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New Two Parameter Gamma Function

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Abstract

In this paper we introduce the New/Generalized two parameter Gamma function and Pochhammer symbol. We named them, as Generalized p - k Gamma Function and Generalized p - k Pochhammer symbol and denoted as ${}^{a}_{p}\Gamma_k(x)$ and ${}^{a}_{p}(x)_{n,k}$ respectively. We prove the several identities for ${}^{a}_{p}\Gamma_k(x)$ and ${}^{a}_{p}(x)_{n,k}$ those satisfied by the classical Gamma function. Also we provide the integral representation for the ${}^{a}_{p}\Gamma_k(x)$.

Mathematics Subject Classification : 33B15, 33E12.

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1 Introduction

The Classical Gamma function is defined as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

Here $e = 2.71....$ is a irrational number.
And we introduce the generalized two parameter Gamma function as,

$${}^{a}_{p}\Gamma_k(x) = \int_0^\infty a^{-\frac{t^k}{p}} t^{x-1} dt.$$

Here $a \in (1, \infty)$ which is far-far general integral than classical Gamma function.
The main aim of this paper is to introduce Generalized two parameter Pochhammer symbol and Generalized two parameter Gamma function. Generalized p - k Gamma function is the deformation of the classical Gamma function, such that ${}^{a}_{p}\Gamma_k(x) \Rightarrow {}_{p}\Gamma_k(x)$ as $a = e$ and ${}^{a}_{p}\Gamma_k(x) \Rightarrow {}_{k}\Gamma_k(x) = \Gamma_k(x)$ as $p = k$; a=e and ${}^{a}_{p}\Gamma_k(x) \Rightarrow {}_{1}\Gamma_1(x) = \Gamma(x)$ as $a = e;\ p, k \to 1$.
In section 2, we defined Generalized two parameter Pochhammer symbol which is denoted as ${}^{a}_{p}(x)_{n,k}$, with its convergent conditions. Generalized two parameter Pochhammer symbol is the deformation of the classical Pochhammer symbol, such that ${}^{a}_{p}(x)_{n,k} \Rightarrow {}_{p}(x)_{n,k}$ as $a = e$; ${}^{a}_{p}(x)_{n,k} \Rightarrow {}_{k}(x)_{n,k} = (x)_{n,k}$ as $p = k$; $a = e$ and ${}^{a}_{p}(x)_{n,k} \Rightarrow {}_{1}(x)_{n,1} = (x)_n$ as $a = e;\ p = k = 1$.
Also we derived Generalized two parameter Pochhammer symbol in terms of the elementary symmetric function. It is most natural to relate the Generalized two parameter Pochhammer

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symbol to the Generalized two parameter Gamma function is defined. We evaluate integral representation of Generalized two parameter Gamma function, also represent Generalized two parameter Gamma function into different infinite product forms and so many recurrence relations are evaluated.
In section 3, we defined Generalized two parameter Psi function. Also evaluate some recurrence relations and functional relation with classical Psi function.
Section 4, deal with the definition of Hypergeometric function with Generalized two Parameter Pochhammer symbol, known as Generalized p-k Hypergeometric function. Also we evaluate the Differential Equation, Functional relation with Classical Hypergeometric function and integral representation of Generalized p-k Hypergeometric function.
Section 5, this section deal with the definition of Generalized p-k Mittag Liffler function and its usual properties.
Throughout this paper we use the notations as $C, R^+, Re(), Z^-$ and N be the sets of complex numbers, positive real numbers, real part of complex number, negative integer and natural numbers respectively.

2 Generalized p - k Pochhammer symbol and Generalized p - k Gamma function

In this section we introduce Generalized p - k Pochhammer symbol and Generalized p - k Gamma function. We evaluate ${}^a_p\Gamma_k(x)$ in terms of limit, recurrence formulas and infinite products.

2.1 Definition

Let $x \in C; k, p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N, a \in (1, \infty)$ than, the Generalized p - k Pochhammer symbol (i.e. Generalized two parameter Pochhammer symbol), ${}^a_p(x)_{n,k}$ is given by

$$ {}^a_p(x)_{n,k} = (\frac{xp}{k\log a})(\frac{xp}{k\log a} + \frac{p}{\log a})(\frac{xp}{k\log a} + \frac{2p}{\log a})\text{.........}(\frac{xp}{k\log a} + \frac{(n-1)p}{\log a}). \tag{2.1} $$

For $s, n \in N$ with $0 \leq s \leq n$, the s^{th} elementary symmetric function

$$ e^n_s(x_1, x_2, ..., x_n) = \sum_{1\leq i_1 \leq i_2 \leq ... \leq i_s \leq n} x_{i_1}.....x_{i_s} $$

on the variables $x_1, x_2, ..., x_n$.

