 doi:10.20944/preprints202004.0521.v1

$\mathbb{D}$ynamics of $\mathbb{C}$ohomological $\mathbb{E}$xpanding $\mathbb{M}$appings I : First and Second Main Results

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April 29, 2020

Abstract

Let $f : \mathcal{V} \longrightarrow \mathcal{V}$ be a Cohomological Expanding Mapping[1] of a smooth complex compact homogeneous manifold with $dim_{\mathbb{C}}(\mathcal{V}) = k \geq 1$ and Kodaira Dimension ≤ 0. We study the dynamics of such mapping from a probabilistic point of view, that is, we describe the asymptotic behavior of the orbit $O_f(x) = \{f^n(x), n \in \mathbb{N} \quad \text{or} \quad \mathbb{Z}\}$ of a generic point. Using pluripotential methods, we construct a natural invariant canonical probability measure of maximum Cohomological Entropy μ_f such that $\chi_{2l}^{-m}(f^m)^*\Omega \to \mu_f \quad$ as $\quad m \to \infty$ for each smooth probability measure Ω on $\mathcal{V}$. Then we study the main stochastic properties of μ_f and show that μ_f is a measure of equilibrium, smooth, ergodic, mixing, K-mixing, exponential-mixing and the unique measure with maximum Cohomological Entropy. We also conjectured that $\mu_f := T_l^+ \wedge T_{k-l}^-$, $\dim_{\mathcal{H}}(\mu_f) = \Psi h_\chi(f)$ and $\dim_{\mathcal{H}}(\text{Supp}T_l^+) \geq 2(k-l) + \frac{\log \chi_{2l}}{\psi_l}$.

Keywords Complex Dynamics · Cohomological Expanding Mapping · Cohomological Degree · Cohomological Entropy · Cohomological Quotient .

Classification – MSC2020 37C45; 37D05; 37D20; 37D25; 37D35; 37F10; 37F15; 37F80

Contents

[1] cf Definition 1.6.

1 Introduction

Let μ be the equilibrium measure associated with an endomorphism f. If φ is an observable, $(\varphi \circ f^n)_{n\geq 0}$ can be seen as a sequence of dependent random variables. As the measure is invariant, these variables are distributed in an identical way, that is, the Borel sets $\{\varphi \circ f^n < t\}$ have the same measure μ for any fixed constant t. We recall some general facts of ergodic theory and probability theory. We refer to [47, 49] for the general theory.

Consider a dynamic system associated with a map $g : X \to X$, measurable against a σ-algebra $\mathcal{F}$ on X. The direct image of a probability measure ν by g is the measure of probability $g_*(\nu)$ defined by

$$g_*(\nu)(A) := \nu(g^{-1}(A))$$

for each measurable set A. Likewise, for any positive measurable function φ, we have

$$\langle g_*(\nu), \varphi\rangle := \langle \nu, \varphi \circ g\rangle.$$

The measure ν is ***invariant*** if $g_*(\nu) = \nu$. When X is a compact metric space and g is continuous, the set $\mathcal{M}(g)$ of invariant probability measures is convex, compact and not empty: for any sequence of probability measures ν_N, the limit values of

$$\frac{1}{N}\sum_{j=0}^{N-1}(g^n)_*(\nu_N)$$

are invariant probability measures.

A measurable set A is ***totally invariant*** if $\nu(A \setminus g^{-1}(A)) = \nu(g^{-1}(A) \setminus A) = 0$. An invariant probability measure ν is ***ergodic*** if any totally invariant set is of measure ν zero or complete . It is easy to show that ν is ergodic if and only if $\varphi \circ g = \varphi$, for $\varphi \in L^1(\nu)$, then φ is constant. Here, we can replace $L^1(\nu)$ by $L^p(\nu)$ with $1 \leq p \leq +\infty$. The **ergodicity** of ν is also equivalent to the fact that it is extremal on $\mathcal{M}(g)$. We remember Birkhoff's ergodic theorem, which is the analogue of the law of large numbers for independent random variables [49].

Theorem 1.1 (Birkhoff)**.** *Let $g : X \to X$ be a measurable map as above. Suppose that ν is an invariant ergodic probability measure . Let φ be a function on $L^1(\nu)$. Then*

$$\frac{1}{N}\sum_{n=0}^{N-1}\varphi(g^n(x)) \to \langle \nu, \varphi\rangle$$

almost everywhere in relation to ν.

When X is a compact metric space, we can apply Birkhoff's theorem to continuous functions φ and deduce that for ν almost all x

$$\frac{1}{N}\sum_{n=0}^{N-1}\delta_{g^n(x)} \to \nu,$$

where δ_x indicates the mass of Dirac at x. The sum

$$\mathrm{St}_N(\varphi) := \sum_{n=0}^{N-1}\varphi \circ g^n$$

is called ***Birkhoff sum***. Therefore, Birkhoff's theorem describes the behavior of $\frac{1}{N}\mathrm{St}_N(\varphi)$ for an observable φ.

A stronger notion than **ergodicity** is the notion of **mixing**. An invariant probability measure ν is ***mixing*** if for each measurable set A, B

$$\lim_{n\to\infty}\nu(g^{-n}(A)\cap B) = \nu(A)\nu(B).$$

Clearly, **mixing** implies **ergodicity**. It is not difficult to see that ν is mixing if, and only if, for any test functions φ, ψ on $L^\infty(\nu)$ or on $L^2(\nu)$, we have

$$\lim_{n\to\infty}\langle \nu, (\varphi \circ g^n)\psi\rangle = \langle \nu, \varphi\rangle\langle \nu, \psi\rangle.$$

The Quantity

$$I_n(\varphi, \psi) := |\langle \nu, (\varphi \circ g^n)\psi\rangle - \langle \nu, \varphi\rangle\langle \nu, \psi\rangle|$$

is called ***the correlation on time*** n of φ and ψ. Thus, **mixing** is equivalent to the convergence of $I_n(\varphi, \psi)$ to 0. We say that ν is ***K-mixing*** if for each $\psi \in L^2(\nu)$

$$\sup_{\|\varphi\|_{L^2(\nu)} \leq 1} I_n(\varphi, \psi) \to 0.$$

Note that **K-mixing** is equivalent to the fact that the σ -algebra $\mathcal{F}_\infty := \cap g^{-n}(\mathcal{F})$ contains only sets zero and complete measures. This is the strongest form of mixing for observables on $L^2(\nu)$. However, it is of interest to obtain quantitative information about the mixing speed for more regular observables, such as smooth functions or Hölder continuous.

Now consider an endomorphism f of degree $d \geq 2$ of $\mathbb{P}^k$ as above and its equilibrium measure μ. We know that μ is totally invariant: $f^*(\mu) = d^k \mu$. If φ is a continuous function, so

$$\langle \mu, \varphi \circ f \rangle = \langle d^{-k} f^*(\mu), \varphi \circ f \rangle = \langle \mu, d^{-k} f_*(\varphi \circ f) \rangle = \langle \mu, \varphi \rangle.$$

We use the obvious fact that $f_*(\varphi \circ f) = d^k \varphi$. Thus, μ is invariant.

Mixing for measure μ was proved in [45].

Theorem 1.2. *Let f be an endomorphism of degree $d \geq 2$ of $\mathbb{P}^k$. So its measure of Greenμ is K-mixing.*

The equilibrium measure μ satisfies remarkable stochastic properties that are quite difficult to obtain in the real dynamic systems scenario. Pluripotential methods replace the delicate estimates used in some real dynamic systems.

Consider a dynamic system $g : (X, \mathcal{F}, \nu) \to (X, \mathcal{F}, \nu)$ as above, where ν is an invariant probability measure. Therefore, g^* defines a linear operator of norm 1 on $L^2(\nu)$. We say that g has *the Jacobian limited* if there is a constant $\kappa > 0$ such that $\nu(g(A)) \leq \kappa\nu(A)$ for each $A \in \mathcal{F}$.

When X is a complex mamifold, it is necessarily orientable .

Let $\mathcal{V}$ be a smooth complex compact homogeneous manifold with $dim_{\mathbb{C}}(\mathcal{V}) = k \geq 1$ and Kodaira dimension ≤ 0 and $f : \mathcal{V} \longrightarrow \mathcal{V}$ be a dominant surjective meromorphic endomorphism, that is, whose Jacobian is not identically null in any local chart. Let ω be a $(1,1)$-strictly positive Hermitian form on $\mathcal{V}$. Let ℓ be a prime number.

Definition 1.3. The *i-th **Cohomological Degree*** $\chi_i(f)$ of f is defined as the spectral radius of the pullback action f^* in the cohomology group ℓ -adic étale $H^i_{\text{ét}}(\mathcal{V}, \mathbf{Q}_\ell)$ independent of ℓ by: (cf [3] [6] [1] [4] [5] for more details)

$$\chi_i(f) = \rho\big(f^*|_{H^i_{\text{ét}}(\mathcal{V}, \mathbf{Q}_\ell)}\big).$$

Definition 1.4. We define the *(l, n)-th **Cohomological Quotient*** $\xi^n_l(f)$ of f as follows:

$$\xi^n_l(f) = \left[\frac{\chi_{2l-1}(f)}{\chi_{2l}(f)}\right]^n$$

Definition 1.5. The **Cohomological Entropy** of f is defined by

$$h_\chi(f) = \max_i \log \chi_{2i}(f).$$

Definition 1.6. We say that f is a **Cohomological Expanding Mapping** when f is dynamically compatible (that is $(f^n)^* = (f^*)^n$) and there is $l \in \{1, ..., k\}$ such that :

$$\xi^{-1}_l(f) > 1.$$

We will write χ_i for $\chi_i(f)$ and ξ^n_l for $\xi^n_l(f)$ if there is no confusion.

Let $(M, \mathcal{F}, m)$ be a probability space and $g : M \to M$ be a measurable map that preserves m, that is, m is g_* - invariant: $g_* m = m$. The measure m is ***ergodic*** if for any measurable set A such that $g^{-1}(A) = A$, we have $m(A) = 0$ or $m(A) = 1$. This is equivalent to the property that m is extremal on the convex set of invariant probability measures (if m is mixing, so it is ergodic). When m is ergodic, Birkhoff's theorem implies that if ψ is an observable on $L^1(m)$ then

$$\lim_{n\to\infty} \frac{1}{n}\Big[\psi(x) + \psi(g(x)) + \cdots + \psi(g^{n-1}(x))\Big] = \langle m, \psi \rangle$$

for m - almost all x.

Suppose now that $\langle m, \psi\rangle = 0$. Then, the previous limit is equal to 0. The theorem of limit central (TLC), when it occurs, provides the speed of this convergence. We say that ψ *satisfies the TLC* if there is a constant $\sigma > 0$ such that

$$\frac{1}{\sqrt{n}}\Big[\psi(x) + \psi(g(x)) + \cdots + \psi(g^{n-1}(x))\Big]$$

converges in distribution for the Gaussian random variable $\mathcal{N}(0, \sigma)$ of mean 0 and variance σ. Remember that ψ is a *coboundary* whether there is a function ψ' on $L^2(\mu)$ such that $\psi = \psi' - \psi' \circ g$. In that case, it is easy to see that

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\Big[\psi(x) + \psi(g(x)) + \cdots + \psi(g^{n-1}(x))\Big] = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\Big[\psi'(x) - \psi'(g^n(x))\Big] = 0$$

in distribution. Therefore, ψ does not satisfy the TLC (sometimes it is said that ψ satisfies the TLC by $\sigma = 0$).

The TLC can be deduced from strong mixing, see [11, 46, 48]. In the following result, $Et(\psi|\mathcal{F}_n)$ indicates *the expectation* of ψ in relation to $\mathcal{F}_n$, that is, $\psi \mapsto Et(\psi|\mathcal{F}_n)$ is the orthogonal projection of $L^2(m)$ in the subspace generated by the measurable functions $\mathcal{F}_n$.

Theorem 1.7 (Gordin)**.** *Consider the decreasing sequence $\mathcal{F}_n := g^{-n}(\mathcal{F})$, $n \geq 0$, of algebras. Let ψ be a function with real value on $L^2(m)$ such that $\langle m, \psi\rangle = 0$. Suppose that*

$$\sum_{n\geq 0}\|Et(\psi|\mathcal{F}_n)\|_{L^2(m)} < \infty.$$

So, the positive number σ defined by

$$\sigma^2 := \langle m, \psi^2\rangle + 2\sum_{n\geq 1}\langle m, \psi(\psi \circ g^n)\rangle$$

is finite. It vanishes if and only if ψ is a coboundary. Furthermore, when $\sigma \neq 0$, then ψ satisfies the TLC with variance σ.

