $J_\infty \cup J'_\infty$.

Proof. Let x be a point out of $J_\infty \cup J'_\infty$. If x is on $\mathcal{T}$, is clear that ν_n^x does not converge to ν. So assume that $x \notin \mathcal{T}$. We have to show that $\varsigma_x = 0$. Fix a constant $\nu > 0$. Just prove that $\varsigma_x \leq 4\nu$. Define $F := \{\nu(U,\cdot) \geq \nu\}$. By Siu's theorem, this is a proper analytical subset of $\mathcal{V}$. By Lemma 5.2, the case where $x \notin F$ is clear. From now on, assume that $x \in F$. By Lemma 5.4 applied to F, we have $\lambda_m(x) \leq \nu\chi_{2k}^m$ for some integer m big enough. Consider all the backward orbits of x of order $l \leq m$ of form

$$\mathcal{O} := \{x_0, x_{-1}, \dots, x_{-l}\} \quad \text{with} \quad x_0 = x, h(x_{-i-1}) = x_{-i} \text{ for } 0 \leq i \leq l-1$$

such that $x_{-i} \in F$ for $i \leq l-1$ and $x_{-l} \notin F$ unless $l = m$. These orbits are counted with multiplicity. Using that ν_n^x is the measure of probability equidistributed on $h^{-n}(x)$, one can see that

$$\nu_n^x = \sum_{\mathcal{O}} \chi_{2k}^{-l}\nu_{n-l}^{x_{-l}}$$

for each $n \geq m$. When considering the masses of measures in the above identity, we have

$$\sum_{\mathcal{O}} \chi_{2k}^{-l} = 1.$$

We deduce from the same identity that

$$\varsigma_x \leq \sum_{\mathcal{O}} \chi_{2k}^{-l}\varsigma_{x_{-l}}.$$

Let Σ, Σ' be the sets of $\mathcal{O}$ with $x_{-l} \in F$ (consequently $l = m$) and with $x_{-l} \notin F$ respectively. By definition of λ_n, we have

$$\sum_{\mathcal{O}\in\Sigma} \chi_{2k}^{-l} = \chi_{2k}^{-m}\lambda_m(x) \leq \nu.$$

On the other hand, we have $\varsigma_{x_{-l}} \leq 2$ for each $\mathcal{O}$ and by Lemma 5.2, $\varsigma_{x_{-l}} \leq 2\nu$ for $\mathcal{O} \in \Sigma'$. It follows that

$$\varsigma_x \leq \sum_{\mathcal{O}\in\Sigma} \chi_{2k}^{-l}\varsigma_{x_{-l}} + \sum_{\mathcal{O}\in\Sigma'} \chi_{2k}^{-l}\varsigma_{x_{-l}} \leq 2\sum_{\mathcal{O}\in\Sigma} \chi_{2k}^{-l} + 2\nu \sum_{\mathcal{O}\in\Sigma'} \chi_{2k}^{-l} \leq 2\nu + 2\nu = 4\nu.$$

This completes the proof of the theorem. ■

We need the following result obtained in [13] in a more general scenario.

Lemma 5.1. *There is a pluripolar subset E of $\mathcal{V}$ containing J'_∞ so that if x is out of E, then ν_n^x converges to ν when n goes to infinity.*

For each $x \notin J'_\infty$, define $\varsigma_x := \sup \|\nu^x - \nu\|$, where the supreme is assumed over all cluster values ν^x of the sequence ν_n^x. So we have $\nu_n^x \to \nu$ if and only if $\varsigma_x = 0$. We deduce from the above lemma and proposition 4.1 the following property.

Lemma 5.2. *Let x be a point out of J'_∞. So we have $\varsigma_x \leq 2\nu(U,x)$. In particular, $\nu^x_m \to \nu$ if $\nu(U,x) = 0$.*

