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The J-Generalized P - K Mittag-Leffler Function

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Abstract

We know that the classical Mittag-Leffler function play an important role as solution of fractional order differential and integral equations. We introduce the j-generalized p - k Mittag-leffler function. We evaluate the second order differential recurrence relation and four different integral representations and introduce a homogeneous linear differential equation whose one of the solution is the j-generalized p-k Mittag-Leffler function.
Also we evaluate the certain relations that exist between j-generalized p - k Mittag-leffler function and Riemann-Liouville fractional integrals and derivatives.
We evaluate Mellin-Barnes integral representation of j-generalized p-k Mittag-Leffler Function. The relationship of j-generalized p-k Mittag-Leffler Function with Fox H-Function and Wright hypergeometric function is also establish. we obtained its Euler transform, Laplace Transform and Mellin transform.
Finally we derive some particular cases.

MSC(2011): 33E12, 33B10, 26A33.

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1 Introduction

The two parameter pochhammer symbol is recently introduce by [9], equation 2.1, in the form,

1.1 Definition

Let $x \in C; k, p \in R^{+} - \{0\}$ and $Re(x) > 0, n \in N$, the p - k Pochhammer Symbol (i.e. Two Parameter Pochhammer Symbol), ${}_p(x)_{n,k}$ is given by

$$ {}_p(x)_{n,k} = (\frac{xp}{k})(\frac{xp}{k} + p)(\frac{xp}{k} + 2p).........(\frac{xp}{k} + (n-1)p). \quad (1.1)$$

And the Two Parameter Gamma Function is given by [9], some of it's result are,

1.2 Definition

For $x \in C/kZ^-; k,p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$, the p - k Gamma Function (i.e. Two Parameter Gamma Function), ${}_p\Gamma_k(x)$ as

$$ {}_p\Gamma_k(x) = \frac{1}{k} \lim_{n\to\infty} \frac{n! p^{n+1} (np)^{\frac{x}{k}}}{{}_p(x)_{n+1,k}}. \tag{1.2}$$

or

$$ {}_p\Gamma_k(x) = \frac{1}{k} \lim_{n\to\infty} \frac{n! p^{n+1} (np)^{\frac{x}{k}-1}}{{}_p(x)_{n,k}}. \tag{1.3}$$

The integral representation of p - k Gamma Function is given by

$$ {}_p\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{p}} t^{x-1} dt. \tag{1.4}$$

$$ {}_p\Gamma_k(x) = (\frac{p}{k})^{\frac{x}{k}} \Gamma_k(x) = \frac{p^{\frac{x}{k}}}{k} \Gamma(\frac{x}{k}). \tag{1.5}$$

$$ {}_p(x)_{n,k} = (\frac{p}{k})^n (x)_{n,k} = (p)^n (\frac{x}{k})_n. \tag{1.6}$$

Also for Generalized p - k Pochhammer Symbol, we have

$$ {}_p(x)_{nq,k} = (\frac{p}{k})^{nq} (x)_{nq,k} = (p)^{nq} (\frac{x}{k})_{nq} = (pq)^{nq} \prod_{r=1}^{q} (\frac{\frac{x}{k}+r-1}{q})_n. \tag{1.7}$$

$$ {}_p(x)_{n,k} = \frac{{}_p\Gamma_k(x+nk)}{{}_p\Gamma_k(x)}. \tag{1.8}$$

$$ {}_p\Gamma_k(x+k) = \frac{xp}{k}\, {}_p\Gamma_k(x). \tag{1.9}$$

$$ np\, {}_p(x)_{n-1,k} = {}_p(x)_{n,k} - {}_p(x-k)_{n,k}. \tag{1.10}$$

And

$$ {}_p(x)_{n+j,k} = {}_p(x)_{j,k} \times {}_p(x+jk)_{n,k}. \tag{1.11}$$

The Mittag-Leffler function $E_\alpha(z)$ introduced by Gosta Mittag-Leffler [4] in 1903, defined as

$$ E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}. \tag{1.12}$$

Here $z \in C, \alpha \geqslant 0$.

Wiman [2] generalized $E_\alpha(z)$ in 1905 and gave $E_{\alpha,\beta}(z)$ known as Wiman function, defined as

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}. \tag{1.13}$$

Here $z, \alpha, \beta \in C; Re(\alpha) > 0, Re(\beta) > 0$.

Prabhakar [17] in 1971, gave next generalization of Mittag-Leffler function and denoted as $E^\gamma_{\alpha,\beta}(z)$ and defined as

$$ E^\gamma_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}. \tag{1.14}$$

Here $z, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$.

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Shukla and Prajapati [1] in 2007, gave second generalization of Mittag-Leffler function and denoted it as $E^{\gamma,q}_{\alpha,\beta}(z)$ and defined as,

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}. \tag{1.15}$$

Here $z, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0, 1) \cup N$.
The function $E^{\gamma,q}_{\alpha,\beta}(z)$ converges absolutely for all z if $q < Re(\alpha) + 1$ and for $|z| < 1$ if $q = Re(\alpha) + 1$. It is entire function of order $\frac{1}{Re(\alpha)}$.

Gehlot K.S.[8] introduce Generalized k- Mittag-Leffler function in 2012, denoted as $GE^{\gamma,q}_{k,\alpha,\beta}(z)$ and defined for $k \in R; z, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0, 1) \cup N$, as,

$$GE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}\, z^n}{\Gamma_k(n\alpha+\beta)(n!)}, \tag{1.16}$$

where$(\gamma)_{nq,k}$ is the k- pochhammer symbol and $\Gamma_k(x)$ is the k-gamma function given by [15].
The generalized Pochhammer symbol is given as,

$$(\gamma)_{nq} = \frac{\Gamma(\gamma+nq)}{\Gamma(\gamma)} = q^{qn} \prod_{r=1}^{q} (\frac{\gamma+r-1}{q})_n,\ if\ q \in N. \tag{1.17}$$

Gehlot K.S.[9], Introduce The p- k Mittag-Leffler function in 2018, denoted by ${}_pE^{\gamma,q}_{k,\alpha,\beta}(z)$ and defined for $k, p \in R^+ - \{0\}; \alpha, \beta, \gamma \in C/kZ^-; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0, 1) \cup N$.

$${}_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{z^n}{n!}. \tag{1.18}$$

Where ${}_p(\gamma)_{nq,k}$ is two parameter Pochhammer symbol given by equation (1.1) and ${}_p\Gamma_k(x)$ is the two parameter Gamma function given by equation (1.3).

L.L.Luque [14]in the year 2019, introduce the L-mittag-Leffler function defined for $\alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, j \in N_0$ by the series

$$L^{\gamma,j}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j}}{\Gamma(n\alpha+\beta)} \frac{z^n}{(n+j)!}, \qquad (z \in C). \tag{1.19}$$

The Fractional Integral operators ([16], Definition 2.1, Page 33) are defined as,

$$(I^{\vartheta}_{0+}f)(z) = \frac{1}{\Gamma(\vartheta)} \int_0^z \frac{f(t)}{(z-t)^{1-\vartheta}} dt, (z > 0), \tag{1.20}$$

and

$$(I^{\vartheta}_{-}f)(z) = \frac{1}{\Gamma(\vartheta)} \int_z^{\infty} \frac{f(t)}{(t-z)^{1-\vartheta}} dt, (z > 0), \tag{1.21}$$

The Fractional Derivative ([16], Definition 2.2, Page 35) are defined as,

$$(D^{\vartheta}_{0+}f)(z) = (\frac{d}{dz})^{[Re(\vartheta)]+1} (I^{1-\vartheta+[Re(\vartheta)]}_{0+} f)(z)$$

$$= \frac{1}{\Gamma(1-\vartheta+[Re(\vartheta)])} (\frac{d}{dz})^{[Re(\vartheta)]+1} \int_0^z \frac{f(t)}{(z-t)^{\vartheta-[Re(\vartheta)]}} dt, (z > 0), \tag{1.22}$$

and

$$(D^{\vartheta}_{-}f)(z) = (-\frac{d}{dz})^{[Re(\vartheta)]+1} (I^{1-\vartheta+[Re(\vartheta)]}_{-} f)(z)$$

$$= \frac{1}{\Gamma(1-\vartheta+[Re(\vartheta)])}(-\frac{d}{dz})^{[Re(\vartheta)]+1}\int_z^\infty \frac{f(t)}{(t-z)^{\vartheta-[Re(\vartheta)]}}dt, (z>0). \quad (1.23)$$

Where $\vartheta \in C(Re(\vartheta) > 0)$.