Note that σ is equal to the limit of $n^{-1/2}\|\psi + \cdots + \psi \circ g^{n-1}\|_{L^2(m)}$. The last expression is equal to $\|\psi\|_{L^2(m)}$ if the family $(\psi \circ g^n)_{n\geq 0}$ is orthogonal on $L^2(m)$.

We refer to [47, 49] for the notion of Lyapunov exponent.

Definition 1.8. An invariant positive measure is ***hyperbolic*** if its Lyapunov exponents are non-zero.

A function *quasi-p.s.h.* on $\mathcal{V}$ is a function of $\mathcal{V}$ on $[-\infty, \infty)$, which is locally the sum of a plurisubharmonic function and a smooth function. For a given $(1, 1)$ -continuous form η, denote by $\mathrm{PSH}_0(\eta)$ the set of quasi-p.s.h. functions φ such that $dd^c\varphi + \eta \geq 0$ and $\sup_{\mathcal{V}}\varphi = 0$. Equip $\mathrm{PSH}_0(\eta)$ with induced distance of $L^1(\mathcal{V})$ using natural inclusion $\mathrm{PSH}_0(\eta) \subset L^1(\mathcal{V})$.

Remember from [22] that a complex measure μ on $\mathcal{V}$ is considered *PC* if each quasi-p.s.h. function is μ-integrable and for each sequence $(\varphi_n)_{n\in\mathbb{N}}$ of quasi-p.s.h. functions converging to φ on L^1 , so that $dd^c\varphi_n + \eta \geq 0$ for some smooth form η independent of n, we have $\langle\mu, \varphi_n\rangle \to \langle\mu, \varphi\rangle$.

A *pluripolar set* on $\mathcal{V}$ is a subset of $\mathcal{V}$ contained on $\{\varphi = -\infty\}$ for some quasi-p.s.h. function φ. By [29], any locally pluripolar set on $\mathcal{V}$ is pluripolar. This result implies in particular that there are abundantly quasi-p.s.h singular functions on $\mathcal{V}$. Note that every PC measure has no mass on pluripolar sets.

Next, we will consider the dynamics of f with $\xi_l^{-1}(f) > 1$.

Here is the first Main Result.

Theorem 1.9. *Let $\mathcal{V}$ be a smooth compact complex homogeneous manifold with $\dim_{\mathbb{C}}(\mathcal{V}) = k \geq 1$ and Kodaira dimension ≤ 0 and $f : \mathcal{V} \longrightarrow \mathcal{V}$ a* ***Cohomological Expanding Mapping****. Let ν be a complex measure with density L^{2k+1} on $\mathcal{V}$ such that $\nu(\mathcal{V}) = 1$. Let ω be a $(1, 1)$-strictly positive Hermitian form on $\mathcal{V}$. So the sequence $\frac{1}{\chi_{2l}^m}(f^m)^*\nu$ converges weakly to a measure of probability PC μ_f with* ***Cohomological Entropy*** *$\geq \log\chi_{2l}$ independent of ν as $m \to \infty$ so that $\chi_{2l}^{-1}f^*\mu_f = \mu_f = f_*\mu_f$ and if f is holomorphic, then for each Hermitian metric ω on $\mathcal{V}$, μ_f is Hölder continuous on $\mathrm{PSH}_0(\omega)$.*

The Hölder continuity of μ_f on $\mathrm{PSH}_0(\omega)$ for f holomorphic implies that μ_f is ***moderate*** in the sense that there are constants $\varepsilon, M > 0$ such that for each $\varphi \in \mathrm{PSH}_0(\omega)$, we have

$$\int_{\mathcal{V}} e^{-\varepsilon\varphi}d\mu_f \leq M.$$

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We remember a new class of functions called *weakly d.s.h.* that replace the role of d.s.h functions (differences of two functions quasi-psh) in case of Kähler. These functions enjoy a compactness property similar to that of the d.s.h functions and the pull-backs of d.s.h functions by meromorphic maps are weakly d.s.h. We obtain the property of **exponential mixing** of μ_f

Here is the second Main Result.

Theorem 1.10. *Let $\mathcal{V}, f, \chi_{2l}, \mu_f$ be as in Theorem 1.9. So μ_f is **exponential mixing** in the sense that for each constant $0 < \alpha \leq 1$, there is a constant A_α such that*

$$\left|\langle \mu_f, (\psi \circ f^m)\varphi\rangle - \langle \mu_f, \psi\rangle\langle \mu_f, \varphi\rangle\right| \leq A_\alpha \xi_l^{\frac{m\alpha}{2}} \|\psi\|_\infty \|\varphi\|_{C^\alpha}$$

*for each $m \geq 0$, each $\psi \in L^\infty(\mathcal{V})$ and every function Hölder continuous φ of order α. In particular, μ_f is **K-mixing**.*

*If a real function Hölder continuous φ is not a **coboundary**,* i.e, *there is not $\psi \in L^2(\mathcal{V})$ with $\varphi = \psi \circ f - \psi$, and satisfies $\langle \mu, \varphi\rangle = 0$, then μ_f satisfies the **central limit theorem**, which means that there is a constant $\sigma > 0$ such that for each interval $I \subset \mathbb{R}$, we have*

$$\lim_{p\to\infty} \mu_f\left\{\frac{1}{\sqrt{p}}\sum_{j=0}^{p-1} \varphi \circ f^j \in I\right\} = \frac{1}{\sqrt{2\pi}\sigma}\int_I e^{-x^2/(2\sigma^2)}dx.$$

The expression $\langle \mu_f, (\psi \circ f^m)\varphi\rangle - \langle \mu_f, \psi\rangle\langle \mu_f, \varphi\rangle$ is called ***the Correlation of order*** m between the observables φ and ψ. The measure μ_f is said ***mixing*** if this correlation converges to 0, when m tends to infinity, for smooth observables (or equivalently, observables continuous, limited or $L^2(\mu_f)$).

Remember that $f_*\varphi$ is defined by

$$f_*\varphi(x) := \sum_{y\in f^{-1}(x)} \varphi(y)$$

where the points on $f^{-1}(x)$ are counted with multiplicities (there are exactly χ_{2k} points). Also define the *Perron-Frobenius Operator* by

$$\Lambda\varphi := \chi_{2k}^{-1} f_*\varphi.$$

As μ_f is totally invariant, this is the adjoint operator of f^* on $L^2(\mu_f)$.

2 First Main Result

In this section, we will prove the Theorem 1.9. For a current T of order 0 defined in a manifold $\mathcal{V}$, we denote by $\|T\|_\mathcal{V}$ the mass of T on $\mathcal{V}$. Let's write $\lesssim$ (resp. $\gtrsim$) for $\leq$ (resp. $\geq$) module a multiplicative constant independent of involving terms in inequality.

Theorem 2.1 (Theorem 1.9 " **First Main Result** ")**.** *Let $\mathcal{V}$ be a smooth compact complex homogeneous manifold with $\dim_\mathbb{C}(\mathcal{V}) = k \geq 1$ and Kodaira dimension ≤ 0 and $f : \mathcal{V} \longrightarrow \mathcal{V}$ a **Cohomological Expanding Mapping**. Let ν be a complex measure with density L^{2k+1} on $\mathcal{V}$ such that $\nu(\mathcal{V}) = 1$. Let ω be a $(1,1)$-strictly positive Hermitian form on $\mathcal{V}$. So the sequence $\frac{1}{\chi_{2l}^m}(f^m)^*\nu$ converges weakly to a measure of probability PC μ_f with **Cohomological Entropy** $\geq \log \chi_{2l}$ independent of ν as $m \to \infty$ so that $\chi_{2l}^{-1} f^*\mu_f = \mu_f = f_*\mu_f$ and if f is holomorphic, then for each Hermitian metric ω on $\mathcal{V}$, μ_f is Hölder continuous on* $\mathrm{PSH}_0(\omega)$.

Let $\mathcal{B}_r$ be the ball centered on 0 of radius r of $\mathbb{C}^k$, where $r \in \mathbb{R}^+$. For $r := 1$ we put $\mathcal{B} := \mathcal{B}_1$. The following result is very important.

Lemma 2.2. *(Classical) Let $r \in (0,1)$. So, for each $(1,1)$-closed real current R of order 0 defined on $\mathcal{B}$, there is a function U_R on $L^{1+1/(2k)}(\mathcal{B}_r)$ so that the following three properties are verified:*

(i)

$$R = dd^c U_R$$

on $\mathcal{B}_r$,

(ii)

$$\|U_R\|_{L^{1+1/(2k)}(\mathcal{B}_r)} \leq c_r\|R\|_\mathcal{B}$$

for some constant c_r independent of R, *(iii) if $(R_n)_{n\in\mathbb{N}}$ is a sequence of $(1,1)$-closed real currents of order 0 of uniformly limited mass, converging weakly to R on $\mathcal{B}$ so $U_{R_n} \to U_R$ on $L^{1+1/(2k)}(\mathcal{B}_r)$.*

Proof. The new point is the estimate for the norm $L^{1+1/(2k)}$ of the potential U_R and its continuity on R. These properties will be obtained by carefully examining the steps in the usual construction of U_R, cf [15, p. 135] [5], [3], [8], [7] for example.

Let R be a $(1,1)$-real current closed on $\mathcal{B}$. Let $x \in \mathbb{C}^k$ be the canonical coordinate system. Let ρ be a smooth function supported compactly on $\mathcal{B}$ and $\int_{\mathcal{B}} \rho dx = 1$. For $y \in \mathcal{B}$, let $A_y : \mathcal{B} \to \mathcal{B}$ be the diffeomorphism defined by

$$A_y(x) := x + \frac{1}{2}(1 - \|x\|)y$$

for $x \in \mathcal{B}$. Since A_y is homotopic to $A_0 :=$ id through homotopy $H_y : [0,1] \times \mathcal{B} \to \mathcal{B}$ defined by $H_y(t,x) := A_{ty}(x)$ for $t \in [0,1]$, the average

$$R' := \int_{\mathcal{B}} (A_y^* R)\rho(y)dy$$

is a smooth closed form that is cohomologous to R. Precisely, by the formula of homotopy, we have

$$R - R' = dL_1, \quad \text{where } L_1 = L_1(R) := \int_B (H_y)_*([0,1] \otimes R)\rho(y)dy.$$

Note that

$$\|R'\|_{L^\infty(\mathcal{B})} \lesssim \|R\|_{\mathcal{B}}, \quad \|L_1\|_{\mathcal{B}} \lesssim \|R\|_{\mathcal{B}}. \tag{2.1}$$

Since R' is a smooth closed form on $\mathcal{B}$, we can use an explicit formula (cf [15, p. 13]) to define a smooth form $L_2 = L_2(R')$ on $\mathcal{B}$ such that

$$R' = dL_2, \quad \|L_2\|_{L^\infty(\mathcal{B})} \lesssim \|R'\|_{L^\infty(\mathcal{B})}.$$

This combined with (2.1) shows that for $L_3 := L_1 + L_2$, we have

$$R = dL_3, \quad \|L_3\|_{\mathcal{B}} \lesssim \|R\|_{\mathcal{B}} \tag{2.2}$$

and L_3 continuously depends on R. So if $(R_n)_{n\in\mathbb{N}}$ is a sequence of $(1,1)$-currents of order 0 with uniformly limited mass, converging towards R so $L_3(R_n)$ is also of uniformly limited mass and converges to $L_3(R)$.

Since R is a $(1,1)$-real form, L_3 is a 1-real form. We decompose L_3 in the sum of one $(1,0)$-form and a $(0,1)$-form as

$$L_3 = L_3^{(1,0)} + L_3^{(0,1)} \tag{2.3}$$

such that $L_3^{(1,0)} = \overline{L_3^{(0,1)}}$ and $L_3^{(1,0)}, L_3^{(0,1)}$ are currents of order 0. We deduce from (2.2) that

$$\|L_3^{(0,1)}\|_{\mathcal{B}} \lesssim \|R\|_{\mathcal{B}} \tag{2.4}$$

For a bidirectional reason and the fact that $R = dL_3$, we have $\bar\partial L_3^{(0,1)} = 0$. It is known that there is a distribution v defined in an open neighborhood of $\overline{\mathcal{B}}_r$ with $\bar\partial v = L_3^{(0,1)}$. We will briefly remember how to build such a v as a function of $L_3^{(0,1)}$. The reference is [15, p. 28].

