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Article

No-Arbitrage Principle in Conic Finance

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Abstract: In a one price economy, the Fundamental Theorem of Asset Pricing (FTAP) establishes that no-arbitrage is equivalent to the existence of an equivalent martingale measure. Such an equivalent measure can be derived as the normal unit vector of the hyperplane that separates the attainable gain subspace and the convex cone representing arbitrage opportunities. However, in two-price financial models (where there is a bid-ask price spread), the set of attainable gains is not a subspace anymore. We use convex optimization, and the conic property of this region to characterize the "No-Arbitrage" principle in financial models with the bid-ask price spread present. This characterization will lead us to the generation of a set of price factor random variables. Under such a set, we can find the lower and upper bounds (supper-hedging and sub-hedging bounds) for the price of any future cash flow. We will show that for any given cash flow, for which the price is outside the above range, we can build a trading strategy that provides one with an arbitrage opportunity. We will generalize this structure to any two-price finite-period financial model.

Keywords: Conic Finance; Convex Optimization; Arbitrage Pricing

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1. Introduction

It has been known for a long time (Harrison & Kreps [15], Harrison & Pliska [16], Delbaen & Schachermayer [9]) that no arbitrage opportunity in a one price economy is equivalent to the existence of an equivalent martingale measure. This result is usually referred to as the Fundamental Theorem of Asset Pricing (FTAP). Rogers, L.C.G. [23] used a directional derivative argument to provide a discrete-time proof for FTAP. The case of trading continuously, in particular, was discussed by Harrison, J. M., and S. R. Pliska [16].

In practice, the law of one price often does not hold. Recently, the literature on transaction costs and the study of no-arbitrage in the presence of bid-ask spreads has been expanding (Jouini and Kallal [18], Bion-Nadal [2], Guasoni, Lepinette and Rasonyi [14]). Madan [21] considers financial equilibrium where there are two separate prices at which one may buy from or sell to the market known as ask and bid prices. This is a realistic situation in financial markets. Inside an efficient market, at a given time, a financial security trades at a unique price. The equilibrium conditions that are needed to provide a unique price all depend on arbitrage opportunities that are quickly exploited. A wide category of financial securities trade in vast and diverse markets and few of them meet the equilibrium conditions we mentioned earlier. Market clearing becomes troubling, which leads to prices that are not unique for equivalent securities in different markets. Whenever the above conditions are not satisfied, the "law of one price" fails to hold. On the other hand, an incomplete market can also take place even under the assumption of "one price". Incomplete condition explains that there exists the presence of some residual risks that can not be eliminated regardless of the existence of the best hedging (Eberlein, Gehrig & Madan [10], and Jacka SD [17]). Also, the markets establish a phenomenon that is not anticipated in the one price theory called illiquidity. Illiquidity can be explained as the lack of ability of the market to establish a unique price, that is, to eliminate the spread between the bid and ask prices. This happens when there is an absence of information and/or a lack of interested parties. In these situations, the bid and ask prices are the only real market information that can be observed. Thus, the theoretical framework that has been used to explain the one price market is not accurate enough to deal with many situations one faces in practice.

We adopt the framework of "conic finance" in this paper. Some earlier studies that use the theory of conic finance include Madan [21], Cherny & Madan [6], Eberlein, Eberlein & Madan [12], Eberlein & Madan [11], Madan & Schoutens [13]. The direction of trade is what distinguishes the theory of two price economies from that of one price economy. Now there are two types of prices, one price is used for buying from the market called the ask price, and the other price is used for selling to the market called the bid price. In the situation of one price economy, the market plays the role of an auctioneer, who clears the trades and decides the prices. However, in the two-price economy, the financial market plays a role of a passive counterparty to all transactions, who buys at the ask price and sells at the bid price. The spread that is between the bid and ask prices becomes a measure of illiquidity. It also evaluates the principal required to support a position and the cost of unraveling a position.

We also find it necessary to mention that the one-period case discussed in this paper was included in the book by Carr-Zhu [4] with reference to this paper (a working paper back then).

1.1. Bid-Ask Spreads

A collection of theoretical approaches exists that are trying to model bid-ask spreads. Cherny & Madan [6] offers a few of the current approaches that have been used to model bid-ask spreads. Copeland & D.Galai [8] have discussed the order processing and inventory costs of providers of liquidity. Constantinides & Lapied [7] and Jouini & Kallal [18] investigated the spreads that involve transaction costs of trading in liquid markets. However, the researches mentioned above are comparatively satisfactory to liquid markets where a transaction cost can happen at a price that a contrary trade direction can take place with no price effect. The price-spreads that are related to the theory of two-price economies are discussed in Carr, Geman & Madan [3] and Cherny & Madan [6]. These are mostly related to locating actual long-term counterparties that are ready to establish a position for an extended period of time. The spread between the bid and ask prices can be observed as a holding of the charge-exerted while the market does not clear rapidly, as to find a counterparty is going to require time and effort because there is no possibility of doing trades in both directions at any transaction price being observed. Specifically, there is not any possibility of complete replication. Also, the spread between the bid and ask prices becomes a reflection of the cost of holding the residual risk (Eberlein, Madan & Schoutens [13]). Therefore, transactions happen close to or at either the ask price or the bid price, conditional on the direction of the trade. The conic finance is trying to model the bid-ask spread by applying the concept of acceptable cash flows to the market (Madan [22]). However, it is assumed that the market requires a minimal level of acceptability for a position to be profitable. Because of the market competition, the bid-price is being raised, and the asked-price is being lowered to establish an acceptable position. Therefore the bid-ask spread is tried to be narrowed so that the risk of a position will be minimized. This spread can be seen as the cost of unraveling a position.

1.2. Two Price Economy

In a two price economy, a comparatively classical view of markets consistent with its role in the traditional competitive breakdown, where markets play the role of counterparties to transactions, is being assumed. The only difference from the traditional aspect is that the length of a trade depends on the direction of the trade, while the market is buying at the bid price and selling at the ask price. Now regard the classical market when trading is performed in either direction at the current price. The market is always willing to sell at a higher price or similarly buy at a lower price and welcomes all zero-cost random cash streams that have a positive expectation under the equilibrium pricing essentials. Notice that this creates a very large set of market risks that are accepted by the classical financial market within a risk-neutral measure. The two-price financial market is more antagonistic as to which trades it will accept. The collection of zero-cost risks acceptable to the financial market is much smaller as a set. A modeling process of this set of acceptable risks can be observed at Artzner, Delbaen, Eber & Heath [1]. , that was further expanded in Cherny & Madan [5] and Cherny & Madan [6]. Especially, the set of zero-cost risks that are acceptable to the financial market as a set of random variables is being modeled as a convex cone, including all the nonnegative random variables. The theoretical structure required for supporting the two-price economy has been given a huge amount of attention in the past few years. The theory was popularized into the field of mathematical finance by Constantinides & Lapied [7] and was introduced as coherent risk measures by Artzner et al. [1]. The research that connects the theory of two-price economies to concave distortions was performed by Cherny & Madan [5] and Cherny & Madan [6]. They are the researches that gave a perceptive of how to establish the bid-ask prices admissible to the theory of two-prices. After that, the theory of two-prices was given the name "conic finance". In the following sections, we present the theory of two prices in an abstract manner as it was set out in Madan [22].

2. The Model (Multi-Period)

In this section, we first define the general multi-period model along with all of the necessary components of the model. We establish the FTAP theorem for the multi-period model in conic finance (two price economy) by considering a utility maximization problem and its dual. We also explain the relation between solutions to our primal and dual problems, along with the price factors derived from the solution of the dual problem, by considering a utility maximization problem and its dual. We use this price factor to find a super-hedging and sub-hedging for any acceptable cash flow.

2.1. The Model Definitions

We start by defining the general model and it's components following Madan (2015) [22].

Definition 1. *Let* $\mathbb{F} = \{\{\emptyset, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T = \mathcal{F}\}$ *be an information filtration on the probability space* $(\Omega, \mathcal{F}, P)$ *, where* $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$ *is a finite sample space, representing the economic states. We consider a T-period financial model where* $T \geq 1$ *and for any* $0 \leq t \leq T$*, cash flows (in\out) are being traded at time t.*

Definition 2. *Let* $\mathcal{X}$ *denote the space of all* $\mathcal{F}-$*adapted cash flows* $x = (x_t)_{t=0}^{T}$ *with the inner product*

$$< x, y >= E\left[\sum_{t=o}^{T} x_t y_t\right]$$

Then $\mathcal{X}$ *is a finite dimensional Hilbert space.*

The market consists of M risky assets $S^m \in \mathcal{X}, m = 1, 2, \ldots, M$ and T risk-free zero coupon bonds $1^u, u = 1, 2, \ldots, T$ where $1^u_u = 1$ and $1^u_t = 0$ for $t \neq u$. At time t, there is a bid and ask price pair $b^i_t \leq a^i_t$. Paying a^i_t, one will receive the income flow $(S^i_t)_{s=t+1}^{T}$ and receiving b^i_t one will get the income flow $(-S^i_t)_{s=t+1}^{T}$. For risk-less bonds, the bid and ask prices of 1^u, paid/received at time $t < u$ is denoted by $g^u_t \leq h^u_t \leq 1$. There are two zero cost cash flows associated with each pair of bid and ask prices.

$$S^{it}_s = \begin{cases} 0 & s < t \\ -a^i_t & s = t \\ S^i_s & s > t \end{cases} \qquad \text{and} \qquad \tilde{S}^{it}_s = \begin{cases} 0 & s < t \\ b^i_t & s = t \\ -S^i_s & s > t \end{cases}$$

Similarly the bond maturing at time u, creates two zero cost cash flows as below:

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$$1_s^{ut} = \begin{cases} 0 & s \neq u,t \\ -h_t^u & s = t \\ 1 & s = u \end{cases} \qquad \text{and} \qquad \tilde{1}_s^{ut} = \begin{cases} 0 & s \neq u,t \\ g_t^u & s = t \\ -1 & s = u \end{cases}$$

2.2. Set of Zero-Cost Cash Flows

Assuming that one can trade any non-negative multiple of above zero-cost cash flows, and suppose $\alpha_t^i, \tilde{\alpha}_t^i, \beta_t^u, \tilde{\beta}_t^u$, for $i = 1, \ldots, M, u = 1, \ldots, T$ are non-negative $\mathcal{F}$-adapted random variables, then $\mathcal{Z}$ is defined to be the set of all zero cost cash flows of the form

$$z = \sum_{t=0}^{T} \left[\sum_{i=1}^{M} \left(\alpha_t^i S^{it} + \tilde{\alpha}_t^i \tilde{S}^{it} \right) + \sum_{u=1}^{T} \left(\beta_t^u 1_s^{ut} + \tilde{\beta}_t^u \tilde{1}_s^{ut} \right) \right] \tag{1}$$

It can be seen from the definition that $\mathcal{Z}$ is a cone. Define $\mathcal{C}$ to be the set of all cash flows $c \in \mathcal{X}$ such that there is $z \in \mathcal{Z}$ with $z \geq c$. Then $\mathcal{C}$ is the set of adapted processes, super-replicable at zero cost. It is clear that $\mathcal{C}$ a closed convex cone and $\mathcal{Z} \subset \mathcal{C}$.

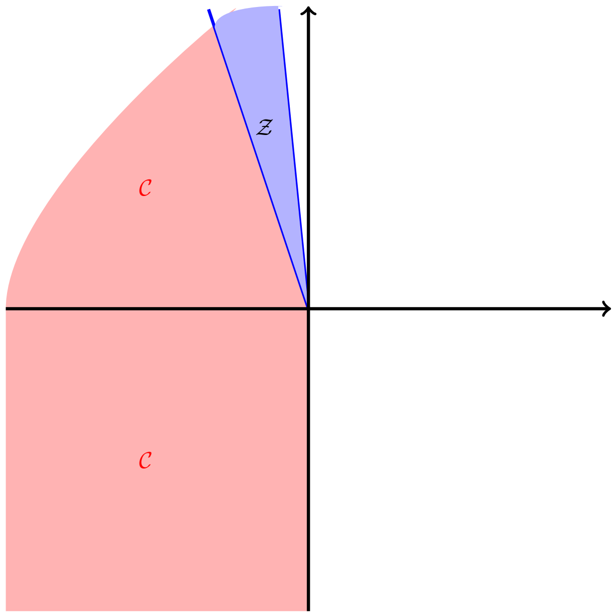

Figure 1. Set of Zero Cost Cash Flows $\mathcal{Z}$, and the cone $\mathcal{C}$

2.3. The Characterization of No Arbitrage

We now define arbitrage and will be characterizing no-arbitrage using a utility maximization problem.

Definition 3. *We say that a cash flow $c = (c_t)_{t=0}^T \in \mathcal{C} \backslash \{0\}$ is an* arbitrage *if $c_t \geq 0$ for all $t = 0, 1, \ldots, T$.*

Definition 4. *(Utility Function) An extended real valued function $u : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is called a utility function if it satisfies the following three characteristics:*

1. *(Risk Aversion) u is strictly concave,*
2. *(Profit Seeking) u is strictly increasing and $\lim_{t \to +\infty} u(t) = +\infty$,*
3. *(Bankruptcy Forbidden) for any $t < 0$, $u(t) = -\infty$.*

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Let $\mathcal{X}^+$ denote the non-negative cone in $\mathcal{X}$, also $w_0 \in \mathcal{X}^+$ be the initial endowment and $(c_t)_{t=0}^T = (c_0, c_1, ..., c_T) \in \mathcal{C}$ be a cash flow. Consider the following optimal trading problem

$$p = \max \left\{ E\left[\sum_{t=0}^{T} u(c_t)\right] ; \ c = (c_t)_{t=0}^T \in w_0 + \mathcal{C} \right\}$$

Where u is a utility function. We are able to characterize the no-arbitrage principle in terms of this utility maximization problem.

Theorem 1 (Characterization of Utility Maximization)**.** *The financial market has no arbitrage opportunity if and only if the optimal trading problem above has a finite value $p < \infty$.*

Proof. ($\Rightarrow$) If there is some arbitrage $w_0 + (c_t)_{t=0}^T \in w_0 + \mathcal{C}$, then, for any $r > 0$, since $\mathcal{C}$ is a convex cone, $w_0 + (rc_t)_{t=0}^T \in w_0 + \mathcal{C}$. But when $r \to +\infty$, we know that

$$\lim_{r \to +\infty} u(w_0 + rc_t) = +\infty$$

therefore $p = +\infty$.

