The Umbrella Function

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May 1, 2020

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Abstract

In this paper we introduce the Umbrella function and its recurrence relations. Also we provide the integral representation for the this newly defined function.

Mathematics Subject Classification : 33B15, 33E12.

Keywords: The Umbrella function, the Umbrella symbol.

1 Introduction

In this paper we introduce a new integral named as Umbrella function, defined as

$$_p\,\text{II}_k\left[f(x);x\right]=\int_0^\infty \Big[f(x)\Big]^{-\frac{t^k}{p}}t^{x-1}dt.$$

Where $x \in C/kZ^-$ and $Re[f(x)] \in (1,\infty), Re(x)>0,\ k,p \in R^+ - \{0\}$.

So many integrals over zero to infinity has been defined in last three centuries with different kernels. The most important integral over zero to infinity was gamma function, defined by Euler in 18^{th} century and gave two different definitions, the first is an infinite product and second is in the form of integral. later Euler discover the reflection formula.
In 19^{th} century, Gauss rewrote Euler's product and consider the complex number first time for the factorial of complex number. Gauss also proved the multiplication theorem of the gamma function and connection between gamma function and elliptic integrals. Weierstrass established the role of the gamma function in complex analysis in terms of infinite product. Also he establish the Weierstrass factorization theorem that "any entire function can be written as a product over its zeros in the complex plane".
19^{th} and 20^{th} centuries, A large number of definitions has been given for the gamma function. Holder's in 1887, proved that the gamma function at least does not satisfy any algebraic differential equation by showing that a solution to such an equation could not satisfy the gamma function recurrence formula, is known as Holder's theorem. In 1922, Harald Bohr and Johannes Mollerup, proved that the gamma function is the unique solution to the factorial recurrence relation that is positive and logarithmically convex for positive number.
Throughout this paper we use the notations as $C, R^+, Re(), Z^-$ and N be the sets of complex numbers, positive real numbers, real part of complex number, negative integer and natural numbers respectively and we use the terminology and symbols of [2] and [3].

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2 The Umbrella function and Umbrella symbol

In this section we introduce The Umbrella function and Umbrella symbol. also we evaluate ${}_p\amalg_k\left[f(x);x\right]$ in terms of limit, recurrence formulas and infinite products.

2.1 Definition

Let $x \in C/kZ^-$ and $Re[f(x)] \in (1,\infty), Re(x) > 0,\ k, p \in R^+ - \{0\}$, then the Umbrella function is denoted as ${}_p\amalg_k\left[f(x);x\right]$ and defined as

$$ {}_p\amalg_k\left[f(x);x\right] = \int_0^\infty \Big[f(x)\Big]^{-\frac{t^k}{p}} t^{x-1}dt. \tag{2.1}$$

2.2 Definition

Let $x \in C$ and $Re[f(x)] \in (1,\infty), Re(x) > 0,\ n \in N; k, p \in R^+ - \{0\}$, then the Umbrella symbol is denoted as ${}_p[f(x);x]_{n,k}$ and defined as

$$ {}_p[f(x);x]_{n,k} = \Big[\frac{\log f(x)}{\log f(x+nk)}\Big]^{\frac{x}{k}} \times \Big[\frac{p}{k\log f(x+nk)}\Big]^{n} x(x+k)(x+2k).....(x+(n-1)k). \tag{2.2}$$

Theorem 2.1 Given $x \in C/kZ^-$ and $Re[f(x)] \in (1,\infty), Re(x) > 0,\ k, p \in R^+ - \{0\}$, then the infinite limit form of the Umbrella function is given as

$$ {}_p\amalg_k\left[f(x);x\right] = \frac{1}{k}\lim_{n\to\infty}\Big[\frac{p}{\log f(x+nk)}\Big]^{\frac{x}{k}+n} \times \frac{n!\, n^{\frac{x}{k}-1}}{{}_p[f(x);x]_{n,k}}. \tag{2.3}$$

Proof: Use the equation (2.1), we get the result.

