

On inner expansions for a singularly perturbed Cauchy problem with confluent fuchsian singularities

S. Malek

University of Lille

59655 Villeneuve d'Ascq cedex, France,

stephane.malek@univ-lille.fr

Abstract

A nonlinear singularly perturbed Cauchy problem with confluent fuchsian singularities is examined. This problem involves coefficients with polynomial dependence in time. A similar initial value problem with logarithmic reliance in time has recently been investigated by the author in [19], for which sets of holomorphic inner and outer solutions were built up and expressed as a Laplace transform with logarithmic kernel. Here, a family of holomorphic inner solutions are constructed by means of exponential transseries expansions containing infinitely many Laplace transforms with special kernel. Furthermore, asymptotic expansions of Gevrey type for these solutions relatively to the perturbation parameter are established.

Key words: asymptotic expansion, Borel-Laplace transform, Cauchy problem, formal power series, integro-differential equation, partial differential equation, singular perturbation. 2010 MSC: 35R10, 35C10, 35C15, 35C20.

1 Introduction

This work falls in the continuance of the recent paper [19], where families of singularly perturbed initial value problems with the following shape

$$(1) \quad Q(\partial_z)y(t, z, \epsilon) = Q_1(\partial_z)y(t, z, \epsilon)Q_2(\partial_z)y(t, z, \epsilon) + (\mathcal{D}_{\epsilon, \alpha}(\partial_t))^{\delta_D}R_D(\partial_z)y(t, z, \epsilon) + P(1/a(\epsilon t, \epsilon), z, \epsilon, \mathcal{D}_{\epsilon, \alpha}(\partial_t), \partial_z)y(t, z, \epsilon) + f(1/a(\epsilon t, \epsilon), z, \epsilon)$$

for vanishing initial data $y(0, z, \epsilon) \equiv 0$ were considered. There, Q, Q_1, Q_2, R_D stand for polynomials with complex coefficients and $\delta_D \geq 2$ is an integer. The operator

$$\mathcal{D}_{\epsilon, \alpha}(\partial_t) := (\epsilon t^2 - \epsilon^\alpha)\partial_t$$

is a fuchsian differential operator at the points $t = \pm\epsilon^{\frac{\alpha-1}{2}}$ for some odd integer $\alpha \geq 3$ and $\epsilon \in \mathbb{C}^*$ is a non vanishing complex parameter. This operator unfolds the basic singularly perturbed irregular operator $\epsilon t^2\partial_t$ of rank 1 at $t = 0$. Recent references about this so-called confluence process of Fuchsian singularities can be found in our work [19]. The function $a(\epsilon t, \epsilon)$ (unveiled in (13)) represents a well chosen logarithmic function in its arguments. The linear differential operator $P(T_1, T_2, T_3, \mathcal{D}_{\epsilon, \alpha}(\partial_t), \partial_z)$ is chosen to be analytic in T_1, T_3 near the origin in \mathbb{C} and

holomorphic w.r.t T_2 on a strip $H_\beta = \{z \in \mathbb{C} / |\text{Im}(z)| < \beta\}$ for some width $\beta > 0$; moreover the forcing term $f(T_1, z, \epsilon)$ is analytic near the origin in \mathbb{C} relatively to T_1, ϵ and holomorphic in z on H_β . Notice that this function $a(\epsilon t, \epsilon)$ is introduced in order to be able to construct nice representable solutions $y(t, z, \epsilon)$ to (1) as Laplace transforms in time t from which parametric asymptotic properties can be analyzed. The fact that both coefficients and forcing term in (1) are holomorphic maps in $1/a(\epsilon t, \epsilon)$ is a strong technical condition, but they turn out to be good approximations of general analytic functions on appropriate domains in time t , for $z \in H_\beta$, provided that ϵ remains small enough.

Two distinguished finite sets of holomorphic solutions $y(t, z, \epsilon)$ to (1) were constructed. The first family consists of the so-called outer solutions $y_p^{\text{out}}(t, z, \epsilon)$, $0 \leq p \leq \iota - 1$ for some integer $\iota \geq 2$, that are holomorphic on domains $\mathcal{A} \times H_\beta \times \mathcal{E}_p^{\text{out}}$, where \mathcal{A} is a fixed bounded sectorial annulus confined apart of the origin in \mathbb{C} and $\mathcal{E}_p^{\text{out}}$ is a bounded sector centered at 0 which belongs to a set $\underline{\mathcal{E}}^{\text{out}} = \{\mathcal{E}_p^{\text{out}}\}_{0 \leq p \leq \iota - 1}$ that covers a full neighborhood of 0 in \mathbb{C}^* called *good covering* in \mathbb{C}^* (see Definition 2). The second family is comprised in the so-called inner solutions $y_p^{\text{in}}(t, z, \epsilon)$, $0 \leq p \leq \eta - 1$ for some integer $\eta \geq 2$, that are constructed on a domain $\mathcal{T}_\epsilon^{\text{in}} = \{\epsilon^{\frac{\alpha-1}{2}} x / x \in \chi\}$ w.r.t time t for some fixed bounded sectorial annulus χ far enough from the origin in \mathbb{C} , $z \in H_\beta$ and ϵ in a sector $\mathcal{E}_p^{\text{in}}$ that is part of a good covering in \mathbb{C}^* .

Both families $y_p^{\text{out/in}}(t, z, \epsilon)$ could be expressed as special Laplace transform and Fourier integrals

$$(2) \quad y_p^{\text{out/in}}(t, z, \epsilon) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\gamma_p}^{\text{out/in}}} W_p^{\text{out/in}}(\tau, m, \epsilon) \exp(\tau a(\epsilon t, \epsilon)) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm$$

for suitable *Borel maps* $W_p^{\text{out/in}}$. Furthermore, from these integral representations, asymptotic behaviour could be extracted. Indeed, the outer solutions $y_p^{\text{out}}(t, z, \epsilon)$ (resp. inner solutions $y_p^{\text{in}}(t, z, \epsilon)$) share a common power series $\hat{O}(\epsilon) = \sum_{k \geq 0} O_k \epsilon^k$ for bounded holomorphic coefficients O_k on $\mathcal{A} \times H_\beta$ as asymptotic expansion of Gevrey order 1 (resp. $\hat{I}(\epsilon) = \sum_{k \geq 0} I_k \epsilon^k$ for bounded holomorphic coefficients I_k on $\mathcal{T}_\epsilon^{\text{in}} \times H_\beta$ as asymptotic expansion of Gevrey order $\frac{2}{\alpha+1}$), meaning that constants $A_p^{\text{out/in}}, B_p^{\text{out/in}} > 0$ can be found with

$$(3) \quad \sup_{t \in \mathcal{A}, z \in H_\beta} \left| y_p^{\text{out}}(t, z, \epsilon) - \sum_{k=0}^{n-1} O_k \epsilon^k \right| \leq A_p^{\text{out}} (B_p^{\text{out}})^n n! |\epsilon|^n$$

for all $n \geq 1$, $\epsilon \in \mathcal{E}_p^{\text{out}}$ and

$$(4) \quad \sup_{x \in \chi, z \in H_\beta} \left| y_p^{\text{in}}(\epsilon^{\frac{\alpha-1}{2}} x, z, \epsilon) - \sum_{k=0}^{n-1} I_k \epsilon^k \right| \leq A_p^{\text{in}} (B_p^{\text{in}})^n \Gamma(1 + \frac{2}{\alpha+1} n) |\epsilon|^n$$

provided that $n \geq 1$, $\epsilon \in \mathcal{E}_p^{\text{in}}$.

In this paper, we turn our attention at a closely related singularly perturbed nonlinear Cauchy problem

$$(5) \quad P_2(\mathcal{D}_{\epsilon, \alpha}(\partial_t)) \partial_z^S u(t, z, \epsilon) = P_1(t, z, \epsilon, \mathcal{D}_{\epsilon, \alpha}(\partial_t), \partial_z) u(t, z, \epsilon) + g(t, \epsilon)$$

under given Cauchy data

$$(6) \quad (\partial_z^j u)(t, 0, \epsilon) = \varphi_j(t, \epsilon) \quad , \quad 0 \leq j \leq S-1.$$

Like in (1), the forcing term $g(t, \epsilon)$ is chosen to be a polynomial in the function $1/a(\epsilon t, \epsilon)$. A similar assumption is put on the Cauchy data (6). The choice of the function $a(\epsilon t, \epsilon)$ is made for the same reason as in our former study (1). Besides, P_2 stands for a polynomial with complex coefficients and $S \geq 2$ is an integer. The novel feature is that the coefficients in the main part P_1 of (5) are assumed to be merely polynomials in t . As we will see in the work, this property has a deep impact on the structure of solutions to (5), (6) if compared with the ones $y_p^{\text{out/in}}$ built up for (1).

For technical reasons that will be clear later in the paper, we impose that the operator $P_2(\mathcal{D}_{\epsilon, \alpha}(\partial_t))$ can be factored out in the main part P_1 which allows us to reduce the problem (5), (6) to a Cauchy problem of Kowalevski type stated in (11), (12). Such a reduction is mandatory within our approach as explained in Section 5 of the work. The forcing term $f(t, \epsilon)$ of the resulting equation (11) is asked to solve a simple singularly perturbed ODE

$$(7) \quad P_2(\mathcal{D}_{\epsilon, \alpha}(\partial_t))f(t, \epsilon) = g(t, \epsilon).$$

As a result, the single equation (5) is written as a coupling of a Kowalevski type equation (11) and a singularly perturbed ODE (7). It is worthwhile noting that a more general forcing term $g(t, z, \epsilon)$ relying holomorphically on z and Cauchy data $\varphi_j(t, \epsilon)$, $0 \leq j \leq S - 1$ that are not only polynomials in $1/a(\epsilon t, \epsilon)$ but also polynomials in t could be treated in a similar manner. However, such a general choice would lead to even more cumbersome and heavy computations which may avoid the reader to have a clear idea of the main purpose of the study.

The main striking difference with our previous work [19] is that now the analytic solutions fail to be constructed as a single special Laplace transform for the kernel $\exp(\tau a(\epsilon t, \epsilon))$. Nevertheless, they can be expressed as exponential sums which involve infinitely many of Laplace transforms, which are called *transseries* in the literature (for an explanation of this terminology we refer to Chapters 4 and 5 of the excellent textbook [6]). Namely, we can provide a finite set of analytic solutions $(t, z) \mapsto u_p(t, z, \epsilon)$, $0 \leq p \leq \iota - 1$ for some integer $\iota \geq 2$ on some domains $\mathcal{T}_\epsilon \times D(0, r)$. Here, $D(0, r)$ represents a small disc centered at 0 with radius $r > 0$ and $\mathcal{T}_\epsilon = \{\epsilon^{\frac{\alpha-1}{2}} x / x \in \chi_1\}$ (which is similar to the domain $\mathcal{T}_\epsilon^{\text{in}}$ introduced above) is a set where χ_1 stands for a tiny bounded sectorial annulus close to 1 in \mathbb{C} , whenever $\epsilon \in \mathcal{E}_p$, where \mathcal{E}_p is a bounded sector centered at 0 that belongs to a good covering in \mathbb{C}^* . Since the domain \mathcal{T}_ϵ remains next to the Fuchsian singular point $\epsilon^{\frac{\alpha-1}{2}}$ and borders the origin as ϵ tends to 0, we call the elements of this set *inner solutions*. Each solution $u_p(t, z, \epsilon)$ has a convergent exponential transseries expansion

$$u_p(t, z, \epsilon) = \sum_{n \geq 0} \left(\int_{L_{\gamma_p}} W_n(\tau, z, \epsilon) \exp(\tau a(\epsilon t, \epsilon)) \frac{d\tau}{\tau} \right) \exp(n\epsilon^{\frac{\alpha+1}{2}} a(\epsilon t, \epsilon))$$

on $\mathcal{T}_\epsilon \times D(0, r)$ for $\epsilon \in \mathcal{E}_p$ (see Theorem 1). Actually, the appearance of such transseries stems from the very expansion of the monomials $(\epsilon t)^l$, $l \geq 1$ as sums of special Laplace transforms as shown in Propositions 1 and 3. In the proof, an interesting small divisor phenomenon occurs which gives rise to the appearance of a special series (52) in the expansion of the basic monomial T , see (46). This special series turn out to carry an exponential series expansion displayed in Lemma 8. It is worthwhile noting that due to the specific arrangement of these expansions, this approach does not allow us to exhibit the so-called outer solutions as in the case of our previous study [19]. In a second main result (Theorem 2), we analyze the parametric asymptotic expansions of these inner solutions. It turns out that the functions $\epsilon \mapsto u_p(t, z, \epsilon)$, $0 \leq p \leq \iota - 1$, share a common formal series $\hat{I}(\epsilon) = \sum_{k \geq 0} I_k \epsilon^k$ with bounded holomorphic coefficients I_k on

$\mathcal{T}_\epsilon \times D(0, r)$ as Gevrey asymptotic expansion of order $\frac{2}{\alpha+1}$. This outcome is comparable to the one obtained in [19].