Theorem 2.1 Formula for the Generalized p - k Pochhammer symbol (i.e.Generalized two parameter Pochhammer symbol) in terms of the elementary symmetric function is given by

$$ {}^a_p(x)_{n,k} = \sum_{i=0}^{n-1} \frac{x}{k} \frac{p^n}{\log a} e^{n-1}_{n-1-i}(1, 2, ..., (n-1))(\frac{x}{k\log a})^i. \tag{2.2} $$

Where $x \in C; k, p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$, $a \in (1, \infty)$.

Proof: The well known identity for elementary symmetric polynomials appear when expand a linear factorization of a monic polynomial

$$ \prod_{j=1}^{n-1}(\lambda + x_j) = \lambda^{n-1}e^{n-1}_0 + \lambda^{n-2}e^{n-1}_1 + ... + e^{n-1}_{n-1} = \sum_{i=0}^{n-1} e^{n-1}_{n-1-i}(1, 2, ..., (n-1))(\lambda)^i. \tag{2.3} $$

Using equation (2.1), we have the desired result.

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2.2 Definition

For $x \in C/kZ^-; k,p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N,\ a \in (1,\infty)$, the Generalized p - k Gamma function (i.e.the Generalized two parameter Gamma function), ${}^a_p\Gamma_k(x)$ is given by

$$ {}^a_p\Gamma_k(x) = \frac{1}{k}\lim_{n\to\infty}\frac{p^{n+1}n!}{(\log a)^{n+1}}\frac{1}{{}^a_p(x)_{n,k}}\left(\frac{np}{\log a}\right)^{\frac{x}{k}-1}. \tag{2.4}$$

Theorem 2.2 Given $x \in C/kZ^-; k,p,s,r \in R^+ - \{0\}$ and $Re(x) > 0, n \in N,\ a,b \in (1,\infty)$, the following identities holds,

$$ {}^a_p(x)_{n,s} = \left(\frac{\log b}{\log a}\right)^n\ {}^b_p(x)_{n,s}. \tag{2.5}$$

$$ {}^a_p(x)_{n,s} = {}^a_p(\frac{kx}{s})_{n,k}. \tag{2.6}$$

$$ {}^a_p(x)_{n,s} = (\frac{p}{s})^n\ {}^a_s(\frac{kx}{s})_{n,k}. \tag{2.7}$$

$$ {}^a_p(x)_{n,k} = (\frac{p}{s})^n\ {}^a_s(x)_{n,k}. \tag{2.8}$$

$$ {}^a_p\Gamma_s(x) = \left(\frac{\log b}{\log a}\right)^{\frac{x}{k}}\ {}^b_p\Gamma_s(x)\ . \tag{2.9}$$

$$ {}^a_p\Gamma_s(x) = \frac{k}{s}\ {}^a_p\Gamma_k(\frac{kx}{s}). \tag{2.10}$$

$$ {}^a_r\Gamma_s(x) = \frac{k}{s}\ (\frac{r}{p})^{\frac{x}{s}}\ {}^a_p\Gamma_k(\frac{kx}{s}). \tag{2.11}$$

$$ {}^a_r\Gamma_k(x) = (\frac{r}{p})^{\frac{x}{k}}\ {}^a_p\Gamma_k(x). \tag{2.12}$$

Proof: Property (2.5), (2.6), (2.7),(2.8) follows directly from definition (2.1) and the results (2.9), (2.10),(2.11),(2.12) will follow directly by using equation (2.4).

Theorem 2.3 Given $x \in C/kZ^-; k,p \in R^+ - \{0\}$ and $Re(x) > 0,\ a \in (1,\infty)$, then the integral representation of New/Generalized p - k Gamma function is given by

$$ {}^a_p\Gamma_k(x) = \int_0^\infty a^{-\frac{t^k}{p}}t^{x-1}dt. \tag{2.13}$$

Proof: Consider the right hand side integral and ([4], Page 2) Tannery's Theorem and equation (2.13), we have

$$\int_0^\infty a^{-\frac{t^k}{p}}t^{x-1}dt = \lim_{n\to\infty}\int_0^{(\frac{np}{\log a})^{\frac{1}{k}}}(1-\frac{t^k\log a}{np})^n\ t^{x-1}dt,$$

let $A_{n,i}(x), i = 0,1,...,n;$ be given by

$$A_{n,i}(x) = \int_0^{(\frac{np}{\log a})^{\frac{1}{k}}}(1-\frac{t^k\log a}{np})^i\ t^{x-1}dt,$$

integration by parts we have the following recurrence formula,

$$A_{n,i}(x) = \frac{ki\log a}{pxn}A_{n,i-1}(x+k).$$

Also,

$$A_{n,0}(x) = \int_0^{(\frac{np}{\log a})^{\frac{1}{k}}}t^{x-1}dt = \frac{1}{x}(\frac{np}{\log a})^{\frac{x}{k}}.$$

Therefor,

$$A_{n,n}(x) = \frac{1}{k} \lim_{n\to\infty} \frac{p^{n+1} n!}{(\log a)^{n+1}} \frac{1}{{}^{a}_{p}(x)_{n,k}(1+\frac{x}{np})} \Big(\frac{np}{\log a}\Big)^{\frac{x}{k}-1},$$

$${}^{a}_{p}\Gamma_k(x) = \lim_{n\to\infty} A_{n,n}(x) = \frac{1}{k} \lim_{n\to\infty} \frac{p^{n+1}}{(\log a)^{n+1}} \frac{n!}{{}^{a}_{p}(x)_{n,k}} \Big(\frac{np}{\log a}\Big)^{\frac{x}{k}-1}.$$

Which complete the proof.