Let ρ be the function as above. We can assume $\rho \equiv 1$ on an open neighborhood of $\overline{\mathcal{B}}_r$. By the Koppelman formula, we have

$$\rho L_3^{(0,1)}(x) = \bar\partial \int_{\mathcal{B}} K_1(x,y) \wedge \rho(y) L_3^{(0,1)}(y) + \int_{\mathcal{B}} K_2(x,y) \wedge \bar\partial\rho(y) \wedge L_3^{(0,1)}(y). \tag{2.5}$$

We do not give explicit formulas here for K_1, K_2 but we emphasize only that K_1, K_2 are the products of $\|x-y\|^{-2k+1}$ with smooth forms on $\mathbb{C}^k$.

Denote by I_1, I_2 the first and second integrals, respectively, on the right side of (2.5). We have

$$\bar{\partial} I_1 + I_2 = \rho L_3^{(0,1)}$$

which is equal to $L_3^{(0,1)}$ on $\mathcal{B}_r$.

By the type of singularity of K_1 and the fact that $L_3^{(0,1)}$ is of order 0, we see that I_1 is a form with coefficients in $L^{1+1/(2k)}(\mathcal{B})$ with

$$\|I_1\|_{L^{1+1/(2k)}(\mathcal{B})} \lesssim \|L_3^{(0,1)}\|_{\mathcal{B}} \lesssim \|R\|_{\mathcal{B}} \tag{2.6}$$

by (2.4). On the other hand, as $\bar{\partial}\rho \equiv 0$ on an open neighborhood of $\overline{\mathcal{B}}_r$, the current I_2 is smooth on $\mathcal{B}_{r'}$ for some $r' > r$. Following exactly the arguments in [15, p. 29], we get a smooth function I_3 on $\mathcal{B}_{r'}$ for some $r' > r$ such that $I_2 = \bar{\partial} I_3$ on $\mathcal{B}_r$ and

$$\|I_3\|_{L^\infty(\mathcal{B}_r)} \le \|L_3^{(0,1)}\|_{\mathcal{B}} \lesssim \|R\|_{\mathcal{B}} \tag{2.7}$$

by (2.4) and $I_3 : R \mapsto I_3(R) \in L^\infty(\mathcal{B}_r)$ is continuous. So if $v := (I_1 + I_3)$ then

$$L_3^{(0,1)} = \bar{\partial} v$$

on $\mathcal{B}_r$. This together with (2.3) gives

$$L_3 = \bar{\partial} v + \partial \bar{v}.$$

We deduce from this and (2.2) that

$$R = dL_3 = \partial\bar{\partial}(v - \bar{v}).$$

Consequently $U_R := 2\pi \operatorname{Im} v$ satisfies $R = dd^c U_R$ (remember that $dd^c = (i/\pi)\partial\bar{\partial}$) and

$$\|U_R\|_{L^{1+1/(2k)}(\mathcal{B}_r)} \lesssim \|I_1\|_{L^{1+1/(2k)}(\mathcal{B}_r)} + \|I_3\|_{L^{1+1/(2k)}(\mathcal{B}_r)} \lesssim \|R\|_{\mathcal{B}}$$

by (2.6) and (2.7).

It remains to prove the property of continuity of U_R. We saw that I_3, L_3 are continuous on R. We just need to check this property to I_1. Let (R_n) be the sequence as defined above. Let's show that $I_1(R_n) \to I_1(R)$ on $L^{1+1/(2k)}(\mathcal{B})$. For the continuity property above of L_3, we have that $S_n := \rho L_3^{(0,1)}(R_n)$ is of uniformly limited mass and converges to $S := \rho L_3^{(0,1)}(R)$ when $n \to \infty$. Write

$$K_1(x,y) = \|x-y\|^{-2k+1} K_1'(x,y),$$

where $K_1'(x,y)$ is a smooth form. For every small constant $\varepsilon > 0$, let

$$K_{1,\varepsilon}(x,y) := \max\{\|x-y\|, \varepsilon\}^{-2k+1} K_1'(x,y)$$

which is a continuous form. Since $\varepsilon \to 0$, we have $K_{1,\varepsilon}(\cdot,y) \to K_1(\cdot,y)$ on $L^{1+1/(2k)}(\mathcal{B})$ uniformly on $y \in \mathcal{B}$. So when $n \to \infty$,

$$\int_{\{y\in\mathcal{B}\}} \big(K_{1,\varepsilon}(x,y) - K_1(x,y)\big) \wedge \big(S_n(y) - S(y)\big) \to 0$$

on $L^{1+1/(2k)}(\mathcal{B})$ because the mass of S_n is uniformly limited. On the other hand,

$$\int_{\{y\in\mathcal{B}\}} K_{1,\varepsilon}(x,y) \wedge \big(S_n(y) - S(y)\big)$$

converges uniformly to 0 as ε is fixed because $K_{1,\varepsilon}$ is continuous. We deduce that $I_1(R_n) \to I_1(R)$ on $L^{1+1/(2k)}(\mathcal{B})$. This completes the proof. ■

Definition 2.3. Let $\mathcal{V}$ be a complex manifold. A function of $\mathcal{V}$ to $[-\infty, \infty)$ is said *function quasi-p.s.h.* if it can be written locally as the sum of a plurisubharmonic function (p.s.h.) and other smooth. For each $(1,1)$-continuous form η, a function quasi-p.s.h. φ is *η-p.s.h.* if $dd^c\varphi + \eta \ge 0$. Through the partition of the unit , each function quasi-p.s.h. is η-p.s.h. for some smooth form η. For a given form η, denote by $\mathrm{PSH}(\eta)$ the set of functions quasi-p.s.h. φ for which $dd^c\varphi + \eta \ge 0$.

Definition 2.4. A locally integrable function φ on $\mathcal{V}$ is said *weakly d.s.h.* if $dd^c\varphi$ is a current of order 0 on $\mathcal{V}$. Let $\mathcal{W}$ be the complex vector space of all functions weakly d.s.h. on $\mathcal{V}$.

Definition 2.5. Every function quasi-p.s.h is weakly d.s.h.. A subset of $\mathcal{V}$ is a *pluripolar set* if it is contained on $\{\varphi = -\infty\}$ for some function quasi-p.s.h. φ. If $\mathcal{V}$ is compact, each locally pluripolar set is pluripolar by [29]. We use a specific case of this result: each analytic proper subset of a compact manifold $\mathcal{V}$ is pluripolar, cf Lemma 2.11 above.

Now consider that $\mathcal{V}$ is *compact*. Let μ_0 be a smooth probability measure on $\mathcal{V}$. We use this measure to define norms L^p on $\mathcal{V}$. For $\varphi \in \mathcal{W}$, put

$$\|\varphi\|_{\mathcal{W}} := \left| \int_{\mathcal{V}} \varphi d\mu_0 \right| + \|dd^c\varphi\|_{\mathcal{V}}, \tag{2.8}$$

where $\|\cdot\|_{\mathcal{V}}$ is the mass of a current on $\mathcal{V}$. Let's write from now $\|\cdot\|$ instead of $\|\cdot\|_{\mathcal{V}}$ if there is no confusion. The function $\|\cdot\|_{\mathcal{W}}$ is a norm on $\mathcal{W}$ because if $dd^c\varphi = 0$ then φ must be a constant. The norm $\|\cdot\|_{\mathcal{W}}$ is similar to the norm of the space of functions d.s.h. in case of Kähler introduced by Dinh-Sibony [22]. However, we do not know whether these two norms are equivalent in this case.

We introduce *the topology* on $\mathcal{W}$ in the following way: we say that $\varphi_n \in \mathcal{W}$ converges to $\varphi \in \mathcal{W}$ when $n \to \infty$ if $\varphi_n \to \varphi$ as current and $\|\varphi_n\|_{\mathcal{W}}$ is uniformly limited.

We have the following compactness result.

Lemma 2.6. *Let $\mathcal{V}$ be a compact complex manifold . There is a constant c so that for each function weakly d.s.h φ on $\mathcal{V}$ with $\int_{\mathcal{V}} \varphi d\mu_0 = 0$, we have*

$$\|\varphi\|_{L^{1+1/(2k)}(\mathcal{V})} \le c\|dd^c\varphi\|_{\mathcal{V}}. \tag{2.9}$$

Furthermore, given a positive constant A, the set $\mathcal{W}_0$ of functions weakly quasi-p.s.h. φ with $\int_{\mathcal{V}} \varphi d\mu_0 = 0$ such that $\|dd^c\varphi\| \le A$ is compact on $L^{1+1/(2k)}(\mathcal{V})$.

A direct consequence of Lemma 2.6 is that if $\varphi_n \to \varphi$ on $\mathcal{W}$ then $\varphi_n \to \varphi$ on $L^{1+1/(2k)}$. In case of Kähler, a similar version of inequality (2.9) for functions d.s.h. with norm L^p in place of norm $L^{1+1/(2k)}$ and $\|\cdot\|_*$ in place of $\|\cdot\|_{\mathcal{V}}$ was proven on [22] using cohomological tools for functions d.s.h. . His proof uses cohomological arguments that are not applicable to prove (2.9) for weakly functions quasi-p.s.h. .

Proof. Consider a function weakly quasi-p.s.h. φ with $\|dd^c\varphi\| \le A$. Let (W_j) be an open (finite) cover of $\mathcal{V}$ where the W_j are local charts of $\mathcal{V}$ biholomorph to the unit ball of $\mathbb{C}^k$. Since $\|dd^c\varphi\| \le A$, by Lemma 2.2, we have $\tau_j \in L^{1+1/(2k)}(W_j)$ for which $dd^c\tau_j = dd^c\varphi$ on W_j and

$$\|\tau_j\|_{L^{1+1/(2k)}(W_j)} \lesssim A. \tag{2.10}$$

Therefore, $\varphi - \tau_j$ can be represented by a pluriharmonic function on W_j. For simplicity, we identified this function with $(\varphi - \tau_j)$. We deduce that $\varphi \in L^{1+1/(2k)}(\mathcal{V})$.

We now assume, on the contrary, that (2.9) is not valid, it means that there is a sequence of non-null functions weakly quasi-p.s.h. φ_n with $\int_{\mathcal{V}} \varphi_n d\mu_0 = 0$ and

$$\infty > \|\varphi_n\|_{L^{1+1/(2k)}(\mathcal{V})} \ge n\|dd^c\varphi_n\|_{\mathcal{V}}.$$

Multiplying φ_n by a positive constant, we can assume that

$$\|\varphi_n\|_{L^{1+1/(2k)}(\mathcal{V})} = 1. \tag{2.11}$$

So we have

$$\|dd^c\varphi_n\| \le 1/n. \tag{2.12}$$

Note that we still have $\int_{\mathcal{V}} \varphi_n d\mu_0 = 0$. Let τ_j^n be the function τ_j for φ_n in place of φ. Put $T_n := dd^c\varphi_n$. These currents of order 0 are of uniformly limited mass and converge to 0 by (2.12). The Lemma 2.2 tells us that τ_j^n converges to 0 on $L^{1+1/(2k)}(W_j')$, for each $W_j' \Subset W_j$. We can also provide that (W_j') continue to be a cover of $\mathcal{V}$. For simplicity, we can assume that $W_j' = W_j$ for each j.