During the last two decades, exponential transseries expansions appear to be a central tool in the study of differential, partial differential and difference equations in the complex domain. Indeed, we refer to the seminal work by O. Costin and R. Costin, see [7], where these class of expansions have shown to be essential in the study of formation of complex singularities along Stokes directions for systems of nonlinear ODEs of the form

$$(8) \quad x^2 y'(x) = A(x)y(x) + B(x, y(x))$$

where $y(x) \in \mathbb{C}^n$, for an integer $n \geq 1$, $x \in \mathbb{C}$, $A(x)$ is an analytic diagonal matrix and for a nonlinearity $B(x, Y)$ analytic near the origin in \mathbb{C}^{n+1} . Later on, the transseries approach was extended by B. Braaksma and R. Kuik, in [4], to nonlinear systems of difference equations

$$(9) \quad y(x+1) = \Lambda(x)y(x) + g(x, y(x))$$

for $y(x) \in \mathbb{C}^n$, with integer $n \geq 1$, $\Lambda(x)$ an analytic diagonal matrix and $g(x, Y)$ analytic near $x = \infty$ and $Y = 0$. Adjustments of this approach applied to partial differential equations (beyond the integrable case) have been initiated in the paper [10]. More recently, transseries expansions (conjointly with a KAM like approach) have been applied to the location of complex singularities of general first order nonlinear scalar equations $y' = F(y(x), 1/x)$ as x tends to infinity, see [9]. Similar strategies have been implemented on nonlinear second order ODEs such as the Painlevé equation P_1 in order to compute in closed form connection coefficients between solutions on sectors called Stokes multipliers, see [8]. For problems related to obstruction for analytic integrability of Hamiltonian systems and transseries expansions of first integrals, we refer to [2]. Another aspect for which transseries turn out to be a powerful tool is the resurgence property of formal power series solutions to differential or more general functional equations (i.e analytic continuation of their Borel transforms). For systems of the form (8) and (9) resurgence properties stemming from transseries expansions of actual holomorphic solutions on sectors have been exhibited in the papers [3], [5]. For parametric resurgence for WKB solutions of 1D complex Schrödinger equations

$$\epsilon^2 y''(x) = x^2(1-x)^2 y(x)$$

w.r.t ϵ , based on exponential series techniques, we mention the work by A. Fruchard and R. Schäfke [13].

Our paper is arranged as follows.

In section 2, we present the main problems (11), (12) and (23), (12) of the work. In the technical propositions 1 and 3, we express the coefficients of (11) as convergent exponential transseries expansions that contain Laplace transforms with specific kernel, which leads to seek for solutions to (11), (12) expressed in the same manner as exponential series involving Laplace transforms of infinitely many Borel maps W_n , $n \geq 0$. The principal accomplishment of this section is the layout of a sequence of convolution problems (100), (101) fulfilled by W_n , $n \geq 0$.

In Section 3, the sequence of convolution problems is solved by means of a majorant series approach, the use of sequences of Banach spaces of sectorial holomorphic functions with exponential growth and the application of the classical Cauchy-Kowalevski theorem.

In Section 4, the two main results of the paper are stated. A set of inner holomorphic solutions to (11), (23), (12) are constructed, which are defined w.r.t ϵ on a good covering in \mathbb{C}^* and relatively to time t on a domain that remains close to the moving fuchsian singularities of (23) (Theorem 1). In Theorem 2, the parametric asymptotic behaviour of the latter solutions is analyzed by means of the classical Ramis-Sibuya approach.

The last section is devoted to the conclusion of the work where insights for prospective works are outlined.

2 Statement of the main problem and related auxiliary equations

Let $\alpha \geq 3$ be an odd natural number. We set $\epsilon \in \mathbb{C}^*$ as a non vanishing complex number. We define the differential operator

$$\mathcal{D}_{\epsilon,\alpha}(\partial_t) = (\epsilon t^2 - \epsilon^\alpha) \partial_t$$

and for any integer $l \geq 1$ we denote $(\mathcal{D}_{\epsilon,\alpha}(\partial_t))^l$ the iteration of order l of the operator $\mathcal{D}_{\epsilon,\alpha}(\partial_t)$.

We consider a finite set I of \mathbb{N}^3 , integers $S, b \geq 2$ and $\Delta_{\mathbf{l}} \geq 0$ that fulfill the next conditions

$$(10) \quad \Delta_{\mathbf{l}} \geq l_1 \quad , \quad S > l_3 \quad , \quad S > bl_2 + l_3$$

for all $\mathbf{l} = (l_1, l_2, l_3) \in I$. We state the main nonlinear Cauchy problem of our study

$$(11) \quad \partial_z^S u(t, z, \epsilon) = \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I} \epsilon^{\Delta_{\mathbf{l}}} d_{\mathbf{l}}(z, \epsilon) t^{l_1} (\mathcal{D}_{\epsilon,\alpha}(\partial_t))^{l_2} \partial_z^{l_3} u(t, z, \epsilon) + e(z, \epsilon) u^2(t, z, \epsilon) + f(t, \epsilon)$$

for given Cauchy data

$$(12) \quad (\partial_z^j u)(t, 0, \epsilon) = \varphi_j(t, \epsilon) \quad , \quad 0 \leq j \leq S - 1.$$

The coefficients $d_{\mathbf{l}}(z, \epsilon)$ and $e(z, \epsilon)$ represent bounded holomorphic functions on a polydisc $D(0, r) \times D(0, \epsilon_0)$ centered at the origin with radii $r, \epsilon_0 > 0$. The forcing term f and the Cauchy data are displayed as follows. Let $P_1(\tau)$ and $P_2(\tau)$ two polynomials with complex coefficients. We assume that the set \mathcal{R}_2 of roots of $P_2(\tau)$ is located apart of some closed disc $\bar{D}(0, \rho)$, for $\rho > 0$, and that $P_1(\tau)$ is vanishing at 0. Let $a(T, \epsilon)$ be the function already considered in our previous study [19],

$$(13) \quad a(T, \epsilon) = \frac{1}{2\epsilon^{\frac{\alpha+1}{2}}} \log \left(\frac{T - \epsilon^{\frac{\alpha+1}{2}}}{T + \epsilon^{\frac{\alpha+1}{2}}} \right)$$

which represents a primitive of the rational function $1/(s^2 - \epsilon^{\alpha+1})$. We define the integral

$$(14) \quad F(T, \epsilon) = \int_{L_d} \frac{P_1(\tau)}{P_2(\tau)} \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau}$$

along a halfline $L_d = \mathbb{R}_+ e^{\sqrt{-1}d}$ in direction $d \in \mathbb{R}$ which avoids the set \mathcal{R}_2 and for which the integral is assumed to make sense. We set $f(t, \epsilon)$ as a time rescaled version of F , namely

$$(15) \quad f(t, \epsilon) = F(\epsilon t, \epsilon).$$

The Cauchy data are built up in a similar manner. Namely, for $0 \leq j \leq S - 1$, let

$$Q_j(\tau) = \sum_{n=1}^{\deg(Q_j)} q_{j,n} \tau^n$$

be polynomials with complex coefficients that vanish at $\tau = 0$. We introduce the integrals

$$(16) \quad \Phi_j(T, \epsilon) = \int_{L_d} Q_j(\tau) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau}$$

and set $\varphi_j(t, \epsilon)$ as a time rescaled version of Φ_j ,

$$(17) \quad \varphi_j(t, \epsilon) = \Phi_j(\epsilon t, \epsilon)$$

for $0 \leq j \leq S - 1$. The domains where F , f , Φ_j and φ_j are well defined and holomorphic will be specified later on in the work.

In this work, the forcing term f is chosen in a way that it solves a simple ODE in the singular operator $\mathcal{D}_{\epsilon, \alpha}(\partial_t)$. Indeed, by construction of $a(T, \epsilon)$, the action of $\mathcal{D}_{\epsilon, \alpha}(\partial_t)$ on f reads as

$$(18) \quad \mathcal{D}_{\epsilon, \alpha}(\partial_t) f(t, \epsilon) = \int_{L_d} \tau \frac{P_1(\tau)}{P_2(\tau)} \exp(\tau a(\epsilon t, \epsilon)) \frac{d\tau}{\tau}.$$

On the other hand, a direct computation shows that for $n \geq 1$,

$$(19) \quad (a(\epsilon t, \epsilon))^{-n} = \frac{(-1)^n}{\Gamma(n)} \int_{L_d} \tau^n \exp(\tau a(\epsilon t, \epsilon)) \frac{d\tau}{\tau}$$

for any direction d for which the integral makes sense. If one expands $P_1(\tau) = \sum_{n=1}^{\deg(P_1)} p_{1,n} \tau^n$, then we set

$$g(t, \epsilon) = \sum_{n=1}^{\deg(P_1)} p_{1,n} \Gamma(n) (-1)^n (a(\epsilon t, \epsilon))^{-n}.$$

From (18) and (19), we deduce that $f(t, \epsilon)$ solves the next singularly perturbed inhomogeneous ODE

$$(20) \quad P_2(\mathcal{D}_{\epsilon, \alpha}(\partial_t)) f(t, \epsilon) = g(t, \epsilon).$$

Besides, the Cauchy data $\varphi_j(t, \epsilon)$ can be expressed as polynomials in the logarithmic function $a(\epsilon t, \epsilon)^{-1}$, by using the representation (19), namely

$$(21) \quad \varphi_j(t, \epsilon) = \sum_{n=1}^{\deg(Q_j)} q_{j,n} \Gamma(n) (-1)^n (a(\epsilon t, \epsilon))^{-n}$$

for any $0 \leq j \leq S - 1$. As a result, let us introduce the next nonlinear differential map

$$(22) \quad \begin{aligned} & \mathcal{P}(t, z, \epsilon, \mathcal{D}_{\epsilon, \alpha}(\partial_t), \partial_z) u(t, z, \epsilon) \\ & := P_2(\mathcal{D}_{\epsilon, \alpha}(\partial_t)) \left(\sum_{\mathbf{l}=(l_1, l_2, l_3) \in I} \epsilon^{\Delta_{\mathbf{l}}} d_{\mathbf{l}}(z, \epsilon) t^{l_1} (\mathcal{D}_{\epsilon, \alpha}(\partial_t))^{l_2} \partial_z^{l_3} u(t, z, \epsilon) + e(z, \epsilon) u^2(t, z, \epsilon) \right). \end{aligned}$$

Then, if $u(t, z, \epsilon)$ solves the main equation (11), it also solves the next singularly perturbed nonlinear Cauchy problem

$$(23) \quad P_2(\mathcal{D}_{\epsilon, \alpha}(\partial_t)) \partial_z^S u(t, z, \epsilon) = \mathcal{P}(t, z, \epsilon, \mathcal{D}_{\epsilon, \alpha}(\partial_t), \partial_z) u(t, z, \epsilon) + g(t, \epsilon)$$

under the Cauchy data (12), with forcing term g that represents a polynomial in the logarithmic function $a(\epsilon t, \epsilon)^{-1}$.

