

Characterization of relationships between the domains of two linear matrix-valued functions with applications

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Abstract. One of the typical forms of linear matrix expressions (linear matrix-valued functions) is given by $A + B_1 X_1 C_1 + \dots + B_k X_k C_k$, where X_1, \dots, X_k are independent variable matrices of appropriate sizes, which include almost all matrices with unknown entries as its special cases. The domain of the matrix expression is defined to be all possible values of the matrix expressions with respect to X_1, \dots, X_k . In this article, we approach some problems on the relationships between the domains of two linear matrix expressions by means of the block matrix method (BMM), the matrix rank method (MRM), and the matrix equation method (MEM). As application, we discuss some topics on the relationships among general solutions of some linear matrix equations and their reduced equations.

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1 Introduction

Throughout this article, we denote by $\mathbb{C}^{m \times n}$ the set of all $m \times n$ complex matrices; by A^* , $r(A)$, and $\mathcal{R}(A)$ the conjugate transpose, the rank, and the range (column space) of a matrix $A \in \mathbb{C}^{m \times n}$, respectively; by I_m the identity matrix of order m ; and $[A, B]$ be a row block matrix consisting of A and B . A matrix $A \in \mathbb{C}^{m \times m}$ is said to be EP (or range Hermitian) if $\mathcal{R}(A^*) = \mathcal{R}(A)$ holds. We next introduce the definition and notation of generalized inverses of a matrix. The Moore–Penrose inverse of $A \in \mathbb{C}^{m \times n}$, denoted by A^\dagger , is the unique matrix $X \in \mathbb{C}^{n \times m}$ satisfying the four Penrose equations

$$(i) AXA = A, \quad (ii) XAX = X, \quad (iii) (AX)^* = AX, \quad (iv) (XA)^* = XA. \quad (1.1)$$

A matrix X is called an $\{i, \dots, j\}$ -generalized inverse of A , denoted by $A^{(i, \dots, j)}$, if it satisfies the i th, \dots , j th equations in (1.1). The collection of all $\{i, \dots, j\}$ -generalized inverses of A is denoted by $\{(A^{(i, \dots, j)})\}$. There are all 15 types of $\{i, \dots, j\}$ -generalized inverses for a given matrix A by definition, but people are mainly interested in the types that involve the first equation:

$$A^\dagger, \quad A^{(1,3,4)}, \quad A^{(1,2,4)}, \quad A^{(1,2,3)}, \quad A^{(1,4)}, \quad A^{(1,3)}, \quad A^{(1,2)}, \quad A^{(1)}, \quad (1.2)$$

which are usually called the eight commonly-used types of generalized inverses of A in the literature; see e.g., [4, 5, 18]. In addition, we also denote by $P_A = I_m - AA^\dagger$ and $Q_A = I_n - A^\dagger A$ the orthogonal projectors (Hermitian idempotent matrices) induced from A . The Kronecker product of any two matrices A and B is defined to be $A \otimes B = (a_{ij}B)$. The vectorization operator of a matrix $A = [\mathbf{a}_1, \dots, \mathbf{a}_n]$ is defined to be $\text{vec}(A) = \vec{A} = [\mathbf{a}_1^\top, \dots, \mathbf{a}_n^\top]^\top$. A well-known property on the vec operator of a triple matrix product is $\vec{AXB} = (B^\top \otimes A)\vec{X}$; see e.g., [3, 20].

Linear matrix expressions that involve variable matrices arise in a variety of problems in pure and applied mathematics. In the present paper we pursue our study of a general linear matrix expressions of the form

$$f(X_1, X_2, \dots, X_k) = A + B_1 X_1 C_1 + B_2 X_2 C_2 + \dots + B_k X_k C_k, \quad (1.3)$$

where $A \in \mathbb{C}^{m \times n}$, $B_i \in \mathbb{C}^{m \times p_i}$, and $C_i \in \mathbb{C}^{q_i \times n}$ are given, and $X_i \in \mathbb{C}^{p_i \times q_i}$ are variable matrices, $i = 1, 2, \dots, k$. Eq. (1.3) is usually called a Linear Matrix-Valued Function (LMVF), while the collection of all possible matrix values of (1.3), called the domain of (1.3), is denoted schematically by

$$\mathcal{D}_f = \{f(X_1, X_2, \dots, X_k) \mid X_i \in \mathbb{C}^{p_i \times q_i}, \quad i = 1, 2, \dots, k\}. \quad (1.4)$$

Eq. (1.3) includes many kinds of well-known matrix expressions with variable entries as its special cases, such as, $A + BX$, $A + BXC$, $A + BX + YC$, see e.g., [25, 27], as well as various partially specified matrices, such as, $\begin{bmatrix} A & B \\ C & ? \end{bmatrix}$, $\begin{bmatrix} A & ? \\ ? & D \end{bmatrix}$, $\begin{bmatrix} A & ? \\ ? & ? \end{bmatrix}$, etc, see e.g., [1, 7, 8, 11, 12]. There are many natural modifications of considering the LMVF in mathematics and applications. Here we mention a few:

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- (I) The general solution of a consistent linear matrix equation $AX = B$ is $X = A^{-}B + (I - A^{-}A)U$, where A^{-} denotes a g -inverse of A and the matrix U is arbitrary; the general solution of a consistent linear matrix equation $AXB = C$ can be written as $X = A^{-}CB^{-} + (I - A^{-}A)U_1 + U_2(I - BB^{-})$, where U_1 and U_2 are two arbitrary matrices.
- (II) The general expression of g -inverse A^{-} can be written as $A^{-} = A^{\dagger} + (I - A^{\dagger}A)U_1 + U_2(I - AA^{\dagger})$, where A^{\dagger} is the Moore-Penrose inverse of A , and U_1 and U_2 are two arbitrary matrices.
- (III) Consider a Gauss-Markov model $\{\mathbf{y}, X\beta, \sigma^2\Sigma\}$, where Σ is a known symmetric nonnegative definite matrix and σ^2 is an unknown positive parameter. The general expression of the weighted least-squares estimator (WLSE) of β with respect to a given weight matrix W can be written as $\hat{\beta} = ((X'WX)^{\dagger}X'W + [I_m - (X'WX)^{\dagger}(X'WX)]U)\mathbf{y}$, where U is an arbitrary matrix.
- (IV) The best linear unbiased estimator (BLUE) of $X\beta$ in $\mathcal{M} = \{\mathbf{y}, X\beta, \sigma^2\Sigma\}$ is $G\mathbf{y}$, where $G = [X, 0][X, \Sigma Q_X]^{\dagger} + U(I_m - [X, \Sigma Q_X][X, \Sigma Q_X]^{\dagger})$, in which, U is an arbitrary matrix and $Q_X = I_m - XX^{\dagger}$.

All these matrices are in fact LMVF that involve one or two variable matrices. In such cases, people wish to know properties of these matrix expressions, for example, uniqueness (invariance), maximum and minimum possible ranks, range inclusions, norms, etc. The results obtained can be used to describe and solve the original problems.

As is known to all, one of the fundamental tasks in algebra is to establish and describe various algebraic equalities for operations of elements in the algebra. Assume that two matrix-valued functions $f(X_1, X_2, \dots, X_k)$ and $g(Y_1, Y_2, \dots, Y_l)$ of the same size are given, and one wish to know the connections between the two domains \mathcal{D}_f and \mathcal{D}_g . In this situations, we may divide the work into the following four situations

$$\mathcal{D}_f \cap \mathcal{D}_g \neq \emptyset, \quad \mathcal{D}_f \supseteq \mathcal{D}_g, \quad \mathcal{D}_f \subseteq \mathcal{D}_g, \quad \mathcal{D}_f = \mathcal{D}_g. \quad (1.5)$$

Here we mention some examples on relations between two linear matrix expressions:

- (a) When do two solvable linear matrix equations $A_1X_1B_1 = C_1$ and $A_2X_2B_2 = C_2$, where X_1 and X_2 have the same size, have a common solution?
- (b) When do the set inclusions $\{A^{-}\} \cap \{B^{-}\} = \emptyset$, $\{D_1 - C_1A_1^{-}B_1\} \cap \{D_2 - C_2A_2^{-}B_2\} = \emptyset$, and $\{A^{-} + B^{-}\} \cap \{C^{-}\} = \emptyset$, as well as the set equalities $\{A^{-}\} = \{B^{-}\}$, $\{D_1 - C_1A_1^{-}B_1\} = \{D_2 - C_2A_2^{-}B_2\}$, and $\{A^{-} + B^{-}\} = \{C^{-}\}$ hold?
- (c) When do OLSEs and BLUES under a $\{\mathbf{y}, X\beta, \sigma^2\Sigma\}$ coincide? and when OLSEs and BLUES under two competing linear regression models $\{\mathbf{y}, X_1\beta, \sigma_1^2\Sigma_1\}$ and $\{\mathbf{y}, X_2\beta, \sigma_2^2\Sigma_2\}$ coincide?