Now remember that $\varphi_n - \tau_j^n$ is pluriharmonic on W_j. The last function is of $L^{1+1/(2k)}$-norm limited on W_j because of (2.10) and (2.11). The average equality for pluriharmonic functions implies that $(\varphi_n - \tau_j^n)$ is of $\mathcal{C}^l$-norm uniformly limited on compact subsets of W_j on $n \in \mathbb{N}$ for each $l \in \mathbb{N}$. We deduce that, extracting a subsequence, we can assume that $\varphi_n - \tau_j^n$ converging uniformly to a pluriharmonic function τ_j^∞ on compact subsets of W_j when $n \to \infty$. Since $\|\tau_j^n\|_{L^{1+1/(2k)}(W_j)} \to 0$, we get that

$$\varphi_n \to \tau_j^\infty \quad \text{em } L^{1+1/(2k)}(W_j).$$

This produces this function $\tau^\infty := \tau_j^\infty$ on W_j for each j is a well-defined pluriharmonic function on $\mathcal{V}$. Since $\mathcal{V}$ is compact, τ^∞ is a constant. This combined with $\int_{\mathcal{V}} \varphi_n d\mu_0 = 0$ gives $\tau^\infty = 0$. We proved that $\varphi_n \to 0$ on $L^{1+1/(2k)}(\mathcal{V})$, consequently $\|\varphi_n\|_{L^{1+1/(2k)}} \to 0$, a contradiction. Therefore, (2.9) is verified.

To prove the second desired statement, we again use the function τ_j above. We have that $\varphi - \tau_j$ is pluri-harmonic on W_j and by (2.9), the $L^{1+1/(2k)}$-norm of φ is also $\lesssim A$. Then the $L^{1+1/(2k)}$-norm of the pluri-harmonic function $(\varphi - \tau_j)$ is $\lesssim A$. It follows that its $\mathcal{C}^l$-norm is also $\lesssim A$. Therefore, we can extract a convergent subsequence of $(\varphi - \tau_j)$ for $\varphi \in \mathcal{W}$ on $\mathcal{C}^l$. This combined with the $L^{1+1/(2k)}$ continuity of τ_j on T implies the desired statement. This completes the proof.

■

We equip the vector space $\mathcal{B}$ of Borel's measurable functions on $\mathcal{V}$ with the pointwise convergence topology: $h_n \to h$ if h_n converges pointwise to h at almost all points (with respect to the Lebesgue measure). Let P be a continuous linear endomorphism of the last vector space. Define $\mathcal{W}_P$ to be the set of $\varphi \in \mathcal{W}$ for which $P\varphi \in \mathcal{W}$.

Lemma 2.7. *There is a constant c such that*

$$\|P\varphi\|_{L^{1+1/(2k)}} \leq c\big(\|\varphi\|_{\mathcal{W}} + \|dd^c(P\varphi)\|\big), \tag{2.13}$$

for any $\varphi \in \mathcal{W}_P$. In particular, there is a constant c' such that

$$\|P\varphi\|_{L^{1+1/(2k)}} \leq c\big(\|dd^c\varphi\| + \|dd^c(P\varphi)\|\big) \tag{2.14}$$

for each $\varphi \in \mathcal{W}_P \cap \mathcal{W}_0$. Furthermore, if $\varphi_n \in \mathcal{W}_P \cap \mathcal{W}_0 \to \varphi$ as currents when $n \to \infty$ such that $\big(\|dd^c\varphi_n\| + \|dd^c(P\varphi_n)\|\big)$ are uniformly bounded, then $P\varphi_n \to P\varphi$ on $L^{1+1/(2k)}$.

Proof. The Inequality (2.14) is a direct consequence of (2.13) and of Lemma 2.6. Now suppose there is a sequence $(\varphi_n) \subset \mathcal{W}_P$ for which

$$\|P\varphi_n\|_{L^{1+1/(2k)}} = 1, \quad \|\varphi\|_{\mathcal{W}} + \|dd^c(P\varphi_n)\| \leq 1/n. \tag{2.15}$$

Applying compactness property in Lema 2.6 for the sequence $(P\varphi_n)_{n\in\mathbb{N}}$, we see that by extracting a sub-sequence from φ_n if necessary, the sequence $P\varphi_n$ converges on $L^{1+1/(2k)}$ for a function weakly d.s.h φ'_∞. Consequently,

$$\|\varphi'_\infty\|_{L^{1+1/(2k)}} = 1, \quad \|dd^c\varphi'_\infty\| = 0. \tag{2.16}$$

Therefore φ'_∞ is a constant. As the convergence on L^1 implies the convergence almost always of a subse-quence, we can also assume that $P\varphi_n$ converges almost always to φ'_∞.

On the other hand, the inequality of (2.15) allows us to use the compactness property in the Lemma 2.6 again for (φ_n). Therefore, we can extract a subsequence of (φ_n) converging to $\varphi_\infty := 0$ on $L^{1+1/(2k)}$ and almost always. Thus $P\varphi_n$ converges almost always to $P\varphi_\infty$ because of the continuity of P. It follows that $\varphi'_\infty = P\varphi_\infty = 0$, note here $P(0) = 0$ by the linearity of P. This is a contradiction because of (2.16). Thus (2.13) follows. The last desired statement follows directly from the arguments above. This completes the proof.

■

Let $a \in \mathbb{C}^*$, r be a constant on $(0, |a|)$ and $\delta > 0$ a constant. Assume that $P(1) = a$, where 1 is the constant function equal to 1 on $\mathcal{V}$. Define $\mathcal{W}^\infty_{P,r,\delta}$ to be the set of all $\varphi \in \mathcal{B}$ such that $P^n\varphi \in \mathcal{W}$ for each $n \geq 0$ and

$$\|dd^c(P^n\varphi)\| \leq \delta r^n$$

for each $n \geq 0$, here P^0 denotes the identity map. By the linearity of P, every constant function belongs to $\mathcal{W}^\infty_{P,r,\delta}$. We equip $\mathcal{W}^\infty_{P,r,\delta}$ with the topology induced from there on $\mathcal{W}$. Note that $\mathcal{W}^\infty_{P,r,\delta}$ is closed on $\mathcal{W}$ and

$$r^{-m}P^m(\mathcal{W}^\infty_{P,r,\delta}) \subset \mathcal{W}^\infty_{P,r,\delta}$$

for every positive integer m. So $\mathcal{W}^\infty_{P,r,\delta} \cap \mathcal{W}_0$ is compact and $P^m(\mathcal{W}^\infty_{P,r,\delta})$ is contained in the complex vector subspace $\widetilde{\mathcal{W}}^\infty_{P,r,\delta}$ of $\mathcal{W}$ generated by $\mathcal{W}^\infty_{P,r,\delta}$.

Proposition 2.8. *There is a continuous linear functional function $\mu_P : \widetilde{\mathcal{W}}^\infty_{P,r,\delta} \to \mathbb{C}$ such that for each complex measure ν with density L^{2k+1} on $\mathcal{V}$, $\nu(\mathcal{V}) = 1$ and for each $\varphi \in \widetilde{\mathcal{W}}_{P,r,\delta}$, we have*

$$\langle a^{-n}(P^n)_*\nu, \varphi\rangle \to \langle \mu_P, \varphi\rangle. \tag{2.17}$$

Here for $Q : \mathcal{B} \to \mathcal{B}$, by definition, $\langle Q_*\nu, \varphi\rangle := \langle \nu, Q\varphi\rangle$ for $\varphi \in \mathcal{B}$ such that $Q\varphi$ is ν-integrable.

Proof. Remember that μ_0 is a form of smooth probability volume on $\mathcal{V}$. We just need to construct μ_P on $\mathcal{W}^{\infty}_{P,r,\delta}$ and prove (2.17) for $\varphi \in \mathcal{W}^{\infty}_{P,r,\delta}$. The extension of μ_P to $\widetilde{\mathcal{W}}^{\infty}_{P,r,\delta}$ is done automatically using the linearity of $(P^n)_*\nu$ and (2.17).

Let $\varphi \in \mathcal{W}^{\infty}_{P,r,\delta}$. Put $b_0 := \int_X \varphi d\mu_0$ and $\varphi_0 := \varphi - b_0$. We define two sequences φ_n, b_n as follows. Put

$$b_n = b_n(\varphi) := \int_X (P\varphi_{n-1})d\mu_0, \quad \varphi_n := P\varphi_{n-1} - b_n$$

for $n \geq 1$. We have $r^{-n}\varphi_n \in \mathcal{W}_0 \cap \mathcal{W}^{\infty}_{P,r,\delta}$ and $dd^c(P^m\varphi_n) = dd^c(P^{m+n}\varphi)$ for each n, m. By Lemma 2.7, we have

$$\|\varphi_n\|_{L^{1+1/(2k)}} \leq c\big((\|dd^c(P\varphi_{n-1})\| + \|dd^c\varphi_{n-1}\|\big), \quad |b_n| \leq c\big(\|dd^c(P\varphi_{n-1})\| + \|dd^c\varphi_{n-1}\|\big) \quad (2.18)$$

for some constant c independent of n, φ. It follows that

$$\|\varphi_n\|_{L^{1+1/(2k)}} \leq c\big(\|dd^c(P^n\varphi)\| + \|dd^c(P^{n-1}\varphi)\|\big) \leq c\delta(r+1)r^{n-1}, \quad |b_n| \leq c\delta(r+1)r^{n-1} \quad (2.19)$$

for $n \geq 1$. Since $P(1) = a$ we have $P(b_n) = ab_n$ for each n. Using this, it gives

$$a^{-n}P^n\varphi = b_0 + a^{-n}P^n\varphi_0 = b_0 + a^{-n}P^{n-1}(P\varphi_0) = b_0 + a^{-1}b_1 + a^{-n}P^{n-1}\varphi_1 \quad (2.20)$$

$$= \cdots = b_0 + a^{-1}b_1 + \cdots a^{-n}b_n + a^{-n}\varphi_n. \quad (2.21)$$

Put $b'_n = b'_n(\varphi) := b_0 + a^{-1}b_1 + \cdots a^{-n}b_n$ that converges to a number b'_∞ (depending on φ) by (2.19) and the fact that $|a| > r$. We deduce from (2.20) that

$$|a^{-n}P^n\varphi - b'_n| \leq |a|^{-n}|\varphi_n|.$$

This combined with the first inequality of (2.19) implies that $a^{-n}P\varphi$ converges to b'_∞ on $L^{1+1/(2k)}$. Precisely, we have

$$\|a^{-n}P^n\varphi - b'_n\|_{L^{1+1/(2k)}} \lesssim \delta|a|^{-n}r^n. \quad (2.22)$$

Since $\nu(X) = 1$, we have

$$\langle a^{-n}(P^n)_*\nu, \varphi\rangle - b'_n = \langle \nu, a^{-n}P^n\varphi - b'_n\rangle.$$

Using this, (2.22) and Hölder's inequality imply that $\langle a^{-n}(P^n)_*\nu, \varphi\rangle$ converges to $b'_\infty = b'_\infty(\varphi)$ because ν has L^{2k+1} density. Define $\langle \mu_P, \varphi\rangle := b'_\infty(\varphi)$ that is independent of ν. Then, we obtain the desired convergence for μ_P.

Consider a sequence $\widetilde{\varphi}_m \to \varphi$ on $\mathcal{W}^{\infty}_{P,r,\delta}$. Let $\widetilde{b}_{nm}, \widetilde{\varphi}_{nm}$ respectively the b_n and φ_n for $\widetilde{\varphi}_m$ in place of φ. By the last statement of the Lemma 2.7, $\widetilde{b}_{nm} \to b_n$ when $m \to \infty$ for each n and (2.19) still applies to $\widetilde{b}_{nm}, \widetilde{\varphi}_{nm}$ in place of b_n, φ_n. We infer that $\widetilde{b}'_{nm} \to b'_n$ and $a^{-n}\widetilde{\varphi}_{nm} \to 0$ on $L^{1+1/(2k)}$ when $m \to \infty$. Thus, $\langle \mu_P, \widetilde{\varphi}_m\rangle \to \langle \mu_P, \varphi\rangle$ when $m \to \infty$. In other words, μ_P is continuous. This completes the proof.

■

Let $\mathcal{V}$ be a complex compact manifold and f be a meromorphic self-map on $\mathcal{V}$. Denote by Γ the graph of f on $\mathcal{V} \times \mathcal{V}$ and π_1, π_2 the restrictions to Γ of natural projections of $\mathcal{V} \times \mathcal{V}$ for the first and second components respectively.