Theorem 2.4 Given $x \in C/kZ^-; k, p \in R^+ - \{0\}$ and $Re(x) > 0$, $a \in (1, \infty)$, then we have,

$${}^{a}_{p}\Gamma_k(x) = \frac{p^{\frac{x}{k}}}{x(\log a)^{\frac{x}{k}}} \prod_{n=1}^{\infty} \Big[(1+\frac{1}{n})^{\frac{x}{k}}(1+\frac{x}{nk})^{-1}\Big]. \tag{2.14}$$

Proof: Using equation (2.1) and (2.4), we immediately get the desire result.

Theorem 2.5 Given $x \in C/kZ^-; k, p \in R^+ - \{0\}$ and $Re(x) > 0$, $a \in (1, \infty)$, then we have,

$$\frac{1}{{}^{a}_{p}\Gamma_k(x)} = (\log a)^{\frac{x}{k}} \frac{x}{kp^{\frac{x}{k}}} \lim_{n\to\infty} \Big[n^{-\frac{x}{k}} \prod_{r=1}^{n}(1+\frac{x}{rk})\Big]. \tag{2.15}$$

Proof: Using equation (2.1) and (2.4), we immediately get the desire result.

Theorem 2.6 The relation between Generalized p - k Gamma function, p - k Gamma function, k-Gamma function and classical Gamma function is given by,

$${}^{a}_{p}\Gamma_k(x) = \frac{{}_{p}\Gamma_k(x)}{(\log a)^{\frac{x}{k}}} = (\frac{p}{k\log a})^{\frac{x}{k}}\, \Gamma_k(x) = \frac{1}{k}(\frac{p}{\log a})^{\frac{x}{k}}\, \Gamma(\frac{x}{k}). \tag{2.16}$$

Where $x \in C/kZ^-; k, p \in R^+ - \{0\}$ and $Re(x) > 0$, $a \in (1, \infty)$.
Proof: Using (2.13) and equation (2.19) of [9], we get the desire result.

Theorem 2.7 For $x \in C/kZ^-; n, q \in N; k, p \in R^+ - \{0\}$ and $Re(x) > 0$, $a \in (1, \infty)$, then the relation between Generalized p - k Pochhammer symbol, p - k Pochhammer symbol, k-Pochhammer symbol and classical Pochhammer symbol is given by,

$${}^{a}_{p}(x)_{n,k} = \frac{1}{(\log a)^n}\, {}_{p}(x)_{n,k} = \frac{1}{(\log a)^n}(\frac{p}{k})^n (x)_{n,k} = \frac{1}{(\log a)^n}(p)^n(\frac{x}{k})_n. \tag{2.17}$$

Also for q-Generalized p - k Pochhammer symbol, we have

$${}^{n}_{p}(x)_{nq,k} = \frac{{}_{p}(x)_{nq,k}}{(\log a)^{nq}} = (\frac{p}{k\log a})^{nq}(x)_{nq,k} = (\frac{p}{\log a})^{nq}(\frac{x}{k})_{nq} = (\frac{pq}{\log a})^{nq}\prod_{r=1}^{q}(\frac{\frac{x}{k}+r-1}{q})_n. \tag{2.18}$$

Proof: Using (2.1) and equation (2.20) of [9], we get the desire result.

Theorem 2.8 For $x \in C/kZ^-; k, p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$. $a \in (1, \infty)$, the fundamental equations satisfied by New p - k Gamma function, ${}^{a}_{p}\Gamma_k(x)$ are,

$${}^{a}_{p}\Gamma_k(x+k) = \frac{xp}{k\log a}\, {}^{a}_{p}\Gamma_k(x). \tag{2.19}$$

$${}^{a}_{p}(x)_{n,k} = \frac{{}^{a}_{p}\Gamma_k(x+nk)}{{}^{a}_{p}\Gamma_k(x)}. \tag{2.20}$$

$$\frac{{}^{a}_{p}\Gamma_k(x)}{{}^{a}_{p}\Gamma_k(x-nk)} = \frac{p^n}{k^n(\log a)^n}(x-k)(x-2k).....(x-nk). \tag{2.21}$$

$$ {}_p^a\Gamma_k(1) = \frac{1}{k}\,(\frac{p}{\log a})^{\frac{1}{k}}\;\Gamma(\frac{1}{k}). \tag{2.22}$$