Wright generalized hypergeometric function [15];

$${}_p\psi_q\left[\begin{array}{c}(\alpha_1,A_1),\ldots,(\alpha_p,A_p);\\ \\ (\beta_1,B_1),\ldots,(\beta_q,B_q);\end{array} z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\alpha_i+A_i n)z^n}{\prod_{j=1}^{q}\Gamma(\beta_i+B_i n)n!}. \quad (1.24)$$

$${}_p\psi_q\left[\begin{array}{c}(\alpha_1,A_1),\ldots,(\alpha_p,A_p);\\ \\ (\beta_1,B_1),\ldots,(\beta_q,B_q);\end{array} z\right] = H^{1,p}_{p,q+1}\left[-z \,|\, \begin{array}{c}(1-\alpha_1,A_1),\ldots,(1-\alpha_p,A_p);\\ \\ (0,1),(1-\beta_1,B_1),\ldots,(1-\beta_q,B_q);\end{array}\right]. \quad (1.25)$$

Where $H^{m,n}_{p,q}[.]$ denotes the Fox H-function.
Euler Beta transform,[6],

$$B[f(z):a,b] = \int_0^1 z^{a-1}(1-z)^{b-1}f(z)dz. \quad (1.26)$$

Laplace transform,([7],equation3.1.1),

$$L[f(z):s] = \int_0^\infty e^{-sz}f(z)dz. \quad (1.27)$$

Mellin transform,([7],equation 4.1.1),

$$M[f(z):s] = \int_0^\infty z^{s-1}f(z)dz = f^*(s), Re(s) > 0, \quad (1.28)$$

then

$$f(z) = M^{-1}[f^*(s):x] = \int f^*(s)x^{-s}ds, \quad (1.29)$$

Throughout this paper Let $C, R^+, Re(), Z^-, N_0$ and N be the sets of complex numbers, positive real numbers, real part of complex number, negative integer, whole number and natural numbers respectively.

2 The j-generelized p - k Mittag-Leffler function

In this section we introduce the j-generelized p - k Mittag-Leffler function and prove some of its properties.

2.1 Definition

Let $k,p \in R^+ - \{0\}; \alpha,\beta,\gamma \in C/kZ^-; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0,\ j \in N_0$ and $q \in (0,1) \cup N$. The j-generelized p - k Mittag-Leffler function denoted by ${}^j_pE^{\gamma,q}_{k,\alpha,\beta}(z)$ and defined as

$${}^j_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)}\frac{z^n}{(n+j)!}, \qquad z \in C. \quad (2.1)$$

Where ${}_p(\gamma)_{nq,k}$ is two parameter Pochhammer symbol given by equation (1.1) and ${}_p\Gamma_k(x)$ is the two parameter Gamma function given by equation (1.3).

Particular cases : For some particular values of the parameters $j, p, q, k, \alpha, \beta, \gamma$ we can obtain certain defined and undefined Mittag-Leffler functions:

(a) For $j = 0$ equation (2.1), reduces in the p-k Mittag-Leffler functions defined by [10],

$$ {}_p^0E_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{z^n}{n!}, \qquad z \in C. \tag{2.2}$$

(b) For $q = 1$ equation (2.1), reduces in j form of p-k Mittag-Leffler functions defined as,

$$ {}_p^jE_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j),k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{z^n}{(n+j)!}, \qquad z \in C. \tag{2.3}$$

(c) For $q = 1, p = k$ equation (2.1), reduces in j form of k- Mittag-Leffler functions defined as,

$$ {}_k^jE_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{(n+j),k}}{\Gamma_k(n\alpha+\beta)} \frac{z^n}{(n+j)!}, \qquad z \in C. \tag{2.4}$$

(d) For $q = 1, j = 0$ equation (2.1), reduces in generalized form of k- Mittag-Leffler functions defined as.

$$ {}_pE_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n,k}\, z^n}{{}_p\Gamma_k(n\alpha+\beta)(n!)}. \tag{2.5}$$

(e) For $p = k, j = 0$ equation (2.1), reduces in Generalized k- Mittag-Leffler functions defined by [8].

$$ {}_kE_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{{}_k(\gamma)_{nq,k}\, z^n}{{}_k\Gamma_k(n\alpha+\beta)(n!)} = GE_{k,\alpha,\beta}^{\gamma,q}(z). \tag{2.6}$$

(f) For $p = k, q = 1, j = 0$ equation (2.1), reduces in k - Mittag-Leffler functions defined by [3].

$$ {}_kE_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\, z^n}{\Gamma_k(n\alpha+\beta)(n!)} = E_{k,\alpha,\beta}^{\gamma}(z). \tag{2.7}$$

(g) For $p = k$ and $k = 1, j = 0$ equation (2.1), reduces in Mittag-Leffler functions defined by [1].

$$ {}_1E_{1,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}\, z^n}{\Gamma(n\alpha+\beta)(n!)} = E_{\alpha,\beta}^{\gamma,q}(z). \tag{2.8}$$

(h) For $p = k = q = 1$ equation (2.1), reduces in L-Mittag-Leffler functions defined by [14].

$$ {}_1E_{1,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j}\, z^n}{\Gamma(n\alpha+\beta)(n+j)!} = L_{\alpha,\beta}^{\gamma,j}(z). \tag{2.9}$$

(i) For $p = k, q = 1, j = 0$ and $k = 1$ equation (2.1), reduces in Mittag-Leffler functions defined by [17].

$$ {}_1E_{1,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n}\, z^n}{\Gamma(n\alpha+\beta)(n!)} = E_{\alpha,\beta}^{\gamma}(z), \tag{2.10}$$

(j) For $p = k, q = 1, k = 1, j = 0$ and $\gamma = 1$ equation (2.1), reduces in Mittag-Leffler functions defined by [3].

$$ {}_1E_{1,\alpha,\beta}^{1,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)} = E_{\alpha,\beta}(z), \tag{2.11}$$

(k) For $p = k, q = 1, k = 1, \gamma = 1, j = 0$ and $\beta = 1$ equation (2.1), reduces in Mittag-Leffler functions defined by [4].

$$ {}_{1}E^{1,1}_{1,\alpha,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)} = E_\alpha(z). \tag{2.12} $$

Theorem 2.1: The j-generelized p - k Mittag-Leffler function defined by equation (2.1) is an entire function of order

$$ \frac{1}{\rho} = Re(\frac{\alpha}{k}) - q + 1. \tag{2.13} $$

Proof: Let R is the radius of convergence of the j-generelized p - k Mittag-Leffler function. The asymptotic Striling formula for Gamma function and factorial are given by,[5]

$$ \Gamma(az + b) = \sqrt{2\pi} e^{-az} (az)^{az+b-\frac{1}{2}} \Big[1 + o(\frac{1}{z})\Big], (arg(az + b) < \pi; \mid z \mid \to \infty). \tag{2.14} $$

and

$$ n! = \sqrt{2\pi} e^{-n} (n)^{n+\frac{1}{2}} \Big[1 + o(\frac{1}{n})\Big], (n \in N; n \to \infty). \tag{2.15} $$

From equation (2.1), we have

$$ {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_{p}(\gamma)_{(n+j)q,k}}{{}_{p}\Gamma_k(n\alpha + \beta)} \frac{z^n}{(n + j)!} = \sum_{n=0}^{\infty} C_n z^n, $$

since

$$ R = \limsup_{n\to\infty} \Big| \frac{C_n}{C_{n+1}} \Big|, $$

$$ \Big| \frac{C_n}{C_{n+1}} \Big| = \Big| \frac{{}_{p}(\gamma)_{(n+j)q,k}}{{}_{p}\Gamma_k(n\alpha + \beta)} \frac{1}{(n + j)!} \times \frac{{}_{p}\Gamma_k(n\alpha + \alpha + \beta)(n + 1 + j)!}{{}_{p}(\gamma)_{(n+1+j)q,k}} \Big| $$

using equations (2.19) and (2.20) of [9], we have

$$ \Big| \frac{C_n}{C_{n+1}} \Big| = (n + 1 + j) \Big| p^{\frac{\alpha - qk}{k}} \Big| \Big| \frac{\Gamma(nq + jq + \frac{\gamma}{k})}{\Gamma(nq + jq + q + \frac{\gamma}{k})} \Big| \Big| \frac{\Gamma(\frac{n\alpha+\alpha+\beta}{k})}{\Gamma(\frac{n\alpha+\beta}{k})} \Big|, $$