($\Leftarrow$) If $p = +\infty$, there exists a sequence $c^n = (c_t^n)_{t=0}^T \in \mathcal{C}$ for which

$$\lim_{n \to \infty} E\left[\sum_{t=0}^{T} u(c_t^n + w_0)\right] \to +\infty$$

Since u is continuous and increasing, we have $||c^n|| \to \infty$. For $0 \le t \le T$, let $b_t^n = \dfrac{c_t^n}{||c^n||}$ and suppose

$$b^n = (b_t^n)_{t=0}^T \to b = (b_t)_{t=0}^T \in \mathcal{C}\backslash\{0\}, \text{ since } \ 1 = ||b^n|| \to ||b||$$

then $b = (b_t)_{t=0}^T \neq 0$. Also by the condition on the domain of utility function $u(c_t^n + w_0)$, we need to have $c_t^n + w_0 > 0$ or $c_t^n > -w_0$. Dividing both sides by $||c^n||$ we have

$$b_t^n = \frac{c_t^n}{||c^n||} > -\frac{w_0}{||c^n||} \to 0$$

so that $b_t \ge 0$ for $0 \le t \le T$. This means that $b = (b_t)_{t=0}^T$ is an arbitrage. $\square$

3. FTAP for Multi-Period Model

In this section, we establish FTAP for conic finance model, characterizing no-arbitrage in terms of the extension of pricing factors. A first version of this theorem was proven by M. Harrison and D. Kreps in 1979 [15]. More general versions of the theorem were proven in 1981 by M. Harrison [16] and S. Pliska and in 1994 by F. Delbaen and W. Schachermayer [9].

Theorem 2 (The First Fundamental Theorem of Asset Pricing)**.** *A financial market with time horizon T and price processes of the risky asset and risk-less bond given by $S_1, ..., S_T$ and $S_1^0, ..., S_T^0$, respectively, is arbitrage-free under the probability P if and only if there exists another probability measure Q such that*

i. For any event A, $P(A) = 0$ if and only if $Q(A) = 0$ (We say in this case that P and P are equivalent probability measures).

ii. The discounted price process, $X_1 := \frac{S_1}{S_1^0}, ..., XT := \frac{S_T}{S_T^0}$ is a martingale under Q.

A measure Q that satisfies (i) and (ii) is known as a risk neutral measure.

We now start by considering dual of the utility maximization problem in section 2. First, we rephrase the utility maximization problem in section 2 as an abstract convex programming problem.

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For $x = (x_t)_{t=0}^T \in \mathcal{X}$, define $g(x) = \iota_{w_0+\mathcal{C}}(x)$ and $f(x) = E\left[\sum_{t=0}^T(-u)(x_t)\right]$. Then we can rewrite the optimal trading problem (1) as

$$\begin{aligned} p &= -\min\left\{E\left[\sum_{t=0}^T(-u)(c_t)\right] \mid c = (c)_{t=0}^T \in w_0 + \mathcal{C}\right\} \\ &= -\min_{x\in\mathcal{X}}\left\{E\left[\sum_{t=0}^T(-u)(x_t)\right] + \iota_{w_0+\mathcal{C}}(x)\right\} \\ &= -\min_{x\in\mathcal{X}}\{f(x)+g(x)\} \end{aligned}$$

The dual representation of the problem above gives us an extension of the Fundamental Theorem of Asset Pricing (FTAP) in conic finance. We approach FTAP via an utility optimization problem.

Definition 5 (Pricing Factor). *For $t < u \le T$, we say f_t^u is a pricing factor from time t to time u and compatible with financial market, if the two following conditions are satisfied*

$$g_t^u \le \mathbf{E}_t\left[\sum_{s=t+1}^T f_t^u 1_s^{ut}\right] \le h_t^u, \tag{2}$$

and also for $1 \le i \le M$

$$b_t^i \le \mathbf{E}_t\left[\sum_{s=t+1}^T f_t^u S_t^i\right] \le a_t^i.$$

Theorem 3. *(FTAP in conic finance) Let $u(t)$ be a utility function and consider a two price financial market consisting of price processes of the risky assets and risk-free bonds. Then the market is arbitrage-free under the probability P if and only if there exists a pricing factor.*

Proof. The sufficient condition of the theorem is clear. For the necessary part, consider the primal problem above. We know that the initial endowment $\varpi \ge 0$, also $dom(g) = \varpi + \mathcal{C}$ and $dom(f) = \mathbb{R}_+^2$ so that $\varpi \in dom(g) \cap cont(f) \neq \emptyset$ and the Constraint Qualification is satisfied, where $cont(f)$ is the support of f on which f is continuous. Since Constraint Qualification holds, by applying the *Strong Duality* for the primal p and dual problem

$$d = -\max_{x\in\mathcal{X}^*}\{-f^*(-x^*) - g^*(x^*)\}$$

we have $p = d < \infty$. Therefore,

$$\begin{aligned} p &= -\max_{z\in\mathcal{X}^*}\{-f^*(-z) - g^*(z)\} \\ &= \min_{z\in\mathcal{X}^*}\{f^*(-z) + g^*(z)\} \\ &= \min_{z\in\mathcal{X}^*}\left\{E\left[\sum_{t=0}^T(-u)^*(-z_t)\right] + \langle z, w_0\rangle + \sigma_{\mathcal{C}}(z)\right\} \end{aligned}$$

If $\bar{x}, \bar{z}$ are the solutions to the primal and dual problems, then $0 < \bar{z} \in -\partial(-u)(\bar{x})$. Also we need to have $\bar{z} \in \mathcal{C}^o = \{z | \langle z, c\rangle \le 0 \text{ for all } c \in \mathcal{C}\}$. Let $\mathbf{E}_t$ be the conditional expectation with respect to $\mathcal{F}_t$. Since for any set $O \in \mathcal{F}_t$, $\chi_O 1^{ut}, \chi_O \tilde{1}^{ut} \in \mathcal{Z} \subset \mathcal{C}$, we have

$$\langle \bar{z}, \chi_O 1^{ut}\rangle \le 0 \text{ and } \langle \bar{z}, \chi_O \tilde{1}^{ut}\rangle \le 0$$

which implies $g_t^u \bar{z}_t \le \mathbf{E}_t\left[\bar{z}_u\right] \le h_t^u \bar{z}_t$ or

$$g_t^u \le \mathbf{E}_t\left[\sum_{s=t+1}^{T} \frac{\bar{z}_u}{\bar{z}_t} 1_s^{ut}\right] \le h_t^u \tag{3}$$

Using the definition above and by letting $f_t^u = \dfrac{\bar{z}_u}{\bar{z}_t}$ as the pricing factor, equation (3) can be written as

$$g_t^u \le \mathbf{E}_t\left[\sum_{s=t+1}^{T} f_t^u 1_s^{ut}\right] \le h_t^u \tag{4}$$

Also for any $O \in \mathcal{F}_t$, $\chi_O S^{it}, \chi_O \tilde{S}^{ut} \in \mathcal{Z} \subset \mathcal{C}$ (here χ_O is the characteristic function on set O), so that

$$b_t^i \bar{z}_t \le \mathbf{E}_t\left[\sum_{s=t+1}^{T} \bar{z}_s S_s^i\right] \le a_t^i \bar{z}_t, \quad \text{wihch implies}$$

$$b_t^i \le \mathbf{E}_t\left[\sum_{s=t+1}^{T} f_t^u S_t^i\right] \le a_t^i.$$

□

Remark 1. *Through the proof we gave above, there are a few things we would like to point out here:*

1. *Pricing factor is not unique. This is clear since the existence of a pricing factor in FTAP comes from the existence of a solution to the dual problem, and we know the dual problem does not necessary have a unique solution.*
2. *The pricing factor is related to the utility function u via the duality. In fact we saw specifically that $\bar{z} \in -\partial(-u)(\bar{x})$.*
3. *The pricing factors can be used to generate prices for cash flows $c \in \mathcal{C}$ with no arbitrage existing. We will explain this more in detail on the coming sub-section.*

4. Price Bounds and Their Estimates

A distinct feature of conic finance is that prices of assets are not unique. In fact, we see that every price factor will provide a non-arbitrage price. We will see that the set of price factors will provide us price bounds, outside of which arbitrage arises.

4.1. Definition of the Bounds

Define $\mathcal{PF}$ to be the set of all price factors f_t^s that are compatible with the financial market which is

$$\mathcal{PF} = \left\{ f_t^s \,|\, g_t^u \le \mathbf{E}_t\left[\sum_{s=t+1}^{T} f_t^u 1_s^{ut}\right] \le h_t^u,\ b_t^i \le \mathbf{E}_t\left[\sum_{s=t+1}^{T} f_t^s S_s^i\right] \le a_t^i \right\} \tag{5}$$

Let $c = (c_t) \in \mathcal{C}$. We define

$$u_t(c) = \sup\left\{ \mathbf{E}_t\left[\sum_{s=t+1}^{T} f_t^s c_s\right] \,|\, f_t^s \in \mathcal{PF} \right\} \tag{6}$$

and

$$l_t(c) = \inf\left\{ \mathbf{E}_t\left[\sum_{s=t+1}^{T} f_t^s c_s\right] \,|\, f_t^s \in \mathcal{PF} \right\} \tag{7}$$

Remark 2. *The values of $u_t(c)$ and $l_t(c)$ are the lower and upper bounds for bid and ask prices of the cash flow $c = (c_t) \in \mathcal{C}$, equivalent to existing no arbitrage. Equivalently, $[l_t(c), u_t(c)]$ is the no arbitrage region for the price of a given cash flow $c = (c_t) \in \mathcal{C}$.*

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Remark 3. *As we know, any of $f_t^u \in \mathcal{PF}$ has a one to one correspondence relation to a solution of the dual problem say $\bar{z} \in \mathcal{C}^o$ so that $\bar{z}_t = \Gamma_t M_t = \Gamma_t \dfrac{Q}{P}$, where Q is a P-equivalent martingale measure defined as $Q = M_t P$. Since f_t^u are in direct relation to the solution of dual problem, this leads to*

$$\mathbf{E}_t \left[\sum_{s=t+1}^{T} \bar{z}_s c_s \right] = \mathbf{E}_t \left[\bar{z}_t \sum_{s=t+1}^{T} \frac{\bar{z}_s}{\bar{z}_t} c_s \right] = \bar{z}_t \mathbf{E}_t \left[\sum_{s=t+1}^{T} f_t^s c_s \right]$$

which is a linear combination in terms of the pricing factor we defined above, so that the corresponding maximization and minimization above (and later in our text) becomes a linear programming problem.

A question that one can ask is, if the market (bid and ask) prices of a cash flow $c = (c_t) \in \mathcal{C}$, are outside of the no arbitrage region $[l_t(c), u_t(c)]$, will any arbitrage opportunity become possible? We will answer this question in the next section.

4.2. Computations in One-Period

To illustrate the general pattern, we start with a simple example.

Example 1. *Consider a one period model ($T = 1$) with a sample space $\Omega = \{\omega_1, \omega_2\}$ containing only two elements and let $M = 1$ (only one risky asset, $S^1 = S$). Also consider a bond*

$$1^{1,0} = (-h_0^1, 1) \qquad\qquad \tilde{1}^{1,0} = (g_0^1, -1)$$

where the ask and prices are $h_0^1 = 0.9$ and $g_0^1 = 0.8$ respectively. Assume the asset S_t over times $t = 0, 1$ has standard initial price 1 and payoff as following

$$\begin{tikzcd}
 & S_1(\omega_1) = 2 \\
S_0 = 1 \arrow[ur] \arrow[dr] & \\
 & S_1(\omega_2) = 0.5
\end{tikzcd}$$

Now consider the following primal maximization problem

$$p_1 := \begin{cases} u_0 = \max \mathbf{E}\left[f_0^1 S_1\right] \\ g_0^1 \leq \mathbf{E}\left[f_0^1\right] \leq h_0^1 \end{cases} = \begin{cases} u_0 = \max \left[f_0^1(\omega_1)P(\omega_1)S_1(\omega_1) + f_0^1(\omega_2)P(\omega_2)S_1(\omega_2)\right] \\ g_0^1 \leq f_0^1(\omega_1)P(\omega_1) + f_0^1(\omega_2)P(\omega_2) \leq h_0^1 \end{cases} \tag{8}$$

For simplicity we let $r_1 := f_0^1(\omega_1)P(\omega_1)$ and $r_2 := f_0^1(\omega_2)P(\omega_2)$ so that the linear programming problem (8) becomes

$$\begin{aligned} p_1 &= \begin{cases} u_0 = \max\left[r_1 S_1(\omega_1) + r_2 S_1(\omega_2)\right] \\ g_0^1 \leq r_1 + r_2 \leq h_0^1 \end{cases} \\ &= \begin{cases} u_0 = \max\left[r_1 S_1(\omega_1) + r_2 S_1(\omega_2)\right] \\ r_1 + r_2 \leq h_0^1 \\ -r_1 - r_2 \leq -g_0^1 \end{cases} \end{aligned} \tag{9}$$

A systematical way of deriving the arbitrage strategy is to use linear programming duality. By using the linear programming duality, we can write the dual problem to (9) as

$$d_1 := \begin{cases} u_0 = \min\left(-g_0^1 t_1 + h_0^1 t_2\right) \\ t_2 - t_1 \geq S_1(\omega_1) \\ t_2 - t_1 \geq S_1(\omega_2) \\ t_1, t_2 \geq 0 \end{cases}$$

$$= \begin{cases} u_0 = \min\left(-g_0^1 t_1 + h_0^1 t_2\right) \\ t_2 \ \geq \ t_1 + \max\limits_{\omega\in\Omega} \ S_1(\omega) \\ t_1, t_2 \geq 0 \end{cases} \tag{10}$$

If we use the constraint condition in (10) and substitute in the objective function, we have

$$u_0 = \min_{t_1 \geq 0}\left[-g_0^1 t_1 + h_0^1\left(t_1 + \max_{\omega\in\Omega} \ S_1(\omega)\right)\right] = \min_{t_1\geq 0}\left[t_1\left(h_0^1 - g_0^1\right) + h_0^1 \max_{\omega\in\Omega} \ S_1(\omega)\right]$$

but since $h_0^1 - g_0^1 \geq 0$, minimum happens when $t_1 = 0$ so that

$$u_0 = h_0^1\left(\max_{\omega\in\Omega} \ S_1(\omega)\right) = 0.9\ (2) = 1.8 \tag{11}$$

Therefore 1.8 is the highest ask price for S_1 offered at time $t = 0$ under no-arbitrage condition. That means for any offered price higher than 1.8 one should be able to build a portfolio creating an arbitrage.

In fact for a bid price $b_0 > 1.8$, since the solution to (10) is $t_1 = 0, t_2 = 2$, one can buy 2 unites of bond $1^{1,0} = (-0.9, 1)$ at time $t = 0$ for the price of 1.8 and receive 2 at time $t = 1$ so that S_1 can be delivered and still making a profit of $\dfrac{b_0 - 1.8}{0.9} > 0$.

Similarly we can find a lowest bid price l_0 for which one can construct a portfolio providing arbitrage if any ask price lower than l_0 is available.