$$ {}_p^a\Gamma_k(k) = \frac{p}{k\log a}. \tag{2.23}$$

$$ {}_p^a\Gamma_k(p) = \frac{1}{k}(\frac{p}{\log a})^{\frac{p}{k}}\;\Gamma(\frac{p}{k}). \tag{2.24}$$

$$ {}_p^a\Gamma_k(x)\,{}_p^a\Gamma_k(-x) = \frac{\pi}{xk}\,\frac{1}{\sin(\frac{\pi x}{k})}. \tag{2.25}$$

$$ {}_p^a\Gamma_k(x)\,{}_p^a\Gamma_k(k-x) = \frac{p}{k^2\log a}\,\frac{\pi}{\sin(\frac{\pi x}{k})}. \tag{2.26}$$

Proof: All the results follow directly from using equation (2.1), (2.4) and (2.13).

Theorem 2.9 For $x \in C/kZ^-; k,p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$. $a \in (1,\infty)$, then the recurrence relations for Generalized p - k Pochhammer symbol are given by,

$$ np \times {}_p^a(x)_{n-1,k} = \log a \times \Big[{}_p^a(x)_{n,k} - {}_p^a(x-k)_{n,k}\Big]. \tag{2.27}$$

And

$$ {}_p^a(x)_{n+j,k} = {}_p^a(x)_{j,k} \times {}_p^a(x+jk)_{n,k}. \tag{2.28}$$

Proof: Using equation (2.17) and basic relations $n(x)_{n-1} = (x)_n - (x-1)_n$, $(x)_{n+j} = (x)_j(x+j)_n$, we get the desired result.

3 Generalized p - k Psi function

In this section, we introduce Generalized p - k Psi function ${}_p^a\psi_k(x,y)$. We evaluate explicit formula that relate the ${}_p^a\psi_k(x)$ to classical Psi function $\psi(x)$. Also prove some identities.

3.1 Definition

For $x \in C/kZ^-; k,p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$, $a \in (1,\infty)$, the logarithmic derivative of the Generalized p - k Gamma function is known as Generalized p - k Psi function,

$$ {}_p^a\psi_k(x) = \frac{d}{dx}\log[{}_p^a\Gamma_k(x)] = \frac{1}{{}_p^a\Gamma_k(x)}\frac{d}{dx}[{}_p^a\Gamma_k(x)]. \tag{3.1}$$

$$ \log[{}_p^a\Gamma_k(x)] = \int_1^x {}_p^a\psi_k(x)dx. \tag{3.2}$$

Theorem 3.1 Some properties of ${}_p^a\psi_k(x)$ are given by

$$ {}_p^a\psi_k(x) = \frac{1}{k}\log(\frac{p}{\log a}) + \psi(\frac{x}{k}). \tag{3.3}$$

$$ {}_p^a\psi_k(x) = \frac{\log p}{k} - \frac{log(loga)}{k} - \gamma - \frac{k}{x} + x\sum_{n=1}^{\infty}\frac{1}{n(x+nk)}. \tag{3.4}$$

$$ {}_p^a\psi_k(x) = \frac{\ln p}{k} - \frac{log(loga)}{k} - \gamma + (x-k)\sum_{n=0}^{\infty}\frac{1}{(n+1)(x+nk)}. \tag{3.5}$$

Where γ is Euler's constant and $\psi(x)$ is classical Psi function.

Proof: Using the definition (3.1) and equation (3.9), (3.10) of [9], we have immediately get above results.

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4 Hypergeometric function

In this section we define the Hypergeometric function using Generalized p - k Pochhammer symbols. Here we will use the notation of [3].

4.1 Definition

Given $x \in C$, $c = (c_1, ..., c_r) \in C^r; k, p \in (R^+)^r; s, t \in (R^+)^q, d = (d_1, d_2, ..., d_q) \in C^q$ such that $d_j \in C/s_j Z^-$, $a_i, b_j \in (1, \infty)$, the Generalized p-k Hypergeometric function ${}^a_rF^b_q(c, p, k; d, t, s; x)$ is given by

$$ {}^a_rF^b_q(c, p, k; d, t, s; x) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} {}^{a_i}_{p_i}(c_i)_{n,k_i}}{\prod_{j=1}^{q} {}^{b_j}_{t_j}(d_j)_{n,s_j}} \frac{x^n}{n!}. \tag{4.1} $$

By using Ratio Test we can show that the series (4.1) converges for all finite x if $r \leq q$. If $r > q+1$, the series diverges and if $r = q+1$, it converges for all x such that $|x| < |\frac{(t_1 t_2 .....t_q)(\log a_1 ... \log a_r)}{(p_1 p_2 .....p_r)(\log b_1 ... \log b_r)}|$.