Let Φ be a form with measurable coefficients on $\mathcal{V}$. We say that $\Phi \in L^1$ if its coefficients are L^1 functions (in relation to the Lebesgue measure on $\mathcal{V}$). If Ω is a dense open subset of Zariski of $\mathcal{V}$ such that π_2 is a unrestricted cover on Ω, the form $f_*\Phi := (\pi_2|_{\pi_2^{-1}(\Omega)})_*(\pi_1^*\Phi)$ is a measurable form on Ω. Consequently $f_*\Phi$ is a measurable form on $\mathcal{V}$ independent of Ω. We can verify that $f_* : \mathcal{B} \to \mathcal{B}$ is continuous. Consequently, f_* is an example of the map P considered above.

If $f_*\Phi \in L^1$, then we can define $f_*\Phi$ to be a current of order 0 induced by $f_*\Phi$ on $\mathcal{V}$. This definition is independent of the choice of Ω. Note that the pull-back by f of smooth functions or smooth forms is always on L^1. The following is similar to the results on [9, 23].

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Lemma 2.9. *For each quasi-p.s.h. function φ on $\mathcal{V}$, we have $f_*\varphi \in L^1$ and if $dd^c\varphi + \eta \geq 0$ for some $(1,1)$-continuous form $\eta > 0$, then $dd^c(f_*\varphi) + f_*\eta \geq 0$. In particular,*

$$(f^n)_*\varphi \in \mathcal{W}_{f_*} \cap \mathcal{W}. \tag{2.23}$$

The inclusion (2.23) explains the crucial roles of $\mathcal{W}_{f_*}, \mathcal{W}$ in this study.

Proof. Let $\sigma : \Gamma' \to \Gamma$ be a desingularization of Γ. Let Ω be as above. Put $\pi'_j := \pi_j \circ \sigma$ for $j = 1, 2$. Since φ is quasi-p.s.h., $\varphi \circ \pi'_1$ is also. Thus, $\varphi \circ \pi_1 = \sigma_*(\varphi \circ \pi'_1)$ is on $L^1(\Gamma_f)$. Since

$$\|f_*\varphi\|_{L^1(\Omega)} = \|(\pi_2)_*(\varphi \circ \pi_1)\|_{L^1(\Omega)} \lesssim \|\varphi \circ \pi_1\|_{L^1(\Gamma)},$$

we get the first desired statement.

By [2], [4] and the fact that $\eta > 0$, there is a decreasing sequence of smooth functions quasi-p.s.h φ_n converging pointwise to φ such that $dd^c\varphi_n + \eta \geq 0$ for each n. By Lebesgue's dominated convergence theorem, the sequence $\varphi_n \circ \pi'_1$ converges on L^1 to $\varphi \circ \pi'_1$. It follows that the sequence of positive smooth forms $dd^c(\varphi_n \circ \pi'_1) + \pi'^*_1\eta$ converges weakly to $dd^c(\varphi \circ \pi'_1) + \pi'^*_1\eta$. Thus, the last current is also positive. Now note that

$$(\pi'_2)_*(dd^c(\varphi \circ \pi'_1) + \pi'^*_1\eta) = dd^c\big((\pi'_2)_*\pi'^*_1\varphi\big) + (\pi'_2)_*\pi'^*_1\eta = dd^c\big((\pi_2)_*\pi^*_1\varphi\big) + (\pi_2)_*\pi^*_1\eta$$

because $\pi^*_1\varphi$ and $\pi^*_1\eta$ have no mass in zero Lebesgue measure sets. Therefore $dd^c(f_*\varphi) + f_*\eta \geq 0$.

Note that $f_*\eta$ has finite mass on $\mathcal{V}$. We infer that $f_*\varphi \in \mathcal{W}$. In other words, $\varphi \in \mathcal{W}_{f_*} \cap \mathcal{W}$. Applying this to f^n instead of f and using the formula that $(f^n)_*\varphi = f_*(f^{n-1})_*\varphi$ as functions in some suitable open dense subset of $\mathcal{V}$, we get (2.23). This completes the proof.

■

Lemma 2.10. *Let $\mathcal{V}$ be a compact complex manifold of dimension k and $f : \mathcal{V} \longrightarrow \mathcal{V}$ be a* ***Cohomological Expanding Mapping****. Let φ be a function quasi-p.s.h. on $\mathcal{V}$ with $dd^c\varphi + \eta \geq 0$ for some $(1,1)$-continuous form η. So there is a constant A independent of φ, η for which*

$$\|dd^c(f^n)_*\varphi\| \leq A\chi^n_{2l-1}\|\eta\|_{L^\infty} \tag{2.24}$$

for each $n \geq 1$.

Proof. Replacing η by a strictly positive smooth form that dominates it, we can assume that $\eta > 0$. Let ω be a metric of Gauduchon on $\mathcal{V}$, this means that ω is a Hermitian metric and $dd^c\omega^{k-1} = 0$, cf [**?**]. Let Γ_n be the graph of f^n and $\pi_{1,n}, \pi_{2,n}$ the natural maps of Γ_n for the first and second components of $\mathcal{V} \times \mathcal{V}$. By Lemma 2.9, the current $dd^c(f^n)_*\varphi + (f^n)_*\eta$ is positive. So, using $dd^c\omega^{k-1} = 0$ gives

$$\|dd^c(f^n)_*\varphi + (f^n)_*\eta\| \lesssim \langle dd^c(f^n)_*\varphi + (f^n)_*\eta, \omega^{k-1}\rangle = \langle (f^n)_*\eta, \omega^{k-1}\rangle \lesssim \langle (f^n)_*\omega, \omega^{k-1}\rangle$$

This combined with the definition of $\chi_{2l-1}(f)$ gives

$$\|dd^c(f^n)_*\varphi + (f^n)_*\eta\| \leq A(\chi_{2l-1})^n\|\eta\|_{L^\infty}.$$

The desired inequality follows immediately. This completes the proof.

■

We come now to the end of the proof of the first main result.

***End of Proof of Theorem** 1.9.* $\xi_l^{-1}(f) > 1$. Put

$$P := f_*, \quad a := \chi_{2l}, \quad r := \chi_{2l-1}, \quad \delta := A,$$

where A is the constant on Lemma 2.10. Let φ be a function quasi-p.s.h. whith $dd^c\varphi + \eta \geq 0$ for some $(1,1)$-continuous form $\eta > 0$ such that $\|\eta\|_{L^\infty} \leq 1$. We have $P(1) = a$ and $\varphi \in \mathcal{W}^\infty_{P,r,\delta}$ by Lemma 2.10. Every function quasi-p.s.h. is on $\widetilde{\mathcal{W}}^\infty_{P,r,\delta}$. Since ν does not have mass in proper analytical subsets of $\mathcal{V}$, Note that

$$\langle (f^m)^*\nu, \varphi\rangle = \langle \nu, (f^m)_*\varphi\rangle = \langle \nu, P^m\varphi\rangle \tag{2.25}$$

because we only need to consider integrals on a dense open subset of Zariski of $\mathcal{V}$. Applying Proposition 2.8 for P, we get a continuous functional μ_P on $\widetilde{\mathcal{W}}^{\infty}_{P,r,\delta}$ such that

$$\langle \chi_{2l}^{-m}(f^m)^*\nu, \varphi\rangle \to \langle \mu_P, \varphi\rangle,$$

for each $\varphi \in \widetilde{\mathcal{W}}^{\infty}_{P,r,\delta}$. Choosing $\nu \geq 0$, we see that $\langle \mu_P, \varphi\rangle \geq 0$ if $\varphi \geq 0$. Let μ_f be the probability measure on $\mathcal{V}$ defined by $\langle \mu_f, \varphi\rangle := \langle \mu_P, \varphi\rangle$ for each smooth function φ. Remember here that smooth functions are quasi-p.s.h. on $\mathcal{V}$. Let's prove that $\mu_f = \mu_P$ for each function quasi-p.s.h. φ.

Consider a sequence of smooth functions quasi-p.s.h φ'_m with $dd^c\varphi'_m + \eta \geq 0$ decreasing to φ, we have $\langle \mu_f, \varphi'_m\rangle = \langle \mu_P, \varphi'_m\rangle$ and $\langle \mu_f, \varphi'_m\rangle \to \langle \mu_f, \varphi\rangle$ by Lebesgue's monotonous convergence theorem. This combined with the continuity of μ_P gives $\langle \mu_f, \varphi\rangle = \langle \mu_P, \varphi\rangle$. So we have

$$\lim_{m\to\infty} \langle \chi_{2l}^{-m}(f^m)^*\nu - \mu_f, \varphi\rangle = 0 \tag{2.26}$$

for each function quasi-p.s.h. φ on $\mathcal{V}$.

As the functions quasi-p.s.h are μ_f- integrable, μ_f has no mass on pluripolar sets. By Lemma 2.11 below, proper analytic subsets of $\mathcal{V}$ are pluripolar. This implies that μ_f has no mass on proper analytic subsets of $\mathcal{V}$. We deduce that the pull-back $f^*\mu_f$ is well defined. Here we just take the pull-back of μ_f on an open subset of Zariski Ω of $\mathcal{V}$ where π_2 is a non-branched cover. It can be seen that this definition is independent of the choice of Ω and if $(\Phi_m)_{m\in\mathbb{N}}$ is a sequence of positive measures without mass on the proper analytical subsets of $\mathcal{V}$ and converging to μ_f, then $f^*\Phi_m$ converges to $f^*\mu_f$ because the mass of $f^*\Phi_m$ converges to that of $f^*\mu_f$, cf for example [28, Lema 3.6]. The Equality

$$\chi_{2l}^{-1} f^*\mu_f = \mu_f \tag{2.27}$$

is obtained by applying the pull-back f^* for convergence $\chi_{2l}^{-m}(f^m)^*\nu \to \mu_f$, where ν is a smooth measure of probability. Once we have $f_*f^* = \chi_{2l}$ on Borel's measurable functions, we get $f_*\mu_f = \mu_f$, in other words, μ_f is invariant by f.

Let I_f be the indeterminacy set of f. Put $Z := \cup_{m\in\mathbb{Z}} f^m(I_f)$. The measure μ_f has no mass on Z. The cohomological entropy of μ_f is by definition $\mathbf{1}_{\mathcal{V}\setminus Z}\mu_f$ in relation to $f|_{\mathcal{V}\setminus Z}$. For Parry's inequality [24, 27], using $f^*\mu_f = \chi_{2l}\mu_f$, we deduce that the cohomological entropy of μ_f is at least $\log \chi_{2l}$.

Suppose now that f is holomorphic. To prove that μ_f is Hölder continuous on $\mathrm{PSH}(\omega)$, we use a known idea of [24]. Without loss of generality, we can assume that $\|\omega\|_{L^\infty} \leq 1$. Let φ, ψ be two functions quasi-p.s.h. on $\mathrm{PSH}(\omega)$. Remember that they are on $\mathcal{W}^{\infty}_{P,r,\delta}$.