Throughout this work, we are searching for ϵ -time rescaled solutions of (11) with the shape

$$u(t, z, \epsilon) = U(\epsilon t, z, \epsilon).$$

As a matter of fact, if we define the next differential operator

$$\mathcal{D}_{\epsilon, \alpha}(\partial_T) = (T^2 - \epsilon^{\alpha+1})\partial_T$$

the function $U(T, z, \epsilon)$, by means of the change of variable $T = \epsilon t$, is required to solve the next Cauchy problem

$$(24) \quad \begin{aligned} \partial_z^S U(T, z, \epsilon) &= \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I} \epsilon^{\Delta_1 - l_1} d_{\mathbf{l}}(z, \epsilon) T^{l_1} (\mathcal{D}_{\epsilon, \alpha}(\partial_T))^{l_2} \partial_z^{l_3} U(T, z, \epsilon) + e(z, \epsilon) U^2(T, z, \epsilon) + F(T, \epsilon) \end{aligned}$$

assuming the Cauchy data

$$(25) \quad (\partial_z^j U)(T, 0, \epsilon) = \Phi_j(T, \epsilon) \quad , \quad 0 \leq j \leq S - 1.$$

We undertake a similar strategy as in our previous contribution [19], where solutions to the related problem (1) were asked to be expressed through special Laplace transforms that involve the function $a(T, \epsilon)$ and Fourier integrals.

We first need to describe the domains in time T on which we plan to construct our solutions. Namely, let us fix a bounded sectorial annulus (centered at 0)

$$\chi_0 = \{x \in \mathbb{C} / r_1 < |x| < r_2, \alpha_1 < \arg(x) < \alpha_2\}$$

for some given radii $r_1, r_2 > 0$ and with angles $\alpha_1 < \alpha_2$. We set the next sectorial annulus (centered at 1)

$$(26) \quad \chi_1 = 1 + \chi_0 = \{1 + x_0 / x_0 \in \chi_0\}$$

and introduce the next open sectorial domain

$$(27) \quad \mathcal{T}_\epsilon = \{\epsilon^{\frac{\alpha-1}{2}} x / x \in \chi_1\}$$

for all $\epsilon \in \mathbb{C}^*$.

In the next lemma, bounds estimates are supplied for the map $a(T, \epsilon)$.

Lemma 1 *For any given $\delta > 0$, a small outer radius $r_2 > 0$ can be selected in a way that*

$$(28) \quad a(T, \epsilon) = \left(\frac{1}{2\epsilon^{\frac{\alpha+1}{2}}} \log \left| \frac{x-1}{x+1} \right| \right) \times (1 + a_{\text{in}}(x))$$

holds provided that $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$, $T = \epsilon^{(\alpha+1)/2} x \in \epsilon \mathcal{T}_\epsilon$, where $a_{\text{in}}(s)$ defines a function on χ_1 that suffers

$$(29) \quad \sup_{s \in \chi_1} |a_{\text{in}}(s)| \leq \delta.$$

Proof For $T = \epsilon^{(\alpha+1)/2}x$ where $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$ and $x \in \chi_1$, we can expand

$$a(T, \epsilon) = a(\epsilon^{\frac{\alpha+1}{2}}x, \epsilon) = \frac{1}{2\epsilon^{\frac{\alpha+1}{2}}} \log \left(\frac{x-1}{x+1} \right).$$

Since we can write

$$\begin{aligned} \log \left(\frac{x-1}{x+1} \right) &= \log \left| \frac{x-1}{x+1} \right| + \sqrt{-1} \arg \left(\frac{x-1}{x+1} \right) \\ &= \log \left| \frac{x-1}{x+1} \right| \left(1 + \sqrt{-1} \frac{\arg((x-1)/(x+1))}{\log(|x-1|/|x+1|)} \right) \end{aligned}$$

where

$$(30) \quad \frac{r_1}{2+r_2} < \left| \frac{x-1}{x+1} \right| < \frac{r_2}{2-r_2}$$

is close to 0 provided that $x \in \chi_1$ whenever $0 < r_1 < r_2$ are taken small enough, Lemma 1 follows. \square

In order to be able to build up actual solutions of (24), our first main technical task will be here to express the most basic monomial T in terms of special Laplace transforms. The next subsection 2.1 is devoted to the explanation of the next proposition.

Proposition 1 *There exist*

- a positive radius $\epsilon_0 > 0$ close to 0 and a small outer radius $r_2 > 0$
- a positive real number $a_0 > 0$ and an entire function $z \mapsto \omega_0(z, \epsilon)$ on \mathbb{C} for all $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$ for which one can find two constants $C, M > 0$ with

$$(31) \quad |\omega_0(z, \epsilon)| \leq C \frac{|z|}{1+|z|^2} \exp(M|z|)$$

for all $z \in \mathbb{C}$, all $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$,

for which the next decomposition holds

$$(32) \quad T = \int_{L_{d^-}} \omega_0(\tau, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} + \epsilon^{\frac{\alpha+1}{2}} a_0 + 2\epsilon^{\frac{\alpha+1}{2}} \sum_{n \geq 1} \exp(n\epsilon^{\frac{\alpha+1}{2}} a(T, \epsilon))$$

where $L_{d^-} = \mathbb{R}_+ e^{\sqrt{-1}d^-}$ stands for a halfline in direction

$$d^- = \arg(\epsilon^{\frac{\alpha+1}{2}}) + \Delta^-$$

for any positive real number $\Delta^- > 0$ not too large (actually less than $\pi/2$), provided that $T \in \epsilon\mathcal{T}_\epsilon$ and $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$

2.1 Expression of a monomial as exponential series with special Laplace transforms

We first need to introduce an analytic Borel transform for an analytic function near the origin outside a discrete set of pole singularities and analyze its Laplace transform accordingly. The next proposition turns out to be a variant of the proposition 12 of [16] which dealt with bounded sectorial analytic functions.

Proposition 2 1) Let $\rho > 0$ be a real number and let $F : D(0, \rho) \setminus \{0\} \mapsto \mathbb{C}$ be an analytic function apart of a discrete singular set $\Theta = \{p_n \in \mathbb{C}^* / n \in \mathbb{Z}^*\}$ along which F possesses simple poles. We assume that

- The set Θ is located on a line $\mathcal{L}_0 = l_0 \mathbb{R}$ passing through 0 for some complex number $l_0 \in \mathbb{C}^*$ and is contained in the open disc $D(0, \rho/2)$.
- There exists a complex number F_0 such that, for every open sector W^+ (resp. W^-) centered at 0, with radius ρ , bisecting direction $\arg(l_0) + \frac{\pi}{2}$ (resp. $\arg(l_0) - \frac{\pi}{2}$), such that $W^+ \cap \mathcal{L}_0 = \emptyset$ (resp. $W^- \cap \mathcal{L}_0 = \emptyset$), the next limits

$$(33) \quad \lim_{u \rightarrow 0, u \in W^+} F(u) = F_0 \quad , \quad \lim_{u \rightarrow 0, u \in W^-} F(u) = -F_0$$

hold.

We set the analytic Borel transform (of order 1) of F as the next integral

$$(34) \quad (B_1 F)(z) = \frac{-1}{2\sqrt{-1}\pi} \int_{C(0, \rho/2)} F(u) \exp\left(\frac{z}{u}\right) \frac{z}{u^2} du$$

along a positively oriented circle $C(0, \rho/2)$ centered at 0 with radius $\rho/2$. The function $(B_1 F)(z)$ defines an entire function on \mathbb{C} and moreover we can find two constants $C, M > 0$ with

$$|(B_1 F)(z)| \leq C|z| \exp(M|z|)$$

for all $z \in \mathbb{C}$.

2) Let $B : \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic function with bounds

$$|B(z)| \leq C|z| \exp(M|z|)$$

for some given constants $C, M > 0$, for all $z \in \mathbb{C}$. We define the Laplace transform (of order 1) of B in the direction $d \in \mathbb{R}$ as

$$(L_1^d B)(T) = \int_{L_d} B(\tau) \exp(-\tau/T) \frac{d\tau}{\tau}$$

where $L_d = \mathbb{R}_+ \exp(\sqrt{-1}d)$ stands for a halfline in direction d . The function $(L_1^d B)(T)$ is analytic and bounded on every closed subsector S (centered at 0) of a sector $S(d, \pi, \tilde{\rho}) \subset D(0, \rho/2)$ which represents an open sector with bisecting direction d , opening π and well chosen radius $\tilde{\rho} > 0$.

3) For all $n \in \mathbb{Z}^*$, we denote $\text{Res}_{u=p_n}(F(u))$ the residue of the function $F(u)$ at the pole $u = p_n$. Assume that the next series

$$\psi(T) = \sum_{n \in \mathbb{Z}^*} \text{Res}_{u=p_n}(F(u)) \frac{T}{p_n(T - p_n)}$$

converges on some subsector S_ψ (centered at 0) of $S(d, \pi, \tilde{\rho})$. Then, the next identity

$$(35) \quad (L_1^d(z \mapsto (B_1 F)(z)))(T) = F(T) - \psi(T)$$

holds for all $T \in S_\psi$.

Proof The proofs of the first two parts 1) and 2) are straightforward and can be performed by direct estimates on the integral representations. We focus on the third part 3). Using Fubini's theorem, we can write

$$(36) \quad (L_1^d(u \mapsto (B_1 F)(u)))(T) = \int_{L_d} \left(-\frac{1}{2\sqrt{-1}\pi} \int_{C(0,\rho/2)} F(u) e^{v/u} \frac{v}{u^2} du \right) e^{-v/T} dv/v \\ = -\frac{1}{2\sqrt{-1}\pi} \int_{C(0,\rho/2)} \frac{F(u)}{u^2} \left(\int_{L_d} \exp(v(\frac{1}{u} - \frac{1}{T})) dv \right) du = \frac{1}{2\sqrt{-1}\pi} \int_{C(0,\rho/2)} \frac{F(u)}{u} \frac{T}{T-u} du.$$

On the other hand, provided that $T \in S_\psi$, the function $u \mapsto \frac{F(u)}{u} \frac{T}{T-u}$ is holomorphic on $D(0, \rho/2) \setminus \{0\}$ except at the singular points $\Theta \cup \{T\}$, for which the residue can be computed explicitely

$$(37) \quad \text{Res}_{u=T} \left(\frac{F(u)}{u} \frac{T}{T-u} \right) = -F(T), \quad \text{Res}_{u=p_n} \left(\frac{F(u)}{u} \frac{T}{T-u} \right) = \text{Res}_{u=p_n} (F(u)) \frac{T}{p_n(T-p_n)}$$

for all $n \in \mathbb{Z}^*$. By definition of the residue, and following a similar line of arguments as in the classical proof of the residue formula (see [15], Chap. 6), we can find a holomorphic function $u \mapsto G_T(u)$ on \mathbb{C} except at $\Theta \cup \{T\}$ (where it has simple poles) such that the function

$$H_T(u) = \frac{F(u)}{u} \frac{T}{T-u} - G_T(u)$$

is holomorphic on the punctured disc $D(0, \rho/2) \setminus \{0\}$ and such that

$$(38) \quad \frac{1}{2\sqrt{-1}\pi} \int_{C(0,\rho/2)} \frac{F(u)}{u} \frac{T}{T-u} du = F(T) - \psi(T) + \frac{1}{2\sqrt{-1}\pi} \int_{C(0,\rho/2)} H_T(u) du.$$