Theorem 4.1 Given $x \in C$, $c = (c_1, ..., c_r) \in C^r; k, p \in (R^+)^r; s, t \in (R^+)^q, d = (d_1, d_2, ..., d_q) \in C^q$ such that $d_j \in C/s_j Z^-$, $a_i, b_j \in (1, \infty)$, then the functional relation between Generalized p - k Hypergeometric function and classical Hypergeometric function is given by,

$$ {}^a_rF^b_q(c, p, k; d, t, s; x) = {}_rF_q\Big(\frac{c}{k}; \frac{d}{s}; \frac{\prod_{i=1}^{r} \frac{p_i}{\log a_i}}{\prod_{j=1}^{q} \frac{t_j}{\log b_j}} x\Big). \tag{4.2} $$

Proof: Using definition (2.17), we get above result.

Theorem 4.2 The Differential equation of Generalized p - k Hypergeometric function is given by

$$ [\theta \prod_{j=1}^{q}(\theta + \frac{d_j}{s_j} - 1) - Ax \prod_{i=1}^{r}(\theta + \frac{c_i}{k_i})]W = 0. \tag{4.3} $$

Where $\theta = x\frac{d}{dx}$, $A = \frac{\prod_{i=1}^{r} \frac{p_i}{\log a_i}}{\prod_{j=1}^{q} \frac{t_j}{\log b_j}}$ and $W = {}^a_rF^b_q(c, p, k; d, t, s; x)$.

For $r \leq q+1$, $i = 1, 2, ..., r$ and $j = 1, 2, ..., q$ when no $\frac{d_j}{s_j}$ is a negative integer or zero and no two $\frac{d_j}{s_j}$ is differ by an integer or zero.
Proof: Using Function relation (4.2) and page 74, of [3] we get the desired result.

Theorem 4.3 For any $b \in C; k, \frac{p}{\log a} > 0$ and $|x| < \frac{\log a}{p}$, $a \in (1, \infty)$, the following identity holds

$$ \sum_{n=0}^{\infty} \frac{{}^a_p(b)_{n,k}\, x^n}{n!} = (1 - \frac{xp}{\log a})^{-\frac{b}{k}}. \tag{4.4} $$

Proof: Using (2.17) we get immediately the desired result.

Theorem 4.4 Given $x \in C$, $c = (c_1, ..., c_r) \in C^r; k, p \in (R^+)^r; s, t \in (R^+)^q, d = (d_1, d_2, ..., d_q) \in C^q$ such that $d_j \in C/s_j Z^-$, $a_i, b_j \in (1, \infty)$, the integral representation of Generalized p - k Hypergeometric function is given by,

$$ {}^a_rF^b_q(c, p, k; d, t, s; x) = \prod_{i=1}^{r}\prod_{j=1}^{q} \frac{\Gamma(\frac{d_j}{s_j})}{\Gamma(\frac{c_i}{k_i})\Gamma(\frac{d_j}{s_j} - \frac{c_i}{k_i})} \int_0^1 t^{\frac{c_i}{k_i}-1}(1-t)^{\frac{d_j}{s_j}-\frac{c_i}{k_i}-1} e^{\frac{p_i \log b_j}{t_j \log a_i}xt} dt. \tag{4.5} $$

Proof: Using (4.2), we get immediately the desired result.

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5 New p-k Mittag-Liffler function

In this section we introduce the New/Generelized p - k Mittag-Leffler function and prove some of its properties.

5.1 Definition

Let $k, p \in R^+ - \{0\}; \alpha, \beta, \gamma \in C/kZ^-; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0,\ j \in N_0$ and $q \in (0,1) \cup N,\ a \in (1, \infty)$, the Generelized p - k Mittag-Leffler function denoted by ${}^{j}_{a,p}E^{\gamma,q}_{k,\alpha,\beta}(z)$ and defined as

$$ {}^{j}_{a,p}E^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}^{a}_{p}(\gamma)_{(n+j)q,k}}{{}^{a}_{p}\Gamma_k(n\alpha+\beta)} \frac{z^n}{(n+j)!}, \qquad z \in C. \tag{5.1} $$

Where ${}^{a}_{p}(\gamma)_{nq,k}$ is generalized two parameter Pochhammer symbol given by equation (2.1) and ${}^{a}_{p}\Gamma_k(x)$ is the generalized two parameter Gamma function given by equation (2.4).