The approach illustrated above can also be used in financial market that involves both bonds and stocks. A general result is summarized in the following theorem. We will use the set of all possible pricing factors corresponding to all solutions of the dual problem to provide a no-arbitrage region for a given cash flow. We state the result for super-hedging, i.e. upper bound for no-arbitrage. Sub-hedging can be discussed similarly.

Theorem 4 (Super-Hedging). *Suppose that $(c_0, c_1) \in \mathcal{C}$ is an acceptable cash flow, then*

$$u_0 = \sup_{f_0^1 \in \mathcal{PF}}\left\{\mathbf{E}\left[f_0^1 c_1\right]\right\}$$

$$= \min_{(\Lambda,\gamma)_0 \in \mathbb{R}_+^{2M+2}}\left\{-p_0((\Lambda,\gamma)_0) + \sup_{\omega\in\Omega}\left[g_0^1\left(c_1(\omega) - p_1((\Lambda,\gamma)_0)(\omega)\right)\right]\right\}$$

is a super-hedging bound, where

$$\begin{cases} p_0((\Lambda,\gamma)_0) = -\gamma_0^1 h_0^1 + \tilde{\gamma}_0^1 g_0^1 + \sum_{i=1}^M \left[-\lambda_0^i a_0^i + \tilde{\lambda}_0^i b_0^i\right] \\ p_1((\Lambda,\gamma)_0) = \gamma_0^1 - \tilde{\gamma}_0^1 + \sum_{i=1}^M \left[\lambda_0^i - \tilde{\lambda}_0^i\right] S_1^i \end{cases}$$

If b_0 is a bid price that exceeds a super-hedging bound u_0, then the solution to the minimization problem (dual) is a trading strategy $(\bar{\Lambda}, \bar{\gamma})_0 \in \mathbb{R}_+^{2M+2}$ for which if we acquire the zero cost cash flows $\left(p_0\left((\bar{\Lambda},\bar{\gamma})_0\right), p_1\left((\bar{\Lambda},\bar{\gamma})_0\right)\right)$, then our portfolio has an arbitrage opportunity.

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Proof. Let $c = (c_0, c_1) \in \mathcal{C}$ and

$$u_0 = \sup_{f_0^1 \in \mathcal{P}} \left\{ \mathbf{E}\left[f_0^1 c_1 \right] \right\} = p$$

subject to

$$g_0^1 \leq \mathbf{E}\left[f_0^1 \right] \leq h_0^1 \text{ and}$$

$$b_0^1 \leq \mathbf{E}\left[f_0^1 S_1^i \right] \leq a_0^1 \,,\, 1 \leq i \leq M$$

Here we will be formulating the dual problem. To that purpose define the Lagrangian as

$$\begin{aligned} &L(f_0^1, (\Lambda,\gamma)_0) = \\ &\mathbf{E}\left[f_0^1 c_1 \right] + \left\{ \sum_{i=1}^{M} \lambda_0^i \left(a_0^i - \mathbf{E}\left[f_0^1 S_1^i \right] \right) \right\} \\ &+ \left\{ \sum_{i=1}^{M} \tilde{\lambda}_0^i \left(\mathbf{E}\left[f_0^1 S_1^i \right] - b_0^i \right) \right\} + \left\{ \gamma_0^1 \left(h_0^1 - \mathbf{E}\left[f_0^1 \right] \right) \right\} \\ &+ \left\{ \tilde{\gamma}_0^1 \left(\mathbf{E}\left[f_0^1 \right] - g_0^1 \right) \right\} \end{aligned}$$

where

$$(\Lambda,\gamma)_0 = (\lambda_0^1, \ldots, \lambda_0^M, \tilde{\lambda}_0^1, \ldots, \tilde{\lambda}_0^M, \gamma_0^1, \tilde{\gamma}_0^1) \in \mathbb{R}_+^{2M+2} \tag{12}$$

and all $\lambda_0^i, \tilde{\lambda}_0^i, \gamma_0^1, \tilde{\gamma}_0^1$ for $i = 1, \ldots, M$ are non-negative constants.

First, it can be seen that

$$\inf_{(\Lambda,\gamma)_0 \in \mathbb{R}_+^{2M}} L(f_0^1, (\Lambda,\gamma)_0) = \begin{cases} \mathbf{E}\left[f_0^1 c_1 \right] & f_0^1 \in \mathcal{P} \\ -\infty & otherwise \end{cases}$$

Therefore, we can write

$$\sup_{f_0^1 \in \mathcal{P}} \left\{ \mathbf{E}\left[f_0^1 c_1 \right] \right\} = \sup_{f_0^1 \in \mathcal{P}} \inf_{(\Lambda,\gamma)_0 \in \mathbb{R}_+^{2M+2}} L(f_0^1, (\Lambda,\gamma)_0)$$

On the other hand, for non-negative constants

$$(\Lambda,\gamma)_0 = (\lambda_0^1, \ldots, \lambda_0^M, \tilde{\lambda}_0^1, \ldots, \tilde{\lambda}_0^M, \gamma_0^1, \tilde{\gamma}_0^1) \in \mathbb{R}_+^{2M+2}$$

we can recognize

$$\begin{aligned} &\gamma_0^1 1^{10} + \tilde{\gamma}_0^1 1^{10} + \sum_{i=1}^{M} \left(\lambda_0^i S^{i0} + \tilde{\lambda}_0^i S^{i0} \right) \\ &= \left(-\gamma_0^1 h_0^1, \gamma_0^1 \right) + \left(\tilde{\gamma}_0^1 g_0^1, -\tilde{\gamma}_0^1 \right) + \sum_{i=1}^{M} \left[\left(-\lambda_0^i a_0^i, \lambda_0^i S_1^i \right) + \left(\tilde{\lambda}_0^i b_0^i, -\tilde{\lambda}_0^i S_1^i \right) \right] \\ &= \left(-\gamma_0^1 h_0^1 + \tilde{\gamma}_0^1 g_0^1 + \sum_{i=1}^{M} \left[-\lambda_0^i a_0^i + \tilde{\lambda}_0^i b_0^i \right], \gamma_0^1 - \tilde{\gamma}_0^1 + \sum_{i=1}^{M} \left[\lambda_0^i - \tilde{\lambda}_0^i \right] S_1^i \right) \\ &= (p_0((\Lambda,\gamma)_0), p_1((\Lambda,\gamma)_0)) \end{aligned} \tag{13}$$

Now, under no arbitrage assumption, the primal linear programming problem has a finite value, hence the Constraint Qualification is satisfied (under no-arbitrage assumption the primal problem has a finite value, therefore feasible) so that both primal and dual problems has solutions, therefore by strong duality we have

$$
\begin{aligned}
&\sup_{f_0^1 \geq 0} \inf_{(\Lambda,\gamma)_0 \in \mathbb{R}_+^{2M+2}} L(f_0^1, (\Lambda,\gamma)_0) \\
&= \inf_{(\Lambda,\gamma)_0 \in \mathbb{R}_+^{2M+2}} \sup_{f_0^1 \geq 0} L(f_0^1, (\Lambda,\gamma)_0) \\
&= \inf_{(\Lambda,\gamma)_0 \in \mathbb{R}_+^{2M+2}} \sup_{f_0^1 \geq 0} \mathbf{E}\left[f_0^1 c_1\right] + \left\{\sum_{i=1}^{M} \lambda_0^i \left(a_0^i - \mathbf{E}\left[f_0^1 S_1^i\right]\right)\right\} \\
&+ \left\{\sum_{i=1}^{M} \tilde{\lambda}_0^i \left(\mathbf{E}\left[f_0^1 S_1^i\right] - b_0^i\right)\right\} + \left\{\gamma_0^1 \left(h_0^1 - \mathbf{E}\left[f_0^1\right]\right)\right\} + \left\{\tilde{\gamma}_0^1 \left(\mathbf{E}\left[f_0^1\right] - g_0^1\right)\right\} \\
&= \inf_{(\Lambda,\gamma)_0 \in \mathbb{R}_+^{2M+2}} \sup_{f_0^1 \geq 0} \left\{\mathbf{E}\left[f_0^1 \left(c_1 - \left[\gamma_0^1 - \tilde{\gamma}_0^1 + \sum_{i=1}^{M} \left[\lambda_0^i - \tilde{\lambda}_0^i\right] S_1^i\right]\right)\right]\right\} \\
&\qquad - \left(\gamma_0^1 h_0^1 - \tilde{\gamma}_0^1 g_0^1 + \sum_{i=1}^{M} \left[-\lambda_0^i a_0^i + \tilde{\lambda}_0^i b_0^i\right]\right) \\
&= \inf_{(\Lambda,\gamma)_0 \in \mathbb{R}_+^{2M+2}} \sup_{f_0^1 \geq 0} \left\{\mathbf{E}\left[f_0^1 \left(c_1 - p_1((\Lambda,\gamma)_0)\right)\right] - p_0((\Lambda,\gamma)_0)\right\}
\end{aligned}
$$

If we consider b_0 to be the premium (received at time $t = 0$) for delivering $-c_1$ at time $t = 1$, then simply the problem becomes to build a zero cost portfolio (p_0, p_1) so that the overall transaction, $(b_0 + p_0, p_1 - c_1)$ is an arbitrage. Thus, one should have,

$$
b_0 + p_0 > 0 \quad \text{and} \quad p_1(\omega) - c_1(\omega) \geq 0 \text{ for all } \omega \in \Omega
$$

The last equation then, in terms of the dual linear programming problem can be written as

$$
d = \begin{cases}
\min(-p_0((\Lambda,\gamma)_0)) & (\Lambda,\gamma)_0 \in \mathbb{R}_+^{2M+2} \\
\qquad s.t. & \\
p_1((\Lambda,\gamma)_0)(\omega) \geq c_1(\omega) & \forall \omega \in \Omega
\end{cases}
$$

Under no arbitrage assumption we know this linear programming problem has a finite value and a solution exists. Let $(\bar{\Lambda}, \bar{\gamma})_0 \in \mathbb{R}_+^{2M+2}$ be the solution of this dual problem. Since the feasibility condition is satisfied, there exists $\bar{f}_0^1 \in \mathcal{P}$ (solution to the primal problem) such that

$$
p = L(\bar{f}_0^1, (\bar{\Lambda}, \bar{\gamma})_0) = d.
$$

Now suppose,

$$
b_0 > \inf_{(\Lambda,\gamma)_0 \in \mathbb{R}_+^{2M+2}} \sup_{f_0^1 \geq 0} \left\{\mathbf{E}\left[f_0^1 \left(c_1 - p_1((\Lambda,\gamma)_0)\right)\right] - p_0((\Lambda,\gamma)_0)\right\}
$$

then there are non-negative constants $(\bar{\Lambda}, \bar{\gamma})_0 \in \mathbb{R}_+^{2M+2}$ and $\bar{f}_0^1 \in \mathcal{P}$ by which we can construct a zero cost trading strategy such that

$$
\left(\frac{b_0 + p_0((\bar{\Lambda}, \bar{\gamma})_0)}{\bar{f}_0^1}\right) + p_1((\bar{\Lambda}, \bar{\gamma})_0)(\omega) > c_1(\omega) \quad \text{for all} \quad \omega \in \Omega.
$$

$\square$

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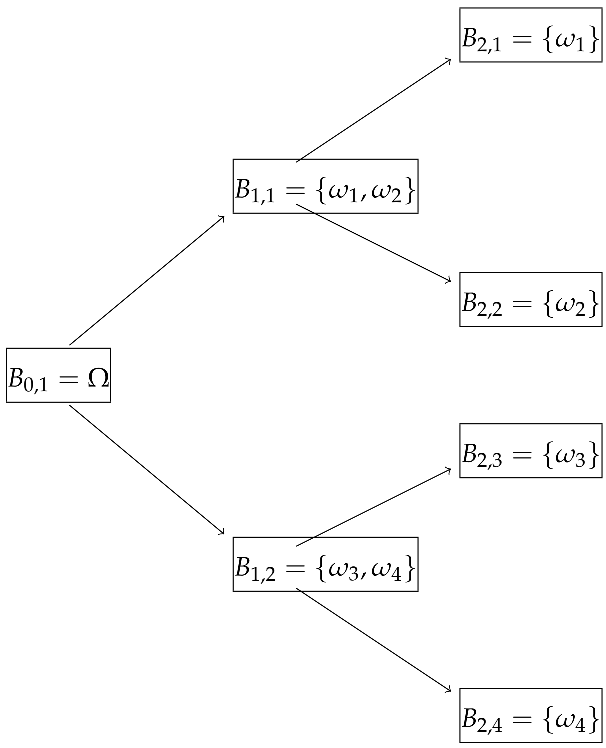
In summary, we can see that a super-hedging bound u_0 can be represented as the value of a pair of dual linear programming problems. The primal provides an easy way of evaluating u_0 with the help of price factor in set $\mathcal{P}$. The dual solution, provides us a super-hedging portfolio which guarantees an arbitrage in case of existence of a favorable price for the cash flow $c \in \mathcal{C}$

4.3. Two-Period Examples

Before we start setting up the problem for a two period model, first we aim to explain a simple example for a two period case. We will discuss and illustrate this two period model with only one risky asset.

4.3.1. Involving Only 1-period Bonds

Example 2. *Let* $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ *and consider a two period model with* $T = 2$ *and* $t = 0$ *and the information structure* $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$ *where for* $k = 0, 1, 2$, *the* σ*-algebra* $\mathcal{F}_k$ *is generated by the set* $\mathcal{P}_k$ *which are defined as* $\mathcal{P}_0 = \{B_{0,1}\}, \mathcal{P}_1 = \{B_{1,1}, B_{1,2}\}, \mathcal{P}_2 = \{B_{2,1}, B_{2,2}, B_{2,3}, B_{2,4}\}$. *The sets* $B_{i,j}$ *are all subsets of* Ω *and they are defined as in the following diagram*



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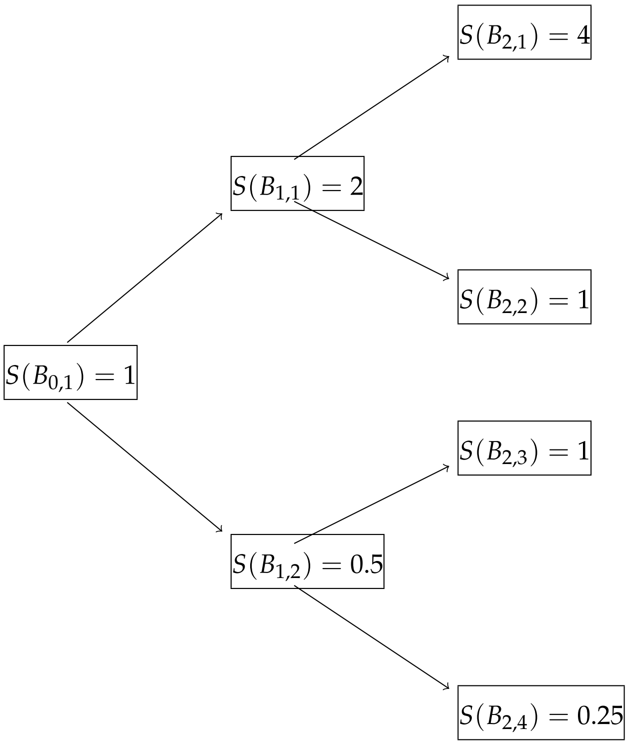

Also consider the bonds

$$1^{1,0} = (-h_0^1, 1, 0) \qquad \tilde{1}^{1,0} = (g_0^1, -1, 0)$$

$$1^{2,0} = (-h_0^2, 0, 1) \qquad \tilde{1}^{2,0} = (g_0^2, 0, -1)$$

$$1^{2,1} = (0, -h_1^2, 1) \qquad \tilde{1}^{2,1} = (0, g_1^2, -1)$$

where $h_0^1 = 0.95$, $g_0^1 = 0.9$, $h_0^2 = 0.9$, $g_0^2 = 0.8$ *and also* $h_1^2(B_{1,1}) = 0.9$, $h_1^2(B_{1,2}) = 0.95$, $g_1^2(B_{1,1}) = 0.85$, $g_1^2(B_{1,2}) = 0.9$.