By Cauchy's formula, one can deform the circle $C(0, \rho/2)$ into any circle $C(0, \delta)$ centered at 0 with radius $0 < \delta \leq \rho/2$ without changing the value of the integral,

$$(39) \quad \frac{1}{2\sqrt{-1}\pi} \int_{C(0,\rho/2)} H_T(u) du = \frac{1}{2\sqrt{-1}\pi} \int_{C(0,\delta)} H_T(u) du.$$

Now, we set $\Delta_0 > 0$ a small positive real number and we split the circle $C(0, \delta)$ into the union of four arcs of circles

$$C(0, \delta) = \bigcup_{j=1}^4 A_{j,\Delta_0}(0, \delta)$$

where

$$\begin{aligned} A_{1,\Delta_0}(0, \delta) &= \{u \in \mathbb{C} / |u| = \delta, \arg(l_0) + \Delta_0 \leq \arg(u) \leq \arg(l_0) + \pi - \Delta_0\}, \\ A_{2,\Delta_0}(0, \delta) &= \{u \in \mathbb{C} / |u| = \delta, \arg(l_0) + \pi - \Delta_0 \leq \arg(u) \leq \arg(l_0) + \pi + \Delta_0\}, \\ A_{3,\Delta_0}(0, \delta) &= \{u \in \mathbb{C} / |u| = \delta, \arg(l_0) + \pi + \Delta_0 \leq \arg(u) \leq \arg(l_0) + 2\pi - \Delta_0\}, \\ A_{4,\Delta_0}(0, \delta) &= \{u \in \mathbb{C} / |u| = \delta, \arg(l_0) + 2\pi - \Delta_0 \leq \arg(u) \leq \arg(l_0) + 2\pi + \Delta_0\}. \end{aligned}$$

Accordingly, we decompose the integral along the circle into four parts

$$(40) \quad \frac{1}{2\sqrt{-1}\pi} \int_{C(0,\delta)} H_T(u) du = \sum_{j=1}^4 \frac{1}{2\sqrt{-1}\pi} \int_{A_{j,\Delta_0}(0,\delta)} H_T(u) du.$$

From the hypotheses of the second item of 1), we can compute the next limit

$$(41) \quad \begin{aligned} \lim_{\delta \rightarrow 0} \frac{1}{2\sqrt{-1}\pi} \int_{A_{1,\Delta_0}(0,\delta)} H_T(u) du &= \lim_{\delta \rightarrow 0} \frac{1}{2\sqrt{-1}\pi} \int_{A_{1,\Delta_0}(0,\delta)} \frac{1}{u} \left(F(u) \frac{T}{T-u} - u G_T(u) \right) du \\ &= \lim_{\delta \rightarrow 0} \frac{1}{2\sqrt{-1}\pi} \int_{\arg(l_0)+\Delta_0}^{\arg(l_0)+\pi-\Delta_0} \frac{1}{\delta e^{\sqrt{-1}\theta}} \left(F(\delta e^{\sqrt{-1}\theta}) \frac{T}{T-\delta e^{\sqrt{-1}\theta}} - \delta e^{\sqrt{-1}\theta} G_T(\delta e^{\sqrt{-1}\theta}) \right) \\ &\quad \times \delta \sqrt{-1} e^{\sqrt{-1}\theta} d\theta = \frac{1}{2\pi} F_0(\pi - 2\Delta_0) \end{aligned}$$

and in a similar manner we get

$$(42) \quad \lim_{\delta \rightarrow 0} \frac{1}{2\sqrt{-1}\pi} \int_{A_{3,\Delta_0}(0,\delta)} H_T(u) du = -\frac{1}{2\pi} F_0(\pi - 2\Delta_0).$$

On the other hand, since $H_T(u)$ is in particular continuous on the circle $C(0, \delta)$, we get that

$$(43) \quad \lim_{\Delta_0 \rightarrow 0} \frac{1}{2\sqrt{-1}\pi} \int_{A_{j,\Delta_0}(0,\delta)} H_T(u) du = 0$$

for $j = 2, 4$, all $\delta > 0$ fixed. As a result, owing to the decomposition (40) and the above estimates (41), (42) together with (43), we observe that

$$\lim_{\delta \rightarrow 0} \frac{1}{2\sqrt{-1}\pi} \int_{C(0,\delta)} H_T(u) du = 0.$$

Therefore, according to (39), we reach that

$$(44) \quad \frac{1}{2\sqrt{-1}\pi} \int_{C(0,\rho/2)} H_T(u) du = 0.$$

As a result, the identity (35) follows from (36), (38) and (44). \square

We now roughly explain the strategy which leads to the expansion (32). Assume that we can find a function $b(u, \epsilon)$ that fulfills the next identity

$$(45) \quad b\left(-\frac{1}{a(T, \epsilon)}, \epsilon\right) = T.$$

We denote $\Theta = \{p_n / n \in \mathbb{Z}^*\}$ the set of poles of $u \mapsto b(u, \epsilon)$. From the identity (35), we get the next decomposition

$$(L_1^d(z \mapsto B_1(u \mapsto b(u, \epsilon))))(T) = b(T, \epsilon) - \sum_{n \in \mathbb{Z}^*} \text{Res}_{u=p_n}(b(u, \epsilon)) \frac{T}{p_n(T - p_n)}.$$

If one replaces T by $-1/a(T, \epsilon)$ in the latter decomposition, we can express T as a sum of a special Laplace transform of an entire function and a special series

$$(46) \quad T = (L_1^d(z \mapsto B_1(u \mapsto b(u, \epsilon))))(-1/a(T, \epsilon)) + \sum_{n \in \mathbb{Z}^*} \text{Res}_{u=p_n}(b(u, \epsilon)) \frac{-1/a(T, \epsilon)}{p_n(-\frac{1}{a(T, \epsilon)} - p_n)}.$$

In a **first step** of the analysis, we compute $b(u, \epsilon)$, describe its singular points and provide bounds w.r.t u and ϵ . We denote $T \mapsto a^{-1}(T, \epsilon)$ the inverse function of $T \mapsto a(T, \epsilon)$ that can be displayed as follows

$$a^{-1}(T, \epsilon) = \epsilon^{\frac{\alpha+1}{2}} \frac{1 + \exp(2\epsilon^{\frac{\alpha+1}{2}} T)}{1 - \exp(2\epsilon^{\frac{\alpha+1}{2}} T)}.$$

According to the requirement (45), we deduce that $b(u, \epsilon)$ can be computed as

$$(47) \quad b(u, \epsilon) = a^{-1}(-1/u, \epsilon) = \epsilon^{\frac{\alpha+1}{2}} \frac{1 + \exp(-2\epsilon^{\frac{\alpha+1}{2}}/u)}{1 - \exp(-2\epsilon^{\frac{\alpha+1}{2}}/u)}.$$

We observe that the function $u \mapsto b(u, \epsilon)$ is holomorphic in \mathbb{C}^* outside a discrete set of simple poles (which relies on ϵ) given by

$$(48) \quad \Theta_\epsilon = \left\{ -\frac{\epsilon^{(\alpha+1)/2}}{\sqrt{-1}\pi n} = p_n/n \in \mathbb{Z}^* \right\}$$

and which is located on the line $\mathcal{L}_{0,\epsilon} = \sqrt{-1}\epsilon^{(\alpha+1)/2}\mathbb{R}$ passing through the origin.

Lemma 2 1) Let $\rho, \epsilon_0 > 0$ be positive real numbers. There exists a constant $C_1 > 0$ (which relies on ρ and ϵ_0) such that

$$(49) \quad |b(u, \epsilon)| \leq C_1$$

for all $u \in C(0, \rho/2)$, $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$, provided that $\epsilon_0 > 0$ is taken small enough.

2) If W_ϵ^+ (resp. W_ϵ^-) is an open sector centered at 0, with radius ρ , with bisecting direction $\arg(\epsilon^{(\alpha+1)/2}) + \pi$ (resp. $\arg(\epsilon^{(\alpha+1)/2})$), such that $W_\epsilon^+ \cap \mathcal{L}_{0,\epsilon} = \emptyset$ (resp. $W_\epsilon^- \cap \mathcal{L}_{0,\epsilon} = \emptyset$), we get

$$\lim_{u \rightarrow 0, u \in W_\epsilon^+} b(u, \epsilon) = -\epsilon^{(\alpha+1)/2}, \quad \lim_{u \rightarrow 0, u \in W_\epsilon^-} b(u, \epsilon) = \epsilon^{(\alpha+1)/2}.$$

Proof 1) We parametrize the circle $C(0, \rho/2)$ by

$$C(0, \rho/2) = \left\{ \frac{\rho}{2} \exp(\sqrt{-1}\theta) / \theta \in [0, 2\pi) \right\}.$$

When u is taken on $C(0, \rho/2)$, the function $b(u, \epsilon)$ is expressed as

$$b(u, \epsilon) = \epsilon^{\frac{\alpha+1}{2}} \frac{1 + \exp(-\frac{4}{\rho}\epsilon^{\frac{\alpha+1}{2}}e^{-\sqrt{-1}\theta})}{1 - \exp(-\frac{4}{\rho}\epsilon^{\frac{\alpha+1}{2}}e^{-\sqrt{-1}\theta})}.$$

On the other hand, we remind that the convexity inequality $|\exp(z)| \leq \exp(|z|)$ holds for all $z \in \mathbb{C}$ and that there exists a holomorphic function $\varepsilon(z)$ with $e^z = 1 + z + z\varepsilon(z)$ such that $\varepsilon(z)$ tends to 0 when z comes close to the origin in \mathbb{C} . As a result, we get that

$$|b(u, \epsilon)| \leq |\epsilon|^{\frac{\alpha+1}{2}} \frac{1 + e^{\frac{4}{\rho}|\epsilon|^{(\alpha+1)/2}}}{\frac{4}{\rho}|\epsilon|^{(\alpha+1)/2} |1 + \varepsilon(-\frac{4}{\rho}\epsilon^{\frac{\alpha+1}{2}}e^{-\sqrt{-1}\theta})|} \leq \frac{\rho}{6} (1 + e^{\frac{4}{\rho}\epsilon_0^{(\alpha+1)/2}})$$

provided that $\epsilon_0 > 0$ is chosen small enough. The first part 1) of the lemma follows.

2) We take $u \in W_\epsilon^+$ and $v \in W_\epsilon^-$. By construction, we can write

$$u = -s_1 \epsilon^{\frac{\alpha+1}{2}} e^{\sqrt{-1}\theta_1}, \quad v = s_2 \epsilon^{\frac{\alpha+1}{2}} e^{\sqrt{-1}\theta_2}$$

for some radii $s_1, s_2 \in [0, \rho/|\epsilon|^{(\alpha+1)/2}]$ and angles $\theta_1, \theta_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Hence, we can compute

$$b(u, \epsilon) = \epsilon^{\frac{\alpha+1}{2}} \frac{1 + \exp(-\frac{2}{s_1}e^{-\sqrt{-1}\theta_1})}{-1 + \exp(-\frac{2}{s_1}e^{-\sqrt{-1}\theta_1})}, \quad b(v, \epsilon) = \epsilon^{\frac{\alpha+1}{2}} \frac{1 + \exp(-\frac{2}{s_2}e^{-\sqrt{-1}\theta_2})}{1 + \exp(-\frac{2}{s_2}e^{-\sqrt{-1}\theta_2})}$$

and since

$$|\exp(-\frac{2}{s_j}e^{-\sqrt{-1}\theta_1})| = \exp(-\frac{2}{s_j} \cos(\theta_j)) \rightarrow 0$$

as $s_j > 0$ tends to 0, for $j = 1, 2$, we deduce

$$\lim_{s_1 \rightarrow 0, s_1 > 0} b(u, \epsilon) = -\epsilon^{\frac{\alpha+1}{2}} \quad , \quad \lim_{s_2 \rightarrow 0, s_2 > 0} b(v, \epsilon) = \epsilon^{\frac{\alpha+1}{2}}.$$

The second point 2) follows. \square

In the next lemma, we exhibit estimates for the analytic Borel transform of $b(u, \epsilon)$.