Particular cases : For some particular values of the parameters $a, j, p, q, k, \alpha, \beta, \gamma$ we can obtain certain defined and undefined Mittag-Leffler functions:

(a) For $q = 1$ equation (5.1), reduces in New j form of p-k Mittag-Leffler function defined as,

$$ {}^{j}_{a,p}E^{\gamma,1}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}^{a}_{p}(\gamma)_{(n+j),k}}{{}^{a}_{p}\Gamma_k(n\alpha+\beta)} \frac{z^n}{(n+j)!}. \tag{5.2} $$

(b) For $q = 1, p = k$ equation (5.1), reduces in New j form of k- Mittag-Leffler function defined as,

$$ {}^{j}_{a,k}E^{\gamma,1}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}^{a}(\gamma)_{(n+j),k}}{{}^{a}\Gamma_k(n\alpha+\beta)} \frac{z^n}{(n+j)!}. \tag{5.3} $$

(c) For $a = e$ equation (5.1), reduces in j-generalized form of p-k Mittag-Leffler function given by [7], defined as.

$$ {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_{p}(\gamma)_{(n+j)q,k}\, z^n}{{}_{p}\Gamma_k(n\alpha+\beta)(n+j)!}. \tag{5.4} $$

(d) For $p = k, j = 0$ equation (5.1), reduces in New Generalized k- Mittag-Leffler function defined as.

$$ {}_{a,k}E^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}^{a}_{k}(\gamma)_{nq,k}\, z^n}{{}^{a}_{k}\Gamma_k(n\alpha+\beta)(n!)}. \tag{5.5} $$

(e) For $p = k, q = 1, j = 0$ equation (5.1), reduces in New k - Mittag-Leffler function defined by [3].

$$ {}_{a,k}E^{\gamma,1}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}^{a}(\gamma)_{n,k}\, z^n}{{}^{a}\Gamma_k(n\alpha+\beta)(n!)} = {}^{a}E^{\gamma}_{k,\alpha,\beta}(z). \tag{5.6} $$

(f) For $j = 0$ equation (5.1), reduces in New form p-k Mittag-Leffler function defined as,

$$ {}^{0}_{a,p}E^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}^{a}_{p}(\gamma)_{nq,k}}{{}^{a}_{p}\Gamma_k(n\alpha+\beta)} \frac{z^n}{(n)!}. \tag{5.7} $$

(g) For $j = 0, a = e$ equation (5.1), reduces in the p-k Mittag-Leffler function defined by [8]as,

$$ {}^{0}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_{p}(\gamma)_{nq,k}}{{}_{p}\Gamma_k(n\alpha+\beta)} \frac{z^n}{(n)!}. \tag{5.8} $$

(h) For $p = k, j = 0, a = e$ equation (5.1), reduces in Generalized k- Mittag-Leffler function defined by [10].

$$ {}_kE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_k(\gamma)_{nq,k}\, z^n}{{}_k\Gamma_k(n\alpha+\beta)(n!)} = GE^{\gamma,q}_{k,\alpha,\beta}(z). \tag{5.9}$$

(i) For $p = k, q = 1, j = 0, a = e$ equation (5.1), reduces in k - Mittag-Leffler function defined by [5].

$$ {}_kE^{\gamma,1}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\, z^n}{\Gamma_k(n\alpha+\beta)(n!)} = E^{\gamma}_{k,\alpha,\beta}(z). \tag{5.10}$$

(j) For $p = k, a = e$ and $k = 1, j = 0$ equation (5.1), reduces in Mittag-Leffler function defined by [1].

$$ {}_1E^{\gamma,q}_{1,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}\, z^n}{\Gamma(n\alpha+\beta)(n!)} = E^{\gamma,q}_{\alpha,\beta}(z). \tag{5.11}$$

(k) For $p = k = q = 1, a = e$ equation (5.1), reduces in L-Mittag-Leffler function defined by [11].

$$ {}_1E^{\gamma,1}_{1,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j}\, z^n}{\Gamma(n\alpha+\beta)(n+j)!} = L^{\gamma,j}_{\alpha,\beta}(z). \tag{5.12}$$

(l) For $a = e, p = k, q = 1, j = 0$ and $k = 1$ equation (5.1), reduces in Mittag-Leffler function defined by [12].

$$ {}_1E^{\gamma,1}_{1,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n}\, z^n}{\Gamma(n\alpha+\beta)(n!)} = E^{\gamma}_{\alpha,\beta}(z), \tag{5.13}$$

(m) For $a = e, p = k, q = 1, k = 1, j = 0$ and $\gamma = 1$ equation (5.1), reduces in Mittag-Leffler function defined by [2].

$$ {}_1E^{1,1}_{1,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)} = E_{\alpha,\beta}(z), \tag{5.14}$$

(n) For $a = e, p = k, q = 1, k = 1, \gamma = 1, j = 0$ and $\beta = 1$ equation (5.1), reduces in Mittag-Leffler function defined by [6].

$$ {}_1E^{1,1}_{1,\alpha,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)} = E_{\alpha}(z). \tag{5.15}$$

Theorem 5.1: The New/Generelized p - k Mittag-Leffler function defined by equation (5.1) is an entire function of order

$$ \frac{1}{\rho} = Re(\frac{\alpha}{k}) - q + 1. \tag{5.16}$$

Proof: Let R is the radius of convergence of the generelized p - k Mittag-Leffler function. The asymptotic Striling formula for Gamma function and factorial are given by,[3]

$$ \Gamma(az+b) = \sqrt{2\pi}e^{-az}(az)^{az+b-\frac{1}{2}}\Big[1 + o(\frac{1}{z})\Big], (arg(az+b) < \pi; \mid z \mid \to \infty). \tag{5.17}$$

and

$$ n! = \sqrt{2\pi}e^{-n}(n)^{n+\frac{1}{2}}\Big[1 + o(\frac{1}{n})\Big], (n \in N; n \to \infty). \tag{5.18}$$