using equation (2.15), we have

$$ \simeq \Big| p^{\frac{\alpha}{k} - q} \Big| \Big| q^{-q} \Big| \Big| (\frac{\alpha}{k})^{\frac{\alpha}{k}} \Big| \Big| n^{\frac{\alpha}{k} + 1 - q} \Big| \to \infty $$

when,

$$ Re(\frac{\alpha}{k} + 1 - q) > 0, $$

Thus, the j-generelized p - k Mittag-Leffler function is an entire function for $q < Re(\frac{\alpha}{k}) + 1$
To determine the order ρ,

$$ \rho = \limsup_{n\to\infty} \frac{n \ln n}{\ln(\frac{1}{|C_n|})}, \tag{2.16} $$

$$ \Big| \frac{1}{C_n} \Big| = \Big| \frac{{}_{p}\Gamma_k(n\alpha + \beta)(n + j)!}{{}_{p}(\gamma)_{(n+j)q,k}} \Big|, $$

using theorem 2.19 and 2.20 of [9], we have

$$ \Big| \frac{1}{C_n} \Big| = \frac{(n + j)!}{k} \Big| p^{\frac{\gamma}{k} + \frac{n\alpha+\beta}{k} - \frac{\gamma+(n+j)qk}{k}} \Big| \Big| \frac{\Gamma(\frac{\gamma}{k})\Gamma(\frac{n\alpha+\beta}{k})}{\Gamma(\frac{\gamma}{k} + (n + j)q)} \Big|, $$

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By using equation (2.12) and (2.13), we get

$$\left|\frac{1}{C_n}\right| = k^{-1}(2\pi)^{\frac{1}{2}}\left|p^{(\frac{\alpha-qk}{k})n+\frac{\beta}{k}-jq}\right|\left|(\frac{\alpha}{k})^{\frac{n\alpha}{k}+\frac{\beta}{k}-\frac{1}{2}}\right|\left|n^{\frac{n\alpha}{k}+\frac{\beta}{k}-\frac{\gamma}{k}-nq-jq+n+j+\frac{1}{2}}\right|\left|e^{-nRe(\frac{\alpha}{k}+1-q)}\right|$$

taking ln of above equation and put in equation (2.13), we have the order of j-generelized p - k Mittag-Leffler function is given by

$$\rho = \frac{k}{Re(\alpha) - qk + k}.$$

Hence.

Theorem 2.2: The functional relation between the j-generelized p - k Mittag-Leffler function given by equation (2.1) with p - k Mittag-Leffler function defined by [10] and generelized Mittag-Leffler function defined by [1] are given by

$$ {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z) = \left(kp^{jq-\frac{\beta}{k}}\right) {}^{j}E^{\frac{\gamma}{k},q}_{\frac{\alpha}{k},\frac{\beta}{k}}(zp^{q-\frac{\alpha}{k}}). \tag{2.17}$$

$$(\frac{d}{dz})^{l}\left[z^{j} \times {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z)\right] = {}_{p}(\gamma)_{lq,k}\ z^{j-l}\ {}^{j-l}_{p}E^{\gamma+lqk,q}_{k,\alpha,\beta}(z),\ for\ l < j. \tag{2.18}$$

$$(\frac{d}{dz})^{l}\left[z^{j} \times {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z)\right] = {}_{p}(\gamma)_{lq,k}\ {}_{p}E^{\gamma+lqk,q}_{k,\alpha,\beta}(z),\ for\ l = j. \tag{2.19}$$

$$(\frac{d}{dz})^{l}\left[z^{j} \times {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z)\right] = {}_{p}(\gamma)_{lq,k}\ {}_{p}E^{\gamma+lqk,q}_{k,\alpha,\beta+l\alpha-j\alpha}(z),\ for\ l > j. \tag{2.20}$$

Proof of equation (2.16)
Using equation (1.5) and (1.6), we get the desired result.
Proof of equation (2.17), (2.18) and (2.19)
Using the equation (2.1), in right hand side of (2.17), we have

$$\frac{d^{l}}{dz^{l}}\left[z^{j} \times {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z)\right] = \sum_{n=0}^{\infty}\frac{{}_{p}(\gamma)_{(n+j)q,k}}{{}_{p}\Gamma_{k}(n\alpha+\beta)}\frac{z^{n+j-l}}{(n+j-l)!},$$

using equation (1.11), we have

$$\frac{d^{l}}{dz^{l}}\left[z^{j} \times {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z)\right] = \sum_{n=0}^{\infty}\frac{{}_{p}(\gamma)_{lq,k}\ {}_{p}(\gamma+lqk)_{(n+j-l)q,k}}{{}_{p}\Gamma_{k}(n\alpha+\beta)}\frac{z^{n+j-l}}{(n+j-l)!},$$

hence we have,

$$= {}_{p}(\gamma)_{lq,k}\ z^{j-l}\ {}^{j-l}_{p}E^{\gamma+lqk,q}_{k,\alpha,\beta}(z),\ for\ l < j.$$

$$= {}_{p}(\gamma)_{jq,k}\ {}_{p}E^{\gamma+lqk,q}_{k,\alpha,\beta}(z),\ for\ l = j.$$

$$= {}_{p}(\gamma)_{lq,k}\ {}_{p}E^{\gamma+lqk,q}_{k,\alpha,\beta+l\alpha-j\alpha}(z),\ for\ l > j.$$

Theorem 2.3: The following elementary properties are satisfied by the j-generelized p - k Mittag-Leffler function defined by equation (2.1),

$$k\ {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z) = p\beta\ {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta+k}(z) + zp\alpha\frac{d}{dz}\ {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta+k}(z). \tag{2.21}$$

$$pq\ {}_{p}(\gamma)_{q-1,k}\ {}^{j-1}_{p}E^{\gamma+kq-k,q}_{k,\alpha,\beta}(z) = {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z) - {}^{j}_{p}E^{\gamma-k,q}_{k,\alpha,\beta}(z). \tag{2.22}$$

$$\sum_{n=0}^{\infty}(x+y)^{n}\ {}^{j}_{p}E^{nqk+k,q}_{k,0,nk+jk+k}(xy) = \sum_{r=0}^{\infty}\frac{{}_{p}\Gamma_{k}(rqk+k)(xyp)^{r}}{{}_{p}\Gamma_{k}(rk+jk+k)} \times {}^{j}_{p}E^{rqk+k,q}_{k,qk,k}(\frac{x+y}{p}). \tag{2.23}$$

Proof of equation (2.20)
Consider the right hand side of equation (2.20),

$$A \equiv p\beta \, {}_p^jE^{\gamma,q}_{k,\alpha,\beta+k}(z) + zp\alpha \frac{d}{dz} \, {}_p^jE^{\gamma,q}_{k,\alpha,\beta+k}(z),$$

using equation (2.1),

$$A \equiv p\beta \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta+k)} \frac{z^n}{(n+j)!} + zp\alpha \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta+k)} \frac{nz^{n-1}}{(n+j)!},$$

$$A \equiv p \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}(n\alpha+\beta)}{{}_p\Gamma_k(n\alpha+\beta+k)} \frac{z^n}{(n+j)!},$$

using the equation (1.9), we have

$$A \equiv k \, {}_p^jE^{\gamma,q}_{k,\alpha,\beta}(z).$$

Proof of equation (2.21)
Consider the right hand side of (2.21),

$$A \equiv \, {}_p^jE^{\gamma,q}_{k,\alpha,\beta}(z) - \, {}_p^jE^{\gamma-k,q}_{k,\alpha,\beta}(z),$$

using equation (2.1), we have

$$A \equiv \sum_{n=0}^{\infty} \frac{z^n}{{}_p\Gamma_k(n\alpha+\beta)(n+j)!} \Big[{}_p(\gamma)_{(n+j)q,k} - {}_p(\gamma-k)_{(n+j)q,k} \Big],$$

using equations (1.10) and (1.11), we have

$$A \equiv pq \, {}_p(\gamma)_{q-1,k} \, {}_p^{j-1}E^{\gamma+kq-k,q}_{k,\alpha,\beta}(z).$$