Theorem 2.2 Given $x \in C/kZ^-$ and $Re[f(x)] \in (1,\infty), Re(x) > 0,\ k, p \in R^+ - \{0\}$, the following identities holds,

$$ {}_p\amalg_s\left[f(x);x\right] = \frac{k}{s}\Big[\frac{\log f(\frac{kx}{s})}{\log f(x)}\Big]^{\frac{x}{s}} {}_p\amalg_k\left[f(\frac{kx}{s});\frac{kx}{s}\right]. \tag{2.4}$$

$$ {}_r\amalg_s\left[f(x);x\right] = \frac{k}{s}\Big[\frac{\log f(\frac{kx}{s})}{\log f(x)}\Big]^{\frac{x}{s}}(\frac{r}{p})^{\frac{x}{s}} {}_p\amalg_k\left[f(\frac{kx}{s});\frac{kx}{s}\right]. \tag{2.5}$$

$$ {}_r\amalg_k\left[f(x);x\right] = (\frac{r}{p})^{\frac{x}{k}} {}_p\amalg_k\left[f(x);x\right]. \tag{2.6}$$

Proof: Property (2.4), (2.5),(2.6) will follow directly by using equation (2.1) and (2.3).

Theorem 2.3 Given $x \in C/kZ^-$ and $Re[f(x)] \in (1,\infty), Re(x) > 0,\ k, p \in R^+ - \{0\}$, then we have,

$$ {}_p\amalg_k\left[f(x);x\right] = \frac{1}{x}\Big(\frac{p}{\log f(x)}\Big)^{\frac{x}{k}} \prod_{n=1}^{\infty}\Big[(1+\frac{1}{n})^{\frac{x}{k}}(1+\frac{x}{nk})^{-1}\Big]. \tag{2.7}$$

Proof: Using equation (2.1) and (2.3), we immediately get the desire result.

Theorem 2.4 Given Given $x \in C/kZ^-$ and $Re[f(x)] \in (1,\infty), Re(x) > 0,\ k, p \in R^+ - \{0\}$, then we have,

$$ \frac{1}{{}_p\amalg_k\left[f(x);x\right]} = \frac{x}{k}\Big(\frac{\log f(x)}{p}\Big)^{\frac{x}{k}} \lim_{n\to\infty}\Big[n^{-\frac{x}{k}}\prod_{r=1}^{n}(1+\frac{x}{rk})\Big]. \tag{2.8}$$

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Proof: Using equation (2.1) and (2.3), we immediately get the desire result.

Theorem 2.5 Given $x \in C/kZ^-$ and $Re[f(x)] \in (1, \infty), Re(x) > 0,\ k, p \in R^+ - \{0\}$, then the relation between the Umbrella function, Generalized p - k Gamma function, p - k Gamma function, k-Gamma function and classical Gamma function is given by,

$$ {}_p \amalg_k [f(x); x] = \Big(\frac{\log a}{\log f(x)}\Big)^{\frac{x}{k}} {}_p^a\Gamma_k(x); \tag{2.9}$$

$$ {}_p \amalg_k [f(x); x] = {}_p^a\Gamma_k(x);\ if\ f(x) = a \in (0, \infty)\ \forall\ x \in C/kZ^-. \tag{2.10}$$

$$ {}_p \amalg_k [f(x); x] = \Big(\frac{1}{\log f(x)}\Big)^{\frac{x}{k}} {}_p\Gamma_k(x). \tag{2.11}$$

$$ {}_p \amalg_k [f(x); x] = \Big(\frac{p}{k \log f(x)}\Big)^{\frac{x}{k}} \Gamma_k(x). \tag{2.12}$$

$$ {}_p \amalg_k [f(x); x] = \frac{1}{k}\Big(\frac{p}{\log f(x)}\Big)^{\frac{x}{k}} \Gamma(\frac{x}{k}). \tag{2.13}$$

Proof: Using (2.1) and equation (2.14) of [4], we get the desire result.