From equation (5.1), we have

$$ {}^{j}_{a,p}E^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}^a_p(\gamma)_{(n+j)q,k}}{{}^a_p\Gamma_k(n\alpha+\beta)} \frac{z^n}{(n+j)!} = \sum_{n=0}^{\infty} C_n z^n,$$

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since

$$R=\limsup_{n\to\infty}\Big|\frac{C_n}{C_{n+1}}\Big|,$$

$$\Big|\frac{C_n}{C_{n+1}}\Big|=\Big|\frac{{}_p^a(\gamma)_{(n+j)q,k}}{{}_p^a\Gamma_k(n\alpha+\beta)}\frac{1}{(n+j)!}\times\frac{{}_p^a\Gamma_k(n\alpha+\alpha+\beta)(n+1+j)!}{{}_p^a(\gamma)_{(n+1+j)q,k}}\Big|$$

using equations (2.16) and (2.20), we have

$$\Big|\frac{C_n}{C_{n+1}}\Big|=(n+1+j)\Big|(\frac{p}{\log a})^{\frac{\alpha-qk}{k}}\Big|\Big|\frac{\Gamma(nq+jq+\frac{\gamma}{k})}{\Gamma(nq+jq+q+\frac{\gamma}{k})}\Big|\Big|\frac{\Gamma(\frac{n\alpha+\alpha+\beta}{k})}{\Gamma(\frac{n\alpha+\beta}{k})}\Big|,$$

using equation (5.17), we have

$$\simeq\Big|(\frac{p}{\log a})^{\frac{\alpha}{k}-q}\Big|\Big|q^{-q}\Big|\Big|(\frac{\alpha}{k})^{\frac{\alpha}{k}}\Big|\Big|n^{\frac{\alpha}{k}+1-q}\Big|\to\infty$$

when,

$$Re(\frac{\alpha}{k}+1-q)>0,$$

Thus, the generelized p - k Mittag-Leffler function is an entire function for $q<Re(\frac{\alpha}{k})+1$
To determine the order ρ,

$$\rho=\limsup_{n\to\infty}\frac{n\ln n}{\ln(\frac{1}{|C_n|})},\tag{5.19}$$

$$\Big|\frac{1}{C_n}\Big|=\Big|\frac{{}_p^a\Gamma_k(n\alpha+\beta)(n+j)!}{{}_p^a(\gamma)_{(n+j)q,k}}\Big|,$$

using equations (2.16) and (2.20), we have

$$\Big|\frac{1}{C_n}\Big|=\frac{(n+j)!}{k}\Big|(\frac{p}{\log a})^{\frac{\gamma}{k}+\frac{n\alpha+\beta}{k}-\frac{\gamma+(n+j)qk}{k}}\Big|\Big|\frac{\Gamma(\frac{\gamma}{k})\Gamma(\frac{n\alpha+\beta}{k})}{\Gamma(\frac{\gamma}{k}+(n+j)q)}\Big|,$$

By using equation (5.17) and (5.18), we get

$$\Big|\frac{1}{C_n}\Big|=k^{-1}(2\pi)^{\frac{1}{2}}\Big|(\frac{p}{\log a})^{(\frac{\alpha-qk}{k})n+\frac{\beta}{k}-jq}\Big|\Big|(\frac{\alpha}{k})^{\frac{n\alpha}{k}+\frac{\beta}{k}-\frac{1}{2}}\Big|\Big|n^{\frac{n\alpha}{k}+\frac{\beta}{k}-\frac{\gamma}{k}-nq-jq+n+j+\frac{1}{2}}\Big|\Big|e^{-nRe(\frac{\alpha}{k}+1-q)}\Big|$$

taking ln of above equation and put in equation (5.19), we have the order of New p - k Mittag-Leffler function is given by

$$\rho=\frac{k}{Re(\alpha)-qk+k}.$$

Hence.