Proof. We saw that the condition $x \notin J'_\infty$ is needed to define ν^x_m. As always we have $\varsigma_x \leq 2$, we just need to consider the case where $\nu(Q,x) < 1$. Fix a constant $\nu(U,x) < \nu \leq 1$. Let $B(x,r)$ be a ball as in the conclusion of the Proposition 4.1. Also choose a point y on $B(x,r) \setminus E$. This choice is always possible, since E is pluripolar. Write

$$h^{-m}(x) = x^{(1)}_{-m}, \ldots, x^{(\chi^m_{2k})}_{-m}$$

and

$$h^{-m}(y) = y^{(1)}_{-m}, \ldots, y^{(\chi^m_{2k})}_{-m}$$

where each point is repeated according to its multiplicity. The Proposition 4.1 implies that we can organize these points so that the distance between $x^{(j)}_{-m}$ and $y^{(j)}_{-m}$ be less than $(\chi_{2k-1}/\chi_{2k} + \varepsilon)^{m/2}$ for at least $(1-\nu)\chi^m_{2k}$ indexes j. Here $\varepsilon > 0$ is a fixed constant small enough. Since $(\chi_{2k-1}/\chi_{2k} + \varepsilon)^{m/2}$ tends to 0, we deduce that any cluster value in the sequence $\nu^x_m - \nu^y_m$ is a measure of mass at most equal to 2ν. The property is valid for all $\nu > \nu(U,x)$. Therefore, the result follows from Lemma 5.1 which implies that $\nu^y_m \to \nu$. ■

The set $\mathcal{T}$ in Theorem 2.2 is given in the following proposition.

Proposition 5.3. *There is a proper analytical subset $\mathcal{T}$ of $\mathcal{V}$, possibly empty, satisfying the following three conditions:*
(1) no component of $\mathcal{T}$ is contained on $J_\infty \cup J'_\infty$;
(2) $h^{-1}(\mathcal{T} \setminus J') \subset \mathcal{T}$;
(3) any proper analytic subset of $\mathcal{V}$ satisfying (1) e (2) is contained on $\mathcal{T}$. In addition, we have

$$\mathcal{T} = \overline{h^{-1}(\mathcal{T} \setminus J')} = \overline{h(\mathcal{T} \setminus J)}.$$

Proof. Consider the set $F_0 := \{\nu(U,x) \geq 1\}$. By Siu's theorem, F_0 is a proper analytic subset of $\mathcal{V}$ [45]. Denote by $n \geq 1$ the analytical set F_n which is the closing of the set

$$\{z \notin J_\infty \cup J'_\infty, \quad h^{-i}(z) \in F \text{ for } 0 \leq i \leq n\}.$$

Since the sequence F_n is decreasing, is stationary: we have $F_n = F_{n+1}$ for n big enough. Denote by $\mathcal{T} := F_n$ for n big enough. Is clear that $\mathcal{T}$ satisfies the property (1). We have by definition

$$\overline{h^{-1}(\mathcal{T} \setminus (J_\infty \cup J'_\infty))} \subset \mathcal{T}.$$

As $h^{-1}(\mathcal{T} \setminus (J_\infty \cup J'_\infty))$ is dense on $h^{-1}(\mathcal{T} \setminus J')$, the set $\mathcal{T}$ satisfies (2). Denote by $\mathcal{T}_n$ the closing of $h^{-n}(\mathcal{T} \setminus (J_\infty \cup J'_\infty))$. This is a decreasing sequence of analytical sets that satisfy the property (1). So it's stationary: we have $\mathcal{T}_n = \mathcal{T}_{n+1}$ for m big enough. As $h(\mathcal{T}_{n+1} \setminus J)$ is dense on $\mathcal{T}_n$, we deduce from the last identity that $\mathcal{T}_{n-1} = \mathcal{T}_n$ and therefore, by induction, $\mathcal{T}_1 = \mathcal{T}$. It follows that $\mathcal{T} = \overline{h^{-1}(\mathcal{T} \setminus J')}$ which also implies that $\mathcal{T} = \overline{h(\mathcal{T} \setminus J)}$. Let $\mathcal{T}'$ be a proper analytic subset of $\mathcal{V}$ satisfying (1) and (2). We have to show that $\mathcal{T}' \subset \mathcal{T}$. Property (2) implies that if x is a point on $\mathcal{T}' \setminus J'_\infty$ then any cluster value of ν^x_n is supported by $\mathcal{T}'$. As ν has no mass on $\mathcal{T}'$, we deduce that $\varsigma_x = 2$. Lemma 5.2 implies that x is on F_0. So we have $\mathcal{T}' \subset F_0$. Property (2) also, together with the definition of $\mathcal{T}$, implies that $\mathcal{T}' \subset \mathcal{T}$. This completes the proof of the proposition. ■

We need the following characterization of the set $\mathcal{T}$.