Also let $S^1 = S$ *be our only risky asset with the above price diagram corresponding to zero-cost cash flows*

$$S^{i0} = (-a_0^2, S_1, S_2) \qquad \tilde{S}^{i0} = (b_0^2, -S_1, -S_2)$$

$$S^{i1} = (0, -a_1^2, S_2) \qquad \tilde{S}^{i1} = (0, b_1^2, -S_2)$$

We aim to find the highest ask price while no arbitrage exists. We need to mention here that we are assuming there is no 2-period bond $(-h_0^2, 0, 1)$ *is available. So we are going to treat the problem as two consecutive 1-period problems. For this reason, first we try to find the best ask price* u_1^2 *(a* $\mathcal{F}_1$*-measurable random variable) of having* S_2 *being paid at time* $t = 1$*. Therefore for* $i = 1, 2$ *we construct the following maximization problem*

$$p : \begin{cases} u_1^2(B_{1,i}) = \max \mathbf{E}_1 \left[f_1^2 S_2 \right] (B_{1,i}) \\ g_1^2(B_{1,i}) \leq \mathbf{E}_1 \left[f_1^2 \right] \leq h_1^2(B_{1,i}) \end{cases} \tag{14}$$

and by definition 2.2 we have

$$p : \begin{cases} u_1^2 = \max < f_1^2, S_2 > \\ g_1^2 \leq \mathbf{E}_1 \left[f_1^2 \right] \leq h_1^2 \end{cases} \tag{15}$$

for which the Lagrangian becomes

$$\mathcal{L}(f_1^2, t) =< f_1^2, S_2 > + t_1 \left(h_1^2 - < f_1^2, S_2 > \right) + t_2 \left(< f_1^2, S_2 > - g_1^2 \right) \tag{16}$$

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$$=< f_1^2, S_2 - t_1 + t_2 > +t_1 h_1^2 - t_2 g_1^2$$

so that we have

$$u_1^2 = \sup_{f_2^2 \geq 0} \inf_{t \geq 0} \mathcal{L}(f_1^2, t) \tag{17}$$

or

$$p: \begin{cases} u_1^2 = \inf t_1 h_1^2 - t_2 g_1^2 \\ t_1 - t_2 \geq \sup_{\omega \in \Omega} S_2(\omega) \end{cases} \tag{18}$$

Solving the latter linear programing we will have:

$$u_1^2(B_{1,i}) = h_1^2(B_{1,i}) \max_{B_{1,i}} S_2\left(B_{1,i} | S_1(B_{1,i})\right) = \begin{cases} 0.9(4) = 3.6 & \text{if } i = 1, \\ 0.95(1) = 0.95 & \text{if } i = 2. \end{cases} \tag{19}$$

Now consider the following diagram and maximization problem

$$\begin{array}{ccc} & & S_1(B_{1,1}) + u_1^2(B_{1,1}) \ = \ 5.6 \\ & & \nearrow \\ S_0 & = & 1 \\ & & \searrow \\ & & S_1(B_{1,2}) + u_1^2(B_{1,2}) \ = \ 1.45 \end{array}$$

$$\begin{cases} u_0^2 = \max \mathbf{E}_0 \left[f_1^2 \left(S_1 + u_1^2 \right) \right] \\ \quad g_0^1 \leq \mathbf{E}_0 \left[f_1^2 \right] \leq h_0^1. \end{cases} \tag{20}$$

And similar to example (1) this problem has the solution

$$u_0^2 = h_0^1 \left[\max_{\omega \in \Omega} (S_1(\omega) + u_1^2(\omega)) \right] = (0.95)(5.6) = 3.42 \tag{21}$$

Therefore 3.42 is the highest ask price for $(\ . \ , S_1, S_2)$ *offered at time* $t = 0$ *to have the no-arbitrage condition satisfied. That means for any offered price bigger than 3.42 one should be able to build a portfolio creating an arbitrage.*

In fact for an offered bid price of $b_0^2 > 3.42$, *since the dual problem solutions are* $t_1 = 0$, $t_2 = 5.6$, $t_1(B_{1,1}) = 0$, $t_2(B_1, 1) = 4$, $t_1(B_{1,2}) = 0$, $t_2(B_{1,2}) = 1$, *one can buy 5.6 unites of bond* $1^{1,0} = (-0.95, 1, 0)$ *at time* $t = 0$ *for the price of 3.42 and receive 5.6 at time* $t = 1$ *so that* S_1 *can be delivered and left with at least* $3.6 = \max(u_1^2)(\omega)$ *so that one is able to deliver* S_2 *at time* $t = 2$ *and still making a profit of at least* $\dfrac{b_0^2 - 3.42}{(0.95)(h_1^2)} > 0$.

Similarly we can find a lowest bid price l_0^2 *for which one can construct a portfolio providing an arbitrage if any ask price smaller than* l_0^2 *is available, where analogous steps to example (1) leads to*

$$l_0^2 = g_0^1 \left[\min_{\omega \in \Omega} (S_1(\omega) + l_1^2(\omega)) \right] = (0.9)(0.725) = 0.6525 \tag{22}$$

Therefore 0.6525 is the lowest bid price (sub-hedging bound) for $(\ . \ , S_1, S_2)$ *offered at time* $t = 0$ *to have the no-arbitrage condition satisfied. That means for any offered ask smaller than 0.6525 one should be able to build a portfolio creating an arbitrage.*

Next, we consider a problem involving options using a similar approach.

Example 3. *Let* $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ *and consider a two period model with the information structure* $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$ *where for* $k = 0, 1, 2$*, the* σ*-algebra* $\mathcal{F}_k$ *is generated by the set* $\mathcal{P}_k$ *which are defined as* $\mathcal{P}_0 = \{B_{0,1}\}, \mathcal{P}_1 = \{B_{1,1}, B_{1,2}\}, \mathcal{P}_2 = \{B_{2,1}, B_{2,2}, B_{2,3}, B_{2,4}\}$*. The sets* $B_{i,j}$ *are all subsets of* Ω *and they are defined as in the following diagram*

$$\begin{tikzcd}
 & & \boxed{B_{2,1} = \{\omega_1\}} \\
 & \boxed{B_{1,1} = \{\omega_1, \omega_2\}} \arrow[ur] \arrow[dr] & \\
 & & \boxed{B_{2,2} = \{\omega_2\}} \\
\boxed{B_{0,1} = \Omega} \arrow[uur] \arrow[ddr] & & \\
 & & \boxed{B_{2,3} = \{\omega_3\}} \\
 & \boxed{B_{1,2} = \{\omega_3, \omega_4\}} \arrow[ur] \arrow[dr] & \\
 & & \boxed{B_{2,4} = \{\omega_4\}}
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \boxed{S(B_{2,1}) = 4} \\
 & \boxed{S(B_{1,1}) = 2} \arrow[ur] \arrow[dr] & \\
 & & \boxed{S(B_{2,2}) = 1} \\
\boxed{S(B_{0,1}) = 1} \arrow[uur] \arrow[ddr] & & \\
 & & \boxed{S(B_{2,3}) = 1} \\
 & \boxed{S(B_{1,2}) = 0.5} \arrow[ur] \arrow[dr] & \\
 & & \boxed{S(B_{2,4}) = 0.25}
\end{tikzcd}$$

We define the risk-free bonds as

$$1^{1,0} = (-h_0^1, 1, 0) \qquad \tilde{1}^{1,0} = (g_0^1, -1, 0)$$

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$$1^{2,0} = (-h_0^2, 0, 1) \qquad \tilde{1}^{2,0} = (g_0^2, 0, -1)$$
$$1^{2,1} = (0, -h_1^2, 1) \qquad \tilde{1}^{2,1} = (0, g_1^2, -1)$$

where $h_0^1 = 0.95$, $g_0^1 = 0.9$, $h_0^2 = 0.9$, $g_0^2 = 0.8$ *and also* $h_1^2(B_{1,1}) = 0.9$, $h_1^2(B_{1,2}) = 0.95$, $g_1^2(B_{1,1}) = 0.85$, $g_1^2(B_{1,2}) = 0.9$. *Also let* $S^1 = S$ *be our only risky asset with the above price diagram with zero-cost cash flows*

$$S^{i0} = (-a_0^2, S_1, S_2) \qquad \tilde{S}^{i0} = (b_0^2, -S_1, -S_2)$$
$$S^{i1} = (0, -a_1^2, S_2) \qquad \tilde{S}^{i1} = (0, b_1^2, -S_2)$$

and also according to our calculations in previous example

$$0.6525 = l_0^2 \leq b_0^2 \leq a_0^2 \leq u_0^2 = 3.42,$$

$$0.85 = l_1^2(B_{1,1}) \leq b_1^2(B_{1,1}) \leq a_1^2(B_{1,1}) \leq u_1^2(B_{1,1}) = 3.6,$$
$$0.225 = l_1^2(B_{1,2}) \leq b_1^2(B_{1,2}) \leq a_1^2(B_{1,2}) \leq u_1^2(B_{1,2}) = 0.95.$$

Now consider an option with the payoff of c_t *at times* $t = 1, 2$ *and* S^1 *as the underlying asset. Also assume the strike price of* $K = 1$ *at time* $t = 2$. *Then the payoffs are* $c_2(B_{2,1}) = 3$, $c_2(B_{2,2}) = c_2(B_{2,3}) = c_2(B_{2,4}) = 0$.

Consider the cash flow $(., 0, c_2)$. *We would like to find the highest ask-price at time* $t = 0$ *for this cash flow. Therefore we consider the following maximization problem*

$$\begin{cases} u_0^2 = \max \mathbf{E}_0 \left[f_0^1 c_1 + f_0^2 c_2 \right] = \max \mathbf{E}_0 \left[f_0^1 \left(c_1 + \mathbf{E}_1 \left[f_1^2 c_2 \right] \right) \right] \\ g_0^1 \leq \mathbf{E}_0 \left[f_0^1 \right] \leq h_0^1, \\ g_0^2 \leq \mathbf{E}_0 \left[f_0^2 \right] \leq h_0^2, \\ b_0^2 \leq \mathbf{E}_0 \left[f_0^1 S_1 + f_0^2 S_2 \right] \leq a_0^2. \end{cases} \tag{23}$$

On the other hand we know that

$$\begin{cases} u_1^2 = \max \mathbf{E}_1 \left[f_1^2 c_2 \right] \\ g_1^2 \leq \mathbf{E}_1 \left[f_1^2 \right] \leq h_1^2, \\ b_1^2 \leq \mathbf{E}_1 \left[f_1^2 S_2 \right] \leq a_1^2. \end{cases} \tag{24}$$

Therefore, similar to our previous example, first we find a super-hedging bound for c_2 *(the random variable* u_1^2*), at time* $t = 1$. *Since* u_1^2 *is a random variable taking valued on* $B_{1,1}$ *and* $B_{1,2}$, *for* $k = 1, 2$ *which corresponds to the two sub-branches of the diagram at time* $t = 1$, *we set up a linear programming maximization as*

$$p_k := \begin{cases} u_1^2(B_{1,k}) = \max\limits_{B_{2,j} \subset B_{1,k}} \left[c_2^T f_1^2 \right] (B_{1,k}) \\ g_1^2(B_{1,k}) \leq (f_1^2)^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq h_1^2(B_{1,k}), \\ b_1^2(B_{1,k}) \leq S_2^T f_1^2 \leq a_1^2(B_{1,k}), \end{cases} \tag{25}$$

which has the values of $u_1^2(B_{1,1}) = 2.7$ *and* $u_1^2(B_{1,2}) = 0$.

Now that we have u_1^2 *values determined and since* $c_1 = 0$, *we use the following problem to calculate* u_0^2

$$\begin{cases} u_0^2 = \max \mathbf{E}_0 \left[f_0^1 \left(c_1 + u_1^2 \right) \right] = \max \mathbf{E}_0 \left[f_0^1 \left(u_1^2 \right) \right] \\ g_0^1 \leq \mathbf{E}_0 \left[f_0^1 \right] \leq h_0^1, \end{cases} \tag{26}$$

which has the maximum value of $u_0^2(c_2) = u_1^2(B_{1,1}) h_0^1 = 2.7(0.95) = 2.565$.

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Now if at time $t = 0$ there is an available cash flow $(b_0, 0, -c_2)$ with $b_0 > 2.565 = u_0^2(c_2)$, then one can use the following zero-cost trading strategy to create an arbitrage. In fact the arbitrage strategy is to invest all of b_0 amount into $1^{10} = (-h_0^1, 1, 0)$ at time $t = 0$ and then invest the revenue received at time $t = 1$ into $S^{i1} = (0, -a_1^2, S_2)$ therefore at time $t = 2$ one will have

$$\frac{b_0}{h_0^1}\frac{S_2}{a_1^2} > \frac{2.565}{0.95}\frac{S_2}{a_1^2} = 2.7\left(\frac{S_2}{a_1^2}\right) = \begin{cases} 2.7\left(\dfrac{S_2(B_{2,1})}{a_1^2(B_{1,1})}\right) = 3 \\ 2.7\left(\dfrac{S_2(B_{2,2})}{a_1^2(B_{1,1})}\right) > 0 \\ 2.7\left(\dfrac{S_2(B_{2,3})}{a_1^2(B_{1,2})}\right) > 0 \\ 2.7\left(\dfrac{S_2(B_{2,4})}{a_1^2(B_{1,2})}\right) > 0 \end{cases}$$

so that an arbitrage is available.