Let $b_n(\varphi), b_n(\psi)$ be as in the proof of the proposition 2.8. Let J_f be the Jacobian of f. We have

$$\|f_*\varphi - f_*\psi\|_{L^1} = \sup_{\|h\|_{L^\infty}\leq 1} |\langle f_*\varphi - f_*\psi, h\mu_0\rangle| = \sup_{\|h\|_{L^\infty}\leq 1} |\langle \varphi - \psi, (h\circ f)f^*\mu_0\rangle|$$

what is

$$\leq \|J_f\|_{L^\infty}\|\varphi - \psi\|_{L^1}.$$

Applying the latest inequality to f^n in place of f gives

$$|b_n(\varphi) - b_n(\psi)| \leq 2^n \|J_f\|^n_{L^\infty}\|\varphi - \psi\|_{L^1}.$$

Put

$$A_1 := \sum_{n=0}^{M+1} \chi_{2l}^{-n}[b_n(\varphi) - b_n(\psi)], \quad A_2 := \sum_{n=M+1}^{\infty} \chi_{2l}^{-n}[b_n(\varphi) - b_n(\psi)].$$

Using (2.20) gives

$$\langle \mu_f, \varphi - \psi\rangle = A_1 + A_2, \quad |A_1| \leq \sum_{n=0}^{M} \chi_{2l}^{-n} 2^n \|J_f\|^n_{L^\infty}\|\varphi - \psi\|_{L^1}, \quad |A_2| \lesssim (\chi_{2l-1})^M \chi_{2l}^{-M}.$$

Consider the case where $2\|J_f\|_{L^\infty} \leq \chi_{2l}$. We have $|A_1| \leq M\|\varphi - \psi\|_{L^1}$. Choosing M to be the smallest integer for which $M \geq -\log\|\varphi - \psi\|_{L^1}/\log\tau$, where $\tau := \chi_{2l}/(\chi_{2l-1})$, we get that

$$|\langle \mu_f, \varphi - \psi\rangle| \leq |A_1| + |A_2| \lesssim \|\varphi - \psi\|^{1-\varepsilon}_{L^1}$$

which implies that μ_f is Hölder continuous in that case. It remains to treat the case $2\|J_f\|_{L^\infty} \geq \chi_{2l}$. We have

$$|A_1| \leq M 2^M \chi_{2l}^{-M}\|J_f\|^M_{L^\infty}\|\varphi - \psi\|_{L^1} + \tau^{-M}.$$

Choose $M := -\log\|\varphi - \psi\|_{L^1}/\log(2\chi_{2l}^{-1}\tau\|J_f\|_{L^\infty})$. We see that

$$|A_1| + |A_2| \lesssim -\log\|\varphi - \psi\|_{L^1}\|\varphi - \psi\|_{L^1}^{\log\tau/\log(2\chi_{2l}^{-1}\tau\|J_f\|_{L^\infty})}.$$

Consequently, μ_f is also Hölder continuous in this case. This completes the proof. ■

Now we would like to say something about Theorem 1.10. If we try to imitate the arguments in the proof of [22, Teorema. 1.3] to prove Theorem 1.10, we are led to estimate $|\langle \mu_f, |\varphi_n|\rangle|$. The measure μ_f still satisfies the property that for each ω-p.s.h. function φ with $\sup_{\mathcal{V}} \varphi = 0$ is of $L^1(\mu_f)$-norm uniformly limited, cf [22, Proposition 2.3]. But unlike the case of Kähler, we don't know if φ_n is the difference of two ω-p.s.h functions. So this explains why we cannot directly apply the approach in [22] to obtain a correlation decay for μ_f.

Lemma 2.11. *Any proper analytic subset V of a complex compact manifold $\mathcal{V}$ is a pluripolar set on $\mathcal{V}$.*

Proof. We use here the idea in [22] where the authors prove the same result when $\mathcal{V}$ is Kähler. Suppose now that V is smooth and $\mathrm{codim} V \geq 2$ (otherwise the problem is trivial). Let $\sigma : \widehat{\mathcal{V}} \to \mathcal{V}$ be the explosion of $\mathcal{V}$ along V. Denote by $\widehat{V}$ the exceptional hypersurface.

Let ω be a positive-defined Hermitian form on $\mathcal{V}$. Let $\widehat{\omega}_h$ be a form of Chern of $\mathcal{O}(-\widehat{V})$ whose restriction to each fiber of $\widehat{V} \approx \mathbb{P}(E)$ is strictly positive. Choosing ω if necessary, we can assume that $\widehat{\omega} := \sigma^*\omega + \widehat{\omega}_h > 0$. Since $\sigma_*\widehat{\omega}_h = \sigma_*\widehat{\omega} - \omega$, the closed current $\sigma_*\widehat{\omega}_h$ is quasi positive. Thus, there is a function quasi-p.s.h. φ on $\widehat{\mathcal{V}}$ such that

$$\sigma_*\widehat{\omega}_h = dd^c\varphi + \eta \tag{2.28}$$

for some smooth closed form η. Multiplying $\widehat{\omega}_h$ by a strictly positive constant, we have $\sigma^*\sigma_*\widehat{\omega}_h = \widehat{\omega}_h + [\widehat{V}]$. Thus $|\varphi \circ \sigma(\widehat{x}) - \log \mathrm{dist}(\widehat{x}, \widehat{V})|$ is a limited function on $\widehat{\mathcal{V}}$. As a consequence,

$$|\varphi(x) - \log \mathrm{dist}(x, V)| \lesssim 1 \tag{2.29}$$

on compact subsets of $\mathcal{V}$. Consequently, V is contained in $\{\varphi = -\infty\}$. Thus V is pluripolar in this case.

By the construction above, we can build a Hermitian metric in the explosion $\widehat{\mathcal{V}}$ of $\mathcal{V}$ alongo V as the sum of a pull-back of a Hermitian on $\mathcal{V}$ and an appropriate form of Chern of $\mathcal{O}(-\widehat{V})$. Thus, if $\sigma' : \widehat{\mathcal{V}}' \to \mathcal{V}$ is a composition of explosions along smooth submanifolds, so there's a form $(1,1)$ closed and smooth η' on $\widehat{\mathcal{V}}'$ and a Hermitian metric ω on $\mathcal{V}$ such that $\widehat{\omega}' = \sigma'^*\omega + \eta'$ is a Hermitian metric on $\widehat{\mathcal{V}}'$.

Now consider the general situation where V is an analytical subset of $\mathcal{V}$. As a finite union of pluripolar sets is again pluripolar, it is enough to prove that the regular part $\mathrm{Reg}V$ of V is a pluripolar set because we can write V as a finite union of the regular parts of suitable analytical subsets of $\mathcal{V}$. By Hironaka's desingularization, there is a composition $\sigma' : \widehat{\mathcal{V}}' \to \mathcal{V}$ of explosions along smooth submanifolds that do not cross $\mathrm{Reg}V$ (or their inverse images) so that the strict transformation $\widehat{V}'$ of V is smooth.

Let $\widehat{\omega}', \omega, \eta$ be as above. For the above arguments, $\widehat{V}' \subset \{\widehat{\varphi}' = -\infty\}$ for some function quasi-p.s.h. $\widehat{\varphi}'$ on $\widehat{\mathcal{V}}'$ and $dd^c\widehat{\varphi}' + \widehat{\omega}' \geq 0$. Put $S := \sigma'_*(dd^c\widehat{\varphi}' + \eta')$ which is a $(1,1)$ - current closed on $\mathcal{V}$ and $S + \omega \geq 0$. We can write

$$S = dd^c\varphi_S + \eta_S, \quad \sigma_*\eta' = dd^c\psi + \eta$$

for some smooth closed forms η_S, η. We have

$$dd^c\varphi_S + \eta_S + \omega \geq 0, \quad dd^c\psi + \eta + \omega \geq 0.$$

Thus φ_S, ψ are quasi-p.s.h. functions on $\mathcal{V}$. In addition, we also have

$$\varphi_S = \sigma'_*(\widehat{\varphi}') + \psi + \text{a smooth function}$$

on an open neighborhood of $\mathrm{Reg}V$ in which σ' is biholomorph. Consequently, $\mathrm{Reg}V \subset \{\varphi'_S = -\infty\}$. This completes the proof.

■

3 Second Main Result

In this section, we prove the Theorem 1.10. Our idea is to consider suitable test functions in the Sobolev space $W^{1,2}$. This approach is inspired by [21].

Fix a smooth volume form μ_0 on $\mathcal{V}$ and we use this form to define the norm in space $L^2(\mathcal{V})$. Let $W^{1,2}$ be the function space with real value $\varphi \in L^2(\mathcal{V})$ such that $d\varphi$ has L^2 coefficients. Remember the following inequality of Poincaré-Sobolev: for $\varphi \in W^{1,2}$ with $\int_{\mathcal{V}} \varphi d\mu_0 = 0$, we have

$$\|\varphi\|_{L^2} \leq c\|d\varphi\|_{L^2}, \tag{3.1}$$

for some constant c independent of φ, cf for example [26] or [25]. Note that the term $\|d\varphi\|^2_{L^2}$ is comparable to the mass of the positive current $i\partial\varphi \wedge \bar{\partial}\varphi$. We have the following lemma.

Lemma 3.1. *([21, Pro. 3.1]) Let I be a compact subset of $\mathcal{V}$ $(2k-1)$- Hausdorff's zero dimensional measure . Let φ be a function with real value $L^1_{loc}(\mathcal{V}\backslash I)$. Suppose that the coefficients of $d\varphi$ are in $L^2(\mathcal{V}\backslash I)$. Then $\varphi \in W^{1,2}$ and there is a compact subset M of $\mathcal{V}\backslash I$ and a constant $c > 0$ both independent of φ such that*

$$\|\varphi\|_{L^1(\mathcal{V})} \le c(\|\varphi\|_{L^1(M)} + \|d\varphi\|_{L^1(\mathcal{V})}).$$

Definition 3.2. Let $W^{1,2}_{*,f}$ be the subset of $W^{1,2}$ consisting of φ such that there are $m_1 \in \mathbb{N}$, a $(1,1)$-continuous form η and a function η-p.s.h. ψ satisfying

$$i\partial\varphi \wedge \bar{\partial}\varphi \le dd^c\big((f^{m_1})_*\psi\big) + (f^{m_1})_*\eta \tag{3.2}$$

as currents. *A size representative* of φ is $\mathbf{m} := (m_0, m_1)$, where m_0 is an upper limit of $\|\eta\|_{L^\infty}$.

If $\mathcal{V}$ is Kähler, $W^{1,2}_{*,f}$ coincides with the space $W^{1,2}_*$ considered in [21] that is independent of f. In this context, the space $W^{1,2}_*$ is studied in detail in [10] and used in [16] for the study of correspondences on Riemann surfaces with two equal dynamic degrees. Let $\xi_l^{-1}(f) > 1$. We have the following observation.

Lemma 3.3. *Let $\varphi \in W^{1,2}_{*,f}$ and $\mathbf{m} = (m_0, m_1)$ be a size representative of φ. So we have*

$$\|d\varphi\|_{L^2} \le A m_0^{1/2}(\chi_{2l-1})^{m_1/2}$$

for some constant A independent of φ.

Proof. Let η be as on (3.2). Let ω be a Hermitian metric on $\mathcal{V}$ with $dd^c\omega^{k-1} = 0$. Testing $dd^c\big((f^{m_1})_*\psi\big) + (f^{m_1})_*\eta$ with this form, we see that the norm of $dd^c\big((f^{m_1})_*\psi\big) + (f^{m_1})_*\eta$ is equal to $\int_{\mathcal{V}}(f^{m_1})_*\eta \wedge \omega^{k-1}$ which is limited by $Am_0(\chi_{2l-1})^{m_1}$ for some constant A independent of η, m_0, m_1. The desired inequality then follows. This completes the proof. ∎

Let $\varphi \in W^{1,2}_{*,f}$. Define $\varphi^+ := \max\{\varphi, 0\}$ a $\varphi^- := \max\{-\varphi, 0\}$. Consider a Lipschitz function $\chi : \mathbb{R} \to \mathbb{R}$. We have $\partial(\chi\circ\varphi) = (\chi'\circ\varphi)\partial\varphi$. This can be seen using a sequence of smooth functions, converging to φ on $W^{1,2}$. We deduce that

$$i\partial(\chi\circ\varphi)\wedge\bar{\partial}(\chi\circ\varphi) = (\chi'\circ\varphi)^2 i\partial\varphi\wedge\bar{\partial}\varphi.$$

Consequently, $\chi\circ\varphi \in W^{1,2}_{*,f}$. In particular, let $\chi(t) := |t|$, $\max\{t, 0\}$ or $\max\{-t, 0\}$ for $t \in \mathbb{R}$, we get the following crucial property.

Lemma 3.4. *For each $\varphi \in W^{1,2}_{*,f}$, if $\mathbf{m} = (m_0, m_1)$ is a representative of size of φ, then $\mathbf{m}$ is also a size representative of $|\varphi|$, φ^+ and φ^-.*

We already know that the pushforward of a function quasi-p.s.h. by f is a function weakly d.s.h. The following result, which explains the role of $W^{1,2}_*$ in this study, provides a more accurate description in the case of functions quasi-p.s.h limited.

Lemma 3.5. *Each function quasi-p.s.h limited is on $W^{1,2}_{*,f}$ and f_* preserves $W^{1,2}_{*,f}$. In addition, for each $\varphi \in W^{1,2}_{*,f}$, if $\mathbf{m} = (m_0, m_1)$ is a size representative of φ, then $\mathbf{m}' := (d_k m_0, m_1 + 1)$ is a size representative of $f_*\varphi$ and*

$$\|f_*\varphi\|_{L^2} \le c(\|\varphi\|_{L^1} + \|d(f_*\varphi)\|_{L^2}) \tag{3.3}$$

for some constant c independent of φ.