Proof of equation (2.22)
Consider the Left hand side of equation (2.22),

$$A \equiv \sum_{n=0}^{\infty} (x+y)^n \, {}_p^jE^{nqk+k,q}_{k,0,(n+j+1)k}(xy),$$

using equation (2.1), we have

$$A \equiv \sum_{n=0}^{\infty} (x+y)^n \sum_{r=0}^{\infty} \frac{{}_p(nqk+k)_{(r+j)q,k}}{{}_p\Gamma_k(nk+jk+k)} \frac{(xy)^r}{(r+j)!}, \tag{2.24}$$

now simplifying, by using equation (1.5) and (1.6), we have

$$\begin{aligned} {}_p(nqk+k)_{(r+j)q,k} &= p^{(r+j)q}(nq+1)_{(r+j)q}, \\ &= p^{(r+j)q} \frac{\Gamma(nq+(r+j)q+1)}{\Gamma(nq+1))}, \\ &= p^{(r+j)q} \frac{\Gamma((r+j)q+1+nq)}{\Gamma((r+j)q+1)} \frac{\Gamma((r+j)q+1)}{\Gamma(nq+1)}, \\ &= {}_p\Gamma_k(rqk+k) \frac{{}_p(rqk+k)_{(r+j)q,k}}{{}_p\Gamma_k(nqk+k)}, \end{aligned}$$

then equation (2.23) becomes by rearranging the terms, we have

$$A \equiv \sum_{r=0}^{\infty} \frac{{}_p\Gamma_k(rqk+k)(xyp)^r}{{}_p\Gamma_k(rk+jk+k)} \sum_{n=0}^{\infty} \frac{{}_p(rqk+k)_{(n+j)q,k}}{{}_p\Gamma_k(qkn+k)(n+j)!} (\frac{x+y}{p})^n.$$

This completes the proof.

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3 Fractional Integral and Differentiation of the j-generalized p - k Mittag-Leffler Function

In this section we evaluate certain relations that exist between the j-generalized p - k Mittag-Leffler Function and Riemann-Liouville fractional integrals and derivatives. It has been shown that the fractional integration and differentiation operators of j-generalized p - k Mittag-Leffler Function with power multipliers into the function of the same form. Also point out some special cases.

Theorem 3.1 The left-side Riemann-Liouville Fractional Integral Operator I^{ϑ}_{0+} of the j-generalized p - k Mittag-Leffler Function is given by,

$$(I^{\vartheta}_{0+}[t^{\frac{\beta}{k}-1}\,{}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(t^{\frac{\alpha}{k}})])(z) = p^{\vartheta} z^{\frac{\beta}{k}+\vartheta-1}\,{}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta+k\vartheta}(z^{\frac{\alpha}{k}}). \tag{3.1}$$

Where

$k,p \in R^{+}-\{0\}; \alpha,\beta,\gamma \in C/kZ^{-}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, j \in N_0,\ q \in (0,1)\cup N \ and \ Re(\vartheta) > 0.$

Proof: Consider left hand side,

$$A \equiv (I^{\vartheta}_{0+}[t^{\frac{\beta}{k}-1}\,{}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(t^{\frac{\alpha}{k}})])(z),$$

by virtue of equation (1.20) and (2.1), we have

$$A \equiv \frac{1}{\Gamma(\vartheta)}\int_0^z \frac{t^{\frac{\beta}{k}-1}}{(z-t)^{1-\vartheta}}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)}\frac{t^{\frac{n\alpha}{k}}}{(n+j)}dt,$$

interchanging the order of integration and summation and evaluate the inner integral by substitute $t = zu$ and using the beta function formula, it gives

$$A \equiv \frac{z^{\frac{\beta}{k}+\vartheta-1}}{\Gamma(\vartheta)}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)}\frac{z^{\frac{n\alpha}{k}}}{(n+j)!}\frac{\Gamma(\frac{n\alpha+\beta}{k})\Gamma(\vartheta)}{\Gamma(\frac{n\alpha+\beta+k\vartheta}{k})},$$

using the equation (1.5), we have

$$A \equiv p^{\vartheta} z^{\frac{\beta}{k}+\vartheta-1}\,{}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta+k\vartheta}(z^{\frac{\alpha}{k}}).$$

Hence, we get the desired result.

Theorem 3.2 The right-side Riemann-Liouville Fractional Integral Operator I^{ϑ}_{-} of the j-generalized p - k Mittag-Leffler Function is given by,

$$(I^{\vartheta}_{-}[t^{-\frac{\beta}{k}-\vartheta}\,{}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(t^{-\frac{\alpha}{k}})])(z) = p^{\vartheta} z^{-\frac{\beta}{k}}\,{}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta+k\vartheta}(z^{-\frac{\alpha}{k}}). \tag{3.2}$$

Where

$k,p \in R^{+}-\{0\}; \alpha,\beta,\gamma \in C/kZ^{-}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, j \in N_0,\ q \in (0,1)\cup N \ and \ Re(\vartheta) > 0.$

Proof: Consider left hand side,

$$A \equiv (I^{\vartheta}_{-}[t^{-\frac{\beta}{k}-\vartheta}\,{}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(t^{-\frac{\alpha}{k}})])(z),$$

by virtue of equation (1.21) and (2.1), we have

$$A \equiv \frac{1}{\Gamma(\vartheta)} \int_{z}^{\infty} \frac{t^{-\frac{\beta}{k}-\vartheta}}{(t-z)^{1-\vartheta}} \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{t^{-\frac{n\alpha}{k}}}{(n+j)!} dt,$$

interchanging the order of integration and summation and evaluate the inner integral by substitute $t = \frac{z}{u}$ and using the beta function formula, it gives

$$A \equiv \frac{z^{-\frac{\beta}{k}}}{\Gamma(\vartheta)} \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{z^{-\frac{n\alpha}{k}}}{(n+j)!} \frac{\Gamma(\frac{n\alpha+\beta}{k})\Gamma(\vartheta)}{\Gamma(\frac{n\alpha+\beta+k\vartheta}{k})},$$

using the equation (1.5), we have

$$A \equiv p^{\vartheta} z^{-\frac{\beta}{k}} \, {}_p^j E_{k,\alpha,\beta+k\vartheta}^{\gamma,q}(z^{-\frac{\alpha}{k}}).$$

Hence, we get the desired result.

Theorem 3.3 The left-side Riemann-Liouville Fractional Derivative Operator D_{0+}^{ϑ} of the j-generalized p - k Mittag-Leffler Function is given by,

$$(D_{0+}^{\vartheta}[t^{\frac{\beta}{k}-1} \, {}_p^j E_{k,\alpha,\beta}^{\gamma,q}(t^{\frac{\alpha}{k}})])(z) = p^{-\vartheta} z^{\frac{\beta}{k}-\vartheta-1} \, {}_p^j E_{k,\alpha,\beta-k\vartheta}^{\gamma,q}(z^{\frac{\alpha}{k}}). \tag{3.3}$$

Where

$k, p \in R^{+}-\{0\}; \alpha, \beta, \gamma \in C/kZ^{-}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, j \in N_0,\ q \in (0,1) \cup N \ and \ Re(\vartheta) > 0.$

Proof: Consider left hand side,

$$A \equiv (D_{0+}^{\vartheta}[t^{\frac{\beta}{k}-1} \, {}_p^j E_{k,\alpha,\beta}^{\gamma,q}(t^{\frac{\alpha}{k}})])(z),$$

by virtue of equation (1.22) and (2.1), we have

$$A \equiv \frac{1}{\Gamma(1-\vartheta+[Re(\vartheta)])} (\frac{d}{dz})^{[Re(\vartheta)]+1} \int_{0}^{z} \frac{t^{\frac{\beta}{k}-1}}{(z-t)^{\vartheta-Re(\vartheta)}} \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{t^{\frac{n\alpha}{k}}}{(n+j)!} dt,$$

interchanging the order of integration and summation and evaluate the inner integral by substitute $t = zu$ and using the beta function formula, it gives

$$A \equiv \frac{z^{\frac{\beta}{k}-\vartheta-1}}{\Gamma(\vartheta)} \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{z^{\frac{n\alpha}{k}}}{(n+j)!} \frac{\Gamma(\frac{n\alpha+\beta}{k})\Gamma(\vartheta)}{\Gamma(\frac{n\alpha+\beta-k\vartheta}{k})},$$

using the equation (1.5), we have

$$A \equiv p^{-\vartheta} z^{\frac{\beta}{k}-\vartheta-1} \, {}_p^j E_{k,\alpha,\beta-k\vartheta}^{\gamma,q}(z^{\frac{\alpha}{k}}).$$

Hence, we get the desired result.