And again we can find the lowest bid-price l_0^2 at time $t = 0$ for the cash flow $(., 0, c_2)$ as

$$l_0^2 = (g_0^1)(l_1^2(B_{1,1})) = (0.9)(0.6375) = 0.57375$$

and this is the lowest bid price possible under no-arbitrage condition. Now, if at time $t = 0$ there is a cash flow $(-a_0, 0, c_2)$ available where $a_0 < 0.57375$ then one should be able to have an arbitrage. In fact, by constructing the following zero-cost trading strategy, that is shorting a_0 units of $\tilde{1}^{10}$ at time $t = 0$ and then shorting $\frac{a_0}{g_0^1}$ units of $\tilde{S}^{i1} = (0, b_1^2, -S_2)$ at time $t = 1$ one would have a negative balance of

$$\frac{a_0}{g_0^1}\frac{S_2}{b_1^2} < c_2$$

smaller than c_2, thus an arbitrage.

4.3.2. Involving 2-period bond

As we noticed above, the arbitrage trading strategy was involved with a bond on the first period and risky asset on second period. This wasn't coincidental since the cash flow c_2 was a very simplified case, also derived from the risky asset which itself was priced by the risk-free bond. In fact we had no need nor had to solve the dual problem. However. in a more general case we need to write the dual problem and solve it to find the solution and optimal value. Thus, let us consider the following maximization problem for the following diagram

$$\begin{tikzcd} & \boxed{S(B_{2,1})} \\ \boxed{S(B_{1,1})} \arrow[ur] \arrow[dr] & \\ & \boxed{S(B_{2,2})} \end{tikzcd}$$

that is

$$p_1 := \begin{cases} u_1^2(B_{1,1}) = \max\left[r_1(B_{1,1})c_2(B_{2,1}) + r_2(B_{1,1})c_2(B_{2,2})\right](B_{1,1}) \\ g_1^2(B_{1,1}) \leq r_1(B_{1,1}) + r_2(B_{1,1}) \leq h_1^2(B_{1,1}), \\ b_1^2(B_{1,1}) \leq r_1(B_{1,1})S_2(B_{2,1}) + r_2(B_{1,1})S_2(B_{2,2}) \leq a_1^2(B_{1,1}). \end{cases} \tag{27}$$

for which the constrains in all one-sided inequalities can be written as

$$p_1 := \begin{cases} u_1^2(B_{1,1}) = \max\left[r_1(B_{1,1})c_2(B_{2,1}) + r_2(B_{1,1})c_2(B_{2,2})\right](B_{1,1}) \\ r_1(B_{1,1}) + r_2(B_{1,1}) \le h_1^2(B_{1,1}), \\ -r_1(B_{1,1}) - r_2(B_{1,1}) \le -g_1^2(B_{1,1}), \\ r_1(B_{1,1})S_2(B_{2,1}) + r_2(B_{1,1})S_2(B_{2,2}) \le a_1^2(B_{1,1}), \\ -r_1(B_{1,1})S_2(B_{2,1}) - r_2(B_{1,1})S_2(B_{2,2}) \le -b_1^2(B_{1,1}), \\ r_1, r_2 \ge 0. \end{cases} \tag{28}$$

and having the objective function and constrains in matrix form we have

$$p_1 := \begin{cases} u_1^2(B_{1,1}) = \max\left(\begin{bmatrix} c_2(B_{2,1}) & c_2(B_{2,2}) \end{bmatrix} \begin{bmatrix} r_1(B_{1,1}) \\ r_2(B_{1,1}) \end{bmatrix}\right) \\ \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ S_2(B_{2,1}) & S_2(B_{2,2}) \\ -S_2(B_{2,1}) & -S_2(B_{2,2}) \end{bmatrix} \begin{bmatrix} r_1(B_{1,1}) \\ r_2(B_{1,1}) \end{bmatrix} \le \begin{bmatrix} h_1^2(B_{1,1}) \\ -g_1^2(B_{1,1}) \\ a_1^2(B_{1,1}) \\ -b_1^2(B_{1,1}) \end{bmatrix}, \\ r_1, r_2 \ge 0. \end{cases} \tag{29}$$

Now we write the dual to this linear programming problem which is

$$d_1 := \begin{cases} u_1^2(B_{1,1}) = \min\left(\begin{bmatrix} h_1^2(B_{1,1}) & -g_1^2(B_{1,1}) & a_1^2(B_{1,1}) & -b_1^2(B_{1,1}) \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix}\right) \\ \begin{bmatrix} 1 & -1 & S_2(B_{2,1}) & -S_2(B_{2,1}) \\ 1 & -1 & S_2(B_{2,2}) & -S_2(B_{2,2}) \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \ge \begin{bmatrix} c_2(B_{2,1}) \\ c_2(B_{2,2}) \end{bmatrix}, \\ t_1, t_2, t_3, t_4 \ge 0. \end{cases} \tag{30}$$

or equivalently

$$d_1 := \begin{cases} u_1^2(B_{1,1}) = \min\left(\begin{bmatrix} h_1^2(B_{1,1}) & -g_1^2(B_{1,1}) & a_1^2(B_{1,1}) & -b_1^2(B_{1,1}) \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix}\right) \\ \begin{bmatrix} -1 & 1 & -S_2(B_{2,1}) & S_2(B_{2,1}) \\ -1 & 1 & -S_2(B_{2,2}) & S_2(B_{2,2}) \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \le \begin{bmatrix} -c_2(B_{2,1}) \\ -c_2(B_{2,2}) \end{bmatrix}, \\ t_1, t_2, t_3, t_4 \ge 0. \end{cases} \tag{31}$$

If we let, as an example, $a_1^2(B_{1,1}) = 3$, $b_1^2(B_{1,1}) = 2$, $c_2(B_{2,1}) = 3$ and $c_2(B_{2,2}) = 2$, then by solving (31) using linear programming with complementary slackness conditions (we used MATLAB for our purpose) we have $t_1 = \frac{5}{3}$, $t_3 = \frac{1}{3}$, $t_2 = t_4 = 0$ (which is the solution to the dual problem d_1 and indicates the portfolio strategy one needs to choose in the case of existence of any arbitrage opportunity) and $u_1^2(B_{1,1}) = 2.5$.

Similarly by solving the dual problem for the other part of the diagram

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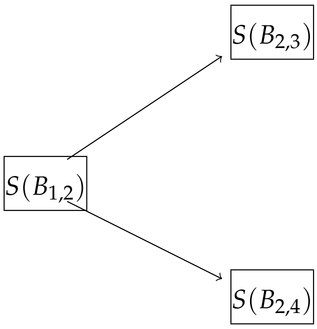
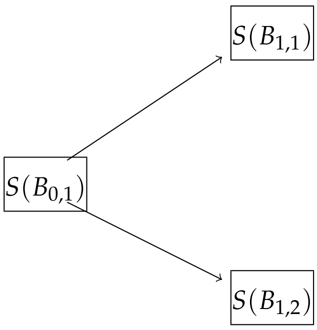


that is

$$d_1 := \begin{cases} u_1^2(B_{1,2}) = \min\left(\begin{bmatrix} h_1^2(B_{1,2}) & -g_1^2(B_{1,2}) & a_1^2(B_{1,2}) & -b_1^2(B_{1,2}) \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \right) \\ \begin{bmatrix} -1 & 1 & -S_2(B_{2,3}) & S_2(B_{2,3}) \\ -1 & 1 & -S_2(B_{2,4}) & S_2(B_{2,4}) \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \leq \begin{bmatrix} -c_2(B_{2,3}) \\ -c_2(B_{2,4}) \end{bmatrix}. \\ t_1, t_2, t_3, t_4 \geq 0. \end{cases} \tag{32}$$

and if we let $a_1^2(B_{1,2}) = 0.7$ and $b_1^2(B_{1,2}) = 0.4$ also let $c_2(B_{2,3}) = 1$ and $c_2(B_{2,2}) = 0.5$ then the solution is $t_1 = \dfrac{1}{3}, t_2 = t_4 = 0, t_3 = \dfrac{2}{3}$ and $u_1^2(B_{1,2}) = 0.7833$.

Now for the last part consider the following picture and it's corresponding linear programming in dual form



with

$$d_1 := \begin{cases} u_0^2(B_{0,1}) = \min\left(\begin{bmatrix} h_0^1(B_{0,1}) & -g_0^1(B_{0,1}) & a_0^1(B_{0,1}) & -b_0^1(B_{0,1}) \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \right) \\ \begin{bmatrix} -1 & 1 & -S_1(B_{1,1}) & S_1(B_{1,1}) \\ -1 & 1 & -S_1(B_{1,2}) & S_1(B_{1,2}) \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \leq \begin{bmatrix} -u_1^2(B_{1,1}) \\ -u_1^2(B_{1,2}) \end{bmatrix}, \\ t_1, t_2, t_3, t_4 \geq 0. \end{cases} \tag{33}$$

for which when $a_0^1(B_{0,1}) = 1.6$, $b_0^1(B_{0,1}) = 1.5$ (here we simply assumed that $c_1 = 0$), the corresponding solution becomes $t_1 = \dfrac{19}{90}$, $t_3 = \dfrac{103}{90}$, $t_2 = t_4 = 0$ and $u_0^2(B_{0,1}) = 2.0317$. So for any bid price offered at time $t = 0$ larger than 2.0317, there is an arbitrage and the trading strategy is as following:

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$$\begin{tikzcd}
 & \frac{5}{3}1^{21} + \frac{1}{3}S^{12} \\
\frac{19}{90}1^{10} + \frac{103}{90}S^{01} \arrow[ur] \arrow[dr] & \\
 & \frac{1}{3}1^{21} + \frac{2}{3}S^{12}
\end{tikzcd}$$

buy $\frac{19}{90}$ units of 1^{10} and $\frac{103}{90}$ units of S^{01} and at time $t = 1$ in either case we are able cover the asked price of c_2 at time $t = 2$. In fact in the case of $B_{1,1}$ possibility, we buy $\frac{5}{3}$ unites of 1^{21} and $\frac{1}{3}$ unites of S^{12}, and in the case of $B_{1,2}$ possibility, we buy $\frac{1}{3}$ unites of 1^{21} and $\frac{2}{3}$ unites of S^{12} and in either case c_2 will be covered to deliver and the difference will be an arbitrage.

As we noted in the last diagram, at least $\frac{1}{3}$ of the risk-free bond 1^{21} is needed to be bought at time $t = 1$. Now we consider a trading strategy which purchases $\frac{1}{3}$ of the bond 1^{20} bought at time $t = 0$. Therefore the trading strategy can be replaced by

$$\begin{tikzcd}
 & \frac{4}{3}1^{21} + \frac{1}{3}S^{12} \\
\frac{1}{3}1^{20} + t_1 1^{10} + t_2 \tilde{1}^{10} + t_3 S^{01} + t_4 \tilde{S}^{01} \arrow[ur] \arrow[dr] & \\
 & \frac{2}{3}S^{12}
\end{tikzcd}$$

where t_1, t_2, t_3, t_4 are the solutions to the new linear programming problem below

$$\begin{cases} \min \left(\begin{bmatrix} h_0^1(B_{0,1}) & -g_0^1(B_{0,1}) & a_0^1(B_{0,1}) & -b_0^1(B_{0,1}) \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \right) \\ \begin{bmatrix} -1 & 1 & -S_1(B_{1,1}) & S_1(B_{1,1}) \\ -1 & 1 & -S_1(B_{1,2}) & S_1(B_{1,2}) \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \leq \begin{bmatrix} -\tilde{u}_1^2(B_{1,1}) \\ -\tilde{u}_1^2(B_{1,2}) \end{bmatrix}, \\ t_1, t_2, t_3, t_4 \geq 0. \end{cases} \tag{34}$$

and

$$\begin{cases} \tilde{u}_1^2(B_{1,1}) = \frac{4}{3}h_1^2(B_{1,1}) + \frac{1}{3}a_1^2(B_{1,1}) = \frac{11}{5} \\ \tilde{u}_1^2(B_{1,2}) = \frac{2}{3}a_1^2(B_{1,2}) = \frac{7}{15} \end{cases} \tag{35}$$

Solution to the linear programming above is $t_2 = t_4 = 0, t_1 = \frac{1}{9}, t_3 = \frac{73}{72}$ so that the largest ask price for c_2 in this case becomes

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$$\frac{1}{3}h_0^2 + \frac{1}{9}\left(h_0^1\right) + \frac{73}{72}a_0^1 = 2.02778$$

and as we see, this upper bound is smaller and much more accurate than the upper bound $u_0^2 = 2.0317$ which we found above without using the 1^{20} bond.

Now consider the cash flow $(., c_1, c_2)$ and let α to be the portion of $1^{20} = (-h_0^2, 0, 1)$ that we carry in the trading strategy at time $t = 0$. We would like to find the highest ask-price at time $t = 0$ for this cash flow. Therefore we consider the following maximization problem and unlike the example above, we try to solve this problem in one step

$$\begin{cases} u_0^2 = \max \mathbf{E}_0\left[f_0^1 c_1 + f_0^2 c_2\right] = \max \mathbf{E}_0\left[f_0^2 \alpha + f_0^1\left(c_1 + \mathbf{E}_1\left[f_0^2\left(c_2 - \alpha\right)\right]\right)\right] \\ g_0^1 \leq \mathbf{E}_0\left[f_0^1\right] \leq h_0^1, \\ g_0^2 \leq \mathbf{E}_0\left[f_0^2\right] \leq h_0^2, \\ b_0^2 \leq \mathbf{E}_0\left[f_0^1 S_1 + f_0^2 S_2\right] \leq a_0^2, \end{cases} \tag{36}$$

and by the definition of expectation and letting $r_{ij} = f_0^i(B_{i,j})Q(B_{i,j})$ we have

$$u_0^2(\alpha) = \max\{\alpha\left[s_{21} + s_{22} + s_{23} + s_{24}\right] + r_{11}\left[c_1(B_{11}) + r_{21}c_2(B_{21}) + r_{22}c_2(B_{22})\right]$$

$$+ r_{12}\left[c_1(B_{12}) + r_{23}c_2(B_{23}) + r_{24}c_2(B_{24})\right] - \alpha\left[r_{11}(r_{21} + r_{22}) + r_{12}(r_{23} + r_{24})\right]\}$$

where

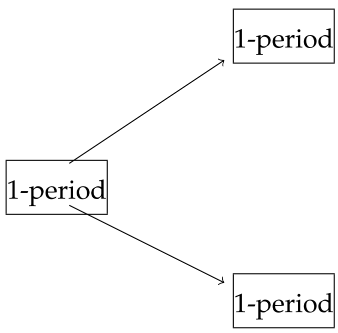
$$\begin{cases} r_{11} + r_{12} \leq h_0^1, \\ -r_{11} - r_{12} \leq -g_0^1, \\ s_{21} + s_{22} + s_{23} + s_{24} \leq h_0^2, \\ -s_{21} - s_{22} - s_{23} - s_{24} \leq -g_0^2, \\ r_{11}\left[S_1(B_{11}) + r_{21}S_2(B_{21}) + r_{22}S_2(B_{22})\right] + r_{12}\left[S_1(B_{12}) + r_{23}S_2(B_{23}) + r_{24}S_2(B_{24})\right] \leq a_0^2, \\ -r_{11}\left[S_1(B_{11}) + r_{21}S_2(B_{21}) + r_{22}S_2(B_{22})\right] - r_{12}\left[S_1(B_{12}) + r_{23}S_2(B_{23}) + r_{24}S_2(B_{24})\right] \leq b_0^2. \end{cases} \tag{37}$$

As we can see, the above linear programming is a ten variable linear programming with one parameter α and this is only for a binary priced market where at each time there are only two possibilities for the stock and cash flow price. Also we are only considering one risky asset.