These facts show that algebraic features and performances of the matrix set in (1.4) are necessarily worth for investigation from both theoretical and applied points of view. In fact, a class of fundamental and meaningful problems that have been identified in the matrix calculus is the characterization of relationships between two given LMVF under various assumptions. In view of the above facts, the present author intends to investigate the relationships between two domains \mathcal{D}_1 and \mathcal{D}_2 generated from some special cases of (1.3) using the matrix range and rank methodology. We also discuss the connections among general solutions of some linear matrix equations and their reduced linear matrix equations.

2 Preliminaries

Block matrix, rank of matrix, and matrix equation are basic concepts in linear algebra, while the block matrix method (BMM), the matrix rank method (MRM), and the matrix equation method (MEM) are three fundamental and traditional analytic methods that are widely used in matrix theory and applications because they give one the ability to construct and analyze various complicated matrix expressions and matrix equalities in a subtle and computationally tractable way.

We next present a group of well-known results on ranks of matrices and matrix equations that are described by way of generalized inverses, MRM and BMM, which we shall use to deal with various matrix expressions and matrix equalities.

Lemma 2.1 ([13]). Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times k}$, and $C \in \mathbb{C}^{l \times n}$. Then

$$r[A, B] = r(A) + r(P_A B) = r(B) + r(P_B A), \quad (2.1)$$

$$r \begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C Q_A) = r(C) + r(A Q_C), \quad (2.2)$$

$$r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(P_B A Q_C). \quad (2.3)$$

In particular, the following results hold.

$$(a) r[A, B] = r(A) \Leftrightarrow \mathcal{R}(A) \supseteq \mathcal{R}(B) \Leftrightarrow A A^\dagger B = B \Leftrightarrow P_A B = 0.$$

$$(b) r \begin{bmatrix} A \\ C \end{bmatrix} = r(A) \Leftrightarrow \mathcal{R}(C^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow C A^\dagger A = C \Leftrightarrow C Q_A = 0.$$

$$(c) r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) \Leftrightarrow P_B A Q_C = 0.$$

Lemma 2.2 ([26]). Let $A_i \in \mathbb{C}^{m \times n_i}$, and denote $\widehat{A}_i = [A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_k]$, $i = 1, 2, \dots, k$. Then

$$(k-1)r[A_1, A_2, \dots, A_k] + \dim[\mathcal{R}(\widehat{A}_1) \cap \mathcal{R}(\widehat{A}_2) \cap \dots \cap \mathcal{R}(\widehat{A}_k)] = r(\widehat{A}_1) + r(\widehat{A}_2) + \dots + r(\widehat{A}_k). \quad (2.4)$$

In particular, the following three statements are equivalent:

$$(a) r[A_1, A_2, \dots, A_k] = r(A_1) + r(A_2) + \dots + r(A_k).$$

$$(b) (k-1)r[A_1, A_2, \dots, A_k] = r(\widehat{A}_1) + r(\widehat{A}_2) + \dots + r(\widehat{A}_k).$$

$$(c) \mathcal{R}(\widehat{A}_1) \cap \mathcal{R}(\widehat{A}_2) \cap \dots \cap \mathcal{R}(\widehat{A}_k) = \{0\}.$$

We also use the following well-known results in the sequel.

Lemma 2.3 ([19]). Let

$$AX = B \quad (2.5)$$

be a given linear matrix equation, where $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{m \times p}$ are known matrices, and $X \in \mathbb{C}^{n \times p}$ is an unknown matrix. Then, the following statements are equivalent:

(i) Eq. (2.5) is consistent.

(ii) $\mathcal{R}(A) \supseteq \mathcal{R}(B)$.

(iii) $r[A, B] = r(A)$.

(iv) $A A^\dagger B = B$.

In this case, the general solution of the equation can be written in the parametric form

$$X = A^\dagger B + Q_A U, \quad (2.6)$$

where $U \in \mathbb{C}^{n \times p}$ is an arbitrary matrix. In particular, (2.5) holds for all matrices $X \in \mathbb{C}^{n \times p}$ if and only if both $A = 0$ and $B = 0$, or equivalently, $[A, B] = 0$.

Lemma 2.4 ([19]). Let

$$AXB = C \quad (2.7)$$

be a given linear matrix equation, where $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{p \times q}$, and $C \in \mathbb{C}^{m \times q}$ are given. Then, the following statements are equivalent:

(i) Eq. (2.7) is consistent.

(ii) Both $\mathcal{R}(A) \supseteq \mathcal{R}(C)$ and $\mathcal{R}(B^*) \supseteq \mathcal{R}(C^*)$.

(iii) Both $r[A, C] = r(A)$ and $r\begin{bmatrix} B \\ C \end{bmatrix} = r(B)$.

(iv) $AA^\dagger CB^\dagger B = C$.

In this case, the general solution of (2.7) can be written in the parametric form

$$X = A^\dagger CB^\dagger + Q_A U + V P_B, \quad (2.8)$$

where $U, V \in \mathbb{C}^{n \times p}$ are arbitrary matrices. In particular, (2.7) holds for all matrices $X \in \mathbb{C}^{n \times p}$ if and only if

$$\text{either } [A, C] = 0 \text{ or } \begin{bmatrix} B \\ C \end{bmatrix} = 0. \quad (2.9)$$

Lemma 2.5 ([2]). The matrix equation

$$A_1 X_1 + X_2 B_2 = C \quad (2.10)$$

is consistent if and only if

$$r\begin{bmatrix} C & A_1 \\ B_2 & 0 \end{bmatrix} = r(A_1) + r(B_2), \quad (2.11)$$

or equivalently,

$$P_{A_1} C Q_{B_2} = 0. \quad (2.12)$$

Eq. (2.10) holds for all matrices X_1 and X_2 if and only if one of the following four block matrix equalities holds

$$\begin{bmatrix} C & A_1 \\ B_2 & 0 \end{bmatrix} = 0. \quad (2.13)$$

Lemma 2.6 ([14]). The matrix equation

$$A_1 X_1 B_1 + A_2 X_2 B_2 = C \quad (2.14)$$

is consistent if and only if the following four conditions hold

$$r[C, A_1, A_2] = r[A_1, A_2], \quad r\begin{bmatrix} C & A_1 \\ B_2 & 0 \end{bmatrix} = r(A_1) + r(B_2), \quad (2.15)$$

$$r\begin{bmatrix} C & A_2 \\ B_1 & 0 \end{bmatrix} = r(A_2) + r(B_1), \quad r\begin{bmatrix} C \\ B_1 \\ B_2 \end{bmatrix} = r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad (2.16)$$

or equivalently,

$$P_A C = 0, \quad P_{A_1} C Q_{B_2} = 0, \quad P_{A_2} C Q_{B_1} = 0, \quad C Q_B = 0, \quad (2.17)$$

where $A = [A_1, A_2]$ and $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$.

Lemma 2.7 ([10, 22]). Equation (2.14) holds for all matrices X_1 and X_2 if and only if one of the following four block matrix equalities holds

$$(i) \quad [C, A_1, A_2] = 0, \quad (ii) \quad \begin{bmatrix} C & A_1 \\ B_2 & 0 \end{bmatrix} = 0, \quad (iii) \quad \begin{bmatrix} C & A_2 \\ B_1 & 0 \end{bmatrix} = 0, \quad (iv) \quad \begin{bmatrix} C \\ B_1 \\ B_2 \end{bmatrix} = 0. \quad (2.18)$$

Lemma 2.8 ([21]). The matrix equation

$$A_1 X_1 + X_2 B_2 + A_3 X_3 B_3 + A_4 X_4 B_4 = C \quad (2.19)$$

is consistent iff the following four conditions hold

$$r \begin{bmatrix} C & A_1 & A_3 & A_4 \\ B_2 & 0 & 0 & 0 \end{bmatrix} = r[A_1, A_3, A_4] + r(B_2), \quad (2.20)$$

$$r \begin{bmatrix} C & A_1 & A_3 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} B_2 \\ B_4 \end{bmatrix} + r[A_1, A_3], \quad (2.21)$$

$$r \begin{bmatrix} C & A_1 & A_4 \\ B_2 & 0 & 0 \\ B_3 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} B_2 \\ B_3 \end{bmatrix} + r[A_1, A_4], \quad (2.22)$$

$$r \begin{bmatrix} C & A_1 \\ B_2 & 0 \\ B_3 & 0 \\ B_4 & 0 \end{bmatrix} = r \begin{bmatrix} B_2 \\ B_3 \\ B_4 \end{bmatrix} + r(A_1). \quad (2.23)$$

3 Some fundamental results on relationships between domains of two linear matrix-valued functions

We start with two groups of known results on the relationships between two matrix sets generated from the two simplest cases in (1.3).