Proof. Let φ be a function quasi-p.s.h limited and $f : \mathcal{V} \to \mathcal{V}$ a dominant meromorphic map. Using the identity

$$2i\partial\varphi\wedge\bar{\partial}\varphi = i\partial\bar{\partial}\varphi^2 - 2\varphi i\partial\bar{\partial}\varphi$$

we see that there is a $(1,1)$-continuous form η and a function η-p.s.h. ψ for which $i\partial\varphi\wedge\bar{\partial}\varphi \le dd^c\psi + \eta$. Consequently $\varphi \in W^{1,2}_{*,f}$.

Now let φ be an arbitrary element of $W^{1,2}_{*,f}$. Let η and ψ be such that (3.2) holds. Fix a dense open subset of Zariski Ω of $\mathcal{V}$ in which $f_*\varphi, (f^{m_1})_*\psi, (f^{m_1})_*\eta$ are well-defined functions or forms and π_1 is an unbranched cover on $f^{-1}(\Omega)$. We have $f_*\varphi \in L^1_{loc}(\Omega)$ and

$$\|f_*\varphi\|_{L^1(K)} \le c\|\varphi\|_{L^1}, \tag{3.4}$$

for any compact K on Ω and some constant c independent of φ. Note that $\mathcal{V}\backslash\Omega$ is a proper analytical subset of $\mathcal{V}$, Thus , is of Hausdorff $(2k-1)$-dimensional and zero measure. On Ω, by Cauchy-Schwarz inequality, we have

$$i\partial(f_*\varphi)\wedge\bar\partial(f_*\varphi)\& \leq \chi_{2l} f_*(i\partial\varphi\wedge\bar\partial\varphi) \leq \chi_{2l} f_*\big[dd^c\big((f^{m_1})_*\psi\big)+(f^{m_1})_*\eta\big]$$

$$=\chi_{2l}[dd^c\big((f^{m_1+1})_*\psi\big)+(f^{m_1+1})_*\eta].$$

It follows that $d(f_*\varphi)\in L^2(\Omega)$. For this and by Lemma 3.1, we get $f_*\varphi\in W^{1,2}$. Thus, $i\partial(f_*\varphi)\wedge\bar\partial(f_*\varphi)$ has no mass on $\mathcal{V}\backslash\Omega$. It follows that

$$i\partial(f_*\varphi)\wedge\bar\partial(f_*\varphi)\leq\chi_{2l}\mathbf{1}_\Omega[dd^c\big((f^{m_1+1})_*\psi\big)+(f^{m_1+1})_*\eta]\leq\chi_{2l}[dd^c\big((f^{m_1+1})_*\psi\big)+(f^{m_1+1})_*\eta]$$

because the last current is positive by Lemma 2.9. Combining this with (3.1) and (3.4) gives (3.3). The desired statement then follows. This completes the proof. ■

Let $\varphi\in W^{1,2}_{*,f}$ and $\mathbf{m}=(m_0,m_1)$ be a size representative of φ. Consider f_* acting on Borel's measurable functions. Remember that f_* preserves the set of constant functions. As in the last section, let $b_0:=\int_{\mathcal{V}}\varphi d\mu_0$, and $\varphi_0:=\varphi-b_0$. We define two sequences φ_n,b_n as follows. Put

$$b_n=b_n(\varphi):=\int_{\mathcal{V}}(f_*\varphi_{n-1})d\mu_0,\quad \varphi_n:=f_*\varphi_{n-1}-b_n$$

for $n\geq1$. Note that φ_n differs from $((f^n)_*\varphi)$ by a constant. Lemma 3.5 implies that $\mathbf{m}_n:=(\chi_{2l}^n m_0,m_1+n)$ is a size representative of φ_n. This together with Lemma 3.4 imply that

Lemma 3.6. $\mathbf{m}_n:=(\chi_{2l}^n m_0,m_1+n)$ *is also a size representative of* $|\varphi_n|,\varphi_n^+$ *and* φ_n^-.

By Lemma 3.3, we get

$$\|d\varphi_n\|_{L^2}Am_0^{1/2}\chi_{2l}^{n/2}(\chi_{2l-1})^{(n+m_1)/2}(3.5)$$

Using (3.5), (3.1) and (3.3) give

$$\|\varphi_n\|_{L^2}\leq Am_0^{1/2}\chi_{2l}^{n/2}(\chi_{2l-1})^{(n+m_1)/2},\quad |b_n|\leq Am_0^{1/2}\chi_{2l}^{n/2}(\chi_{2l-1})^{(n+m_1)/2}\tag{3.6}$$

for $n\geq1$ and some possible different constant A. Now we are in a situation very similar to the one in the last section. Using arguments similar to those in the last section, we can show that $\lim_{n\to\infty}\langle\chi_{2l}^{-n}(f^n)^*\omega^k,\varphi\rangle$ exists and denote by $b'_\infty(\varphi)$ its limit. In fact, we have

$$b'_\infty=\sum_{j=0}^\infty\chi_{2l}^{-j}b_j.$$

It follows that

$$|b'_\infty(\varphi)|\leq\|\varphi\|_{L^1}+Am_0^{1/2}(\chi_{2l-1})^{m_1/2}\tag{3.7}$$

for some constant A independent of φ. Clearly, if φ is a function quasi-p.s.h limited , b'_∞ is equal to the same number defined in the last section. So we have

$$\langle\mu_f,\varphi\rangle=b'_\infty(\varphi)$$

for function quasi-p.s.h limited φ. Let $W^{1,2}_{**,f}$ the subset of $W^{1,2}_{*,f}$ consisting of functions that are continuous outside a closed pluripolar set. Note that f_* preserves $W^{1,2}_{**,f}$ because f is a covering outside an analytical subset of $\mathcal{V}$. We now affirm that

Lemma 3.7. *For* $\varphi\in W^{1,2}_{**,f}$, *we have* $\langle\mu_f,\varphi\rangle=b'_\infty(\varphi)$.

Proof. The proof is similar to that on [21, Lemma 5.5]. We proved first that φ is μ_f-integrable. We assume for a moment that $\varphi\geq0$. Let V be a closed pluripolar set such that φ is continuous outside of V. Remember that μ_f has no mass on pluripolar sets, therefore, on V. Since $\chi_{2l}^{-n}(f^n)^*\omega^k$ converges to μ_f as positive measures and $\mathcal{V}\backslash V$ is open, we have

$$\langle\mu_f,\varphi\rangle\leq\liminf_{n\to\infty}\langle\chi_{2l}^{-n}(f^n)^*\omega^k,\varphi\rangle=\lim_{n\to\infty}\sum_{j=0}^n\chi_{2l}^{-j}b_j+\liminf_{n\to\infty}\langle\omega^k,\chi_{2l}^{-n}\varphi_n\rangle$$

which is equal to $b'_\infty(\varphi)$. Thus φ is μ_f-integrable if $\varphi \geq 0$. In general, write $\varphi = \varphi^+ - \varphi^-$ and applying the last property, show that φ is μ_f-integrable. If $\mathbf{m} = (m_0, m_1)$ is a size representative of φ, then we also get that

$$|\langle \mu_f, \varphi| \leq |b'_\infty(\varphi^+)| + |b'_\infty(\varphi^-)| \leq A(\|\varphi\|_{L^1} + m_0^{1/2}(\chi_{2l-1})^{m_1/2}), \tag{3.8}$$

for some constant c independent of φ. Now using $f^*\mu_f = \chi_{2l}\mu_f$ gives

$$|\langle \mu_f, \varphi\rangle - b'_\infty(\varphi)| = |\langle \mu_f, \chi_{2l}^{-n}(f^n)_*\varphi - b'_\infty(\varphi)\rangle| \leq |c_n| + |\langle \mu_f, \chi_{2l}^{-n}\varphi_n\rangle|,$$

where $c_n := -\sum_{j\geq n+1} \chi_{2l}^{-j}|b_j|$. Note that the first term on the right side of the last inequality tends to 0 because of (3.6). On the other hand, by (3.8) and Lemma 3.6, the second term is limited by

$$A\chi_{2l}^{-n}(\|\varphi_n\|_{L^1} + m_0^{1/2}\chi_{2l}^{n/2}(\chi_{2l-1})^{(m_1+n)/2})$$

which tends to 0 when $n \to \infty$. This produces the desired equality. This completes the proof.

■

Theorem 3.8. *Let $\mathcal{V}, f, \chi_{2l}, \chi_{2l-1}$ be as above with $\xi_l^{-1}(f) > 1$. So there is a constant $A > 0$ such that*

$$I_n(\psi, \varphi) := |\langle \mu_f, (\psi \circ f^n)\varphi\rangle - \langle \mu_f, \psi\rangle\langle \mu_f, \varphi\rangle| \leq A\|\psi\|_\infty A_n(\varphi),$$

where

$$A_n(\varphi) := \left[\|\varphi\|_{L^1} + m_0^{1/2}(\chi_{2l-1})^{m_1/2}\right]\chi_{2l}^{-n/2}(\chi_{2l-1})^{n/2},$$

for each $\psi \in L^\infty(\mu_f)$, $\varphi \in W^{1,2}_{**,f}$ and (m_0, m_1) a size representative of φ,

Note that if φ is a function η-p.s.h. limited for some $(1,1)$-continuous form η of L^∞-norm ≤ 1, then there is a constant $\widetilde{m}_0$ independent of φ such that $(\widetilde{m}_0, 1)$ is a size representative of φ. Therefore, the above theorem gives a uniform correlation decay for each φ.

Proof. Let the annotations be as above. $I_n(\psi, \varphi + c) = I_n(\psi, \varphi)$ for each constant c because of the invariance of μ_f. We can assume that $\langle \mu_f, \varphi\rangle = 0$. By Lemma 3.7, we get $b'_\infty(\varphi) = 0$. Consequently, $\chi_{2l}^{-n}(f^n)_*(\varphi) = c_n + \chi_{2l}^{-n}\varphi_n$. Using $f^*\mu_f = \chi_{2l}\mu_f$ gives

$$I_n(\psi, \varphi) = \chi_{2l}^{-n}|\langle \mu_f, \psi(f^n)_*(\varphi)\rangle| = |\langle \mu_f, \psi(c_n + \chi_{2l}^{-n}\varphi_n)\rangle| \leq |c_n| + \chi_{2l}^{-n}|\langle \mu_f, |\varphi_n|\rangle|. \tag{3.9}$$

Note that, as before, we have

$$|c_n| \leq AA_n(\varphi)$$

for some constant A independent of φ. On the other hand, f_* preserves $W^{1,2}_{**,f}$, thus $\varphi_n \in W^{1,2}_{**,f}$ and so is $|\varphi_n|$. By Lemma 3.6, $(\chi_{2l}^n m_0, m_1 + n)$ is a size representative of $|\varphi_n|$ if (m_0, m_1) is a size representative of φ. Arguing as in the proof of Lemma 3.7 gives that

$$\chi_{2l}^{-n}|\langle \mu_f, |\varphi_n|\rangle| \leq AA_n(\varphi)$$

for some constant A independent of φ. Hence the desired inequality follows. This completes the proof.

■

***End of Proof of Theorem** 1.10.* The central limit theorem for μ_f is a direct consequence of its correlation decay as shown in [21]. Therefore, it remains to prove the property of the correlation decay. By Theorem 3.8, for each $\mathcal{C}^1$ function φ on $\mathcal{V}$, we have

$$I(\psi, \varphi) \leq A\|\psi\|_\infty\|\varphi\|_{\mathcal{C}^1}\chi_{2l}^{-n/2}(\chi_{2l-1})^{n/2}.$$

This combined with the interpolation inequality for functional in Banach spaces $\mathcal{C}^1, \mathcal{C}^0$ provides the desired correlation decay for μ_f, cf [21].