Theorem 2.6 For Given $x \in C/kZ^-$ and $Re[f(x)] \in (1, \infty), Re(x) > 0,\ k, p \in R^+ - \{0\}$, then the relation between the Umbrella symbol, Generalized p - k Pochhammer symbol, p - k Pochhammer symbol, k-Pochhammer symbol and classical Pochhammer symbol is given by,

$$ {}_p[f(x); x]_{n,k} = \Big[\frac{\log f(x)}{\log f(x+nk)}\Big]^{\frac{x}{k}} \times \Big[\frac{\log a}{\log f(x+nk)}\Big]^{n} {}_p^a(x)_{n,k} \tag{2.14}$$

$$ {}_p[f(x); x]_{n,k} = {}_p^a(x)_{n,k},\ \ if\ f(x) = a \in (0, \infty)\ \forall\ x \in C/kZ^-. \tag{2.15}$$

$$ {}_p[f(x); x]_{n,k} = \Big[\frac{\log f(x)}{\log f(x+nk)}\Big]^{\frac{x}{k}} \times \Big[\frac{1}{\log f(x+nk)}\Big]^{n} {}_p(x)_{n,k}. \tag{2.16}$$

$$ {}_p[f(x); x]_{n,k} = \Big[\frac{\log f(x)}{\log f(x+nk)}\Big]^{\frac{x}{k}} \times \Big[\frac{p}{k \log f(x+nk)}\Big]^{n} (x)_{n,k}. \tag{2.17}$$

$$ {}_p[f(x); x]_{n,k} = \Big[\frac{\log f(x)}{\log f(x+nk)}\Big]^{\frac{x}{k}} \times \Big[\frac{p}{\log f(x+nk)}\Big]^{n} (\frac{x}{k})_n. \tag{2.18}$$

Proof: Using (2.2) and equation (2.15) of [64], we get the desire result.

Theorem 2.7 For Given $x \in C/kZ^-$ and $Re[f(x)] \in (1, \infty), Re(x) > 0,\ k, p \in R^+ - \{0\}$, the fundamental equations satisfied by the Umbrella function, ${}_p^a\Gamma_k(x)$ are,

$$ {}_p \amalg_k [f(x+k); x+k] = \frac{xp}{k \log f(x+k)}\Big(\frac{\log f(x)}{\log f(x+k)}\Big)^{\frac{x}{k}} {}_p \amalg_k [f(x); x]. \tag{2.19}$$

$$ {}_p[f(x); x]_{n,k} = \frac{{}_p \amalg_k [f(x+nk); x+nk]}{{}_p \amalg_k [f(x); x]}. \tag{2.20}$$

$$ {}_p[f(x); x]_{n,k} \times {}_p[f(k-x); k-x]_{n,k} = \frac{p}{k^2}\Big(\frac{\log f(k-x)}{\log f(x)}\Big)^{\frac{x}{k}} \frac{\pi}{\sin(\frac{\pi x}{k})}. \tag{2.21}$$

Proof: All the results follow directly from using equation (2.1)and (2.2).

3 particular cases

For particular values of function $f(x)$, we can get different known and new integrals.
[1] If we substitute $if\ f(x) = a \in (0, \infty)\ \forall\ x \in C/kZ^-$, then the Umbrella function will convert into New two parameter gamma function given by [4].
[2] If we substitute $if\ f(x) = e\ \forall\ x \in C/kZ^-$, then the Umbrella function will convert into two parameter gamma function given by [5].
[3] If we substitute $if\ f(x) = e\ \forall\ x \in C/kZ^-$ and $p = k$, then the Umbrella function will convert into k-gamma function given by [1].
[4] If we substitute $if\ f(x) = e\ \forall\ x \in C/kZ^-$ and $p = k = 1$, then the Umbrella function will convert into gamma function as defined in [2] and [3].

References

[1] Diaz, R. and Pariguan, E. On hypergeometric functions and Pochhammer k-symbol. Divulgaciones Mathematicas, Vol. 15 No. 2 (2007) 179-192.
[2] Earl D. Rainville, Special Function, The Macmillan Company, New york,1963.
[3] Erdelyi, A., Higher Transcendental Function Vol. 1, McGraw-Hill Book Company, New York, 1953.
[4] Kuldeep Singh Gehlot, New Two Parameter Gamma Function, 30 April 2020 doi:10.20944/preprints202004.0537.v1.
[5] Kuldeep Singh Gehlot, Two Parameter Gamma Function and it's Properties, arXiv:1701.01052v1[math.CA] 3 Jan 2017.