Theorem 3.4 The right-side Riemann-Liouville Fractional Derivative Operator D_{-}^{ϑ} of the j-generalized p - k Mittag-Leffler Function is given by,

$$(D_{-}^{\vartheta}[t^{-\frac{\beta}{k}+\vartheta} \, {}_p^j E_{k,\alpha,\beta}^{\gamma,q}(t^{-\frac{\alpha}{k}})])(z) = p^{-\vartheta} z^{-\frac{\beta}{k}} \, {}_p^j E_{k,\alpha,\beta-k\vartheta}^{\gamma,q}(z^{-\frac{\alpha}{k}}). \tag{3.4}$$

Where

$k, p \in R^{+}-\{0\}; \alpha, \beta, \gamma \in C/kZ^{-}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, j \in N_0,\ q \in (0,1) \cup N \ and \ Re(\vartheta) > 0.$

Proof: Consider left hand side,

$$A \equiv (D_{-}^{\vartheta}[t^{-\frac{\beta}{k}+\vartheta}\ {}_{p}^{j}E_{k,\alpha,\beta}^{\gamma,q}(t^{-\frac{\alpha}{k}})])(z),$$

by virtue of equation (1.23) and (2.1), we have

$$A \equiv \frac{1}{\Gamma(1-\vartheta+[Re(\vartheta)])}(-\frac{d}{dz})^{[Re(\vartheta)]+1}\int_{z}^{\infty}\frac{z^{-\frac{\beta}{k}+\vartheta}}{(t-z)^{\vartheta-[Re(\vartheta)]}}\sum_{n=0}^{\infty}\frac{{}_{p}(\gamma)_{(n+j)q,k}}{{}_{p}\Gamma_{k}(n\alpha+\beta)}\frac{t^{-\frac{n\alpha}{k}}}{(n+j)!}dt,$$

interchanging the order of integration and summation and evaluate the inner integral by substitute $t=\frac{z}{u}$ and using the beta function formula, it gives

$$A \equiv \frac{z^{-\frac{\beta}{k}}}{\Gamma(\vartheta)}\sum_{n=0}^{\infty}\frac{{}_{p}(\gamma)_{(n+j)q,k}}{{}_{p}\Gamma_{k}(n\alpha+\beta)}\frac{z^{-\frac{n\alpha}{k}}}{(n+j)!}\frac{\Gamma(\frac{n\alpha+\beta}{k})\Gamma(\vartheta)}{\Gamma(\frac{n\alpha+\beta-k\vartheta}{k})},$$

using the equation (1.5), we have

$$A \equiv p^{-\vartheta}z^{-\frac{\beta}{k}}\ {}_{p}^{j}E_{k,\alpha,\beta-k\vartheta}^{\gamma,q}(z^{-\frac{\alpha}{k}}).$$

Hence, we get the desired result.

4 Recurrence Relation and Integral Representation of the j-generalized p - k Mittag-Leffler Function

In this section we evaluate the recurrence relations and integral representations of the j-generalized p - k Mittag-Leffler function.

Theorem 4.1 For $k,p\in R^{+}-\{0\}$; $\alpha+r,\beta+s+k,\gamma\in C/kZ^{-}$; $R(\alpha+r)>0, R(\beta+s+k)>0, R(\gamma)>0, q\in(0,1)\cup N, j\in N_{0}$, we get

$$\begin{aligned}{}_{p}^{j}E_{k,\alpha+r,\beta+s+k}^{\gamma,q}(z)-p\ {}_{p}^{j}E_{k,\alpha+r,\beta+s+2k}^{\gamma,q}(z) &= \frac{p^{2}}{k^{2}}\Big[(\alpha+r)^{2}z^{2}\ {}_{p}^{j}\ddot{E}_{k,\alpha+r,\beta+s+3k}^{\gamma,q}(z)\\ &+\Big\{(\alpha+r)^{2}+(\alpha+r)(2\beta+2s+2k)z\Big\}\ {}_{p}^{j}\dot{E}_{k,\alpha+r,\beta+s+3k}^{\gamma,q}(z)\\ &+(\beta+s)(\beta+s+2k)\ {}_{p}^{j}E_{k,\alpha+r,\beta+s+3k}^{\gamma,q}(z)\Big],\end{aligned} \tag{4.1}$$

where ${}_{p}^{j}\dot{E}_{k,\alpha,\beta}^{\gamma,q}(z)=\frac{d}{dz}\ {}_{p}^{j}E_{k,\alpha,\beta}^{\gamma,q}(z)$ and ${}_{p}^{j}\ddot{E}_{k,\alpha,\beta}^{\gamma,q}(z)=\frac{d^{2}}{dz^{2}}\ {}_{p}^{j}E_{k,\alpha,\beta}^{\gamma,q}(z)$.

Proof: The j-generalized p-k Mittag-Leffler function, from equation (2.1),

$${}_{p}^{j}E_{k,\alpha+r,\beta+s+k}^{\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{{}_{p}(\gamma)_{(n+j)q,k}\ z^{n}}{{}_{p}\Gamma_{k}(n(\alpha+r)+\beta+s+k)((n+j)!)},$$

using equation (1.9), we have

$${}_{p}^{j}E_{k,\alpha+r,\beta+s+k}^{\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{k}{p}\frac{{}_{p}(\gamma)_{(n+j)q,k}\ z^{n}}{{}_{p}\Gamma_{k}(n(\alpha+r)+\beta+s)\{n(\alpha+r)+\beta+s\}((n+j)!)}, \tag{4.2}$$

and

$${}_{p}^{j}E_{k,\alpha+r,\beta+s+2k}^{\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{{}_{p}(\gamma)_{(n+j)q,k}\ z^{n}}{{}_{p}\Gamma_{k}(n(\alpha+r)+\beta+s+2k)((n+j)!)}, \tag{4.3}$$

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using equation (1.9), we have

$$ {}_{p}^{j}E_{k,\alpha+r,\beta+s+2k}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{{}_{p}(\gamma)_{(n+j)q,k}\, z^n}{{}_{p}\Gamma_k(n(\alpha+r)+\beta+s)((n+j)!)} $$

$$ \times \frac{k^2}{p^2} \frac{1}{\{n(\alpha+r)+\beta+s)\}\{n(\alpha+r)+\beta+s+k)\}}, $$

$$ {}_{p}^{j}E_{k,\alpha+r,\beta+s+2k}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{k}{p^2}[\frac{1}{(n(\alpha+r)+\beta+s)} - \frac{1}{(n(\alpha+r)+\beta+s+k)}] $$

$$ \times \frac{{}_{p}(\gamma)_{(n+j)q,k}\, z^n}{{}_{p}\Gamma_k(n(\alpha+r)+\beta+s)((n+j)!)}, $$

$$ {}_{p}^{j}E_{k,\alpha+r,\beta+s+2k}^{\gamma,q}(z) = \frac{k}{p^2}[\frac{p}{k}\, {}_{p}^{j}E_{k,\alpha+r,\beta+s+k}^{\gamma,q}(z) - S], $$

$$ S = \frac{p}{k}\, {}_{p}^{j}E_{k,\alpha+r,\beta+s+k}^{\gamma,q}(z) - \frac{p^2}{k}\, {}_{p}^{j}E_{k,\alpha+r,\beta+s+2k}^{\gamma,q}(z), \tag{4.4} $$

where

$$ S = \sum_{n=0}^{\infty} \frac{{}_{p}(\gamma)_{(n+j)q,k}\, z^n}{{}_{p}\Gamma_k(n(\alpha+r)+\beta+s)\{n(\alpha+r)+\beta+s+k)\}((n+j)!)}, \tag{4.5} $$

applying the simple identity $\frac{1}{u} = \frac{k}{u(u+k)} + \frac{1}{u+k}$; for $u = n(\alpha+r)+\beta+s+k$ to (4.5),we obtain,

$$ S = \sum_{n=0}^{\infty} \frac{k\, {}_{p}(\gamma)_{(n+j)q,k}\, z^n}{{}_{p}\Gamma_k(n(\alpha+r)+\beta+s)((n+j)!)} \times \frac{1}{\{n(\alpha+r)+\beta+s+k\}\{n(\alpha+r)+\beta+s+2k\}} $$

$$ + \sum_{n=0}^{\infty} \frac{{}_{p}(\gamma)_{(n+j)q,k}\, z^n}{{}_{p}\Gamma_k(n(\alpha+r)+\beta+s)\{n(\alpha+r)+\beta+s+2k)\}((n+j)!)}, $$

$$ S = \sum_{n=0}^{\infty} \frac{k\{n(\alpha+r)+\beta+s\}_{p}(\gamma)_{(n+j)q,k}\, z^n}{{}_{p}\Gamma_k(n(\alpha+r)+\beta+s)((n+j)!)} $$

$$ \times \frac{1}{\{n(\alpha+r)+\beta+s\}\{n(\alpha+r)+\beta+s+k\}\{n(\alpha+r)+\beta+s+2k\}} $$