4.4. Complexity of Multi-Period Model

Consider a 2-period model and let v_2 be the number of variables in the linear programming associated with this model and p_2 be the number of parameters associated with 2-period bonds in this model. We already know that $v_1 = 2$ and $p_1 = 0$. Also we can see that adding a single 1-period bond into the model, adds 2 variables and one parameter to the linear programming problem.

On the other hand a 2-period model is equivalent to a composition of a 1-period model followed by two 1-period models as illustrated in following diagram



Since a 2-period model is the composition of a 1-period followed by two 1-period models, then $\begin{pmatrix} v_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ p_1 \end{pmatrix} + 2\begin{pmatrix} v_1 \\ p_1 \end{pmatrix} + \text{2-period bond} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 1 \end{pmatrix}$ which coincides with what we found before. And this is the case for a over simplified case, binary model with a two states at each time.

Now consider a T-period model and let v_T be the number of variables in the linear programming associated with this model and p_T be the number of parameters associated with multi-period bonds in this model. A T-period model is equivalent to a composition of a 1-period model followed by two $(T-1)$-period models as illustrated in following diagram

$$\begin{tikzcd} & \boxed{(T-1)\text{-period}} \\ \boxed{1\text{-period}} \arrow[ur] \arrow[dr] & \\ & \boxed{(T-1)\text{-period}} \end{tikzcd}$$

For example, a 3-period model is the composition of a 1-period followed by two 2-period models, so that

$$\begin{aligned} \begin{pmatrix} v_3 \\ p_3 \end{pmatrix} &= \begin{pmatrix} v_1 \\ p_1 \end{pmatrix} + 2\begin{pmatrix} v_2 \\ p_2 \end{pmatrix} + \text{2-period bond} + \text{3-period bond} \\ &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 10 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 8 \\ 1 \end{pmatrix} = \begin{pmatrix} 34 \\ 4 \end{pmatrix} \end{aligned}$$

Now for a T-period model we have

$$\begin{pmatrix} v_T \\ p_T \end{pmatrix} = \begin{pmatrix} v_1 \\ p_1 \end{pmatrix} + 2\begin{pmatrix} v_{T-1} \\ p_{T-1} \end{pmatrix} + \text{2-period bond} + \text{3-period bond} + \ldots + T\text{-period bond} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 2\begin{pmatrix} v_{T-1} \\ p_{T-1} \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 8 \\ 1 \end{pmatrix} + \ldots + \begin{pmatrix} 2^T \\ 1 \end{pmatrix} = \begin{pmatrix} 2v_{T-1} + 2 + 4 + 8 + \cdots + 2^T \\ 2p_{T-1} + T - 1 \end{pmatrix}$$

or

$$\begin{pmatrix} v_T \\ p_T \end{pmatrix} = \begin{pmatrix} 2v_{T-1} + 2^{T+1} - 2 \\ 2p_{T-1} + T - 1 \end{pmatrix}$$

As it can be seen, when the periods of the model increases, the number of variables and parameters increase at an exponential rate. Considering the fact that this calculation was only done for the simplest case of a model (binary prices and one risky asset) we notice that for a $T-period$ model, the parametrized linear programming with p_T parameters and v_T variables make the problem very complicated to solve.

Lemma 1. *For a T-period model with v_T as the number of variables in the linear programming associated with the model and p_T as the number of parameters associated with multi-period bonds, we have*

$$\begin{pmatrix} v_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} v_T \\ p_T \end{pmatrix} = \begin{pmatrix} 2v_{T-1} + 2^{T+1} - 2 \\ 2p_{T-1} + T - 1 \end{pmatrix} \tag{38}$$

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Because of the reason explained above, we suggest solving the problem one step at a time, which means (similar to the 2-period linear programming problem that we solved) starting form the end and solving multiple 1-period problems on each step and continuing this process backward until we reach the initial time on the problem.

4.5. *Estimate of Multi-Period Bounds (breaking into one periods)*

Consider the following portfolio of zero-cost cash flows

$$\begin{aligned}(p_0, p_1, p_2) &= t_1 1^{10} + t_2 1^{20} + t_3 S^{10} + t_4 S^{20} + t_5 S^{21} + t_6 1^{21} \\ &= t_1\left(-h_0^1, 1, 0\right) + t_2\left(-h_0^2, 0, 1\right) + t_3\left(-a_0^1, S_1, 0\right) + t_4(-a_0^2, 0, S_2) \\ &\quad + t_5\left(0, -a_1^2, 1\right) + t_6\left(0, -h_1^2, 1\right)\end{aligned} \tag{39}$$

where

$$\begin{aligned}p_0 &= -t_1 h_0^1 - t_2 h_0^2 - t_3 a_0^1 - t_4 a_0^2 \\ p_1 &= t_1 + t_3 S_1 - t_5 a_1^2 - t_6 h_1^2 \\ p_2 &= t_2 + t_4 S_2 + t_5 S_2 + t_6\end{aligned} \tag{40}$$

also $t_1, t_2, t_3, t_4 \in \mathcal{F}_0$ and $t_5, t_6 \in \mathcal{F}_1$. In fact we have the following trading diagram corresponding to the above portfolio at time $t = 0$ and $t = 1$

$$\begin{tikzcd}[column sep=small] & t_5(B_{11})S^{21}(B_{11}) + t_6(B_{11})1^{21}(B_{11}) \\ t_1 1^{10} + t_2 1^{20} + t_3 S^{10} + t_4 S^{20} \arrow[ur] \arrow[dr] & \\ & t_5(B_{12})S^{21}(B_{12}) + t_6(B_{12})1^{21}(B_{12})\end{tikzcd}$$

Since we are assuming to deliver $c_2 \in \mathcal{F}_2$ at $t = 2$, indeed we have to solve the linear programing problem

$$\begin{cases} \min(-p_0) \\ \text{such that;} \\ p_1 \geq 0 \\ p_2 - c_2 \geq 0 \\ t_1, \ldots, t_6 \geq 0 \end{cases} \tag{41}$$

We can define the corresponding Lagrangian as

$$\mathcal{L}(t_1, \ldots, t_6, \lambda_1, \lambda_2) = (-p_0) - \lambda_1 p_1 - \lambda_2(p_2 - c_2) \tag{42}$$

where $\lambda_1 \in \mathcal{F}_1$ and $\lambda_2 \in \mathcal{F}_2$ are the random variable Lagrange multipliers. Therefore our minimization problem can be stated as

$$\inf_{t \geq 0} \sup_{\lambda \geq 0} \mathcal{L}(t_1, \ldots, t_6, \lambda_1, \lambda_2) \tag{43}$$

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where

$$\begin{aligned}
&\mathcal{L}(t_1,\ldots,t_6,\lambda_1(B_{11}),\lambda_1(B_{12}),\lambda_2(B_{21}),\lambda_2(B_{22}),\lambda_2(B_{24}),\lambda_2(B_{24})) = \\
&t_1h_0^1 + t_2h_0^2 + t_3a_0^1 + t_4a_0^2 \\
&-\lambda_1(B_{11})\left[t_1 + t_3S_1(B_{11}) - t_5(B_{11})a_1^2(B_{11}) - t_6(B_{11})h_1^2(B_{11})\right] \\
&-\lambda_1(B_{12})\left[t_1 + t_3S_1(B_{12}) - t_5(B_{12})a_1^2(B_{12}) - t_6(B_{12})h_1^2(B_{12})\right] \\
&-\lambda_2(B_{21})\left[t_2 + t_4S_2(B_{21}) + t_5(B_{11})S_2(B_{21}) + t_6(B_{11}) - c_2(B_{21})\right] \\
&-\lambda_2(B_{22})\left[t_2 + t_4S_2(B_{22}) + t_5(B_{11})S_2(B_{22}) + t_6(B_{11}) - c_2(B_{22})\right] \\
&-\lambda_2(B_{23})\left[t_2 + t_4S_2(B_{23}) + t_5(B_{12})S_2(B_{23}) + t_6(B_{12}) - c_2(B_{23})\right] \\
&-\lambda_2(B_{24})\left[t_2 + t_4S_2(B_{24}) + t_5(B_{12})S_2(B_{24}) + t_6(B_{12}) - c_2(B_{24})\right]
\end{aligned} \tag{44}$$

Now by Linearity Constraint Qualification the problem is feasible so that by Lagrangian strong duality we have

$$\inf_{t\geq 0}\sup_{\lambda\geq 0}\mathcal{L}(t_1,\ldots,t_6,\lambda_1,\lambda_2) = \sup_{\lambda\geq 0}\inf_{t\geq 0}\mathcal{L}(t_1,\ldots,t_6,\lambda_1,\lambda_2) \tag{45}$$

hence $\mathcal{L}(t_1,\ldots,t_6,\lambda_1,\lambda_2)$ can be written as

$$\begin{aligned}
&\mathcal{L}(t_1,\ldots,t_6,\lambda_1(B_{11}),\lambda_1(B_{12}),\lambda_2(B_{21}),\lambda_2(B_{22}),\lambda_2(B_{24}),\lambda_2(B_{24})) \\
&= t_1\left[h_0^1 - \sum_{j=1}^{2}\lambda_1(B_{1j})\right] + t_2\left[h_0^2 - \sum_{k=1}^{4}\lambda_2(B_{2k})\right] \\
&+ t_3\left[a_0^1 - \sum_{j=1}^{2}\lambda_1(B_{1j})S_1(B_{1j})\right] + t_4\left[a_0^2 - \sum_{k=1}^{4}\lambda_2(B_{2k})S_2(B_{2k})\right] \\
&+ t_5(B_{11})\left[\lambda_1(B_{11})a_1^2(B_{11}) - \lambda_2(B_{21})S_2(B_{21}) - \lambda_2(B_{22})S_2(B_{22})\right] \\
&+ t_5(B_{12})\left[\lambda_1(B_{12})a_1^2(B_{12}) - \lambda_2(B_{23})S_2(B_{23}) - \lambda_2(B_{24})S_2(B_{24})\right] \\
&+ t_6(B_{11})\left[\lambda_1(B_{11})h_1^2(B_{11}) - \lambda_2(B_{21}) - \lambda_2(B_{22})\right] \\
&+ t_6(B_{12})\left[\lambda_1(B_{12})h_1^2(B_{12}) - \lambda_2(B_{23}) - \lambda_2(B_{24})\right] \\
&+ \sum_{k=1}^{4}\lambda_2(B_{2k})c_2(B_{2k})
\end{aligned} \tag{46}$$

We can see that the corresponding maximization problem to the Lagrangian above is

$$\begin{cases} \max\left(\sum_{k=1}^{4}\lambda_2(B_{2k})c_2(B_{2k})\right) \\ \text{such that;} \\ \lambda_1(B_{11})+\lambda_1(B_{12})\le h_0^1 \\ \lambda_2(B_{21})+\lambda_2(B_{22})+\lambda_2(B_{23})+\lambda_2(B_{24})\le h_0^2 \\ \lambda_1(B_{11})S_1(B_{11})+\lambda_1(B_{12})S_1(B_{12})\le a_0^1 \\ \lambda_2(B_{21})S_2(B_{21})+\lambda_2(B_{22})S_2(B_{22})+\lambda_2(B_{23})S_2(B_{23})+\lambda_2(B_{24})S_2(B_{24})\le a_0^2 \\ \dfrac{\lambda_2(B_{21})}{\lambda_1(B_{11})}S_2(B_{21})+\dfrac{\lambda_2(B_{22})}{\lambda_1(B_{11})}S_2(B_{22})\le a_1^2(B_{11}) \\ \dfrac{\lambda_2(B_{23})}{\lambda_1(B_{12})}S_2(B_{23})+\dfrac{\lambda_2(B_{24})}{\lambda_1(B_{12})}S_2(B_{24})\le a_1^2(B_{12}) \\ \dfrac{\lambda_2(B_{21})}{\lambda_1(B_{11})}+\dfrac{\lambda_2(B_{22})}{\lambda_1(B_{11})}\le h_1^2(B_{11}) \\ \dfrac{\lambda_2(B_{23})}{\lambda_1(B_{12})}+\dfrac{\lambda_2(B_{24})}{\lambda_1(B_{12})}\le h_1^2(B_{12}) \\ \lambda_1,\lambda_2\ge 0 \end{cases} \tag{47}$$

Unsurprisingly, we observe that λ_1, λ_2 are the pricing factors in the expectation form. In fact as we defined before, if we let

$$\begin{aligned} \lambda_1(B_{1j}) &= f_0^1(B_{1j}) = r_{1j} && j = 1,2 \\ \lambda_2(B_{1k}) &= f_0^2(B_{1k}) = u_k && k = 1,2,3,4 \\ \lambda_2(B_{1k}) &= f_1^2(B_{2k}) = r_{2k} && k = 1,2,3,4 \end{aligned} \tag{48}$$

then the linear programming (47) becomes

$$\begin{cases} \max\left(u_1c_2(B_{21})+u_2c_2(B_{22})+u_3c_2(B_{23})+u_4c_2(B_{24})\right) \\ \text{such that;} \\ r_{11}+r_{12}\le h_0^1 \\ u_1+u_2+u_3+u_4\le h_0^2 \\ r_{11}S_1(B_{11})+r_{12}S_1(B_{12})\le a_0^1 \\ u_1S_2(B_{21})+u_2S_2(B_{22})+u_3S_2(B_{23})+u_4S_2(B_{24})\le a_0^2 \\ r_{21}S_2(B_{21})+r_{22}S_2(B_{22})\le a_1^2(B_{11}) \\ r_{23}S_2(B_{23})+r_{24}S_2(B_{24})\le a_1^2(B_{12}) \\ r_{21}+r_{22}\le h_1^2(B_{11}) \\ r_{23}+r_{24}\le h_1^2(B_{12}) \\ r_{11},r_{12},r_{21},r_{22},r_{23},r_{24},u_1,u_2,u_3,u_4\ge 0 \end{cases} \tag{49}$$

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which, as we can see, is the primal maximization problem (the dual of the dual in this case) corresponding to our initial portfolio. We can also observe that the problem (49) is indeed the expectation form of a super-hedging problem as

$$\begin{cases} \max \mathbf{E}_0\left[f_0^2 c_2\right] \\ \text{such that;} \\ \mathbf{E}_0\left[f_0^1\right] \le h_0^1 \\ \mathbf{E}_0\left[f_0^2\right] \le h_0^2 \\ \mathbf{E}_0\left[f_0^1 S_1\right] \le a_0^1 \\ \mathbf{E}_0\left[f_0^2 S_2\right] \le a_0^2 \\ \mathbf{E}_1\left[f_1^2 S_2\right] \le a_1^2 \\ \mathbf{E}_1\left[f_1^2\right] \le h_1^2 \end{cases} \tag{50}$$

Solving the linear programming problem (49) we have the following solutions

t_1	1.3327	$t_5(B_{11})$	0.1371
t_2	0.5256	$t_5(B_{12})$	0.4847
t_3	0.2725	$t_6(B_{11})$	1.5377
t_4	0.0969	$t_6(B_{12})$	0.6964

with a super-hedging value of

$$u_0^2 = 2.5649$$

As we notice, there is slightly small decrease on the value of upper bound for the asked price comparing with the case where we calculated this upper bound one step at a time. This is completely justified because the minimization above has more constraints since we are using the two period bond and asset to formulate our problem. But as we stated before, for multi-period cases (that requires us to solve linear models with multiple variables and parameters with a very small adjustments) we will find the upper-bound and lower-bound by solving linear programming problems one at a time.