Lemma 3 *We set*

$$(50) \quad \omega_1(z, \epsilon) = B_1(u \mapsto b(u, \epsilon))(z)$$

as the analytic Borel transform of $u \mapsto b(u, \epsilon)$. The function $z \mapsto \omega_1(z, \epsilon)$ defines an entire function on \mathbb{C} for all $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$, moreover one can select two constants $C, M > 0$ (which rely on ρ and C_1 introduced in Lemma 2 1) such that

$$(51) \quad |\omega_1(z, \epsilon)| \leq C|z| \exp(M|z|)$$

for all $z \in \mathbb{C}$, all $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$, provided that ϵ_0 is close enough to 0.

Proof By Definition of the analytic Borel transform (34) and the bounds (49), we get that

$$|\omega_1(z, \epsilon)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| b\left(\frac{\rho}{2}e^{\sqrt{-1}\theta}, \epsilon\right) \right| \left| \exp\left(\frac{z}{\rho e^{\sqrt{-1}\theta}/2}\right) \right| \frac{|z|}{\rho/2} d\theta \leq \frac{2C_1}{\rho} |z| \exp\left(\frac{2}{\rho}|z|\right)$$

for all $z \in \mathbb{C}$, all $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$, whenever $\epsilon_0 > 0$ is small enough. \square

In a **second step**, we focus on the special series

$$(52) \quad \psi(T, \epsilon) = \sum_{n \in \mathbb{Z}^*} \text{Res}_{u=p_n}(b(u, \epsilon)) \frac{T}{p_n(T - p_n)}.$$

We first need to compute the residue of the function $u \mapsto b(u, \epsilon)$ at the singular points of the set Θ_ϵ .

Lemma 4 *The residue of $u \mapsto b(u, \epsilon)$ at the singular points p_n of (48) are given by*

$$(53) \quad \text{Res}_{u=p_n}(b(u, \epsilon)) = -p_n^2$$

for all $n \in \mathbb{Z}^*$.

Proof If one sets $f(u) = 1 - \exp(-2\epsilon^{(\alpha+1)/2}/u)$ and $g(u) = 1 + \exp(-2\epsilon^{(\alpha+1)/2}/u)$, we observe that

$$\text{Res}_{u=p_n}(b(u, \epsilon)) = \epsilon^{(\alpha+1)/2} \frac{g(p_n)}{f'(p_n)} = -p_n^2$$

for all $n \in \mathbb{Z}^*$. \square

In the next lemma, we observe that each linear fractional map $T \mapsto \frac{T}{p_n(T - p_n)}$ can be expressed as a Laplace transform through an explicit formula. However, the direction of integration depends on the sign of n .

Lemma 5 1) Let $n \geq 1$ be a positive integer. We choose a direction $d^+ \in \mathbb{R}$ such that

$$(54) \quad d^+ = \arg(\epsilon^{\frac{\alpha+1}{2}}) - \Delta^+$$

for some $\Delta^+ > 0$ not too large (less than $\pi/2$). Then, the next formula

$$(55) \quad \frac{T}{p_n(T - p_n)} = \int_{L_{d^+}} -\frac{\tau}{p_n^2} \exp\left(\frac{\tau}{p_n}\right) \exp\left(-\frac{\tau}{T}\right) \frac{d\tau}{\tau}$$

where $L_{d^+} = \mathbb{R}_+ e^{\sqrt{-1}d^+}$, is valid whenever $T \in \mathbb{C}^*$ satisfies

$$(56) \quad \cos(d^+ - \arg(T)) > 0.$$

2) Let $n \leq -1$ be a negative integer. We take a direction $d^- \in \mathbb{R}$ with

$$(57) \quad d^- = \arg(\epsilon^{\frac{\alpha+1}{2}}) + \Delta^-$$

for some $\Delta^- > 0$ not too large (less than $\pi/2$). Then, the next identity

$$(58) \quad \frac{T}{p_n(T - p_n)} = \int_{L_{d^-}} -\frac{\tau}{p_n^2} \exp\left(\frac{\tau}{p_n}\right) \exp\left(-\frac{\tau}{T}\right) \frac{d\tau}{\tau}$$

holds provided that $T \in \mathbb{C}^*$ fulfills

$$(59) \quad \cos(d^- - \arg(T)) > 0.$$

According to Lemma 4 and 5, we notice that the special series $\psi(T, \epsilon)$ can be expressed through the next limit formula

$$(60) \quad \psi(T, \epsilon) = \lim_{\delta \rightarrow 0, \delta > 0} \psi_\delta(T, \epsilon)$$

for

$$\psi_\delta(T, \epsilon) = \sum_{n \geq 1} \int_{L_{d^+, \delta}} \exp\left(\frac{\tau}{p_n}\right) \exp\left(-\frac{\tau}{T}\right) d\tau + \sum_{n \geq 1} \int_{L_{d^-, \delta}} \exp\left(\frac{\tau}{p_n}\right) \exp\left(-\frac{\tau}{T}\right) d\tau$$

where

$$L_{d^-, \delta} = [\delta, +\infty) e^{\sqrt{-1}d^-}, \quad L_{d^+, \delta} = [\delta, +\infty) e^{\sqrt{-1}d^+}$$

represent halflines at distance δ of the origin, whenever T is subjected to both (56) and (59).

In the last part of the proof, our goal will be to provide an explicit formula for the right handside of the identity (60) as exponential series involving Laplace transforms along the single direction d^- . We set

$$(61) \quad K_1(\tau, \epsilon) = \sum_{n \geq 1} \exp\left(\frac{\tau}{p_n}\right) = \sum_{n \geq 1} (e^{\tau/p_1})^n = \frac{e^{\tau/p_1}}{1 - e^{\tau/p_1}}$$

provided that $|e^{\tau/p_1}| < 1$. The function $\tau \mapsto K_1(\tau, \epsilon)$ can be analytically extended on the whole plane \mathbb{C} (as a function again denoted $K_1(\tau, \epsilon)$) except at the discrete set of points

$$(62) \quad \mathcal{S}_\epsilon = \{-2l\epsilon^{\frac{\alpha+1}{2}}/l \in \mathbb{Z}\}.$$

We also define

$$(63) \quad K_{-1}(\tau, \epsilon) = \sum_{n \geq 1} e^{\tau/p_{-n}} = \sum_{n \geq 1} (e^{\tau/p_{-1}})^n = \frac{e^{\tau/p_{-1}}}{1 - e^{\tau/p_{-1}}} = \frac{e^{-\tau/p_1}}{1 - e^{-\tau/p_1}}$$

whenever $\tau \in L_{d^-, \delta}$. The next lemma holds

Lemma 6 Let $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$ and δ chosen such that

$$(64) \quad 0 < \delta < 2|\epsilon|^{\frac{\alpha+1}{2}}.$$

The next formula holds

$$(65) \quad \begin{aligned} \psi_\delta(T, \epsilon) &= \int_{L_{d^+, \delta}} K_1(\tau, \epsilon) \exp(-\tau/T) d\tau + \int_{L_{d^-, \delta}} K_{-1}(\tau, \epsilon) \exp(-\tau/T) d\tau \\ &= 2\epsilon^{\frac{\alpha+1}{2}} \sum_{n \geq 1} \exp(-2n\epsilon^{\frac{\alpha+1}{2}}/T) + \int_{L_{d^-, \delta}} (-1)e^{-\tau/T} d\tau - \int_{C_{d^-, d^+, \delta}} K_1(\tau, \epsilon) e^{-\tau/T} d\tau \end{aligned}$$

where $C_{d^-, d^+, \delta}$ stands for (a well oriented) arc of circle of radius $\delta > 0$ centered at 0 that joins the two halflines $L_{d^-, \delta}$ and $L_{d^+, \delta}$, provided that $T \in \mathbb{C}^*$ fulfills (56) and (59).

Proof The first equality of (65) is a direct consequence of the uniform convergence of the series (61) (resp. (63)) on every compact segments of the halflines $L_{d^+, \delta}$ (resp. $L_{d^-, \delta}$).

We discuss now the second equality. We integrate the function $\tau \mapsto K_1(\tau, \epsilon) \exp(-\tau/T)$ along the oriented path formed by the union $U_{d^-, d^+, \delta}$ of the segments $-L_{d^-, \delta}$, $C_{d^-, d^+, \delta}$ and $L_{d^+, \delta}$. By construction, the path $U_{d^-, d^+, \delta}$ encloses the set (which represents a subset of \mathcal{S}_ϵ described in (62))

$$\mathcal{S}_\epsilon^- = \{2l\epsilon^{\frac{\alpha+1}{2}}/l \geq 1\}$$

of poles of the map $\tau \mapsto K_1(\tau, \epsilon) \exp(-\tau/T)$, under the constraint (64). The residue theorem implies that the next identity holds

$$(66) \quad \int_{U_{d^-, d^+, \delta}} K_1(\tau, \epsilon) \exp(-\tau/T) d\tau = 2\sqrt{-1}\pi \sum_{n \geq 1} \text{Res}_{\tau=2n\epsilon^{\frac{\alpha+1}{2}}} (K_1(\tau, \epsilon) \exp(-\tau/T))$$

for all $T \in \mathbb{C}^*$ under the restriction (56) and (59).

On the other hand, the residue of the function $\tau \mapsto K_1(\tau, \epsilon) \exp(-\tau/T)$ at $\tau = 2n\epsilon^{\frac{\alpha+1}{2}}$ can be explicitly computed. Namely, set $g(\tau) = e^{\tau/p_1} \exp(-\tau/T)$ and $f(\tau) = 1 - e^{\tau/p_1}$. Then,

$$(67) \quad \begin{aligned} \text{Res}_{\tau=2n\epsilon^{\frac{\alpha+1}{2}}} (K_1(\tau, \epsilon) \exp(-\tau/T)) &= \text{Res}_{\tau=2n\epsilon^{\frac{\alpha+1}{2}}} \left(\frac{g(\tau)}{f(\tau)} \right) = \frac{g(2n\epsilon^{(\alpha+1)/2})}{f'(2n\epsilon^{(\alpha+1)/2})} \\ &= -p_1 \exp(-2n\epsilon^{\frac{\alpha+1}{2}}/T). \end{aligned}$$

Finally, a direct computation shows that the next relation

$$(68) \quad K_1(\tau, \epsilon) + K_{-1}(\tau, \epsilon) = -1$$

takes place for every $\tau \in L_{d^-, \delta}$. At last, gathering (66), (67) and (68) yields the expected expansion (65). \square

The next lemma computes the limit when δ tends to the origin of the integral along the arc of circle appearing in the right handside of the decomposition (65).

Lemma 7 For all fixed $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$, the next limit

$$(69) \quad \lim_{\delta \rightarrow 0, \delta > 0} \int_{C_{d^-, d^+, \delta}} K_1(\tau, \epsilon) e^{-\tau/T} d\tau = -\sqrt{-1}p_1(d^+ - d^-)$$

holds if $T \in \mathbb{C}^*$ suffers (56) and (59).