Theorem 5.2: The functional relation between the New/ Generalized p - k Mittag-Leffler function given by equation (5.1) with j-generalizes p - k Mittag-Leffler function defined by [7].

$${}_{a,p}^{j}E_{k,\alpha,\beta}^{\gamma,q}(z)=(\log a)^{\frac{\beta}{k}-jq}\times{}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z(\log a)^{\frac{\alpha}{k}-q}).\tag{5.20}$$

$$(\frac{d}{dz})^l\Big[z^j\times{}_{a,p}^{j}E_{k,\alpha,\beta}^{\gamma,q}(z)\Big]={}_p^a(\gamma)_{lq,k}\ z^{j-l}\times{}_{a,p}^{j-l}E_{k,\alpha,\beta}^{\gamma+lqk,q}(z),\ for\ l<j.\tag{5.21}$$

$$(\frac{d}{dz})^l\Big[z^j\times{}_{a,p}^{j}E_{k,\alpha,\beta}^{\gamma,q}(z)\Big]={}_p^a(\gamma)_{lq,k}\times{}_{a,p}E_{k,\alpha,\beta}^{\gamma+lqk,q}(z),\ for\ l=j.\tag{5.22}$$

$$(\frac{d}{dz})^l\Big[z^j\times{}_{a,p}^{j}E_{k,\alpha,\beta}^{\gamma,q}(z)\Big]={}_p^a(\gamma)_{lq,k}\times{}_{a,p}E_{k,\alpha,\beta+l\alpha-j\alpha}^{\gamma+lqk,q}(z),\ for\ l>j.\tag{5.23}$$

Proof of equation (5.20)
Using equation (2.16) and (2.17), we get the desired result.
Proof of equation (5.21), (5.22) and (5.23)
Using the equation (5.1), in right hand side of (5.21), we have

$$\frac{d^l}{dz^l}\left[z^j \times {}^{j}_{a,p}E^{\gamma,q}_{k,\alpha,\beta}(z)\right] = \sum_{n=0}^{\infty} \frac{{}^{a}_{p}(\gamma)_{(n+j)q,k}}{{}^{a}_{p}\Gamma_k(n\alpha+\beta)} \frac{z^{n+j-l}}{(n+j-l)!},$$

using equation (2.28), we have

$$\frac{d^l}{dz^l}\left[z^j \times {}^{j}_{a,p}E^{\gamma,q}_{k,\alpha,\beta}(z)\right] = \sum_{n=0}^{\infty} \frac{{}^{a}_{p}(\gamma)_{lq,k}\;{}^{a}_{p}(\gamma+lqk)_{(n+j-l)q,k}}{{}^{a}_{p}\Gamma_k(n\alpha+\beta)} \frac{z^{n+j-l}}{(n+j-l)!},$$

hence we have,

$$= {}^{a}_{p}(\gamma)_{lq,k}\;\; z^{j-l}\;\; {}^{j-l}_{a,p}E^{\gamma+lqk,q}_{k,\alpha,\beta}(z),\;\; for\; l<j.$$

$$= {}^{a}_{p}(\gamma)_{jq,k}\;\; {}_{a,p}E^{\gamma+lqk,q}_{k,\alpha,\beta}(z),\;\; for\; l=j.$$

$$= {}^{a}_{p}(\gamma)_{lq,k}\;\; {}_{a,p}E^{\gamma+lqk,q}_{k,\alpha,\beta+l\alpha-j\alpha}(z),\;\; for\; l>j.$$

Theorem 5.3: The following elementary properties are satisfied by the New/Generelized p - k Mittag-Leffler function defined by equation (5.1),

$$k\log a \times {}^{j}_{a,p}E^{\gamma,q}_{k,\alpha,\beta}(z) = p\beta \times {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta+k}(z) + zp\alpha\frac{d}{dz}\left[{}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta+k}(z)\right]. \tag{5.24}$$

$$\frac{pq}{\log a} \times {}_{p}(\gamma)_{q-1,k} \times {}^{j-1}_{p}E^{\gamma+kq-k,q}_{k,\alpha,\beta}(z) = {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z) - {}^{j}_{p}E^{\gamma-k,q}_{k,\alpha,\beta}(z). \tag{5.25}$$

Proof of equation (5.24)
Consider the right hand side of equation (5.24),

$$A \equiv p\beta\, {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta+k}(z) + zp\alpha\frac{d}{dz}\;{}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta+k}(z),$$

using equation (5.1),

$$A \equiv p\beta \sum_{n=0}^{\infty} \frac{{}_{p}(\gamma)_{(n+j)q,k}}{{}_{p}\Gamma_k(n\alpha+\beta+k)} \frac{z^n}{(n+j)!} + zp\alpha \sum_{n=0}^{\infty} \frac{{}_{p}(\gamma)_{(n+j)q,k}}{{}_{p}\Gamma_k(n\alpha+\beta+k)} \frac{nz^{n-1}}{(n+j)!},$$

$$A \equiv p \sum_{n=0}^{\infty} \frac{{}_{p}(\gamma)_{(n+j)q,k}(n\alpha+\beta)}{{}_{p}\Gamma_k(n\alpha+\beta+k)} \frac{z^n}{(n+j)!},$$

using the equation (2.19), we have

$$A \equiv k\log a \times {}^{j}_{a,p}E^{\gamma,q}_{k,\alpha,\beta}(z).$$

6 Particular cases

Putting some particular values of $a, j, p, q, k, \alpha, \beta, \gamma$ we obtain all the results given by the research papers [1], [7],[8],[9],[10] and [11].

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