Lemma 3.1 ([25]). *Given two domains of LMVFs:*

$$\mathcal{D}_1 = \{ A_1 + B_1 X_1 \mid X_1 \in \mathbb{C}^{p_1 \times n} \} \text{ and } \mathcal{D}_2 = \{ A_2 + B_2 X_2 \mid X_2 \in \mathbb{C}^{p_2 \times n} \}, \quad (3.1)$$

where $A_1, A_2 \in \mathbb{C}^{m \times n}$, $B_1 \in \mathbb{C}^{m \times p_1}$, and $B_2 \in \mathbb{C}^{m \times p_2}$ are known matrices, and $X_1 \in \mathbb{C}^{p_1 \times n}$ and $X_2 \in \mathbb{C}^{p_2 \times n}$ are variable matrices, we have the following results:

- (a) $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \emptyset$, i.e., there exist X_1 and X_2 such that $A_1 + B_1 X_1 = A_2 + B_2 X_2$ if and only if $\mathcal{R}(A_1 - A_2) \subseteq \mathcal{R}[B_1, B_2]$.
- (b) $\mathcal{D}_1 \subseteq \mathcal{D}_2$ if and only if $\mathcal{R}[A_1 - A_2, B_1] \subseteq \mathcal{R}(B_2)$.
- (c) $\mathcal{D}_1 = \mathcal{D}_2$ if and only if $\mathcal{R}(A_1 - A_2) \subseteq \mathcal{R}(B_1) = \mathcal{R}(B_2)$.

Lemma 3.2 ([25]). *Given two domains of LMVFs:*

$$\mathcal{D}_1 = \{ A_1 + B_1 X_1 C_1 \mid X_1 \in \mathbb{C}^{p_1 \times q_1} \} \text{ and } \mathcal{D}_2 = \{ A_2 + B_2 X_2 C_2 \mid X_2 \in \mathbb{C}^{p_2 \times q_2} \}, \quad (3.2)$$

where $A_i \in \mathbb{C}^{m \times n}$, $B_i \in \mathbb{C}^{m \times p_i}$, and $C_i \in \mathbb{C}^{q_i \times n}$ are given, and $X_i \in \mathbb{C}^{p_i \times q_i}$ are variable matrices, $i = 1, 2$, we have the following results:

- (a) $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \emptyset$ if and only if the following four conditions hold

$$\begin{aligned} \mathcal{R}(A_1 - A_2) &\subseteq \mathcal{R}[B_1, B_2], & \mathcal{R}(A_1^* - A_2^*) &\subseteq \mathcal{R}[C_1^*, C_2^*], \\ r \begin{bmatrix} A_1 - A_2 & B_1 \\ C_2 & 0 \end{bmatrix} &= r(B_1) + r(C_2), & r \begin{bmatrix} A_1 - A_2 & B_2 \\ C_1 & 0 \end{bmatrix} &= r(B_2) + r(C_1). \end{aligned}$$

- (b) $\mathcal{D}_1 \subseteq \mathcal{D}_2$ if and only if one of the following three conditions holds

- (i) $\mathcal{R}[A_1 - A_2, B_1] \subseteq \mathcal{R}(B_2)$ and $\mathcal{R}[A_1^* - A_2^*, C_1^*] \subseteq \mathcal{R}(C_2^*)$.
- (ii) $B_1 = 0$, $\mathcal{R}(A_1 - A_2) \subseteq \mathcal{R}(B_2)$, and $\mathcal{R}(A_1^* - A_2^*) \subseteq \mathcal{R}(C_2^*)$.
- (iii) $C_1 = 0$, $\mathcal{R}(A_1 - A_2) \subseteq \mathcal{R}(B_2)$, and $\mathcal{R}(A_1^* - A_2^*) \subseteq \mathcal{R}(C_2^*)$.

- (c) $\mathcal{D}_1 = \mathcal{D}_2$ if and only if one of the following five conditions holds

- (i) $\mathcal{R}(A_1 - A_2) \subseteq \mathcal{R}(B_1) = \mathcal{R}(B_2)$ and $\mathcal{R}(A_1^* - A_2^*) \subseteq \mathcal{R}(C_1^*) = \mathcal{R}(C_2^*)$.
- (ii) $A_1 = A_2$, $B_1 = 0$, and $B_2 = 0$.

- (iii) $A_1 = A_2, B_1 = 0, \text{ and } C_2 = 0.$
- (iv) $A_1 = A_2, B_2 = 0, \text{ and } C_1 = 0.$
- (v) $A_1 = A_2, C_1 = 0, \text{ and } C_2 = 0.$

As an extension, we have the following result on relationships between domains of two general matrix-valued functions, which we shall use in the latter part of the article.

Theorem 3.3. *Given two domains of LMVFs:*

$$\mathcal{D}_1 = \{ A_1 + B_1 X_1 + Y_1 C_1 \mid X_1 \in \mathbb{C}^{p_1 \times n_1}, Y_1 \in \mathbb{C}^{m_1 \times q_1} \}, \quad (3.3)$$

$$\mathcal{D}_2 = \{ A_2 + B_2 X_2 C_2 + D_2 Y_2 E_2 \mid X_2 \in \mathbb{C}^{s_2 \times t_2}, Y_2 \in \mathbb{C}^{u_2 \times v_2} \}. \quad (3.4)$$

where $A_1 \in \mathbb{C}^{m \times n}, B_1 \in \mathbb{C}^{m \times p_1}, C_1 \in \mathbb{C}^{q_1 \times n}, A_2 \in \mathbb{C}^{m \times n}, B_2 \in \mathbb{C}^{m \times s_2}, C_2 \in \mathbb{C}^{t_2 \times n}, D_2 \in \mathbb{C}^{m \times u_2}, \text{ and } E_2 \in \mathbb{C}^{v_2 \times n}$ are known matrices, we have the following results:

- (a) $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \emptyset$ if and only if the following four conditions hold

$$r \begin{bmatrix} A_2 - A_1 & B_1 & B_2 & D_2 \\ C_1 & 0 & 0 & 0 \end{bmatrix} = r[B_1, B_2, D_2] + r(C_1), \quad (3.5)$$

$$r \begin{bmatrix} A_2 - A_1 & B_1 & B_2 \\ E_2 & 0 & 0 \\ C_1 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} C_1 \\ E_2 \end{bmatrix} + r[B_1, B_2], \quad (3.6)$$

$$r \begin{bmatrix} A_2 - A_1 & B_1 & D_2 \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + r[B_1, D_2], \quad (3.7)$$

$$r \begin{bmatrix} A_2 - A_1 & B_1 \\ C_1 & 0 \\ C_2 & 0 \\ E_2 & 0 \end{bmatrix} = r \begin{bmatrix} C_1 \\ C_2 \\ E_2 \end{bmatrix} + r(B_1). \quad (3.8)$$

- (b) $\mathcal{D}_1 \supseteq \mathcal{D}_2$ if and only if one of the following four conditions holds

$$r \begin{bmatrix} A_2 - A_1 & B_1 & B_2 & D_2 \\ C_1 & 0 & 0 & 0 \end{bmatrix} = r(B_1) + r(C_1), \quad (3.9)$$

$$r \begin{bmatrix} A_2 - A_1 & B_1 & B_2 \\ E_2 & 0 & 0 \\ C_1 & 0 & 0 \end{bmatrix} = r(B_1) + r(C_1), \quad (3.10)$$

$$r \begin{bmatrix} A_2 - A_1 & B_1 & D_2 \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{bmatrix} = r(B_1) + r(C_1), \quad (3.11)$$

$$r \begin{bmatrix} A_2 - A_1 & B_1 \\ C_1 & 0 \\ C_2 & 0 \\ E_2 & 0 \end{bmatrix} = r(B_1) + r(C_1). \quad (3.12)$$

(c) $\mathcal{D}_1 \subseteq \mathcal{D}_2$ if and only if one of the following four conditions holds

$$(i) \quad r[B_2, D_2] = m \text{ or } r \begin{bmatrix} A_1 - A_2 & B_1 & B_2 & D_2 \\ C_1 & 0 & 0 & 0 \end{bmatrix} = r[B_2, D_2], \quad (3.13)$$

$$(ii) \quad r(B_2) = m \text{ or } r(E_2) = n \text{ or } r \begin{bmatrix} A_1 - A_2 & B_1 & B_2 \\ C_1 & 0 & 0 \\ E_2 & 0 & 0 \end{bmatrix} = r(B_2) + r(E_2), \quad (3.14)$$

$$(iii) \quad r(C_2) = n \text{ or } r(D_2) = m \text{ or } r \begin{bmatrix} A_1 - A_2 & B_1 & D_2 \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{bmatrix} = r(C_2) + r(D_2), \quad (3.15)$$

$$(iv) \quad r \begin{bmatrix} C_2 \\ E_2 \end{bmatrix} = n \text{ or } r \begin{bmatrix} A_1 - A_2 & B_1 \\ C_1 & 0 \\ C_2 & 0 \\ E_2 & 0 \end{bmatrix} = r \begin{bmatrix} C_2 \\ E_2 \end{bmatrix}. \quad (3.16)$$

(d) $\mathcal{D}_1 = \mathcal{D}_2$ if and only if both (b) and (c) hold.

Proof. The fact $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \emptyset$ is obviously equivalent to $A_1 + B_1X_1 + Y_1C_1 = A_2 + B_2X_2C_2 + D_2Y_2E_2$ for some X_1, Y_1, X_2 , and Y_2 . Rewrite it as

$$B_1X_1 + Y_1C_1 - B_2X_2C_2 - D_2Y_2E_2 = A_2 - A_1, \quad (3.17)$$

and applying Lemma 2.8 to (3.17) leads to Result (a).