$$ + \sum_{n=0}^{\infty} \frac{\{n(\alpha+r)+\beta+s\}\{n(\alpha+r)+\beta+s+k\}_{p}(\gamma)_{(n+j)q,k}\, z^n}{{}_{p}\Gamma_k(n(\alpha+r)+\beta+s)((n+j)!)} $$

$$ \times \frac{1}{\{n(\alpha+r)+\beta+s\}\{n(\alpha+r)+\beta+s+k\}\{n(\alpha+r)+\beta+s+2k\}}, $$

using equation (1.9), we obtain

$$ S = \sum_{n=0}^{\infty} \frac{k\{n(\alpha+r)+\beta+s\}_{p}(\gamma)_{(n+j)q,k}\, z^n}{\frac{k^3}{p^3}\, {}_{p}\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)} $$

$$ + \sum_{n=0}^{\infty} \frac{\{n(\alpha+r)+\beta+s\}\{n(\alpha+r)+\beta+s+k\}_{p}(\gamma)_{(n+j)q,k}\, z^n}{\frac{k^3}{p^3}\, {}_{p}\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)}, $$

$$ S\frac{k^3}{p^3} = \sum_{n=0}^{\infty} \frac{{}_{p}(\gamma)_{(n+j)q,k}\, z^n}{{}_{p}\Gamma_k(n(\alpha+p)+\beta+s+3k)((n+j)!)} $$

$$ \times[n^2(\alpha+r)^2 + 2n(\alpha+r)(\beta+s+k) + (\beta+s)(\beta+s+2k)]. \tag{4.6} $$

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We now express each summation in the right hand side of (4.6) as follows:

$$\frac{d}{dz}[z\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)] = \sum_{n=0}^{\infty}\frac{(n+1)\,{}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)},$$

$$z\,{}_p^j\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) + {}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) = \sum_{n=0}^{\infty}\frac{(n+1)\,{}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)},$$

$$z\,{}_p^j\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) = \sum_{n=0}^{\infty}\frac{n\,{}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)}. \tag{4.7}$$

Again

$$\frac{d^2}{dz^2}[z^2\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)] = \sum_{n=0}^{\infty}\frac{(n+1)(n+2)\,{}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)}, \tag{4.8}$$

and

$$\frac{d^2}{dz^2}[z^2\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)]$$
$$= z^2\,{}_p^j\ddot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) + 4z\,{}_p^j\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) + 2\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z), \tag{4.9}$$

from equation (4.8) and (4.9) we have

$$\sum_{n=0}^{\infty}\frac{\{n^2\}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)} = z^2\,{}_p^j\ddot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)$$

$$+\,4z\,{}_p^j\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) - 3\sum_{n=0}^{\infty}\frac{\{n\}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)},$$

using equation (4.7), we have

$$\sum_{n=0}^{\infty}\frac{\{n^2\}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)}$$
$$= z^2\,{}_p^j\ddot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) + z\,{}_p^j\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z), \tag{4.10}$$

applying equation (4.4), (4.7) and (4.10) to (4.6), we get

$$\frac{k^3}{p^3}S = (\alpha+r)^2z^2\,{}_p^j\ddot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) + \Big[(\alpha+r)^2 + (\alpha+r)(2\beta+2s+2k)\Big]z$$

$$\times\,{}_p^j\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) + (\beta+s)(\beta+s+2k)\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z),$$

Hence.

Theorem 4.2 For $k,p \in R^+ - \{0\}$; $\alpha+r, \beta+s+k, \gamma \in C/kZ^-$; $R(\alpha+r) > 0, R(\beta+s+k) > 0, R(\gamma) > 0, q \in (0,1) \cup N, j \in N_0$, we get

$$\int_0^1 t^{\beta+s+k-1}\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s}(t^{\alpha+r})dt = \frac{p}{k}\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+k}(1) - \frac{p^2}{k}\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(1). \tag{4.11}$$

Proof: Put $z=1$ in equation (4.4) and (4.5), we have

$$S = \frac{p}{k}\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+k}(1) - \frac{p^2}{k}\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(1)$$

$$= \sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)\{n(\alpha+r)+\beta+s+k)\}((n+j)!)}, \tag{4.12}$$

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now consider the integral,

$$A \equiv \int_0^1 t^{\beta+s+k-1}\, {}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s}(t^{\alpha+r})dt,$$

using the equation (2.1), we have

$$A \equiv \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)((n+j)!)} \int_0^1 t^{n(\alpha+r)+\beta+s+k-1}dt,$$

$$A \equiv \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)\{n(\alpha+r)+\beta+s+k\}((n+j)!)},$$

from equation (4.12), we have the desired result.

Theorem 4.3 For $k, p \in R^+ - \{0\}; \alpha, \beta, \gamma, \delta \in C; R(\alpha) > 0, R(\beta) > 0, R(\gamma) > 0, R(\delta) > 0$ *and* $q \in (0,1) \cup N, j \in N_0$ then

$$p^\delta\, {}_p^jE^{\gamma,q}_{k,\alpha,\beta+\delta k}(z) = \frac{1}{\Gamma(\delta)} \int_0^1 u^{\frac{\beta}{k}-1}(1-u)^{\delta-1}\, {}_p^jE^{\gamma,q}_{k,\alpha,\beta}(z\, u^{\frac{\alpha}{k}})du. \tag{4.13}$$

Proof : Consider the right side integral and using equation (2.1), we have

$$A \equiv \frac{1}{\Gamma(\delta)} \int_0^1 u^{\frac{\beta}{k}-1}(1-u)^{\delta-1}\, {}_p^jE^{\gamma,q}_{k,\alpha,\beta}(z\, u^{\frac{\alpha}{k}})du,$$

$$A \equiv \frac{1}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}\, z^n}{{}_p\Gamma_k(n\alpha+\beta)((n+j)!)} \int_0^1 u^{\frac{\alpha n+\beta}{k}-1}(1-u)^{\delta-1}du,$$

using the definition of Beta function, we have

$$A \equiv \frac{1}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}\, z^n}{{}_p\Gamma_k(n\alpha+\beta)((n+j)!)} \frac{\Gamma(\frac{\alpha n+\beta}{k})\Gamma(\delta)}{\Gamma(\frac{\alpha n+\beta}{k}+\delta)},$$

applying equation (1.5), we have

$$A \equiv \sum_{n=0}^{\infty} \frac{p^\delta\, {}_p(\gamma)_{(n+j)q,k}\, z^n}{{}_p\Gamma_k(n\alpha+\beta+\delta k)((n+j)!)} = p^\delta\, {}_p^jE^{\gamma,q}_{k,\alpha,\beta+\delta k}(z),$$

Hence.

Theorem 4.4 For $k, p \in R^+ - \{0\}; \beta, \gamma \in C; R(\beta) > 0, R(\gamma) > 0$ *and* $\alpha, q \in N, j \in N_0$, then

$${}_p^jE^{\gamma,q}_{k,k\alpha,\beta}(z) = \frac{{}_p(\gamma)_{jq,k}}{{}_p\Gamma_k(\beta)} \prod_{i=1}^{q}\prod_{l=1}^{\alpha} \frac{\Gamma(b_l)}{\Gamma(a_i)\Gamma(b_l-a_i)} \int_0^1 u^{a_i-1}(1-u)^{b_l-a_i-1}E_{1,j+1}\Big(uz\frac{p^{q-\alpha}q^q}{\alpha^\alpha}\Big)du. \tag{4.14}$$

Where $a_i = \dfrac{\frac{\gamma}{k}+qj+i-1}{q}$ and $b_l = \dfrac{\frac{\beta}{k}+l-1}{\alpha}$.