Proof We parametrize the arc of circle

$$C_{d^-, d^+, \delta} = \{\delta e^{\sqrt{-1}\theta} / \theta \in (d^-, d^+) \}$$

which yields the equality

$$\int_{C_{d^-, d^+, \delta}} K_1(\tau, \epsilon) e^{-\tau/T} d\tau = \int_{d^-}^{d^+} \frac{\exp(\delta e^{\sqrt{-1}\theta} / p_1)}{1 - \exp(\delta e^{\sqrt{-1}\theta} / p_1)} \exp(-\delta e^{\sqrt{-1}\theta} / T) \sqrt{-1} \delta e^{\sqrt{-1}\theta} d\theta.$$

From the expansion $e^z = 1 + z + z\varepsilon(z)$ for a holomorphic function $\varepsilon(z)$ near 0 such that $\lim_{z \rightarrow 0} \varepsilon(z) = 0$, we get in particular that

$$\frac{\delta e^{\sqrt{-1}\theta}}{1 - \exp(\delta e^{\sqrt{-1}\theta} / p_1)} = \frac{-p_1}{1 + \varepsilon(\delta e^{\sqrt{-1}} / p_1)}$$

for any fixed $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$ and $\delta > 0$ small enough. As a result, for any fixed $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$ we get that

$$\lim_{\delta \rightarrow 0, \delta > 0} \int_{C_{d^-, d^+, \delta}} K_1(\tau, \epsilon) e^{-\tau/T} d\tau = \int_{d^-}^{d^+} -\sqrt{-1} p_1 d\theta = -\sqrt{-1} p_1 (d^+ - d^-).$$

□

In the following closing lemma we provide an expression of the special series $\psi(T, \epsilon)$ as an exponential series involving Laplace transforms.

Lemma 8 *Let $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$. The next formula*

$$(70) \quad \psi(T, \epsilon) = 2\epsilon^{\frac{\alpha+1}{2}} \sum_{n \geq 1} \exp(-2n\epsilon^{\frac{\alpha+1}{2}} / T) + \int_{L_{d^-}} (-1) \exp(-\tau/T) d\tau + \epsilon^{\frac{\alpha+1}{2}} \frac{d^- - d^+}{\pi}$$

is valid provided that T satisfies (56) and (59).

Proof The lemma is a direct consequence of (60) together with Lemma 6 and 7. □

As a consequence of the two steps above, one can apply Proposition 2 in order to get the next decomposition

$$(71) \quad b(T, \epsilon) = \int_{L_{d^-}} (\omega_1(\tau, \epsilon) - \tau) \exp(-\tau/T) \frac{d\tau}{\tau} + \epsilon^{\frac{\alpha+1}{2}} \frac{d^- - d^+}{\pi} + 2\epsilon^{\frac{\alpha+1}{2}} \sum_{n \geq 1} \exp(-n\epsilon^{\frac{\alpha+1}{2}} / T)$$

which is valid whenever T fulfills (56) and (59). From Lemma 1, we know that

$$-1/a(T, \epsilon) = \frac{2\epsilon^{\frac{\alpha+1}{2}}}{-\log \left| \frac{x-1}{x+1} \right|} (1 + a_{\text{in}}(x))^{-1}$$

with $\sup_{s \in \chi_1} |a_{\text{in}}(s)|$ close to 0, whenever $T = \epsilon^{(\alpha+1)/2} x \in \epsilon\mathcal{T}_\epsilon$, provided that the outer radius $r_2 > 0$ of χ_0 is close enough to 0. From the constraint (30), we observe in particular that

$$\cos(d^+ - \arg(-1/a(T, \epsilon))) > 0 \quad , \quad \cos(d^- - \arg(-1/a(T, \epsilon))) > 0$$

hold, where $\Delta^+, \Delta^- > 0$ are taken not too large (actually less than $\pi/2$). Therefore, both conditions (56) and (59) where T is replaced by $-1/a(T, \epsilon)$ must hold.

As a result, we can substitute T by $-1/a(T, \epsilon)$ in the identity (71) in order to get a decomposition of the monomial

$$T = \int_{L_{d^-}} (\omega_1(\tau, \epsilon) - \tau) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} + \epsilon^{\frac{\alpha+1}{2}} \frac{d^- - d^+}{\pi} + 2\epsilon^{\frac{\alpha+1}{2}} \sum_{n \geq 1} \exp(n\epsilon^{\frac{\alpha+1}{2}} a(T, \epsilon))$$

that holds whenever $T \in \epsilon\mathcal{T}_\epsilon$ and $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$. Furthermore, from the estimates (51), we deduce in particular that the entire function $z \mapsto \omega_0(z, \epsilon) = \omega_1(z, \epsilon) - z$ is subjected to the next constraint : we can select two constants $C, M > 0$ such that

$$|\omega_0(z, \epsilon)| \leq C \frac{|z|}{1 + |z|^2} \exp(M|z|)$$

for all $z \in \mathbb{C}$, all $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$. Proposition 1 follows.

2.2 Construction of a family of convolution equations

Our next assignment is the expression of the fundamental building blocks T^l in the equation (24) for any integer $l \geq 2$ in terms of special Laplace transforms and exponential transseries. The next proposition holds

Proposition 3 *Let $l \geq 1$. There exists*

- *a sequence of entire functions $\tau \mapsto \omega_{l,n}(\tau, \epsilon)$ on \mathbb{C} , for all $n \geq 0$, all $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$ which suffer the upper bounds*

$$(72) \quad |\omega_{l,n}(\tau, \epsilon)| \leq K_l(L_l)^n \frac{|\tau|}{1 + |\tau|^2} \exp(M|\tau|)$$

for all $\tau \in \mathbb{C}$, all $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$, all $n \geq 0$, for some well chosen real numbers $K_l, L_l > 0$ and $M > 0$ chosen as in (31),

- *a sequence of polynomials $\epsilon \mapsto a_{l,n}(\epsilon)$, for all $n \geq 0$ with the bounds*

$$(73) \quad \sup_{\epsilon \in D(0, \epsilon_0)} |a_{l,n}(\epsilon)| \leq A_l(B_l)^n$$

for all $n \geq 0$, for well selected positive numbers $A_l, B_l > 0$

such that

$$(74) \quad T^l = \sum_{n \geq 0} \left(\int_{L_{d^-}} \omega_{l,n}(\tau, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} + a_{l,n}(\epsilon) \right) \exp(n\epsilon^{\frac{\alpha+1}{2}} a(T, \epsilon)).$$

Proof The construction of the sequences $(\omega_{l,n}(\tau, \epsilon))_{n \geq 0}$ and $(a_{l,n}(\epsilon))_{n \geq 0}$ is obtained by induction on l . Namely, according to Proposition 1, for $l = 1$, we can set

$$(75) \quad \omega_{1,0}(\tau, \epsilon) = \omega_0(\tau, \epsilon) \quad , \quad \omega_{1,n}(\tau, \epsilon) = 0$$

for $n \geq 1$ and

$$(76) \quad a_{1,0}(\epsilon) = \epsilon^{\frac{\alpha+1}{2}} a_0 \quad , \quad a_{1,n}(\epsilon) = 2\epsilon^{\frac{\alpha+1}{2}}$$

for $n \geq 1$. According to the bounds (31) and the choice of initial data (75) and (76), we deduce the existence of $K_1, L_1 > 0$ and $A_1, B_1 > 0$ for which (72) and (73) hold.

Assume that the identity (74) holds for a given $l \geq 1$. Then, we can compute the product

$$\begin{aligned}
 (77) \quad T^{l+1} &= T \times T^l = \left(\sum_{n \geq 0} \left(\int_{L_{d^-}} \omega_{1,n}(\tau, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} + a_{1,n}(\epsilon) \exp(n\epsilon^{\frac{\alpha+1}{2}} a(T, \epsilon)) \right) \right. \\
 &\quad \times \left. \left(\sum_{n \geq 0} \left(\int_{L_{d^-}} \omega_{l,n}(\tau, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} + a_{l,n}(\epsilon) \exp(n\epsilon^{\frac{\alpha+1}{2}} a(T, \epsilon)) \right) \right) \right. \\
 &= \sum_{n \geq 0} \left(\sum_{p+q=n} \left(\int_{L_{d^-}} \omega_{1,p}(\tau, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} + a_{1,p}(\epsilon) \right) \right. \\
 &\quad \times \left. \left(\int_{L_{d^-}} \omega_{l,q}(\tau, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} + a_{l,q}(\epsilon) \right) \right) \exp(n\epsilon^{\frac{\alpha+1}{2}} a(T, \epsilon)).
 \end{aligned}$$

Besides, one defines the convolution product

$$(78) \quad \omega_{1,p}(\tau, \epsilon) *_\tau \omega_{l,q}(\tau, \epsilon) := \tau \int_0^\tau \omega_{1,p}(\tau - s, \epsilon) \omega_{l,q}(s, \epsilon) \frac{ds}{(\tau - s)s}$$

where $p, q \geq 0$ are integers, for all $\tau \in \mathbb{C}$, all $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$ and an application of Fubini's theorem yields the next equality

$$\begin{aligned}
 &\left(\int_{L_{d^-}} \omega_{1,p}(\tau, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} \right) \times \left(\int_{L_{d^-}} \omega_{l,q}(\tau, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} \right) \\
 &= \int_{L_{d^-}} (\omega_{1,p}(\tau, \epsilon) *_\tau \omega_{l,q}(\tau, \epsilon)) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau}.
 \end{aligned}$$

As a result, we set the family of functions $\omega_{l+1,n}(\tau, \epsilon)$ and polynomials $a_{l+1,n}(\epsilon)$, for all $n \geq 0$, as

$$(79) \quad \omega_{l+1,n}(\tau, \epsilon) := \sum_{p+q=n} \{ \omega_{1,p}(\tau, \epsilon) *_\tau \omega_{l,q}(\tau, \epsilon) + \omega_{1,p}(\tau, \epsilon) a_{l,q}(\epsilon) + a_{1,p}(\epsilon) \omega_{l,q}(\tau, \epsilon) \}$$

and

$$(80) \quad a_{l+1,n}(\epsilon) := \sum_{p+q=n} a_{1,p}(\epsilon) a_{l,q}(\epsilon)$$

in order to obtain the identity

$$\begin{aligned}
 (81) \quad &\int_{L_{d^-}} \omega_{l+1,n}(\tau, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} + a_{l+1,n}(\epsilon) \\
 &= \sum_{p+q=n} \left(\int_{L_{d^-}} \omega_{1,p}(\tau, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} + a_{1,p}(\epsilon) \right) \\
 &\quad \times \left(\int_{L_{d^-}} \omega_{l,q}(\tau, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} + a_{l,q}(\epsilon) \right)
 \end{aligned}$$

for all $n \geq 0$, which gives rise to the expansion (74) for T^{l+1} , owing to (77).

In a last step, we need to provide bounds of the form (72) and (73) for $\omega_{l+1,n}(\tau, \epsilon)$ and $a_{l+1,n}(\epsilon)$. We set $K_{l,n} = K_l(L_l)^n$ and $A_{l,n} = A_l(B_l)^n$. According to (75), we notice that

$$(82) \quad \sum_{p+q=n} \omega_{1,p}(\tau, \epsilon) *_{\tau} \omega_{l,q}(\tau, \epsilon) = \omega_{1,0}(\tau, \epsilon) *_{\tau} \omega_{l,n}(\tau, \epsilon).$$

Now, the bounds (72) for $\omega_{1,0}(\tau, \epsilon)$ and $\omega_{l,n}(\tau, \epsilon)$ implies

$$(83) \quad \begin{aligned} |\omega_{1,0}(\tau, \epsilon) *_{\tau} \omega_{l,n}(\tau, \epsilon)| &= \left| \tau \int_0^\tau \omega_{1,0}(\tau - s, \epsilon) \omega_{l,n}(s, \epsilon) \frac{ds}{(\tau - s)s} \right| \\ &\leq |\tau| \int_0^{|\tau|} K_{1,0} K_{l,n} \frac{1}{(1 + (|\tau| - h)^2)(1 + h^2)} dh \exp(M|\tau|) \leq K_{1,0} K_{l,n} F \frac{|\tau|}{1 + |\tau|^2} \exp(M|\tau|) \end{aligned}$$

where $F > 0$ is a constant defined as

$$(84) \quad F = \sup_{x \geq 0} (1 + x^2) \int_0^x \frac{1}{(1 + (x - h)^2)(1 + h^2)} dh$$

which is finite due to Corollary 4.9 of [11].