By (2.10), (2.12), and (3.17), the fact $\mathcal{D}_1 \supseteq \mathcal{D}_2$ holds iff

$$P_{B_1}(A_2 - A_1)Q_{C_1} + P_{B_1}B_2X_2C_2Q_{C_1} + P_{B_1}D_2Y_2E_2Q_{C_1} = 0 \quad (3.18)$$

holds for all X_2 and Y_2 . By Lemma 2.7, (3.18) holds for all X_2 and Y_2 iff

$$[P_{B_1}(A_2 - A_1)Q_{C_1}, P_{B_1}B_2, P_{B_1}D_2] = 0, \quad (3.19)$$

$$\begin{bmatrix} P_{B_1}(A_2 - A_1)Q_{C_1} & P_{B_1}B_2 \\ E_2Q_{C_1} & 0 \end{bmatrix} = 0, \quad (3.20)$$

$$\begin{bmatrix} P_{B_1}(A_2 - A_1)Q_{C_1} & P_{B_1}D_2 \\ C_2Q_{C_1} & 0 \end{bmatrix} = 0, \quad (3.21)$$

$$\begin{bmatrix} P_{B_1}(A_2 - A_1)Q_{C_1} \\ C_2Q_{C_1} \\ E_2Q_{C_1} \end{bmatrix} = 0, \quad (3.22)$$

which, by Lemma 2.1(c), are equivalent to (3.9)–(3.12).

By (3.17) and Lemma 2.7, the fact $\mathcal{D}_1 \subseteq \mathcal{D}_2$ holds iff one of the following four equations

$$P_G(A_1 - A_2) + P_GB_1X_1 + P_GY_1C_1 = 0, \quad (3.23)$$

$$P_{B_2}(A_1 - A_2)Q_{E_2} + P_{B_2}B_1X_1Q_{E_2} + P_{B_2}Y_1C_1Q_{E_2} = 0, \quad (3.24)$$

$$P_{D_2}(A_1 - A_2)Q_{C_2} + P_{D_2}B_1X_1Q_{C_2} + P_{D_2}Y_1C_1Q_{C_2} = 0, \quad (3.25)$$

$$(A_1 - A_2)Q_H + B_1X_1Q_H + Y_1C_1Q_H = 0 \quad (3.26)$$

hold for all X_1 and Y_1 . Further by Lemma 2.7, (3.23) holds for all X_1 and Y_1 iff one of the following two conditions holds

$$P_G = 0 \text{ or } r \begin{bmatrix} P_G(A_1 - A_2) & P_GB_1 \\ C_1 & 0 \end{bmatrix} = 0,$$

which are equivalent to

$$r[B_2, D_2] = m \text{ or } r \begin{bmatrix} A_1 - A_2 & B_1 & B_2 & D_2 \\ C_1 & 0 & 0 & 0 \end{bmatrix} = r[B_2, D_2];$$

(3.24) holds for all X_1 and Y_1 iff one of the following three conditions holds

$$P_{B_2} = 0 \text{ or } r \begin{bmatrix} P_{B_2}(A_1 - A_2)Q_{E_2} & P_{B_2}B_1 \\ C_1Q_{E_2} & 0 \end{bmatrix} = 0 \text{ or } Q_{E_2} = 0,$$

which are equivalent to

$$r(B_2) = m \text{ or } r \begin{bmatrix} A_1 - A_2 & B_1 & B_2 \\ C_1 & 0 & 0 \\ E_2 & 0 & 0 \end{bmatrix} = r(B_2) + r(E_2) \text{ or } r(E_2) = n;$$

(3.25) holds for all X_1 and Y_1 iff one of the following four conditions holds

$$P_{D_2} = 0 \text{ or } r \begin{bmatrix} P_{D_2}(A_1 - A_2)Q_{C_2} & P_{D_2}B_1 \\ C_1Q_{C_2} & 0 \end{bmatrix} = 0 \text{ or } Q_{C_2} = 0,$$

which are equivalent to

$$r(D_2) = m \text{ or } r \begin{bmatrix} A_1 - A_2 & B_1 & D_2 \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{bmatrix} = r(C_2) + r(D_2) \text{ or } r(C_2) = n;$$

(3.26) holds for all X_1 and Y_1 iff one of the following four conditions holds

$$Q_H = 0 \text{ or } r \begin{bmatrix} (A_1 - A_2)Q_H & B_1 \\ C_1Q_H & 0 \end{bmatrix} = 0,$$

$$r \begin{bmatrix} C_2 \\ E_2 \end{bmatrix} = n \text{ or } r \begin{bmatrix} A_1 - A_2 & B_1 \\ C_1 & 0 \\ C_2 & 0 \\ E_2 & 0 \end{bmatrix} = r \begin{bmatrix} C_2 \\ E_2 \end{bmatrix}.$$

Combining them leads to (3.13)–(3.16). \square

The results in the above three lemmas can be used, as demonstrated below, to solve many concrete problems on the relationships between solutions of matrix equations, as well as relations between generalized inverses of matrices.

4 Relationships between solutions of two fundamental linear matrix equations

It is well known since Penrose [19] that general solutions of linear matrix equations can be represented certain linear matrix expressions composed with the given matrices in the matrix equations and their generalized inverses. In this situation, we can use the previous results to characterize various relationships between solutions of linear matrix equations. There are many linear matrix equations for which the general solution can explicitly be written as certain explicit linear matrix-valued functions as given in (4.1). In this section, we present a variety of results and facts on relationships between linear transformations of solutions of some fundamental linear matrix equations.

Theorem 4.1. *Assume that the following two matrix equations*

$$A_1X_1 = B_1 \text{ and } A_2X_2 = B_2 \quad (4.1)$$

are consistent, respectively, where $A_i \in \mathbb{C}^{m_i \times n_i}$ and $B_i \in \mathbb{C}^{m_i \times p}$ are given, $i = 1, 2$. Also we denote by

$$\mathcal{D}_1 = \{S_1X_1 + T_1 \mid A_1X_1 = B_1\} \text{ and } \mathcal{D}_2 = \{S_2X_2 + T_2 \mid A_2X_2 = B_2\}, \quad (4.2)$$

the domains of two constrained LMVF_s, where $S_i \in \mathbb{C}^{s \times n_i}$ and $T_i \in \mathbb{C}^{s \times p}$ are given, $i = 1, 2$. Then the following results hold.

- (a) $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \emptyset$ if and only if $r \begin{bmatrix} S_1 & S_2 & T_1 - T_2 \\ A_1 & 0 & -B_1 \\ 0 & A_2 & B_2 \end{bmatrix} = r \begin{bmatrix} S_1 & S_2 \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix}$.
- (b) $\mathcal{D}_1 \subseteq \mathcal{D}_2$ if and only if $r \begin{bmatrix} S_1 & S_2 & T_1 - T_2 \\ A_1 & 0 & -B_1 \\ 0 & A_2 & B_2 \end{bmatrix} = r \begin{bmatrix} S_2 \\ A_2 \end{bmatrix} + r(A_1)$.

$$(c) \mathcal{D}_1 = \mathcal{D}_2 \text{ if and only if } r \begin{bmatrix} S_1 & S_2 & T_1 - T_2 \\ A_1 & 0 & -B_1 \\ 0 & A_2 & B_2 \end{bmatrix} = r \begin{bmatrix} S_1 \\ A_1 \end{bmatrix} + r(A_2) = r \begin{bmatrix} S_2 \\ A_2 \end{bmatrix} + r(A_1).$$

Proof. By Lemma 2.3, the general solutions of the two equations in (4.1) can be expressed as

$$X_1 = A_1^\dagger B_1 + S_1 Q_{A_1} U_1, \quad X_2 = A_2^\dagger B_2 + S_2 Q_{A_2} U_2, \quad (4.3)$$

where $U_1 \in \mathbb{C}^{n_1 \times p}$ and $U_2 \in \mathbb{C}^{n_2 \times p}$ are arbitrary matrices. Then the two sets in (4.2) can be represented as

$$\mathcal{D}_1 = \{ S_1 A_1^\dagger B_1 + S_1 Q_{A_1} U_1 + T_1 \} \text{ and } \mathcal{D}_2 = \{ S_2 A_2^\dagger B_2 + S_2 Q_{A_2} U_2 + T_2 \}. \quad (4.4)$$

Applying Lemma 3.1(a) to (4.4), we obtain that $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \emptyset$ if and only if

$$r[S_1 Q_{A_1}, S_2 Q_{A_2}, S_1 A_1^\dagger B_1 - S_2 A_2^\dagger B_2 + T_1 - T_2] = r[S_1 Q_{A_1}, S_2 Q_{A_2}], \quad (4.5)$$

where by (2.2),

$$\begin{aligned} & r[S_1 Q_{A_1}, S_2 Q_{A_2}, S_1 A_1^\dagger B_1 - S_2 A_2^\dagger B_2 + T_1 - T_2] \\ &= r \begin{bmatrix} S_1 & S_2 & S_1 A_1^\dagger B_1 - S_2 A_2^\dagger B_2 + T_1 - T_2 \\ A_1 & 0 & 0 \\ 0 & A_1 & 0 \end{bmatrix} - r(A_1) - r(A_2) \\ &= r \begin{bmatrix} S_1 & S_2 & T_1 - T_2 \\ A_1 & 0 & -B_1 \\ 0 & A_2 & B_2 \end{bmatrix} - r(A_1) - r(A_2), \end{aligned} \quad (4.6)$$

$$r[S_1 Q_{A_1}, S_2 Q_{A_2}] = r \begin{bmatrix} S_1 & S_2 \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix} - r(A_1) - r(A_2). \quad (4.7)$$

Substituting (4.6) and (4.7) into (4.5) yields $r \begin{bmatrix} S_1 & S_2 & T_1 - T_2 \\ A_1 & 0 & -B_1 \\ 0 & A_2 & B_2 \end{bmatrix} = r \begin{bmatrix} S_1 & S_2 \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix}$, establishing (a).