Lemma 5.4. *Let F be a proper analytic subset of $\mathcal{V}$. Let x be a point on F that doesn't belong to $J_\infty \cup J'_\infty$. Let $\lambda_n(x)$ be the number of orbits backwards $x_0, x_{-1}, \ldots, x_{-n}$ counted with multiplicity with $x_0 = x$, $x_{-i-1} \in h^{-1}(x_{-i})$ for $0 \leq i \leq n-1$ and $x_{-i} \in F$ for $0 \leq i \leq n$. If x is not on $\mathcal{T}$ then $\chi^{-n}_{2k}\lambda_n(x) \to 0$ when $n \to \infty$.*

Proof. Note that, as x is out of $J_\infty \cup J'_\infty$ the same property applies to x_{-i}. By definition, the sequence $\chi^{-n}_{2k}\lambda_n(x)$ is decreasing because each orbit backwards of order $n+1$ is one of χ_{2k} orbit extensions backwards of order n. As the functions λ_n are semi-continuous superior in relation to the Zariski topology induced on $F \setminus (J_\infty \cup J'_\infty)$, the function $\lambda := \lim \chi^{-n}_{2k}\lambda_n$ is also semi-continuous superior in relation to this topology. Let m be the maximum value of λ on $F \setminus (J_\infty \cup J'_\infty \cup \mathcal{T})$. Just show that $m = 0$. Suppose that $m > 0$. Denote by Z^* the set of points $x \in F \setminus (J_\infty \cup J'_\infty \cup \mathcal{T})$ such that $\lambda(x) \geq m$. The closing Z of Z^* is an analytical subset of F. No irreducible component of Z is contained on $\mathcal{T}$. Consider a point

$x \in Z^*$. The invariance properties of $\mathcal{T}$ imply that $h^{-1}(x) \cap \mathcal{T} = \varnothing$. Using the definition of λ and of m, we have

$$m = \lambda(x) = \chi_{2k}^{-1} \sum_{y \in h^{-1}(x) \cap F} \lambda(y).$$

As $\lambda(y) \leq m$ and $\# h^{-1}(x) = \chi_{2k}$, we deduce that $h^{-1}(x) \subset Z$ and $\lambda(y) = m$ for $y \in h^{-1}(x)$. Therefore, $h^{-1}(Z^*) \subset Z$ and $h^{-1}(Z \setminus J') \subset Z$ as $h^{-1}(Z^*)$ is dense on $h^{-1}(Z \setminus J')$. The Proposition 5.3 implies that $Z \subset \mathcal{T}$. This is a contradiction. The lemma follows. ■

Remark 1. Set by induction $\mathcal{T}_0 := \mathcal{T}$, $\mathcal{T}_n := h(\mathcal{T}_{n-1})$ and $\mathcal{T}_\infty := \cup_{n \geq 0} \mathcal{T}_n$. If x is a point on $\mathcal{T}_\infty \setminus J'_\infty$, then ν_n^x has positive mass on $\mathcal{T}$ for some $n \geq 0$. It follows from the invariance properties of $\mathcal{T}$ that ν_n^x does not converge to ν as ν has no mass on $\mathcal{T}$. Lemma 5.4 still applies to $x \notin \mathcal{T}_\infty \cup J'_\infty$. We can show that $\nu_n^x \to \nu$ for those points x. This property is little stronger than Theorem 2.2.

We recall the following result from our previous paper [1] in order to formulate ours conjectures.

Theorem 5.5. *Let $\mathcal{V}$ be a smooth compact complex homogeneous manifold with $\dim_{\mathbb{C}}(\mathcal{V}) = k \geq 1$ and Kodaira dimension ≤ 0 and $h : \mathcal{V} \longrightarrow \mathcal{V}$ a* ***Cohomological Expanding Mapping****. Let ν be a complex measure with density L^{2k+1} on $\mathcal{V}$ such that $\nu(\mathcal{V}) = 1$. Let η be a $(1,1)$-strictly positive Hermitian form on $\mathcal{V}$. So the sequence $\frac{1}{\chi_{2l}^m}(h^m)^*\nu$ converges weakly to a measure of probability PC ν_f with* ***Cohomological Entropy*** *$\geq \log \chi_{2l}$ independent of ν as $m \to \infty$ so that $\chi_{2l}^{-1} h^* \nu_h = \nu_h = h_* \nu_h$ and if h is holomorphic, then for each Hermitian metric η on $\mathcal{V}$, ν_h is Hölder continuous on $\mathrm{PSH}^0(\eta)$.*

6 Conjectures

Here is the First Conjecture.