Proof : Using definition of j-generalized p-k Mittag- Leffler function, from equation (2.1),

$$A \equiv {}_p^jE^{\gamma,q}_{k,k\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}\, z^n}{{}_p\Gamma_k(nk\alpha+\beta)((n+j)!)},$$

using relation (1.11),we have

$$A \equiv \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{jq,k}\, {}_p(\gamma + jqk)_{nq,k}\, z^n}{{}_p(\beta)_{n\alpha,k}\, {}_p\Gamma_k(\beta)((n+j)!)} = \sum_{n=0}^{\infty} D \frac{{}_p(\gamma)_{jq,k}\, z^n}{{}_p\Gamma_k(\beta)((n+j)!)}. \tag{4.15}$$

Where $D \equiv \frac{{}_p(\gamma + jqk)_{nq,k}}{{}_p(\beta)_{n\alpha,k}}$,
using equation (1.6), we have

$$D \equiv \frac{{}_p(\gamma + jqk)_{nq,k}}{{}_p(\beta)_{n\alpha,k}} = \frac{p^{n(q-\alpha)}(\frac{\gamma+jqk}{k})_{qn}}{(\frac{\beta}{k})_{n\alpha}},$$

using the relation given by equation (1.7), we have

$$D \equiv \frac{p^{(q-\alpha)n} q^{qn} \prod_{i=1}^{q}(\frac{\frac{\gamma}{k}+jq+i-1}{q})_n}{\alpha^{\alpha n}\prod_{l=1}^{\alpha}(\frac{\frac{\beta}{k}+l-1}{\alpha})_n},$$

$$let\; a_i = \frac{\frac{\gamma}{k}+jq+i-1}{q}\; and\; b_l = \frac{\frac{\beta}{k}+l-1}{\alpha},$$

$$D \equiv (\frac{p^{(q-\alpha)}q^q}{\alpha^\alpha})^n \prod_{i=1}^{q}\prod_{l=1}^{\alpha}\frac{(a_i)_n}{(b_l)_n},$$

$$D \equiv (\frac{p^{(q-\alpha)}q^q}{\alpha^\alpha})^n \prod_{i=1}^{q}\prod_{l=1}^{\alpha}\frac{\Gamma(a_i+n)\Gamma(b_l)}{\Gamma(b_l+n)\Gamma(a_i)},$$

$$D \equiv (\frac{p^{(q-\alpha)}q^q}{\alpha^\alpha})^n \prod_{i=1}^{q}\prod_{l=1}^{\alpha}\frac{\Gamma(b_l)}{\Gamma(b_l-a_i)\Gamma(a_i)}\frac{\Gamma(a_i+n)\Gamma(b_l-a_i)}{\Gamma(b_l-a_i+a_i+n)},$$

using the definition of Beta function, we have

$$D \equiv (\frac{p^{(q-\alpha)}q^q}{\alpha^\alpha})^n \prod_{i=1}^{q}\prod_{l=1}^{\alpha}\frac{\Gamma(b_l)}{\Gamma(b_l-a_i)\Gamma(a_i)}\int_0^1 u^{a_i+n-1}(1-u)^{b_l-a_i-1}du, \tag{4.16}$$

from equation (4.15) and (4.16), we have

$$A \equiv \frac{{}_p(\gamma)_{jq,k}}{{}_p\Gamma_k(\beta)}\prod_{i=1}^{q}\prod_{l=1}^{\alpha}\frac{\Gamma(b_l)}{\Gamma(b_j-a_i)\Gamma(a_i)}\int_0^1 u^{a_i-1}(1-u)^{b_j-a_i-1}\sum_{n=0}^{\infty}\frac{(uz)^n}{(n+j)!}(\frac{p^{(q-\alpha)}q^q}{\alpha^\alpha})^n du,$$

Hence.

Theorem 4.5 For $k, p \in R^+ - \{0\}; \alpha, \beta, \gamma \in C; R(\alpha) > 0, R(\beta) > 0, R(\gamma) > 0$ and $q \in (0,1) \cup N, j \in N_0$ then

$${}^j_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \frac{p^{jq}}{\Gamma(\frac{\gamma}{k})}\int_0^\infty e^{-t}\, t^{(\frac{\gamma}{k}+jq-1)}\, {}^j_pE^{1,0}_{k,\alpha,\beta}(zt^q p^q)dt. \tag{4.17}$$

Proof: Using definition of j-generalized p-k Mittag-Leffler function, equation (2.1), we have

$${}^j_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{(n+j)q,k}\, z^n}{{}_p\Gamma_k(n\alpha+\beta)((n+j)!)},$$

using equation (1.5) and (1.6), we have

$${}^j_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{{}_p\Gamma_k(n\alpha+\beta)((n+j)!)}\frac{p^{(n+j)q}\Gamma(\frac{\gamma}{k}+q(n+j))}{\Gamma(\frac{\gamma}{k})},$$

$$ {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{z^n}{{}_p\Gamma_k(n\alpha+\beta)((n+j)!)} \frac{p^{q(n+j)}}{\Gamma(\frac{\gamma}{k})} \int_0^{\infty} e^{-t}\, t^{(\frac{\gamma}{k}+q(n+j)-1)}dt, $$

$$ {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z) = \frac{p^{jq}}{\Gamma(\frac{\gamma}{k})} \int_0^{\infty} e^{-t}\, t^{(\frac{\gamma}{k}+jq-1)} \sum_{n=0}^{\infty} \frac{z^n\, p^{qn}\, t^{qn}}{{}_p\Gamma_k(n\alpha+\beta)((n+j)!)}dt, $$

$$ {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z) = \frac{p^{jq}}{\Gamma(\frac{\gamma}{k})} \int_0^{\infty} e^{-t}\, t^{(\frac{\gamma}{k}+jq-1)}\, {}_p^jE_{k,\alpha,\beta}^{1,0}(zt^qp^q)dt. $$

Hence.

5 Integral Transform of the j-generalized P-K Mittag-Leffler Function

In this section we evaluate Mellin-Barnes integral representation of j-generalized p-k Mittag-Leffler function, relationship with Fox H-function and Wright hypergeometric function. Also evaluate Euler Beta Transform, Laplace Transform and Mellin Transform of j-generalized p-k Mittag-Leffler function.

Mellin-Barnes integral representation of the j-generalized p-k Mittag-Leffler function.

Theorem 5.1 Let $k, p \in R^+ - \{0\}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, and\, q \in (0,1) \cup N, j \in N_0$, then the j-generalized p-k Mittag-Leffler function is represented by the Mellin-Barnes integral as,

$$ {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z) = \frac{kp^{jq-\frac{\beta}{k}}}{2\pi i\Gamma(\frac{\gamma}{k})} \int_L \frac{\Gamma(s)\Gamma(1-s)\Gamma(\frac{\gamma}{k}+jq-qs)}{\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})\Gamma(1+j-s)}(-zp^{q-\frac{\alpha}{k}})^{-s}ds. \tag{5.1} $$

Where $|\, argz \,| < \pi$; the contour integration beginning at $-i\infty$ and ending at $+i\infty$, and indented to separate the poles of the integrand as $s = -n$ for every $n \in N_0$(to the left) from those at $s = \frac{\frac{\gamma}{k}+n}{q}$ for every $n \in N_0$ (to the right).

Proof Consider the integral on right side of equation(5.1) and use the theorem of calculus of residues,

$$ A \equiv \frac{kp^{jq-\frac{\beta}{k}}}{2\pi i\Gamma(\frac{\gamma}{k})} \int_L \frac{\Gamma(s)\Gamma(1-s)\Gamma(\frac{\gamma}{k}+jq-qs)}{\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})\Gamma(1+j-s)}(-zp^{q-\frac{\alpha}{k}})^{-s}ds $$

$= 2\pi i$[sum of the residues at the poles $s = 0, -1, -2, ...$]

$$ A \equiv \frac{kp^{jq-\frac{\beta}{k}}}{\Gamma(\frac{\gamma}{k})} \sum_{n=0}^{\infty} \underset{s=-n}{Re} \quad (s+n)\Big[\frac{\Gamma(s)\Gamma(1-s)\Gamma(\frac{\gamma}{k}+jq-qs)}{\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})\Gamma(1+j-s)}\Big](-zp^{q-\frac{\alpha}{k}})^{-s} $$

$$ A \equiv \frac{kp^{jq-\frac{\beta}{k}}}{\Gamma(\frac{\gamma}{k})} \sum_{n=0}^{\infty} \lim_{s\to -n} \Big[\frac{\pi(s+n)}{sin\pi s} \frac{\Gamma(\frac{\gamma}{k}+jq-qs)}{\Gamma(1+j-s)\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})}\Big](-zp^{q-\frac{\alpha}{k}})^{-s} $$

$$ A \equiv \frac{kp^{jq-\frac{\beta}{k}}}{\Gamma(\frac{\gamma}{k})} \sum_{n=0}^{\infty} \Big[\frac{\Gamma(\frac{\gamma}{k}+jq+qn)}{\Gamma(1+j+s)\Gamma(\frac{\beta}{k}+\frac{\alpha n}{k})}\Big](zp^{q-\frac{\alpha}{k}})^n $$

using equations (1.5) and (1.6), we have,

$$ A \equiv {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z). $$

Hence.