Applying Lemma 3.1(b) to (4.5), we obtain that $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \emptyset$ if and only if

$$r[S_1 Q_{A_1}, S_2 Q_{A_2}, S_1 A_1^\dagger B_1 - S_2 A_2^\dagger B_2 + T_1 - T_2] = r(S_2 Q_{A_2}), \quad (4.8)$$

where by (2.2),

$$r(S_2 Q_{A_2}) = r \begin{bmatrix} S_2 \\ A_2 \end{bmatrix} - r(A_2). \quad (4.9)$$

Substituting (4.6) and (4.9) into (4.8) yields Result (b). By a similar approach, we obtain that $\mathcal{D}_1 \supseteq \mathcal{D}_2$ if and only if $r \begin{bmatrix} S_1 & S_2 & T_1 - T_2 \\ A_1 & 0 & -B_1 \\ 0 & A_2 & B_2 \end{bmatrix} = r \begin{bmatrix} S_1 \\ A_1 \end{bmatrix} + r(A_2)$. Combining it with Result (b) leads to Result (c). \square

Corollary 4.2. Assume that $A_1 X_1 = B_1$ and $A_2 X_2 = B_2$ in (4.1) are consistent, respectively, and denote by

$$\mathcal{D}_1 = \{ X_1 \mid A_1 X_1 = B_1 \} \text{ and } \mathcal{D}_2 = \{ X_2 \mid A_2 X_2 = B_2 \} \quad (4.10)$$

the sets of all solutions of the two equations, respectively. Then the following results hold.

(a) The two equations in (4.1) have a common solution if and only if $r \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} = r \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, i.e., $\mathcal{R} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subseteq \mathcal{R} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$.

(b) $\mathcal{D}_1 \subseteq \mathcal{D}_2$ if and only if $r \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} = r(A_1)$, i.e., $\mathcal{R} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subseteq \mathcal{R} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ and $\mathcal{R}(A_2^*) \subseteq \mathcal{R}(A_1^*)$.

(c) $\mathcal{D}_1 = \mathcal{D}_2$ if and only if $r \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} = r(A_1) = r(A_2)$, i.e., $\mathcal{R} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subseteq \mathcal{R} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ and $\mathcal{R}(A_2^*) = \mathcal{R}(A_1^*)$.

Corollary 4.3. Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{m \times p}$ be given, and suppose $AX = B$ is consistent. Also denote

$$\mathcal{D}_1 = \{SX \mid AX = B\} \text{ and } \mathcal{D}_2 = \{SX \mid MAX = MB\}, \quad (4.11)$$

where $M \in \mathbb{C}^{t \times m}$ and $S \in \mathbb{C}^{s \times n}$. Then the following results hold.

(a) $\mathcal{D}_1 \subseteq \mathcal{D}_2$ always holds.

(b) $\mathcal{D}_1 = \mathcal{D}_2$ if and only if $r \begin{bmatrix} MA \\ S \end{bmatrix} = r \begin{bmatrix} A \\ S \end{bmatrix} + r(MA) - r(A)$.

Corollary 4.4. Assume that $AX = B$ in (2.5) is consistent, and denote

$$\mathcal{D}_1 = \{X \mid AX = B\} \text{ and } \mathcal{D}_2 = \{X \mid MAX = MB\}, \quad (4.12)$$

where $M \in \mathbb{C}^{s \times m}$. Then, the following results hold.

(a) $\mathcal{D}_1 \subseteq \mathcal{D}_2$ always holds.

(b) $\mathcal{D}_1 = \mathcal{D}_2$ if and only if $r(MA) = r(A)$.

Assume that the matrix equation in (2.5) is consistent, and partition it as

$$AX = A_1X_1 + A_2X_2 + \cdots + A_kX_k = B, \quad (4.13)$$

where $A_i \in \mathbb{C}^{m \times n_i}$, with $A = [A_1, \dots, A_k]$, $X_i \in \mathbb{C}^{n_i \times p}$ are unknown matrices with $X = [X'_1, \dots, X'_k]'$ and $p = p_1 + \cdots + p_k$, and pre-multiplying (4.13) with P_{Y_i} yields the following reduced linear matrix equations

$$P_{Y_i}AX = P_{Y_i}A_iX_i = P_{Y_i}B, \quad i = 1, \dots, k, \quad (4.14)$$

where $Y_i = [A_1, \dots, A_{i-1}, 0, A_{i+1}, \dots, A_k]$, $i = 1, \dots, k$. Then the family of equations in (4.14) are consistent, respectively. In such cases, We denote by

$$\mathcal{D}_i = \{X_i \mid A_1X_1 + A_2X_2 + \cdots + A_kX_k = B\} \text{ and } \mathcal{H}_i = \{X_i \mid E_{Y_i}A_iX_i = E_{Y_i}B\}, \quad i = 1, \dots, k, \quad (4.15)$$

the matrix sets composed by the partial solutions X_i of (4.13) and (4.14) respectively; and denote by

$$\mathcal{D} = \{X \mid AX = B\} \text{ and } \mathcal{H} = \{[X_1^T, X_2^T, \dots, X_k^T]^T \mid E_{Y_i}A_iX_i = E_{Y_i}B, \quad i = 1, \dots, k\}. \quad (4.16)$$

In this section, we first discuss the relationships between \mathcal{D}_i and \mathcal{H}_i in (4.15), $i = 1, \dots, k$, as well as the two sets in (4.16).

Theorem 4.5. Assume that the matrix equation in (4.13) is consistent, and let \mathcal{D}_i and \mathcal{H}_i be as given in (4.15), $i = 1, \dots, k$. Then the following matrix set equalities

$$\mathcal{D}_i = \mathcal{H}_i \quad (4.17)$$

always hold, $i = 1, \dots, k$.

Proof. Set $S = [0 \dots, I_{n_i}, \dots, 0]$ and $M = E_{Y_i}$ in (4.11), $i = 1, \dots, k$. Then we obtain by (2.5) and simplifications that

$$\begin{aligned} r \begin{bmatrix} E_{Y_i}A_i \\ S \end{bmatrix} - r \begin{bmatrix} A \\ S \end{bmatrix} - r(E_{Y_i}A_i) + r(A) &= r \begin{bmatrix} A & Y_i \\ S & 0 \end{bmatrix} - r \begin{bmatrix} A \\ S \end{bmatrix} - r[Z_i, A_i] + r(A) \\ &= r \begin{bmatrix} 0 & Y_i \\ S & 0 \end{bmatrix} - r \begin{bmatrix} Y_i \\ S \end{bmatrix} - r(A) + r(A) = 0. \end{aligned}$$

Thus (4.17) holds by Corollary 4.3(c). \square

Theorem 4.6. Assume that the matrix equation in (4.13) is consistent, and let \mathcal{D} and \mathcal{H} be as given in (5.4). Then the following results hold.

(a) $\mathcal{D} \subseteq \mathcal{H}$ always holds.

(b) The following statements are equivalent:

- (i) $\mathcal{D} = \mathcal{H}$.
- (ii) $(k-1)r(A) = r(Y_1) + r(Y_2) + \cdots + r(Y_k)$.
- (iii) $r(A) = r(A_1) + r(A_2) + \cdots + r(A_k)$.
- (iv) $\mathcal{R}(Y_1) \cap \mathcal{R}(Y_2) \cap \cdots \cap \mathcal{R}(Y_k) = \{0\}$.