Remember that μ_f is K-mixing if for each $\varphi \in L^2(\mu_f)$, we have

$$\sup_{\psi \in L^2(\mu_f)} I_n(\psi, \varphi) \to 0. \tag{3.10}$$

Note that the operator $\chi_{2l}^{-1}f_*$can be extended to be a continuous linear operator on $L^2(\mu_f)$ because $|f_*\varphi|^2 \leq \chi_{2l}f_*(|\varphi|^2)$. As above, to prove (3.10), we can assume that $\langle \mu_f, \varphi\rangle = 0$. Using (3.9) we have

$$I(\psi, \varphi) \leq \|\chi_{2l}^{-n}(f^n)_*\varphi\|_{L^2(\mu_f)}. \tag{3.11}$$

Consider now φ to be a limited function on $W^{1,2}_{**,f}$. The set of these functions is dense on $L^2(\mu_f)$. We have

$$\|\chi_{2l}^{-n}(f^n)_*\varphi\|_{L^2(\mu_f)} \leq \|\varphi\|_\infty\|\chi_{2l}^{-n}(f^n)_*\varphi\|_{L^1(\mu_f)}$$

that tends to 0 by proof of theorem 3.8. This combined with (3.11) gives (3.10). The proof is completed. ■

Remark 1. By inequality (3.6), we see that for each complex measure ν with density L^2 and $\nu(X) = 1$, $\chi_{2l}^{-n}(f^n)^*\nu$ converges weakly to μ_f.

4 Conjectures

Here is the First Conjecture.

Conjecture 4.1. Let $\mathcal{V}, f, \mu_f$ as in Theorem 1.9. Let $\psi_1, \dots, \psi_k$ be the Lyapunov exponents of μ_f and $\Psi = \sum_i \frac{1}{\psi_i}$ its inverse sum. So the Hausdorff dimension of μ_f satisfies

$$\dim_{\mathcal{H}}(\mu_f) = \Psi h_{\chi}(f).$$

Here is the Second Conjecture.

Conjecture 4.2. Let $\mathcal{V}, f, \mu_f$ be as in Theorem 1.9. So there are T_l^+ and T_{k-l}^- such that μ_f is defined by :

$$\mu_f := T_l^+ \wedge T_{k-l}^- \quad ,$$

where T_l^+ is a positive invariant closed current of bidegree (l, l), i.e.

$$\frac{1}{\chi_{2l}^m}(f^m)^*\omega^l \longrightarrow T_l^+$$

and T_{k-l}^- designates a positive invariant closed current of $(k-l, k-l)$, i.e.

$$\frac{1}{\chi_{2(k-l)}^m}(f^m)_*\omega^{k-l} \longrightarrow T_{k-l}^-.$$

Here is the Third Conjecture.

Conjecture 4.3. Let $\mathcal{V}, f, \mu_f$ be as in Theorem 1.9 and T_l^+ as in Conjecture 4.2. Let $\psi_1, \dots, \psi_k$ be the Lyapunov exponents of μ_f with $\psi_l = \max_{1\leq i\leq k}\psi_i$. So the Hausdorff dimension of the **Support** of T_l^+ satisfies

$$\dim_{\mathcal{H}}(\mathrm{Supp}T_l^+) \geq 2(k-l) + \frac{\log \chi_{2l}}{\psi_l}.$$

References

[1] ARMAND AZONNAHIN. Dinâmica de aplicações cohomologicamente hiperbólicas.

[2] ARMAND AZONNAHIN. Conceitos fundamentais e métodos pluripotenciais para aplicações cohomologicamente expansíveis. *researchgate.net from: DOI: 10.13140/RG.2.2.31156.81281* *https://www.researchgate.net/publication/334694628_Conceitos_Fundamentais_e_Metodos_Pluripotenciais_para_Aplicacoes_Cohomologicamente_Expansiveis*, *https://www.researchgate.net/publication/334694628*, 2019.

[3] ARMAND AZONNAHIN. On cohomological stochastic dynamics of higher dimensions. 2019.

[4] ARMAND AZONNAHIN. On cohomological stochastic dynamics of higher dimensions. *32º Colóquio Brasileiro de Matemática - IMPA, Rio de Janeiro* *https://hal.archives-ouvertes.fr/hal-02280087* *https://www.researchgate.net/profile/Armand_Azonnahin/publication/335618281_On_Cohomological_Stochastic_Dynamics_of_Higher_Dimensions/links/5d707f09a6fdcc9961af983d/On-Cohomological-Stochastic-Dynamics-of-Higher-Dimensions.pdf*, 2019.

[5] ARMAND AZONNAHIN. *On Cohomological Stochastic Dynamics of Higher Dimensions*. September 2019.

[6] ARMAND AZONNAHIN. Dinâmica de aplicações cohomologicamente hiperbólicas. https://hal.archives-ouvertes.fr/hal-02281122/, 2019.

[7] J.P. Demailly. Complex analytic geometry. *available at www.fourier.ujf grenoble.fr/demailly*.

[8] J.P. Demailly. Monge ampère operators, lelong numbers and intersection theory in complex analysis and geometry. *Plemum Press*, pages 115–193, 1993.

[9] M. Méo. Image inverse d'un courant positif fermé par une application surjective. *C.R.A.S.*, (322), 1996.

[10] G. Vigny. Dirichlet like space and capacity in complex analysis in several variables. *J. Funct. Anal.*, (1):247–277, 2007.

References

[1] *Azonnahin, Armand*, Dinâmica de Aplicações Cohomologicamente Expansíveis: Treze Novos Resultados, *10.13140/RG.2.2.24202.82880*, **https://www.researchgate.net/publication/338825994** (2020), 1-177.

[2] *H. Alexander*, Projective capacity, *Ann. Math. Studies*, **100** (1981), 3-27.

[3] *A. Blanchard*, Sur les variétés analytiques complexes, *Ann. Sci. Ecole Norm. Sup.* (3), **73** (1956), 157–202.

[4] *J.Y. Briend et J. Duval*, Deux caractérisations de la mesure d'équilibre d'un endomorphisme de $\mathbb{P}^k(\mathbb{C})$, *IHES Publ. Math.*, **93** (2001), 145-159.

[5] *L. Clozel et E. Ullmo*, Correspondances modulaires et mesures invariantes, *prépublication*, (2002).

[6] *J.P. Demailly*, Monge-Ampère Operators, Lelong numbers and Intersection theory in Complex Analysis and Geometry, *Plemum Press* (1993), 115-193, *(V. Ancona and A. Silva editors)*.

[7] *J.P. Demailly*, Regularization of closed positive currents and intersection theory, *J. Algebraic Geometry*, **1** (1992), 361-409.

[8] *T.C. Dinh*, Distribution des préimages et des points périodiques d'une correspondance polynomiale, *prépublication* (2003).

[9] *T.C. Dinh et N. Sibony*, Dynamique des applications d'allure polynomiale, à paraître dans *J.M.P.A.*

[10] *T.C. Dinh et N. Sibony*, Une borne supérieure pour l'entropie topologique d'une application rationnelle, *prépublication* (2003).

[11] Liverani C., Central limit theorem for deterministic systems. International Conference on Dynamical Systems (Montevideo, 1995), 56-75, Pitman Res. Notes Math. Ser., 362, Longman, Harlow, 1996.

[12] ARMAND AZONNAHIN. *Graus Dinâmicos e Cohomológicos em Variedades Abelianas*. Year 2019. Available at `https://www.researchgate.net/publication/336511093`.

[13] Friedland S., Entropy of algebraic maps, Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993), *J. Fourier Anal. Appl.*, 1995, Special Issue, 215-228.

[14] H. DE THÉLIN AND G. VIGNY, *Entropy of meromorphic maps and dynamics of birational maps*, Mém. Soc. Math. Fr. (N.S.), (2010), pp. vi+98.

[15] J.-P. DEMAILLY, *Complex analytic and differential geometry*. `http://www.fourier.ujf-grenoble.fr/~demailly`.

[16] T.-C. DINH, L. KAUFMANN, AND H. WU, *Dynamics of holomorphic correspondences on riemann surfaces*. `arxiv:1808.10130`, 2018.

[17] T.-C. DINH AND V.-A. NGUYÊN, *Characterization of Monge-Ampère measures with Hölder continuous potentials*, J. Funct. Anal., 266 (2014), pp. 67–84.

[18] T.-C. DINH AND N. SIBONY, *Dynamique des applications d'allure polynomiale*, J. Math. Pures Appl. (9), 82 (2003).

[19] ———, *Regularization of currents and entropy*, Ann. Sci. École Norm. Sup. (4), 37 (2004), pp. 959–971.

[20] ———, *Une borne supérieure pour l'entropie topologique d'une application rationnelle*, Ann. of Math. (2), 161 (2005), pp. 1637–1644.

[21] ———, *Decay of correlations and the central limit theorem for meromorphic maps*, Comm. Pure Appl. Math., 59 (2006), pp. 754–768.

[22] ———, *Distribution des valeurs de transformations méromorphes et applications*, Comment. Math. Helv., 81 (2006), pp. 221–258.

[23] ———, *Pull-back of currents by holomorphic maps*, Manuscripta Math., 123 (2007), pp. 357–371.

[24] ———, *Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings*, in Holomorphic dynamical systems, vol. 1998 of Lecture Notes in Math., Springer, Berlin, 2010, pp. 165–294.

[25] L. C. EVANS, *Partial differential equations*, vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, second ed., 2010.

[26] E. HEBEY, *Sobolev spaces on Riemannian manifolds*, vol. 1635 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1996.

[27] W. Parry, *Entropy and generators in ergodic theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[28] D.-V. Vu, *Intersection of positive closed currents of higher bidegrees*, Michigan Math. J., 65 (2016), pp. 863–872.

[29] ——, *Locally pluripolar sets are pluripolar*. arxiv:1812.00465, 2018.

[30] Abate M., The residual index and the dynamics of holomorphic maps tangent to the identity, *Duke Math. J.*, **107** (2001), no. 1, 173-207.

[31] Alexander H., Projective capacity. Recent developments in several complex variables, *Ann. of Math. Stud.* **100**, Princeton Univ. Press, Princeton, N.J., 1981, 3-27.

[32] Amerik E., Campana F., Exceptional points of an endomorphism of the projective plane, *Math. Z.*, **249** (2005), no. 4, 741-754.

[33] ——, Deux caractérisations de la mesure d'équilibre d'un endomorphisme de $\mathrm{P}\mathfrak{sp}k(\mathbf{C})$, *Publ. Math. Inst. Hautes Études Sci.*, **93** (2001), 145-159.

[34] ——, Personal communication.

[35] Brin M., Katok A., On local entropy, *Geometric dynamics Lecture Notes in Math.*, **1007**, Springer Berlin (1983), 30-38.

[36] Brolin H., Invariant sets under iteration of rational functions, *Ark. Mat.*, **6** (1965), 103-144.

[37] Carleson L., Gamelin T.W., *Complex dynamics*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993.

[38] Cerveau D., Lins Neto A., Hypersurfaces exceptionnelles des endomorphismes de $C\mathrm{P}(n)$, *Bol. Soc. Brasil. Mat. (N.S.)*, **31** (2000), no. 2, 155-161.

[39] Chemin J.-Y., *Théorie des distributions et analyse de Fourier*, cours à l'école polytechnique de Paris, 2003.

[40] Chern S.S., Levine H.I., Nirenberg L., Intrinsic norms on a complex manifold, in Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, 1969, 119-139.

[41] Demailly J.-P., Monge-Ampère Operators, Lelong numbers and Intersection theory in Complex Analysis and Geometry, *Plemum Press* (1993), 115-193, *(V. Ancona and A. Silva editors)*.

[42] ——, *Complex analytic geometry*, available at www.fourier.ujf-grenoble.fr/~demailly.

[43] DeMarco L., Dynamics of rational maps: Lyapunov exponents, bifurcations, and capacity, *Math. Ann.*, **326** (2003), no. 1, 43-73.

[44] Dembo A., Zeitouni O., *Large deviations techniques and applications*, Second edition, Applications of Mathematics (New York), **38**, Springer-Verlag, New York, 1998.

[45] ——, Complex dynamics in higher dimensions. Notes partially written by Estela A. Gavosto, *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, **439**, Complex potential theory (Montreal, PQ, 1993), 131-186, *Kluwer Acad. Publ.*, Dordrecht, 1994.

[46] Gordin M.I., The central limit theorem for stationary processes, *Dokl. Akad. Nauk SSSR*, **188** (1969), 739-741.

[47] Katok A., Hasselblatt B., *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, **54**, Cambridge University Press, Cambridge, 1995.

[48] Viana M., Stochastic dynamics of deterministic systems, vol. **21**, IMPA (1997).

[49] Walters P., *An introduction to ergodic theory*, Graduate Texts in Mathematics, **79**, Springer-Verlag, New York-Berlin, 1982.

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