4.6. The General 2-Period Model

In this section we use the results from the examples in this section to summarize the two different cases on a 2-period model. First case is when one can use all possibilities of bonds and assets (both 1-period and 2-periods).

Theorem 5. *Let $T = 2$, $t = 0, 1$, $c = (c_1, c_2) \in \mathcal{C}$ an acceptable cash flow and consider the following linear programing problem*

$$\begin{cases} u_0^2(II) = \sup \mathbf{E}_0\left[f_0^1 c_1 + f_0^2 c_2\right] \\ \textit{subject to} \\ \mathbf{E}_0\left[f_0^1\right] \le h_0^1,\ \mathbf{E}_0\left[f_0^1(-1)\right] \le -g_0^1, \\ \mathbf{E}_0\left[f_0^2\right] \le h_0^2,\ \mathbf{E}_0\left[f_0^2(-1)\right] \le -g_0^2, \\ \mathbf{E}_1\left[f_1^2\right] \le h_1^2,\ \mathbf{E}_1\left[f_1^2(-1)\right] \le -g_1^2, \\ \mathbf{E}_1\left[f_1^2 S_2^i\right] \le a_1^i\ ,\mathbf{E}_1\left[f_1^2\left(-S_2^i\right)\right] \le -b_1^i, \\ \mathbf{E}_0\left[f_0^1 S_1^i + f_0^2 S_2^i\right] \le a_0^i, \\ \mathbf{E}_0\left[f_0^1\left(-S_1^i\right) + f_0^2\left(-S_2^i\right)\right] \le -b_0^i. \end{cases} \tag{51}$$

Then $u_0^2(II)$ is a super-hedging bound and the solution to dual problem is a portfolio by which one is able to construct an arbitrage opportunity if the market bid price of c exceeds $u_0^2(II)$.

Proof. As we saw in section (4.5) the dual problem to linear programming (51) is

$$\begin{cases} u_0^2(II) = \min(-p_0) \\ \quad p_1 - c_1 \geq 0, \\ \quad p_2 - c_2 \geq 0. \end{cases} \tag{52}$$

where

$$(p_0, p_1, p_2) = \sum_{t=0}^{1} \left[\sum_{i=1}^{M} \left(\alpha_t^i S^{it} + \tilde{\alpha}_t^i \tilde{S}^{it} \right) + \sum_{u=t+1}^{2} \left(\beta_t^u 1_s^{ut} + \tilde{\beta}_t^u \tilde{1}_s^{ut} \right) \right] \tag{53}$$

is the zero-cost cash flow (portfolio) as our trading strategy and to be more precise

$$p_0 = \sum_{i=1}^{M} \left(-\alpha_0^i a_0^i + \tilde{\alpha}_0^i b_0^i \right) + \sum_{u=1}^{2} \left(-\beta_0^u h_0^u + \tilde{\beta}_0^u g_0^u \right) \tag{54}$$

is the cost of building such portfolio to be paid at time $t = 0$. Also

$$p_1 = \underbrace{\sum_{i=1}^{M} \left(\alpha_0^i - \tilde{\alpha}_0^i \right) S_1^i + \left(\beta_0^1 - \tilde{\beta}_0^1 \right)}_{\text{the pay-off coming from t=0}} + \underbrace{\sum_{i=1}^{M} \left(-\alpha_1^i a_1^i + \tilde{\alpha}_1^i b_1^i \right) + \left(-\beta_1^2 h_1^2 + \tilde{\beta}_1^2 g_1^2 \right)}_{\text{adjustment at t=1}} \tag{55}$$

where as indicated, p_1 has two parts, one is the pay-off and the other is the adjustment at time $t = 1$. But p_2 is entirely the pay-off where

$$p_2 = \underbrace{\sum_{i=1}^{M} \left(\alpha_0^i - \tilde{\alpha}_0^i \right) S_2^i + \left(\beta_0^2 - \tilde{\beta}_0^2 \right)}_{\text{the pay-off from t=0}} + \underbrace{\sum_{i=1}^{M} \left(\alpha_1^i - \tilde{\alpha}_1^i \right) S_2^i + \left(\beta_1^2 - \tilde{\beta}_1^2 \right)}_{\text{the pay-off from t=1}} \tag{56}$$

A solution to linear programming problem (52) is a set of non-negative coefficients $\alpha_t^i, \tilde{\alpha}_t^i, \beta_t^u, \tilde{\beta}_t^u$ that determine the trading strategy to super-hedge. Specifically $\beta_0^2, \tilde{\beta}_0^2$ tell us about the number of shares of 2-period bonds 1^{20} and $\tilde{1}^{20}$ traded at $t = 0$. □

In this version of the problem one is able to take advantage of the both 2-period bond and assets, however as we saw before there is some complexity in solving and finding the right strategy to construct the portfolio, where as the number of assets and their price possibilities increase, the complexity of finding the right portfolio strategy increases exponentially.

On the other hand we can find a super-hedge value for a 2-period linear programming problem by solving two 1-period problems successively, in which case we have the following theorem.

Theorem 6. *Let $c = (c_1, c_2) \in \mathcal{C}$ be an acceptable cash flow and suppose*

$$\begin{cases} \qquad u_1^2 = \sup \mathbf{E}_1 \left[f_1^2 c_2 \right] \\ \textit{subject to} \\ \quad \mathbf{E}_1 \left[f_1^2 \right] \leq h_1^2, \; \mathbf{E}_1 \left[f_1^2 (-1) \right] \leq -g_1^2, \\ \mathbf{E}_1 \left[f_1^2 S_2^i \right] \leq a_1^i, \; \mathbf{E}_1 \left[f_1^2 \left(-S_2^i \right) \right] \leq -b_1^i. \end{cases} \tag{57}$$

is a super-hedging bound for c_2 paid at time $t = 1$ (the largest ask price for c_2 under no-arbitrage assumption). Now consider the following linear programming problem

$$\begin{cases} u_0^2(I) = \sup \mathbf{E}_0 \left[f_0^1 \left(c_1 + u_1^2 \right) \right] \\ \textit{subject to} \\ \mathbf{E}_0 \left[f_0^1 \right] \le h_0^1,\ \mathbf{E}_0 \left[f_0^1 (-1) \right] \le -g_0^1, \\ \mathbf{E}_0 \left[f_0^1 S_1^i \right] \le a_0^i, \\ \mathbf{E}_0 \left[f_0^1 \left(-S_1^i \right) \right] \le -b_0^i. \end{cases} \tag{58}$$

then the solution to linear programming problems (57) and (58) and their duals gives us a trading strategy by which we can take advantage if any arbitrage opportunity is available.

Proof. The dual problem of (57) is given by

$$\begin{cases} u_1^2 = \min(-p_1) \\ p_2 - c_2 \ge 0. \end{cases} \tag{59}$$

where

$$\begin{aligned} (p_1, p_2) &= \sum_{i=1}^{M} \left(\alpha_1^i S^{i1} + \tilde{\alpha}_1^i \tilde{S}^{i1} \right) + \left(\beta_1^2 1_1^{21} + \tilde{\beta}_1^2 \tilde{1}_1^{21} \right) \\ &= \left(\sum_{i=1}^{M} \left(-\alpha_1^i a_1^i + \tilde{\alpha}_1^i b_1^i \right) + \left(-\beta_1^2 h_1^2 + \tilde{\beta}_1^2 g_1^2 \right), \sum_{i=1}^{M} \left(\alpha_1^i - \tilde{\alpha}_1^i \right) S_2^i + \left(\beta_1^2 - \tilde{\beta}_1^2 \right) \right) \end{aligned} \tag{60}$$

and the non-negative $\mathcal{F}_1$-measurable random variables $\alpha_1^i, \tilde{\alpha}_1^i, \beta_1^2, \tilde{\beta}_1^2$ determine the trading strategy form $t = 1$ to $t = 2$.

Similarly the dual problem of (58) is

$$\begin{cases} u_0^2(I) = \min(-q_0) \\ q_1 - (c_1 + u_1^2) \ge 0. \end{cases} \tag{61}$$

where

$$\begin{aligned} (q_0, q_1) &= \sum_{i=1}^{M} \left(\alpha_0^i S^{i0} + \tilde{\alpha}_0^i \tilde{S}^{i0} \right) + \left(\beta_0^1 1_0^{10} + \tilde{\beta}_0^1 \tilde{1}_0^{10} \right) \\ &= \left(\sum_{i=1}^{M} \left(-\alpha_0^i a_0^i + \tilde{\alpha}_0^i b_0^i \right) + \left(-\beta_0^1 h_0^1 + \tilde{\beta}_0^1 g_0^1 \right), \sum_{i=1}^{M} \left(\alpha_0^i - \tilde{\alpha}_0^i \right) S_1^i + \left(\beta_0^1 - \tilde{\beta}_0^1 \right) \right) \end{aligned} \tag{62}$$

is a zero-cost cash flow and the non-negative $\mathcal{F}_0$-measurable random variables $\alpha_0^i, \tilde{\alpha}_0^i, \beta_0^1, \tilde{\beta}_0^1$ determine the trading strategy form $t = 0$ to $t = 1$. We notice that

$$q_0 = \sum_{i=1}^{M} \left(-\alpha_0^i a_0^i + \tilde{\alpha}_0^i b_0^i \right) + \left(-\beta_0^1 h_0^1 + \tilde{\beta}_0^1 g_0^1 \right) \tag{63}$$

is the cost of this portfolio paid at $t = 0$. □

As we saw above, we can find a super-hedging bound by two different approaches and because of that there is a slight difference in the values of $u_0^2(I)$ on these two approaches. The following theorem summarizes our above discussion on an upper bound for the difference between two values.

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Theorem 7. *Let $u_0^2(II)$ and $u_0^2(I)$ be the two super-hedging values found in Theorem (5) and Theorem (6) respectively. Then*

1. $u_0^2(II) \leq u_0^2(I)$
2. $u_0^2(I) - u_0^2(II) \leq \left[\left(h_0^1 \max_{\omega\in\Omega} h_1^2(\omega) - h_0^2\right) + \sum_{i=1}^{M}\left(\frac{a_0^{i1}\max_{\omega\in\Omega} a_1^{i2}(\omega) - a_0^{i2}}{\min_{\omega\in\Omega} S_2^i(\omega)}\right)\right] \max_{\omega\in\Omega} c_2(\omega)$

Proof. 1. Let $\mathcal{PF}(II)$ be the set of all constrains for linear programming problem (5) and similarly let $\mathcal{PF}(I)$ be the set of all constrains for linear programming problem (6). We notice that $\mathcal{PF}(I) \subset \mathcal{PF}(II)$ so that

$$u_0^2(II) = \sup_{f_t^u \in \mathcal{PF}(II)} \mathbf{E}_0\left[f_0^1 c_1 + f_0^2 c_2\right] \leq \sup_{f_t^u \in \mathcal{PF}(I)} \mathbf{E}_0\left[f_0^1 c_1 + f_0^2 c_2\right] = u_0^2(I) \tag{64}$$

2. We start the argument by looking in a few easier and more concrete cases.

(a) Consider the case where a super-hedging bound of c_2 is found by only zero-coupon bonds (a 2-period bond h_0^2 or two 1-period bonds h_0^1, h_1^2). Then the price difference would be

$$h_0^1 \max_{\omega\in\Omega} h_1^2(\omega) \max_{\omega\in\Omega} c_2(\omega) - h_0^2 \max_{\omega\in\Omega} c_2(\omega) = \left(h_0^1 \max_{\omega\in\Omega} h_1^2(\omega) - h_0^2\right) \max_{\omega\in\Omega} c_2(\omega) \tag{65}$$

(b) If we assume that there is only one asset with two options (a 2-period $S_0^2 \neq 0$ or two 1-period $S_0^1 and S_1^2$) then the difference in the hedging-price values is

$$\left(a_0^1 \max_{\omega\in\Omega} a_1^2(\omega) - a_0^2\right) \max_{\omega\in\Omega}\left(\frac{c_2}{S_2}(\omega)\right) \leq \left(a_0^1 \max_{\omega\in\Omega} a_1^2(\omega) - a_0^2\right)\left(\frac{\max_{\omega\in\Omega} c_2(\omega)}{\min_{\omega\in\Omega} S_2(\omega)}\right) \tag{66}$$

(c) Now if we use both bond and asset then we have an upper bound for the difference as

$$\left[\left(h_0^1 \max_{\omega\in\Omega} h_1^2(\omega) - h_0^2\right) + \left(\frac{a_0^1 \max_{\omega\in\Omega} a_1^2(\omega) - a_0^2}{\min_{\omega\in\Omega} S_2(\omega)}\right)\right] \max_{\omega\in\Omega} c_2(\omega) \tag{67}$$

(d) Therefore for a super-hedging with both bonds and finite number (M) of assets we have

$$u_0^2(I) - u_0^2(II) \leq \left[\left(h_0^1 \max_{\omega\in\Omega} h_1^2(\omega) - h_0^2\right) + \sum_{i=1}^{M}\left(\frac{a_0^{i1}\max_{\omega\in\Omega} a_1^{i2}(\omega) - a_0^{i2}}{\min_{\omega\in\Omega} S_2^i(\omega)}\right)\right] \max_{\omega\in\Omega} c_2(\omega) \tag{68}$$

□

Remark 4. *The upper bound we found above, is an over estimate, but it is small relative to the portfolio price.*

5. Multi-period Case Theorem

The discussion in Section 4 showed that deriving the most accurate super- and sub-hedging bounds is too complex to be practical. An iterative process dealing with one period (bonds and assets) at a time provides a good compromise between accuracy and tractability.