Proof. By Lemma 2.3, the general solutions of (4.14) are given by

$$X_i = (P_{Y_i} A_i)^\dagger P_{Y_i} B + [I_{n_i} - (P_{Y_i} A_i)^\dagger (P_{Y_i} A_i)] U_i, \quad (4.18)$$

where $U_i \in \mathbb{C}^{n_i \times p}$ are arbitrary, $i = 1, \dots, k$. Substituting (2.6) and (4.18) into (4.16) gives

$$\mathcal{D} = \{A^\dagger B + Q_A U\}, \quad (4.19)$$

$$\mathcal{H} = \left\{ \begin{bmatrix} (P_{Y_1} A_1)^\dagger P_{Y_1} B \\ \vdots \\ (P_{Y_k} A_k)^\dagger P_{Y_k} B \end{bmatrix} + \begin{bmatrix} I_{n_1} - (P_{Y_1} A_1)^\dagger (P_{Y_1} A_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{n_k} - (P_{Y_k} A_k)^\dagger (P_{Y_k} A_k) \end{bmatrix} \begin{bmatrix} U_1 \\ \vdots \\ U_k \end{bmatrix} \right\}. \quad (4.20)$$

Applying Lemma 3.1(b) to (4.19) and (4.20), we see that $\mathcal{D} \subseteq \mathcal{H}$ if and only if

$$\begin{aligned} & r \left[A^\dagger B - \begin{bmatrix} (P_{Y_1} A_1)^\dagger P_{Y_1} B \\ \vdots \\ (P_{Y_k} A_k)^\dagger P_{Y_k} B \end{bmatrix}, Q_A, \begin{bmatrix} I_{n_1} - (P_{Y_1} A_1)^\dagger (P_{Y_1} A_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{n_k} - (P_{Y_k} A_k)^\dagger (P_{Y_k} A_k) \end{bmatrix} \right] \\ &= r \begin{bmatrix} I_{n_1} - (P_{Y_1} A_1)^\dagger (P_{Y_1} A_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{n_k} - (P_{Y_k} A_k)^\dagger (P_{Y_k} A_k) \end{bmatrix}, \end{aligned} \quad (4.21)$$

where by

$$\begin{aligned} & r \left[A^\dagger B - \begin{bmatrix} (P_{Y_1} A_1)^\dagger P_{Y_1} B \\ \vdots \\ (P_{Y_k} A_k)^\dagger P_{Y_k} B \end{bmatrix}, Q_A, \begin{bmatrix} I_{n_1} - (P_{Y_1} A_1)^\dagger (P_{Y_1} A_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{n_k} - (P_{Y_k} A_k)^\dagger (P_{Y_k} A_k) \end{bmatrix} \right] \\ &= r \begin{bmatrix} A^\dagger B - \begin{bmatrix} (P_{Y_1} A_1)^\dagger P_{Y_1} B \\ \vdots \\ (P_{Y_k} A_k)^\dagger P_{Y_k} B \end{bmatrix} & I_n & \begin{bmatrix} I_{n_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{n_k} \end{bmatrix} \\ 0 & A & \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} \\ 0 & 0 & \begin{bmatrix} P_{Y_1} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P_{Y_k} A_k \end{bmatrix} \end{bmatrix} - r(A) - r(P_{Y_1} A_1) - \cdots - r(P_{Y_k} A_k) \\ &= r \begin{bmatrix} 0 & I_n & 0 \\ -B & 0 & -A \\ \begin{bmatrix} P_{Y_1} B \\ \vdots \\ P_{Y_k} B \end{bmatrix} & 0 & \begin{bmatrix} P_{Y_1} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P_{Y_k} A_k \end{bmatrix} \end{bmatrix} - r(A) - r(P_{Y_1} A_1) - \cdots - r(P_{Y_k} A_k) \\ &= n - r(P_{Y_1} A_1) - \cdots - r(P_{Y_k} A_k), \end{aligned} \quad (4.22)$$

and

$$r \begin{bmatrix} I_{n_1} - (P_{Y_1} A_1)^\dagger (P_{Y_1} A_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{n_k} - (P_{Y_k} A_k)^\dagger (P_{Y_k} A_k) \end{bmatrix} = n - r(P_{Y_1} A_1) - \cdots - r(P_{Y_k} A_k). \quad (4.23)$$

Both (4.22) and (4.23) mean that (4.21) is an identity, thus establishing (a).

Substituting (4.18) into (4.13) gives

$$\begin{aligned} & [A_1 - A_1(P_{Y_1}A_1)^\dagger(P_{Y_1}A_1)]U_1 + \cdots + [A_k - A_k(P_{Y_k}A_k)^\dagger(P_{Y_k}A_k)]U_k \\ & = B - A_1(P_{Y_1}A_1)^\dagger P_{Y_1}B + \cdots + A_k(P_{Y_k}A_k)^\dagger P_{Y_k}B. \end{aligned} \quad (4.24)$$

It is obvious that $\mathcal{D} \supseteq \mathcal{H}$ holds if and only if the matrix equation in (4.24) holds for all U_1, \dots, U_k , which by Lemma 2.3 is equivalent to

$$\begin{aligned} & [B - A_1(P_{Y_1}A_1)^\dagger P_{Y_1}B + \cdots + A_k(P_{Y_k}A_k)^\dagger P_{Y_k}B, A_1 - A_1(P_{Y_1}A_1)^\dagger(P_{Y_1}A_1), \\ & \dots, A_k - A_k(P_{Y_k}A_k)^\dagger(P_{Y_k}A_k)] = 0, \end{aligned} \quad (4.25)$$

where by (2.5),

$$\begin{aligned} & r[B - A_1(P_{Y_1}A_1)^\dagger P_{Y_1}B - \cdots - A_k(P_{Y_k}A_k)^\dagger P_{Y_k}B, A_1 - A_1(P_{Y_1}A_1)^\dagger(P_{Y_1}A_1), \\ & \dots, A_k - A_k(P_{Y_k}A_k)^\dagger(P_{Y_k}A_k)] \\ & = r \begin{bmatrix} B - A_1(P_{Y_1}A_1)^\dagger P_{Y_1}B - \cdots - A_k(P_{Y_k}A_k)^\dagger P_{Y_k}B & A_1 & \dots & A_k \\ 0 & P_{Y_1}A_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_{Y_k}A_k \end{bmatrix} \\ & - r(P_{Y_1}A_1) - \cdots - r(P_{Y_k}A_k) \\ & = r \begin{bmatrix} B & A_1 & \dots & A_k \\ P_{Y_1}B & P_{Y_1}A_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ P_{Y_k}B & 0 & \dots & P_{Y_k}A_k \end{bmatrix} - r(P_{Y_1}A_1) - \cdots - r(P_{Y_k}A_k) \\ & = r \begin{bmatrix} B & A_1 & \dots & A_k & 0 & \dots & 0 \\ B & A_1 & \dots & 0 & Y_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B & 0 & \dots & A_k & 0 & \dots & Y_k \end{bmatrix} - kr(A) = r \begin{bmatrix} B & A & 0 & \dots & 0 \\ B & A & Y_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & A & 0 & \dots & Y_k \end{bmatrix} - kr(A) \\ & = r \begin{bmatrix} 0 & A & 0 & \dots & 0 \\ 0 & 0 & Y_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & Y_k \end{bmatrix} - kr(A) = r(Y_1) + \cdots + r(Y_k) - (k-1)r(A). \end{aligned}$$

Thus (4.25) is equivalent to $(k-1)r(A) = r(Y_1) + \cdots + r(Y_k)$. Combining this facts with (a) leads to the equivalence of (i) and (ii) in (b). The equivalence of (ii), (iii), and (iv) in (b) follows from Lemma 2.2. \square

5 Relationships among solutions of $A_1X_1B_1 + A_2X_2B_2 = C$ and its four reduced equations

Eq. (2.14) is well known in matrix theory and applications, which solvability condition and general solution were precisely established using the ranks, ranges, and generalized inverses of the given matrices in the equation; see e.g., [2, 14, 22, 23, 28] and the relevant literature quoted there.

It is easy to see that we can construct from (2.14) some small or transformed linear matrix equations. For instance, pre- and post-multiplying (2.14) with P_{A_i} and Q_{B_i} respectively yield the following four reduced matrix equations

$$P_{A_2}(A_1X_1B_1 + A_2X_2B_2) = P_{A_2}A_1X_1B_1 = P_{A_2}C, \quad (5.1)$$

$$P_{A_1}(A_1X_1B_1 + A_2X_2B_2) = P_{A_1}A_2X_2B_2 = P_{A_1}C, \quad (5.2)$$

$$(A_1X_1B_1 + A_2X_2B_2)Q_{B_2} = A_1X_1B_1Q_{B_2} = CQ_{B_2}, \quad (5.3)$$

$$(A_1X_1B_1 + A_2X_2B_2)Q_{B_1} = A_2X_2B_2Q_{B_1} = CQ_{B_1}, \quad (5.4)$$

respectively. Each of (5.1)–(5.4) is consistent as well, if the matrix equation in (2.14) is consistent. Concerning the relationships among the solutions of (2.14) and (5.1)–(5.4), we have the following results.

Theorem 5.1. Assume that the matrix equation in (2.14) is consistent, and denote by

$$\mathcal{D} = \{ (X_1, X_2) \mid A_1 X_1 B_1 + A_2 X_2 B_2 = C \}, \quad (5.5)$$

$$\mathcal{H}_1 = \{ (X_1, X_2) \mid P_{A_2} A_1 X_1 B_1 = P_{A_2} C \text{ and } P_{A_1} A_2 X_2 B_2 = P_{A_1} C \}, \quad (5.6)$$

$$\mathcal{H}_2 = \{ (X_1, X_2) \mid P_{A_2} A_1 X_1 B_1 = P_{A_2} C \text{ and } A_2 X_2 B_2 Q_{B_1} = C Q_{B_1} \}, \quad (5.7)$$

$$\mathcal{H}_3 = \{ (X_1, X_2) \mid A_1 X_1 B_1 Q_{B_2} = C Q_{B_2} \text{ and } P_{A_1} A_2 X_2 B_2 = P_{A_1} C \}, \quad (5.8)$$

$$\mathcal{H}_4 = \{ (X_1, X_2) \mid A_1 X_1 B_1 Q_{B_2} = C Q_{B_2} \text{ and } A_2 X_2 B_2 Q_{B_1} = C Q_{B_1} \}, \quad (5.9)$$

the collections of all pairs of solutions of (2.14) and (5.1)–(5.4), respectively. Then the following results hold.