For $p \geq 1$, all $q \geq 0$, we check that

$$(85) \quad \omega_{1,p}(\tau, \epsilon) a_{l,q} = 0$$

owing to (75). Furthermore, the bounds (72) and (73) on $\omega_{1,0}(\tau, \epsilon)$, $\omega_{l,q}(\tau, \epsilon)$, $a_{l,n}(\epsilon)$, and $a_{1,p}(\epsilon)$ lead to

$$(86) \quad |\omega_{1,0}(\tau, \epsilon) a_{l,n}(\epsilon)| \leq K_{1,0} A_{l,n} \frac{|\tau|}{1 + |\tau|^2} \exp(M|\tau|)$$

and

$$(87) \quad |a_{1,p}(\epsilon) \omega_{l,q}(\tau, \epsilon)| \leq A_{1,p} K_{l,q} \frac{|\tau|}{1 + |\tau|^2} \exp(M|\tau|)$$

for all integers $p, q \geq 0$ with $p + q = n$.

Gathering (79), (82), (83), (85), (86) and (87) gives rise to the next estimates for $\omega_{l+1,n}(\tau, \epsilon)$,

$$(88) \quad |\omega_{l+1,n}(\tau, \epsilon)| \leq \left(K_{1,0} K_{l,n} F + K_{1,0} A_{l,n} + \sum_{p+q=n} A_{1,p} K_{l,q} \right) \frac{|\tau|}{1 + |\tau|^2} \exp(M|\tau|)$$

for all $n \geq 0$.

On the other hand, from the bounds (73) for $a_{1,p}(\epsilon)$ and $a_{l,q}(\epsilon)$ with $p + q = n \geq 0$, the relation (80) implies that the following bounds hold for $a_{l+1,n}$,

$$(89) \quad |a_{l+1,n}(\epsilon)| \leq \sum_{p+q=n} |a_{1,p}(\epsilon)| |a_{l,q}(\epsilon)| \leq \sum_{p+q=n} A_{1,p} A_{l,q}.$$

As a result, we can to choose $A_{l+1}, B_{l+1}, K_{l+1}, L_{l+1} > 0$ such that both inequalities

$$(90) \quad \sum_{p+q=n} A_{1,p} A_{l,q} = \sum_{p+q=n} A_1(B_1)^p (B_l)^q A_l \leq A_1 A_l (n+1) (\max(B_1, B_l))^n \leq A_{l+1} (B_{l+1})^n$$

and

$$(91) \quad K_{1,0}K_{l,n}F + K_{1,0}A_{l,n} + \sum_{p+q=n} A_{1,p}K_{l,q} \leq K_1K_l(L_l)^nF + K_1A_l(B_l)^n + A_1K_l(n+1)(\max(B_1, L_l))^n \leq K_{l+1}(L_{l+1})^n$$

hold for all $n \geq 0$, in order to obtain the bounds (72) and (73) for $\omega_{l+1,n}(\tau, \epsilon)$ and $a_{l+1,n}(\epsilon)$. Namely, we select $B_{l+1} > 0$ and $A_{l+1} > 0$ such that

$$(92) \quad B_{l+1} > \max(B_1, B_l) , \quad \sup_{n \geq 0} (n+1) \left(\frac{\max(B_1, B_l)}{B_{l+1}} \right)^n \leq \frac{A_{l+1}}{A_1 A_l}$$

which implies (90) and we single out $L_{l+1} > 0$ and $K_{l+1} > 0$ under the conditions

$$(93) \quad L_{l+1} > L_l, \quad L_{l+1} > B_l, \quad L_{l+1} > \max(B_1, L_l),$$

$$\sup_{n \geq 0} \left(K_1K_l \left(\frac{L_l}{L_{l+1}} \right)^n F + K_1A_l \left(\frac{B_l}{L_{l+1}} \right)^n + A_1K_l(n+1) \left(\frac{\max(B_1, L_l)}{L_{l+1}} \right)^n \right) \leq K_{l+1}$$

from which (91) follows. \square

At this stage of the proof, we can exhibit the shape of the solution to the equation (24) we are seeking for. Namely, we assume that it can be expressed as an exponential transseries which involves infinitely many special Laplace transforms

$$(94) \quad U_{d^-}(T, z, \epsilon) := \sum_{n \geq 0} \left(\int_{L_{d^-}} W_n(\tau, z, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} \right) \times \exp(n\epsilon^{\frac{\alpha+1}{2}} a(T, \epsilon))$$

where L_{d^-} is the halfline described in Proposition 1. We assume that for each $n \geq 0$, the expression $(\tau, z, \epsilon) \mapsto W_n(\tau, z, \epsilon)$ represents a function on a domain $(S_{d^-} \cup D(0, \rho)) \times D(0, r) \times D(0, \epsilon_0) \setminus \{0\}$, for an unbounded sector S_{d^-} with bisecting direction d^- . We also make the assumption that each integral along L_{d^-} composing U_{d^-} makes sense.

From now on, we select the direction d in the formula (14) to be $d = d^-$. We assume that the sector S_{d^-} and the direction d^- are properly chosen in a way that S_{d^-} avoids the set \mathcal{R}_2 of the roots of the polynomial $P_2(\tau)$, for all $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$.

In the remaining part of this subsection all the computations are made *at a formal level*. They are presented for the purpose to explain the reader how to derive a family of convolution equations that the sequence W_n , $n \geq 0$ is asked to solve in order that the expression $U_{d^-}(T, z, \epsilon)$ fulfills the problem (24), (25). These computations will be justified and made rigorous later on (in the proof of Theorem 1), once we have shown that the resulting convolution equations (100) and (101) possess actual holomorphic solutions subjected to the uniform bounds (149) (This will be the main objective of Section 3).

We first explain the action of the basic differential operator $\mathcal{D}_{\epsilon, \alpha}(\partial_T)$ on each term of the series (94). Indeed, according to the definition of $a(T, \epsilon)$ given in (13), we get

$$(95) \quad \mathcal{D}_{\epsilon, \alpha}(\partial_T) \left[\left(\int_{L_{d^-}} W_n(\tau, z, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} \right) \times \exp(n\epsilon^{\frac{\alpha+1}{2}} a(T, \epsilon)) \right]$$

$$= \left(\int_{L_{d^-}} (\tau + n\epsilon^{\frac{\alpha+1}{2}}) W_n(\tau, z, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} \right) \times \exp(n\epsilon^{\frac{\alpha+1}{2}} a(T, \epsilon)).$$

Furthermore, with the help of the transseries expansion of T^{l_1} displayed in (74), one can expand the next expression $T^{l_1}(\mathcal{D}_{\epsilon,\alpha}(\partial_T))^{l_2}\partial_z^{l_3}U_{d-}(T, z, \epsilon)$ provided that $l_1 \geq 1$. Namely,

$$\begin{aligned}
 (96) \quad & T^{l_1}(\mathcal{D}_{\epsilon,\alpha}(\partial_T))^{l_2}\partial_z^{l_3}U_{d-}(T, z, \epsilon) \\
 &= \left[\sum_{n \geq 0} \left(\int_{L_{d-}} \omega_{l_1,n}(\tau, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} + a_{l_1,n}(\epsilon) \right) \exp(n\epsilon^{\frac{\alpha+1}{2}} a(T, \epsilon)) \right] \\
 &\times \left[\sum_{n \geq 0} \left(\int_{L_{d-}} (\tau + n\epsilon^{\frac{\alpha+1}{2}})^{l_2} \partial_z^{l_3} W_n(\tau, z, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} \right) \times \exp(n\epsilon^{\frac{\alpha+1}{2}} a(T, \epsilon)) \right] \\
 &= \sum_{n \geq 0} \sum_{p+q=n} \left[\left(\int_{L_{d-}} \omega_{l_1,p}(\tau, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} + a_{l_1,p}(\epsilon) \right) \right. \\
 &\quad \left. \times \left(\int_{L_{d-}} (\tau + q\epsilon^{\frac{\alpha+1}{2}})^{l_2} \partial_z^{l_3} W_q(\tau, z, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} \right) \right] \times \exp(n\epsilon^{\frac{\alpha+1}{2}} a(T, \epsilon)).
 \end{aligned}$$

Using Fubini's theorem we get

$$\begin{aligned}
 (97) \quad & \left(\int_{L_{d-}} \omega_{l_1,p}(\tau, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} \right) \times \left(\int_{L_{d-}} (\tau + q\epsilon^{\frac{\alpha+1}{2}})^{l_2} \partial_z^{l_3} W_q(\tau, z, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} \right) \\
 &= \int_{L_{d-}} \left[\omega_{l_1,p}(\tau, \epsilon) *_\tau ((\tau + q\epsilon^{\frac{\alpha+1}{2}})^{l_2} \partial_z^{l_3} W_q(\tau, z, \epsilon)) \right] \times \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau}
 \end{aligned}$$

where

$$\begin{aligned}
 \omega_{l_1,p}(\tau, \epsilon) *_\tau ((\tau + q\epsilon^{\frac{\alpha+1}{2}})^{l_2} \partial_z^{l_3} W_q(\tau, z, \epsilon)) \\
 = \tau \int_0^\tau \omega_{l_1,p}(\tau - s, \epsilon) (s + q\epsilon^{\frac{\alpha+1}{2}})^{l_2} \partial_z^{l_3} W_q(s, z, \epsilon) \frac{ds}{(\tau - s)s}.
 \end{aligned}$$

We now turn to the transseries expansions of the nonlinear term of (24). Indeed,

$$\begin{aligned}
 (98) \quad & U_{d-}^2(T, z, \epsilon) = \left[\sum_{n \geq 0} \left(\int_{L_{d-}} W_n(\tau, z, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} \right) \exp(n\epsilon^{\frac{\alpha+1}{2}} a(T, \epsilon)) \right] \times \\
 & \left[\sum_{n \geq 0} \left(\int_{L_{d-}} W_n(\tau, z, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} \right) \exp(n\epsilon^{\frac{\alpha+1}{2}} a(T, \epsilon)) \right] \\
 &= \sum_{n \geq 0} \sum_{p+q=n} \left[\left(\int_{L_{d-}} W_p(\tau, z, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} \right) \right. \\
 &\quad \left. \times \left(\int_{L_{d-}} W_q(\tau, z, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} \right) \right] \exp(n\epsilon^{\frac{\alpha+1}{2}} a(T, \epsilon)).
 \end{aligned}$$

Again, Fubini's theorem applies and allows us to write

$$\begin{aligned}
 (99) \quad & \left(\int_{L_{d-}} W_p(\tau, z, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} \right) \times \left(\int_{L_{d-}} W_q(\tau, z, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} \right) \\
 &= \int_{L_{d-}} (W_p(\tau, z, \epsilon) *_\tau W_q(\tau, z, \epsilon)) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau}
 \end{aligned}$$

where

$$W_p(\tau, z, \epsilon) *_{\tau} W_q(\tau, z, \epsilon) = \tau \int_0^{\tau} W_p(\tau - s, z, \epsilon) W_q(s, z, \epsilon) \frac{ds}{(\tau - s)s}.$$

We are now ready to display the set of convolution equations which is asked to be fulfilled by the sequence $(W_n)_{n \geq 0}$.