Conjecture 6.1. Let $\mathcal{V}, h, \nu_h$ as in Theorem 5.5. Let $\psi_1, \ldots, \psi_k$ be the Lyapunov exponents of ν_h and $\Psi = \sum_i \frac{1}{\psi_i}$ its inverse sum. So the Hausdorff dimension of ν_h satisfies

$$\dim_{\mathcal{H}}(\nu_h) = \Psi h_\chi(h).$$

Here is the Second Conjecture.

Conjecture 6.2. Let $\mathcal{V}, h, \nu_h$ as in Theorem 5.5. So there are T_l^+ and T_{k-l}^- such that ν_h is defined by :

$$\nu_h := T_l^+ \wedge T_{k-l}^- \quad ,$$

where T_l^+ is a positive invariant closed current of bidegree (l,l), i.e.

$$\frac{1}{\chi_{2l}^m}(h^m)^* \omega^l \longrightarrow T_l^+$$

and T_{k-l}^- designates a positive invariant closed current of $(k-l, k-l)$, i.e.

$$\frac{1}{\chi_{2(k-l)}^m}(h^m)_* \omega^{k-l} \longrightarrow T_{k-l}^-.$$

Here is the Third Conjecture.

Conjecture 6.3. Let $\mathcal{V}, h, \nu_h$ as in Theorem 5.5 and T_l^+ as in Conjecture 6.2. Let $\psi_1, \ldots, \psi_k$ be the Lyapunov exponents of ν_h with $\psi_l = \max_{1 \leq i \leq k} \psi_i$. So the Hausdorff dimension of the **Support** of T_l^+ satisfies

$$\dim_{\mathcal{H}}(\mathrm{Supp} T_l^+) \geq 2(k-l) + \frac{\log \chi_{2l}}{\psi_l}.$$

Here is the Fourth Conjecture.

Conjecture 6.4. Let $\mathcal{V}, h, \nu_h$ be as in Theorem 1.9 and $\mathcal{T}$ a maximal proper **algebraic subset** possibly empty and totally invariant, and $\nu_m^x := \chi_{2k}^{-m}(h^m)^*(\Gamma_x)$. There is a constant $\gamma > 1$ such that if x is a point out of $\mathcal{T}$, so ν_m^x **converges exponentially** to ν_h , this is, if ζ is a function $\mathcal{C}^\beta$ on $\mathcal{V}$ with $0 < \beta \leq 2$, so

$$|\langle \nu_m^x - \nu_h, \zeta \rangle| \leq M \Big[1 + \log^+ \frac{1}{D(x, \mathcal{T})}\Big]^{\beta/2} \|\zeta\|_{\mathcal{C}^\beta} \gamma^{-\beta m/2},$$

where $M > 0$ is a constant independent of m, x , ζ, $\log^+(\cdot) := \max(0, \log(\cdot))$ and $D(x, \mathcal{T}) := \mathrm{dist}(x, \mathcal{T})$

An **algebraic set** is the set of zeros of a family of polynomials.

Here is the Fifth Conjecture.

Conjecture 6.5. Let $\mathcal{V}, h, \nu_h$ be as in Theorem 1.9 and $\nu_m^x := \chi_{2k}^{-m}(h^m)^*(\Gamma_x)$. Let $1 < \gamma < (\chi_{2k})^{1/k}$ be a fixed constant. There is a proper **algebraic subset** E_γ, possibly empty, on $\mathcal{V}$ such that if x is a point out of E_γ and if ζ is a function $\mathcal{C}^\beta$ on $\mathcal{V}$ with $0 < \beta \leq 2$, so

$$|\langle \nu_m^x - \nu_h, \zeta\rangle| \leq M\Big[1 + \log^+ \frac{1}{D(x, E_\gamma)}\Big]^{\beta/2} \|\zeta\|_{\mathcal{C}^\beta} \gamma^{-\beta m/2},$$

where $M > 0$ is a constant independent of m, x, ζ, $\log^+(\cdot) := \max(0, \log(\cdot))$ and $D(x, E_\gamma) := \mathrm{dist}(x, E_\gamma)$

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