- (a) $\mathcal{D} \subseteq \mathcal{H}_i$ always hold, $i = 1, 2, 3, 4$.
- (b) $\mathcal{D} = \mathcal{H}_1$ if and only if $\mathcal{R}(A_1) \cap \mathcal{R}(A_2) = \{0\}$ or $[B_1^*, B_2^*] = 0$.
- (c) $\mathcal{D} = \mathcal{H}_2$ if and only if $A_2 = 0$, or $B_1 = 0$, or $\mathcal{R}(A_1) \cap \mathcal{R}(A_2) = \{0\}$ and $\mathcal{R}(B_1^*) \cap \mathcal{R}(B_2^*) = \{0\}$.
- (d) $\mathcal{D} = \mathcal{H}_3$ if and only if $A_1 = 0$, or $B_2 = 0$, or $\mathcal{R}(A_1) \cap \mathcal{R}(A_2) = \{0\}$ and $\mathcal{R}(B_1^*) \cap \mathcal{R}(B_2^*) = \{0\}$.
- (e) $\mathcal{D} = \mathcal{H}_4$ if and only if $[A_1, A_2] = 0$ or $\mathcal{R}(B_1^*) \cap \mathcal{R}(B_2^*) = \{0\}$.

Proof. Result (a) follows directly from (5.1)–(5.4). By Lemma 2.4, the general solutions of (5.1)–(5.4) are given by

$$X_1 = (P_{A_2} A_1)^\dagger P_{A_2} C B_1^\dagger + [I_{p_1} - (P_{A_2} A_1)^\dagger (P_{A_2} A_1)] U_1 + V_1 (I_{q_1} - B_1 B_1^\dagger), \quad (5.10)$$

$$X_2 = (P_{A_1} A_2)^\dagger P_{A_1} C B_2^\dagger + [I_{p_2} - (P_{A_1} A_2)^\dagger (P_{A_1} A_2)] U_2 + V_2 (I_{q_2} - B_2 B_2^\dagger), \quad (5.11)$$

$$X_1 = A_1^\dagger C Q_{B_2} (B_1 Q_{B_2})^\dagger + (I_{p_1} - A_1^\dagger A_1) U_3 + V_3 [I_{q_1} - (B_1 Q_{B_2}) (B_1 Q_{B_2})^\dagger], \quad (5.12)$$

$$X_2 = A_2^\dagger C Q_{B_1} (B_2 Q_{B_1})^\dagger + (I_{p_2} - A_2^\dagger A_2) U_4 + V_4 [I_{q_2} - (B_2 Q_{B_1}) (B_2 Q_{B_1})^\dagger], \quad (5.13)$$

respectively, where U_i and V_i are arbitrary matrices, $i = 1, 2, 3, 4$. Substituting (5.10)–(5.11) into (2.14) gives the following matrix equation

$$\begin{aligned} & A_1 [I_{p_1} - (P_{A_2} A_1)^\dagger (P_{A_2} A_1)] U_1 B_1 + A_2 [I_{p_2} - (P_{A_1} A_2)^\dagger (P_{A_1} A_2)] U_2 B_2 \\ &= C - A_1 (P_{A_2} A_1)^\dagger P_{A_2} C - A_2 (P_{A_1} A_2)^\dagger P_{A_1} C, \end{aligned} \quad (5.14)$$

$$\begin{aligned} & A_1 [I_{p_1} - (P_{A_2} A_1)^\dagger (P_{A_2} A_1)] U_1 B_1 + A_2 V_4 [I_{q_2} - (B_2 Q_{B_1}) (B_2 Q_{B_1})^\dagger] B_2 \\ &= C - A_1 (P_{A_2} A_1)^\dagger P_{A_2} C - C Q_{B_1} (B_2 Q_{B_1})^\dagger B_2, \end{aligned} \quad (5.15)$$

$$\begin{aligned} & A_1 V_3 [I_{q_1} - (B_1 Q_{B_2}) (B_1 Q_{B_2})^\dagger] B_1 + A_2 [I_{p_2} - (P_{A_1} A_2)^\dagger (P_{A_1} A_2)] U_2 B_2 \\ &= C - C Q_{B_2} (B_1 Q_{B_2})^\dagger B_1 - A_2 (P_{A_1} A_2)^\dagger P_{A_1} C, \end{aligned} \quad (5.16)$$

$$\begin{aligned} & A_1 V_3 [I_{q_1} - (B_1 Q_{B_2}) (B_1 Q_{B_2})^\dagger] B_1 + A_2 V_4 [I_{q_2} - (B_2 Q_{B_1}) (B_2 Q_{B_1})^\dagger] B_2 \\ &= C - C Q_{B_2} (B_1 Q_{B_2})^\dagger B_2 - C Q_{B_1} (B_2 Q_{B_1})^\dagger B_2, \end{aligned} \quad (5.17)$$

respectively. By Lemma 2.5(b), (5.14) holds for all U_1 and U_2 if and only if one of the following four equalities

$$\begin{aligned} & [A_1 [I_{p_1} - (P_{A_2} A_1)^\dagger (P_{A_2} A_1)], A_2 [I_{p_2} - (P_{A_1} A_2)^\dagger (P_{A_1} A_2)]], \\ & C - A_1 (P_{A_2} A_1)^\dagger P_{A_2} C - A_2 (P_{A_1} A_2)^\dagger P_{A_1} C = 0, \end{aligned} \quad (5.18)$$

$$\begin{bmatrix} C - A_1 (P_{A_2} A_1)^\dagger P_{A_2} C - A_2 (P_{A_1} A_2)^\dagger P_{A_1} C & A_1 [I_{p_1} - (P_{A_2} A_1)^\dagger (P_{A_2} A_1)] \\ B_2 & 0 \end{bmatrix} = 0, \quad (5.19)$$

$$\begin{bmatrix} C - A_1 (P_{A_2} A_1)^\dagger P_{A_2} C - A_2 (P_{A_1} A_2)^\dagger P_{A_1} C & A_2 [I_{p_2} - (P_{A_1} A_2)^\dagger (P_{A_1} A_2)] \\ B_1 & 0 \end{bmatrix} = 0, \quad (5.20)$$

$$\begin{bmatrix} C - A_1 (P_{A_2} A_1)^\dagger P_{A_2} C - A_2 (P_{A_1} A_2)^\dagger P_{A_1} C \\ B_1 \\ B_2 \end{bmatrix} = 0. \quad (5.21)$$

It is easy to verify that the ranks of the left-hand sides of (5.18)–(5.21) are given by

$$\begin{aligned} & r[A_1[I_{p_1} - (P_{A_2}A_1)^\dagger(P_{A_2}A_1)], A_2[I_{p_2} - (P_{A_1}A_2)^\dagger(P_{A_1}A_2)], C - A_1(P_{A_2}A_1)^\dagger P_{A_2}C - A_2(P_{A_1}A_2)^\dagger P_{A_1}C] \\ & = r(A_1) + r(A_2) - r[A_1, A_2], \end{aligned} \quad (5.22)$$

$$\begin{aligned} & r \begin{bmatrix} C - A_1(P_{A_2}A_1)^\dagger P_{A_2}C - A_2(P_{A_1}A_2)^\dagger P_{A_1}C & A_1[I_{p_1} - (P_{A_2}A_1)^\dagger(P_{A_2}A_1)] \\ B_2 & 0 \end{bmatrix} \\ & = r(A_1) + r(A_2) - [A_1, A_2] + r(B_2), \end{aligned} \quad (5.23)$$

$$\begin{aligned} & r \begin{bmatrix} C - A_1(P_{A_2}A_1)^\dagger P_{A_2}C - A_2(P_{A_1}A_2)^\dagger P_{A_1}C & A_2[I_{p_2} - (P_{A_1}A_2)^\dagger(P_{A_1}A_2)] \\ B_1 & 0 \end{bmatrix} \\ & = r(A_1) + r(A_2) - [A_1, A_2] + r(B_1), \end{aligned} \quad (5.24)$$

$$r \begin{bmatrix} C - A_1(P_{A_2}A_1)^\dagger P_{A_2}C - A_2(P_{A_1}A_2)^\dagger P_{A_1}C \\ B_1 \\ B_2 \end{bmatrix} = r \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \quad (5.25)$$

Combining (5.18)–(5.21) with (5.22)–(5.25) leads to the equivalence in (b).