Relationship with Fox H-function

Theorem 5.2 Let $k, p \in R^+ - \{0\}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, and\, q \in (0,1) \cup N, j \in N_0$

then the j-generalized p-k Mittag-Leffler function is given in the form of Fox H-function, as.

$$ {}_{p}^{j}E_{k,\alpha,\beta}^{\gamma,q}(z) = \frac{kp^{jq-\frac{\beta}{k}}}{\Gamma(\frac{\gamma}{k})} H_{2,3}^{1,2}\left[-zp^{q-\frac{\alpha}{k}} \middle| \begin{matrix} (0,1),(1-\frac{\gamma}{k}-jq,q); \\ (0,1),(1-\frac{\beta}{k},\frac{\alpha}{k}),(-j,1); \end{matrix}\right]. \tag{5.2} $$

Proof. Using the equations (5.1) and (1.25), we get the desired result.

Relationship with Wright hypergeometric function

Theorem 5.3 Let $k,p \in R^{+}-\{0\}; Re(\alpha)>0, Re(\beta)>0, Re(\gamma)>0, and\, q \in (0,1)\cup N, j \in N_0$, then the j-generalized p-k Mittag-Leffler function is given in the form of Wright hypergeometric function, as.

$$ {}_{p}^{j}E_{k,\alpha,\beta}^{\gamma,q}(z) = \frac{kp^{jq-\frac{\beta}{k}}}{\Gamma(\frac{\gamma}{k})}\ {}_{2}\Psi_{2}\left[\begin{matrix} (1,1)\ (\frac{\gamma}{k}+jq,q); \\ \qquad zp^{q-\frac{\alpha}{k}} \\ (\frac{\beta}{k},\frac{\alpha}{k})(1+j,1); \end{matrix}\right]. \tag{5.3} $$

Proof. Using the equations (5.1) and (1.24), we get the desired result.

Euler Beta Transform, Laplace Transform and Mellin Transform of j-generalized p-k Mittag-Leffler function

Theorem 5.4 Let $k,p \in R^{+}-\{0\}$; $a,b,\sigma \in C$; $Re(\alpha)>0, Re(\beta)>0, Re(\gamma)>0, Re(\sigma)>0$ and $q \in (0,1)\cup N, j \in N_0$, then Euler Beta Transform of j-generalized p-k Mittag-Leffler function, is given by,

$$ \int_0^1 z^{a-1}(1-z)^{b-1}\, {}_{p}^{j}E_{k,\alpha,\beta}^{\gamma,q}(xz^{\sigma})dz = \frac{kp^{jq-\frac{\beta}{k}}\Gamma(b)}{\Gamma(\frac{\gamma}{k})}\ {}_{3}\psi_{3}\left[\begin{matrix} (1,1),(\frac{\gamma}{k}+qj,q),(a,\sigma); \\ \qquad xp^{q-\frac{\alpha}{k}} \\ (\frac{\beta}{k},\frac{\alpha}{k}),(1+j,1),(a+b,\sigma); \end{matrix}\right] \tag{5.4} $$

Proof Consider the left side integral and using equation (5.4), we have

$$ A \equiv \int_0^1 z^{a-1}(1-z)^{b-1}\, {}_{p}^{j}E_{k,\alpha,\beta}^{\gamma,q}(xz^{\sigma})dz $$

$$ A \equiv \sum_{n=0}^{\infty} \frac{{}_{p}(\gamma)_{(n+j)q,k}\, x^n}{{}_{p}\Gamma_k(n\alpha+\beta)\,(n+j)!} \int_0^1 z^{\sigma n+a-1}(1-z)^{b-1}dz, $$

using definition of Beta function, we have

$$ A \equiv \sum_{n=0}^{\infty} \frac{{}_{p}(\gamma)_{(n+j)q,k}\, x^n}{{}_{p}\Gamma_k(n\alpha+\beta)\,(n+j)!} B(\sigma n+a,b) $$

using equation (1.5),(1.6) and (1.24), we have

$$ A \equiv \frac{kp^{jq-\frac{\beta}{k}}\Gamma(b)}{\Gamma(\frac{\gamma}{k})}\ {}_{3}\psi_{3}\left[\begin{matrix} (1,1),(\frac{\gamma}{k}+qj,q),(a,\sigma); \\ \qquad xp^{q-\frac{\alpha}{k}} \\ (\frac{\beta}{k},\frac{\alpha}{k}),(1+j,1),(a+b,\sigma); \end{matrix}\right]. $$

Hence.

Theorem 5.5 The Laplace transform of j-generalizedp-k Mittag-Leffler function, is given by,

$$ \int_0^{\infty} z^{a-1}e^{-zs}\, {}_{p}^{j}E_{k,\alpha,\beta}^{\gamma,q}(xz^{\sigma})dz = \frac{kp^{jq-\frac{\beta}{k}}\, s^{-a}}{\Gamma(\frac{\gamma}{k})}\ {}_{3}\psi_{2}\left[\begin{matrix} (1,1),(\frac{\gamma}{k}+qj,q),(a,\sigma); \\ \qquad \frac{xp^{q-\frac{\alpha}{k}}}{s^{\sigma}} \\ (\frac{\beta}{k},\frac{\alpha}{k})(1+j,1); \end{matrix}\right] \tag{5.5} $$

Where $k, p \in R^+ - \{0\}$; $a, \sigma \in C$; $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\sigma) > 0$ and $q \in (0,1) \cup N, j \in N_0$, and $| \frac{x}{s^\sigma} |< 1$.

Proof Consider the right side integral and using equation(2.1), we have

$$A \equiv \int_0^\infty z^{a-1} e^{-zs} \, {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(xz^\sigma)dz$$

$$A \equiv \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{x^n}{(n+j)!} \int_0^\infty z^{n\sigma+a-1} \, e^{-zs}dz,$$

using definition of gamma function, we have

$$A \equiv s^{-a} \quad \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)\,(n+j)!} \Gamma(\sigma n + a)(\frac{x}{s^\sigma})^n$$

using equation (1.5),(1.6) and (1.24), we have

$$A \equiv \frac{kp^{jq-\frac{\beta}{k}} \, s^{-a}}{\Gamma(\frac{\gamma}{k})} \, {}_3\psi_2 \left[\begin{array}{c} (1,1), (\frac{\gamma}{k}+qj, q), (a,\sigma); \\ \qquad\qquad \frac{xp^{q-\frac{\alpha}{k}}}{s^\sigma} \\ (\frac{\beta}{k}, \frac{\alpha}{k})(1+j,1); \end{array} \right].$$

Hence.

Theorem 5.6 The Mellin transform of j-generalized p-k Mittag-Leffler function, is given by,

$$\int_0^\infty t^{s-1} \, {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(-wt)dt = \frac{kp^{qj-\frac{\beta}{k}}\Gamma(s)\,\Gamma(1-s)\,\Gamma(\frac{\gamma}{k}+qj-qs)}{\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})\Gamma(\frac{\gamma}{k})\Gamma(1+j-s)} (\frac{p^{\frac{\alpha}{k}-q}}{w})^s \tag{5.6}$$

Where $k, p \in R^+ - \{0\}$; $a, \sigma \in C$; $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(s) > 0$ and $q \in (0,1) \cup N, j \in N_0$.

Proof Putting z = -wt in equation (5.1), we have

$${}_p^jE_{k,\alpha,\beta}^{\gamma,q}(-wt) = \frac{kp^{jq-\frac{\beta}{k}}}{2\pi i\Gamma(\frac{\gamma}{k})} \int_L \frac{\Gamma(s)\Gamma(1-s)\Gamma(\frac{\gamma}{k}+jq-qs)}{\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})\Gamma(1+j-s)} (-wtp^{q-\frac{\alpha}{k}})^{-s}ds.$$

$${}_p^jE_{k,\alpha,\beta}^{\gamma,q}(-wt) = \frac{kp^{jq-\frac{\beta}{k}}}{2\pi i\Gamma(\frac{\gamma}{k})} \int_L f^*(s)(t)^{-s}ds. \tag{5.7}$$

where

$$f^*(s) = \frac{\Gamma(s)\Gamma(1-s)\,\Gamma(\frac{\gamma}{k}+jq-qs)}{\Gamma(\frac{\gamma}{k})\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})\Gamma(1+j-s)} (-wp^{q-\frac{\alpha}{k}})^{-s}$$

using equation (1.28),(1.29) and (5.7), which immediately leads to (3.9).

Particular cases: Putting some particular values of $j, p, q, k, \alpha, \beta, \gamma$ we obtain all the results given by [1], [3],[8],[9],[10],[11],[12],[13]and [14].

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