Theorem 8. *Let $0 \leq t \leq T-1$ and consider the following linear programming problem*

$$\begin{cases} u_t^T = \sup \mathbf{E}_t\left[f_t^{t+1}\left(c_{t+1} + \mathbf{E}_{t+1}\left[f_{t+1}^{t+2}\left(c_{t+2} + \cdots + \mathbf{E}_{T-1}\left[f_{T-1}^T c_T\right]\right)\right]\right)\right] \\ \text{subject to} \\ \quad \mathbf{E}_s\left[f_s^{s+1}\right] \leq h_s^{s+1} \;,\; \mathbf{E}_s\left[f_s^{s+1}(-1)\right] \leq -g_s^{s+1} \;,\; s = t, \ldots, T-1 \\ \quad \mathbf{E}_s\left[f_s^{s+1} S_{s+1}^i\right] \leq a_s^i \;,\; \mathbf{E}_s\left[f_s^{s+1}\left(-S_{s+1}^i\right)\right] \leq -b_s^i \;,\; i = 1, \ldots, M \end{cases} \tag{69}$$

Then u_t^T is a super-hedging bound and the solution to dual problem is a trading strategy by which one is able to construct an arbitrage opportunity if the market bid price of c exceeds u_t^T.

Proof. Because of what was discussed in lemma 1 (the complexity in solving the dual linear programming problem) we solve and find the trading strategy starting at $T-1$ going backward and one step at a time. So first let

$$\begin{cases} u_{T-1}^T = \sup \mathbf{E}_{T-1}\left[f_{T-1}^T c_T\right] \\ \textit{subject to} \\ \quad \mathbf{E}_{T-1}\left[f_{T-1}^T\right] \leq h_{T-1}^T \ , \qquad\qquad \mathbf{E}_{T-1}\left[f_{T-1}^T(-1)\right] \leq -g_{T-1}^T \\ \mathbf{E}_{T-1}\left[f_{T-1}^T S_{T-1}^i\right] \leq a_{T-1}^i \ , \quad \mathbf{E}_{T-1}\left[f_{T-1}^T\left(-S_{T-1}^i\right)\right] \leq -b_{T-1}^i \ , \quad i=1,\ldots,M \end{cases} \tag{70}$$

We formulate the dual problem by using the Lagrangian. First let

$$(\Lambda,\gamma)_{T-1} = (\lambda_{T-1}^1,\ldots,\lambda_{T-1}^M,\tilde{\lambda}_{T-1}^1,\ldots,\tilde{\lambda}_{T-1}^M,\gamma_{T-1}^T,\tilde{\gamma}_{T-1}^T) \in RV\left(\mathbb{R}_+^{2M+2},\mathcal{F}_{T-1}\right) \tag{71}$$

be the Lagrange multiplier of our linear programing problem. So we can write the Lagrangian as

$$\begin{aligned} L(f,(\Lambda,\gamma)_{T-1}) = \mathbf{E}_{T-1}\left[f_{T-1}^T c_T\right] + \sum_{i=1}^M \lambda_{T-1}^i\left(a_{T-1}^i - \mathbf{E}_{T-1}\left[f_{T-1}^T S_{T-1}^i\right]\right) + \\ \sum_{i=1}^M \tilde{\lambda}_{T-1}^i\left(\mathbf{E}_{T-1}\left[f_{T-1}^T S_{T-1}^i\right] - b_{T-1}^i\right) + \gamma_{T-1}^T\left(h_{T-1}^T - \mathbf{E}_{T-1}\left[f_{T-1}^T\right]\right) + \\ \tilde{\gamma}_{T-1}^T\left(\mathbf{E}_{T-1}\left[f_{T-1}^T\right] - g_{T-1}^T\right) \end{aligned} \tag{72}$$

Now we can observe that

$$\inf_{(\Lambda,\gamma)_{T-1}\in RV\left(\mathbb{R}_+^{2M+2},\mathcal{F}_{T-1}\right)} L(f,(\Lambda,\gamma)_{T-1}) = \begin{cases} \mathbf{E}_{T-1}\left[f_{T-1}^T c_T\right] & f \in \mathcal{PD} \\ -\infty & \textit{otherwise} \end{cases} \tag{73}$$

So that, by strong linear programming duality we have

$$\begin{aligned} u_{T-1} &= \sup_{f\in\mathcal{PD}} \inf_{(\Lambda,\gamma)_{T-1}\in RV\left(\mathbb{R}_+^{2M+2},\mathcal{F}_{T-1}\right)} L(f,(\Lambda,\gamma)_{T-1}) \\ &= \inf_{(\Lambda,\gamma)_{T-1}\in RV\left(\mathbb{R}_+^{2M+2},\mathcal{F}_{T-1}\right)} \sup_{f\in\mathcal{PD}} L(f,(\Lambda,\gamma)_{T-1}) \end{aligned} \tag{74}$$

On the other hand, the Lagrangian can be rewritten as

$$\begin{aligned} L(f,(\Lambda,\gamma)_{T-1}) = \mathbf{E}_{T-1}\left\{f_{T-1}^T\left(c_T - \left[\sum_{i=1}^M\left(\lambda_{T-1}^T - \tilde{\lambda}_{T-1}^i\right)S_T^i + \left(\gamma_{T-1}^T - \tilde{\gamma}_{T-1}^T\right)\right]\right)\right\} \\ -\left(\sum_{i=1}^M\left(-\lambda_{T-1}^i a_{T-1}^i + \tilde{\lambda}_{T-1}^i b_{T-1}^i\right) + \left(-\gamma_{T-1}^T h_{T-1}^T + \tilde{\gamma}_{T-1}^T g_{T-1}^T\right)\right) \end{aligned} \tag{75}$$

Now for $(\Lambda,\gamma)_{T-1} \in RV\left(\mathbb{R}_+^{2M+2},\mathcal{F}_{T-1}\right)$ consider the zero-cost portfolio of cash flow

$$\left(p_{T-1}\left((\Lambda,\gamma)_{T-1}\right), p_T\left((\Lambda,\gamma)_{T-1}\right)\right) \tag{76}$$

$$= \gamma^T_{T-1} 1^{T,T-1} + \tilde{\gamma}^T_{T-1} \tilde{1}^{T,T-1} + \sum_{i=1}^{M} \left(\lambda^T_{T-1} S^{i,T-1} + \tilde{\lambda}^i_{T-1} \tilde{S}^{i,T-1} \right) \tag{77}$$

so that

$$p_{T-1}\left((\Lambda,\gamma)_{T-1}\right) = \sum_{i=1}^{M} \left(-\lambda^i_{T-1} a^i_{T-1} + \tilde{\lambda}^i_{T-1} b^i_{T-1} \right) + \left(-\gamma^T_{T-1} h^T_{T-1} + \tilde{\gamma}^T_{T-1} g^T_{T-1} \right) \tag{78}$$

and

$$p_T\left((\Lambda,\gamma)_{T-1}\right) = \sum_{i=1}^{M} \left(\lambda^T_{T-1} - \tilde{\lambda}^i_{T-1} \right) S^i_T + \left(\gamma^T_{T-1} - \tilde{\gamma}^T_{T-1} \right) \tag{79}$$

Since the term $p_{T-1}\left((\Lambda,\gamma)_{T-1}\right)$ is an $\mathcal{F}$ measurable random variable and independent of $f \in \mathcal{PD}$, then we see that (75) can be written as

$$L(f,(\Lambda,\gamma)_{T-1}) = \mathbf{E}_{T-1}\left\{ f^T_{T-1}\left(c_T - p_T\left((\Lambda,\gamma)_{T-1}\right)\right)\right\} - p_{T-1}\left((\Lambda,\gamma)_{T-1}\right) \tag{80}$$

Now let

$$\bar{\omega} \in \arg\max \left\{ c_T - p_T\left((\Lambda,\gamma)_{T-1}\right)\right\} \tag{81}$$

Also let $f^T_{T-1} := g^T_{T-1}$ for all $\omega \in \Omega$. Then (74) becomes

$$u^T_{T-1} = \inf_{(\Lambda,\gamma)_{T-1} \in RV\left(\mathbb{R}^{2M+2}_+, \mathcal{F}_{T-1}\right)} \left\{ g^T_{T-1} \sup_{\omega\in\Omega} \left[c_T(\omega) - p_T\left((\Lambda,\gamma)_{T-1}\right)(\omega)\right] - p_{T-1}\left((\Lambda,\gamma)_{T-1}\right)\right\} \tag{82}$$

If $(\bar{\Lambda},\bar{\gamma})_{T-1}$ is a solution to the minimization problem (82) then it determines the trading strategy from time $T-1$ to time T while u^T_{T-1} is the highest price for c_T (with no arbitrage) paid at time $T-1$. Therefore we set up a linear programming problem that determines a super-hedging value of both u^T_{T-1} and c_{T-1} paid at time $T-2$, as following

$$\begin{cases} u^{T-1}_{T-2} = \sup \mathbf{E}_{T-2}\left[f^T_{T-1}\left(c_{T-1} + u^T_{T-1}\right)\right] \\ \quad \textit{subject to} \\ \qquad \mathbf{E}_{T-2}\left[f^{T-1}_{T-2}\right] \le h^{T-1}_{T-2} \ , \qquad \qquad \mathbf{E}_{T-2}\left[f^{T-1}_{T-2}(-1)\right] \le -g^{T-1}_{T-2} \\ \\ \quad \mathbf{E}_{T-2}\left[f^{T-1}_{T-2} S^i_{T-2}\right] \le a^i_{T-2} \ , \qquad \mathbf{E}_{T-2}\left[f^{T-1}_{T-2}\left(-S^i_{T-2}\right)\right] \le -b^i_{T-2} \ , \ i = 1,\dots,M \end{cases} \tag{83}$$

Similar to what we did above on establishing and solving the dual problem, suppose $(\bar{\Lambda},\bar{\gamma})_{T-2}$ is the solution to dual linear programming problem of (83), then it determines the trading strategy form time $T-2$ to $T-1$. Continuing this backward process and one step at a time, we will have a sequence of pairs, super-hedges and non-negative random variable coefficients

$$\left(u^{t+2}_{t+1}, (\bar{\Lambda},\bar{\gamma})_{t+1}\right), \cdots, \left(u^{T-2}_{T-3}, (\bar{\Lambda},\bar{\gamma})_{T-3}\right), \left(u^{T-1}_{T-2}, (\bar{\Lambda},\bar{\gamma})_{T-2}\right), \left(u^{T}_{T-1}, (\bar{\Lambda},\bar{\gamma})_{T-1}\right) \tag{84}$$

where

$$u^r_{r-1} = \sup_{f\in\mathcal{PD}} \mathbf{E}_{r-1}\left[f^r_{r-1}\left(c_r + u^{r+1}_r\right)\right] \tag{85}$$

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and the first term in (84) is the super-hedging value and the corresponding trading strategy coefficients determined at time $t+1$. Now we set-up our last linear programming problem where we find the super-hedging bound for $c = (0, \cdots, 0, c_{t+1}, c_{t+2}, \cdots, c_T)$. So let

$$\begin{cases} u_t^T = \sup \mathbf{E}_t \left[f_t^{t+1} \left(c_{t+1} + u_{t+1}^{t+2} \right) \right] \\ \quad \textit{subject to} \\ \qquad \mathbf{E}_t \left[f_t^{t+1} \right] \leq h_t^{t+1} \ , \qquad \mathbf{E}_t \left[f_t^{t+1} (-1) \right] \leq -g_t^{t+1} \\ \qquad \mathbf{E}_t \left[f_t^{t+1} S_t^i \right] \leq a_t^i \ , \qquad \mathbf{E}_t \left[f_t^{t+1} \left(-S_t^i \right) \right] \leq -b_t^i \ , \ i = 1, \ldots, M \end{cases} \tag{86}$$

with the corresponding Lagrangian

$$L(f, (\Lambda, \gamma)_t) = \mathbf{E}_t \left\{ f_t^{t+1} \left(c_t + u_{t+1}^{t+2} - p_{t+1} \left((\Lambda, \gamma)_t \right) \right) \right\} - p_t \left((\Lambda, \gamma)_t \right) \tag{87}$$

so that

$$u_t^T = \inf_{(\Lambda,\gamma)_t \in RV\left(\mathbb{R}_+^{2M+2}, \mathcal{F}_{T-1}\right)} \sup_{f \in \mathcal{PD}} L(f, (\Lambda, \gamma)_t) \tag{88}$$

$$= \inf_{(\Lambda,\gamma)_t \in RV\left(\mathbb{R}_+^{2M+2}, \mathcal{F}_{T-1}\right)} \left\{ g_t^{t+1} \sup_{\omega \in \Omega} \left[c_{t+1}(\omega) + u_{t+1}^{t+2}(\omega) - p_{t+1} \left((\Lambda, \gamma)_t \right)(\omega) \right] - p_t \left((\Lambda, \gamma)_t \right) \right\} \tag{89}$$

If $(\bar{\Lambda}, \bar{\gamma})_t$ is a solution to the minimization problem (89) then, in fact, it determines the trading strategy from time t to time $t+1$ and along with the other non-negative random variable coefficients $(\bar{\Lambda}, \bar{\gamma})_r$, $r = t+1, \cdots, T-1$ we have created a trading strategy corresponding to

$$(\bar{\Lambda}, \bar{\gamma})_t, (\bar{\Lambda}, \bar{\gamma})_{t+1}, \cdots, (\bar{\Lambda}, \bar{\gamma})_{T-1} \tag{90}$$

with the super-hedging bound u_t^T by which one can create an arbitrage opportunity if any higher price in the market is available. □

Remark 5. *Even though, the value of u_t^T we find in the Theorem above may not be the lowest super-hedging value for the ask price of the cash flow c, it is still a close estimate, as we discussed in Theorem 7 and an upper bound for the exact super-hedging value $u_t(c)$.*

6. Conclusion

We used convex duality in a multi-period conic finance (two-price economy) to extend the refined version of FTAP. This helped us to provide a framework to drive non-arbitrage lower and upper price bounds and corresponding trading strategies which one can take advantage of when a preferable bid (ask) price exists. We also analyzed the computation complexity of implementation in two settings:

a) using only one period bonds,
b) involving all available bonds.

and we showed that involving all available bonds makes the problem unrealistically complex. We provided estimates on the error between the prices for one period and all bonds, which we saw that it is small. Therefore we recommended method dealing only with one period bonds in practice, where the problems becomes pricing one period at a time. Finding a better approximation for the price bounds and shrinking the non-arbitrage region and lowering the computation complexity are two problems that one could consider for further exploration.

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