By Lemma 2.5(b), (5.15) holds for all U_1 and V_4 if and only if one of the following four equalities

$$[C - A_1(P_{A_2}A_1)^\dagger P_{A_2}C - A_2(P_{A_1}A_2)^\dagger P_{A_1}C, A_1[I_{p_1} - (P_{A_2}A_1)^\dagger(P_{A_2}A_1)], A_2] = 0, \quad (5.26)$$

$$\begin{bmatrix} C - A_1(P_{A_2}A_1)^\dagger P_{A_2}CB_1^\dagger B_1 - A_2A_2^\dagger CQ_{B_1}(B_2Q_{B_1})^\dagger B_2 & A_1[I_{p_1} - (P_{A_2}A_1)^\dagger(P_{A_2}A_1)] \\ [I_{q_2} - (B_2Q_{B_1})(B_2Q_{B_1})^\dagger]B_2 & 0 \end{bmatrix} = 0, \quad (5.27)$$

$$\begin{bmatrix} C - A_1(P_{A_2}A_1)^\dagger P_{A_2}CB_1^\dagger B_1 - A_2A_2^\dagger CQ_{B_1}(B_2Q_{B_1})^\dagger B_2 & A_1 \\ B_2 & 0 \end{bmatrix} = 0, \quad (5.28)$$

$$\begin{bmatrix} C - A_1(P_{A_2}A_1)^\dagger P_{A_2}CB_1^\dagger B_1 - A_2A_2^\dagger CQ_{B_1}(B_2Q_{B_1})^\dagger B_2 \\ B_1 \\ [I_{q_2} - (B_2Q_{B_1})(B_2Q_{B_1})^\dagger]B_2 \end{bmatrix} = 0. \quad (5.29)$$

It is easy to verify that the ranks of the left-hand sides of (5.26)–(5.29) are given by

$$r[C - A_1(P_{A_2}A_1)^\dagger P_{A_2}C - A_2(P_{A_1}A_2)^\dagger P_{A_1}C, A_1[I_{p_1} - (P_{A_2}A_1)^\dagger(P_{A_2}A_1)], A_2] = r(A_2), \quad (5.30)$$

$$\begin{aligned} & r \begin{bmatrix} C - A_1(P_{A_2}A_1)^\dagger P_{A_2}CB_1^\dagger B_1 - A_2A_2^\dagger CQ_{B_1}(B_2Q_{B_1})^\dagger B_2 & A_1[I_{p_1} - (P_{A_2}A_1)^\dagger(P_{A_2}A_1)] \\ [I_{q_2} - (B_2Q_{B_1})(B_2Q_{B_1})^\dagger]B_2 & 0 \end{bmatrix} \\ & = r(A_1) + r(A_2) + r(B_1) + r(B_2) - r[A_1, A_2] - r \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \end{aligned} \quad (5.31)$$

$$r \begin{bmatrix} C - A_1(P_{A_2}A_1)^\dagger P_{A_2}CB_1^\dagger B_1 - A_2A_2^\dagger CQ_{B_1}(B_2Q_{B_1})^\dagger B_2 & A_1 \\ B_2 & 0 \end{bmatrix} = r(A_1) + r(B_2), \quad (5.32)$$

$$r \begin{bmatrix} C - A_1(P_{A_2}A_1)^\dagger P_{A_2}CB_1^\dagger B_1 - A_2A_2^\dagger CQ_{B_1}(B_2Q_{B_1})^\dagger B_2 \\ B_1 \\ [I_{q_2} - (B_2Q_{B_1})(B_2Q_{B_1})^\dagger]B_2 \end{bmatrix} = r(B_1). \quad (5.33)$$

Combining (5.26)–(5.29) with (5.30)–(5.33) leads to the equivalence in (c). Results (d) and (e) can be established by a similar approach. \square

Theorem 5.2. *Assume that the matrix equation in (2.14) is consistent, and let*

$$\mathcal{D}_1 = \{X_1 \mid A_1X_1B_1 + A_2X_2B_2 = C\}, \quad (5.34)$$

$$\mathcal{D}_2 = \{X_2 \mid A_1X_1B_1 + A_2X_2B_2 = C\}, \quad (5.35)$$

$$\mathcal{H}_1 = \{X_1 \mid A_1X_1B_1 - A_2A_2^\dagger A_1X_1B_1B_2^\dagger B_2 = C - A_2A_2^\dagger CB_2^\dagger B_2\}, \quad (5.36)$$

$$\mathcal{H}_2 = \{X_2 \mid A_2X_2B_2 - A_1A_1^\dagger A_2X_2B_2B_1^\dagger B_1 = C - A_1A_1^\dagger CB_1^\dagger B_1\}, \quad (5.37)$$

$$\mathcal{D} = \{(X_1, X_2) \mid A_1X_1B_1 + A_2X_2B_2 = C\}, \quad (5.38)$$

$$\mathcal{H} = \{(X_1, X_2) \mid X_1 \in \mathcal{H}_1 \text{ and } X_2 \in \mathcal{H}_2\}. \quad (5.39)$$

Then the following results hold.

- (a) The matrix set equalities $\mathcal{D}_i = \mathcal{H}_i$ always hold, $i = 1, 2$.
- (b) $\mathcal{D} \subseteq \mathcal{H}$ always holds.
- (c) $\mathcal{D} = \mathcal{H}$ if and only if $\mathcal{R}(B_1^\top \otimes A_1) \cap \mathcal{R}(B_2^\top \otimes A_2) = \{0\}$.

Proof. By the vec operation of matrix, (2.14) can equivalently be expressed as

$$(B_1^\top \otimes A_1) \vec{X}_1 + (B_2^\top \otimes A_2) \vec{X}_2 = \vec{C}, \quad (5.40)$$

which is a special case of (4.13). Pre-multiplying (5.40) with $P_{B_i^\top \otimes A_i}$ yields the following reduced linear matrix equation equations

$$P_{B_2^\top \otimes A_2}(B_1^\top \otimes A_1) \vec{X}_1 = P_{B_2^\top \otimes A_2} \vec{C}, \quad P_{B_1^\top \otimes A_1}(B_2^\top \otimes A_2) \vec{X}_2 = P_{B_1^\top \otimes A_1} \vec{C}. \quad (5.41)$$

Now denote

$$\hat{\mathcal{D}}_i = \{\vec{X}_i \mid (B_1^\top \otimes A_1) \vec{X}_1 + (B_2^\top \otimes A_2) \vec{X}_2 = \vec{C}\}, \quad i = 1, 2, \quad (5.42)$$

$$\hat{\mathcal{H}}_1 = \{\vec{X}_1 \mid P_{B_2^\top \otimes A_2}(B_1^\top \otimes A_1) \vec{X}_1 = P_{B_2^\top \otimes A_2} \vec{C}\}, \quad (5.43)$$

$$\hat{\mathcal{H}}_2 = \{\vec{X}_2 \mid P_{B_1^\top \otimes A_1}(B_2^\top \otimes A_2) \vec{X}_2 = P_{B_1^\top \otimes A_1} \vec{C}\}. \quad (5.44)$$

Then by Corollary 4.5,

$$\hat{\mathcal{D}}_i = \hat{\mathcal{H}}_i, \quad i = 1, 2 \quad (5.45)$$

always hold. On the other hand, it is easy to verify that

$$P_{B_i^\top \otimes A_i} = I_{mn} - (B_i^\top \otimes A_i)(B_i^\top \otimes A_i)^\dagger = I_{mn} - (B_i^\top \otimes A_i)[(B_i^\top)^\dagger \otimes A_i^\dagger] = I_{mn} - (B_i^\dagger B_i)^\top \otimes A_i A_i^\dagger, \quad i = 1, 2,$$

and

$$\begin{aligned} P_{B_2^\top \otimes A_2}(B_1^\top \otimes A_1) &= B_1^\top \otimes A_1 - [(B_2^\dagger B_2)^\top \otimes A_2 A_2^\dagger](B_1^\top \otimes A_1) = B_1^\top \otimes A_1 - (B_1 B_2^\dagger B_2)^\top \otimes A_2 A_2^\dagger A_1, \\ P_{B_1^\top \otimes A_1}(B_2^\top \otimes A_2) &= B_2^\top \otimes A_2 - (B_1^\dagger B_1)^\top \otimes A_1 A_1^\dagger (B_2^\top \otimes A_2) = B_2^\top \otimes A_2 - (B_2 B_1^\dagger B_1)^\top \otimes A_1 A_1^\dagger A_2. \end{aligned}$$

Thus the two equations in (5.41) by the vectorization operation of matrix are equivalent to

$$\begin{aligned} A_1 X_1 B_1 - A_2 A_2^\dagger A_1 X_1 B_1 B_2^\dagger B_2 &= C - A_2 A_2^\dagger C B_2^\dagger B_2, \\ A_2 X_2 B_2 - A_1 A_1^\dagger A_2 X_2 B_2 B_1^\dagger B_1 &= C - A_1 A_1^\dagger C B_1^\dagger B_1, \end{aligned}$$

respectively. Thus the two set equalities in (5.45) are equivalent to the set equalities in (a). Results (b) and (c) follow from applying Theorem 4.6 to (5.40). \square

In addition to the LMVF in (1.3), there are many types of multilinear and nonlinear matrix-valued functions that occur in matrix theory and applications, such as,

$$\begin{aligned} f(X_1, \dots, X_k) &= (A_1 + B_1 X_1 C_1)(A_2 + B_2 X_2 C_2) \cdots (A_k + B_k X_k C_k), \\ g(X_1, Y_1, \dots, X_k, Y_k) &= (A_1 + B_1 X_1 C_1 + D_1 Y_1 E_1) \cdots (A_k + B_k X_k C_k + D_k Y_k E_k), \end{aligned}$$

etc. In these cases, it would be of interest but are also challenging to investigate the connections between a pair of such matrix-valued functions under various specified assumptions.

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