The function $W_0(\tau, z, \epsilon)$ is asked to fulfill the next nonlinear convolution equation

$$(100) \quad \partial_z^S W_0(\tau, z, \epsilon) = \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1 \geq 1} \epsilon^{\Delta_{\mathbf{l}} - l_1} d_{\mathbf{l}}(z, \epsilon) \\ \times \left[\omega_{l_1, 0}(\tau, \epsilon) *_{\tau} (\tau^{l_2} \partial_z^{l_3} W_0(\tau, z, \epsilon)) + a_{l_1, 0}(\epsilon) \tau^{l_2} \partial_z^{l_3} W_0(\tau, z, \epsilon) \right] \\ + \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1=0} \epsilon^{\Delta_{\mathbf{l}} - l_1} d_{\mathbf{l}}(z, \epsilon) \tau^{l_2} \partial_z^{l_3} W_0(\tau, z, \epsilon) + e(z, \epsilon) W_0(\tau, z, \epsilon) *_{\tau} W_0(\tau, z, \epsilon) + \frac{P_1(\tau)}{P_2(\tau)}.$$

For $n \geq 1$, $W_n(\tau, z, \epsilon)$ is required to solve the following linear convolution equation

$$(101) \quad \partial_z^S W_n(\tau, z, \epsilon) = \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1 \geq 1} \epsilon^{\Delta_{\mathbf{l}} - l_1} d_{\mathbf{l}}(z, \epsilon) \\ \times \sum_{p+q=n} \left[\omega_{l_1, p}(\tau, \epsilon) *_{\tau} \{(\tau + q\epsilon^{\frac{\alpha+1}{2}})^{l_2} \partial_z^{l_3} W_q(\tau, z, \epsilon)\} + a_{l_1, p}(\epsilon) (\tau + q\epsilon^{\frac{\alpha+1}{2}})^{l_2} \partial_z^{l_3} W_q(\tau, z, \epsilon) \right] \\ + \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1=0} \epsilon^{\Delta_{\mathbf{l}} - l_1} d_{\mathbf{l}}(z, \epsilon) \times (\tau + n\epsilon^{\frac{\alpha+1}{2}})^{l_2} \partial_z^{l_3} W_n(\tau, z, \epsilon) \\ + e(z, \epsilon) \left(\sum_{p+q=n} W_p(\tau, z, \epsilon) *_{\tau} W_q(\tau, z, \epsilon) \right).$$

3 Resolution of the convolution set of equations within Banach spaces of holomorphic functions

We seek for solutions $W_n(\tau, z, \epsilon)$, $n \geq 0$ of the convolution equations (100) and (101) as formal power series w.r.t z , namely

$$(102) \quad W_n(\tau, z, \epsilon) := \sum_{\beta \geq 0} \frac{W_{n, \beta}(\tau, \epsilon)}{\beta!} z^{\beta}.$$

We first disclose a recursion formula, for each $n \geq 0$, for the sequence of expressions $W_{n, \beta}(\tau, \epsilon)$, $\beta \geq 0$. We need to compute each piece of the equations (100) and (101). Namely, for each $n \geq 0$, we get

$$(103) \quad \partial_z^S W_n(\tau, z, \epsilon) = \sum_{\beta \geq 0} \frac{W_{n, \beta+S}(\tau, \epsilon)}{\beta!} z^{\beta}$$

and

$$\begin{aligned}
 (104) \quad & \sum_{p+q=n} \left[\omega_{l_1,p}(\tau, \epsilon) *_{\tau} \{ (\tau + q\epsilon^{\frac{\alpha+1}{2}})^{l_2} \partial_z^{l_3} W_q(\tau, z, \epsilon) \} + a_{l_1,p}(\epsilon) (\tau + q\epsilon^{\frac{\alpha+1}{2}})^{l_2} \partial_z^{l_3} W_q(\tau, z, \epsilon) \right] \\
 &= \sum_{\beta \geq 0} \left(\sum_{p+q=n} \left[\omega_{l_1,p}(\tau, \epsilon) *_{\tau} \left\{ (\tau + q\epsilon^{\frac{\alpha+1}{2}})^{l_2} \frac{W_{q,\beta+l_3}(\tau, \epsilon)}{\beta!} \right\} \right. \right. \\
 & \quad \left. \left. + a_{l_1,p}(\epsilon) (\tau + q\epsilon^{\frac{\alpha+1}{2}})^{l_2} \frac{W_{q,\beta+l_3}(\tau, \epsilon)}{\beta!} \right] \right) z^{\beta}.
 \end{aligned}$$

Let the convergent Taylor expansion of $d_1(z, \epsilon)$ w.r.t z at 0 be

$$(105) \quad d_1(z, \epsilon) = \sum_{\beta \geq 0} \frac{d_{1,\beta}(\epsilon)}{\beta!} z^{\beta}$$

for all $\epsilon \in D(0, \epsilon_0)$. Owing to (104) and (105), we get

$$\begin{aligned}
 (106) \quad & d_1(z, \epsilon) \times \sum_{p+q=n} \left[\omega_{l_1,p}(\tau, \epsilon) *_{\tau} \{ (\tau + q\epsilon^{\frac{\alpha+1}{2}})^{l_2} \partial_z^{l_3} W_q(\tau, z, \epsilon) \} \right. \\
 & \quad \left. + a_{l_1,p}(\epsilon) (\tau + q\epsilon^{\frac{\alpha+1}{2}})^{l_2} \partial_z^{l_3} W_q(\tau, z, \epsilon) \right] \\
 &= \sum_{\beta \geq 0} \left(\sum_{\beta_1+\beta_2=\beta} \left(\sum_{p+q=n} \left[\omega_{l_1,p}(\tau, \epsilon) *_{\tau} \left\{ (\tau + q\epsilon^{\frac{\alpha+1}{2}})^{l_2} \frac{W_{q,\beta_2+l_3}(\tau, \epsilon)}{\beta_2!} \right\} \right. \right. \right. \\
 & \quad \left. \left. \left. + a_{l_1,p}(\epsilon) (\tau + q\epsilon^{\frac{\alpha+1}{2}})^{l_2} \frac{W_{q,\beta_2+l_3}(\tau, \epsilon)}{\beta_2!} \right] \right) \times \frac{d_{1,\beta_1}(\epsilon)}{\beta_1!} \right) z^{\beta}
 \end{aligned}$$

and also

$$\begin{aligned}
 (107) \quad & d_1(z, \epsilon) \times (\tau + n\epsilon^{\frac{\alpha+1}{2}})^{l_2} \partial_z^{l_3} W_n(\tau, z, \epsilon) \\
 &= \sum_{\beta \geq 0} \left(\sum_{\beta_1+\beta_2=\beta} \frac{d_{1,\beta_1}(\epsilon)}{\beta_1!} (\tau + n\epsilon^{\frac{\alpha+1}{2}})^{l_2} \frac{W_{n,\beta_2+l_3}(\tau, \epsilon)}{\beta_2!} \right) z^{\beta}.
 \end{aligned}$$

On the other hand, we check that

$$(108) \quad \sum_{p+q=n} W_p(\tau, z, \epsilon) *_{\tau} W_q(\tau, z, \epsilon) = \sum_{\beta \geq 0} \left(\sum_{p+q=n} \sum_{h_1+h_2=\beta} \frac{W_{p,h_1}(\tau, \epsilon)}{h_1!} *_{\tau} \frac{W_{q,h_2}(\tau, \epsilon)}{h_2!} \right) z^{\beta}$$

and if one expands $e(z, \epsilon)$ at $z = 0$, namely

$$e(z, \epsilon) = \sum_{\beta \geq 0} \frac{e_{\beta}(\epsilon)}{\beta!} z^{\beta}$$

the next power series expansion for the nonlinear term holds

$$\begin{aligned}
 (109) \quad & e(z, \epsilon) \left(\sum_{p+q=n} W_p(\tau, z, \epsilon) *_{\tau} W_q(\tau, z, \epsilon) \right) \\
 &= \sum_{\beta \geq 0} \left(\sum_{\beta_1+\beta_2=\beta} \frac{e_{\beta_1}(\epsilon)}{\beta_1!} \times \left(\sum_{p+q=n} \sum_{h_1+h_2=\beta_2} \frac{W_{p,h_1}(\tau, \epsilon)}{h_1!} *_{\tau} \frac{W_{q,h_2}(\tau, \epsilon)}{h_2!} \right) \right) z^{\beta}.
 \end{aligned}$$

As a result, we require that the sequence of expressions $W_{0,\beta}(\tau, \epsilon)$, $\beta \geq 0$ fulfills the next nonlinear recursive relation

$$(110) \quad \frac{W_{0,\beta+S}(\tau, \epsilon)}{\beta!} = \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1 \geq 1} \epsilon^{\Delta_{\mathbf{l}} - l_1} \left(\sum_{\beta_1 + \beta_2 = \beta} \left(\omega_{l_1, 0}(\tau, \epsilon) *_{\tau} \left\{ \tau^{l_2} \frac{W_{0, \beta_2 + l_3}(\tau, \epsilon)}{\beta_2!} \right\} \right. \right. \\ \left. \left. + a_{l_1, 0}(\epsilon) \tau^{l_2} \frac{W_{0, \beta_2 + l_3}(\tau, \epsilon)}{\beta_2!} \right) \times \frac{d_{\mathbf{l}, \beta_1}(\epsilon)}{\beta_1!} \right) \\ + \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1 = 0} \epsilon^{\Delta_{\mathbf{l}} - l_1} \left(\sum_{\beta_1 + \beta_2 = \beta} \frac{d_{\mathbf{l}, \beta_1}(\epsilon)}{\beta_1!} \tau^{l_2} \frac{W_{0, \beta_2 + l_3}(\tau, \epsilon)}{\beta_2!} \right) \\ + \sum_{\beta_1 + \beta_2 = \beta} \frac{e_{\beta_1}(\epsilon)}{\beta_1!} \times \left(\sum_{h_1 + h_2 = \beta_2} \frac{W_{0, h_1}(\tau, \epsilon)}{h_1!} *_{\tau} \frac{W_{0, h_2}(\tau, \epsilon)}{h_2!} \right) + \frac{P_1(\tau)}{P_2(\tau)} \delta_{0, \beta}$$

where $\delta_{0,0} = 1$ and $\delta_{0,\beta} = 0$ whenever $\beta \geq 1$, under the assumption that

$$(111) \quad W_{0,j}(\tau, \epsilon) = Q_j(\tau) \quad , \quad 0 \leq j \leq S - 1.$$

This latter constraint stems from the assumption (25) on the Cauchy data $(\partial_z^j U_{d-})(T, 0, \epsilon)$ for $0 \leq j \leq S - 1$. Furthermore, for each $n \geq 1$, we ask the sequence of expressions $W_{n,\beta}(\tau, \epsilon)$, for $n \geq 1$, to be subjected to the next recursive relation

$$(112) \quad \frac{W_{n,\beta+S}(\tau, \epsilon)}{\beta!} \\ = \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1 \geq 1} \epsilon^{\Delta_{\mathbf{l}} - l_1} \left(\sum_{\beta_1 + \beta_2 = \beta} \left(\sum_{p+q=n} [\omega_{l_1, p}(\tau, \epsilon) *_{\tau} \{(\tau + q\epsilon^{\frac{\alpha+1}{2}})^{l_2} \frac{W_{q, \beta_2 + l_3}(\tau, \epsilon)}{\beta_2!}\} \right. \right. \\ \left. \left. + a_{l_1, p}(\epsilon) (\tau + q\epsilon^{\frac{\alpha+1}{2}})^{l_2} \frac{W_{q, \beta_2 + l_3}(\tau, \epsilon)}{\beta_2!} \} \right) \times \frac{d_{\mathbf{l}, \beta_1}(\epsilon)}{\beta_1!} \right) \\ + \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1 = 0} \epsilon^{\Delta_{\mathbf{l}} - l_1} \left(\sum_{\beta_1 + \beta_2 = \beta} \frac{d_{\mathbf{l}, \beta_1}(\epsilon)}{\beta_1!} (\tau + n\epsilon^{\frac{\alpha+1}{2}})^{l_2} \frac{W_{n, \beta_2 + l_3}(\tau, \epsilon)}{\beta_2!} \right) \\ + \sum_{\beta_1 + \beta_2 = \beta} \frac{e_{\beta_1}(\epsilon)}{\beta_1!} \left(\sum_{p+q=n} \sum_{h_1 + h_2 = \beta_2} \frac{W_{p, h_1}(\tau, \epsilon)}{h_1!} *_{\tau} \frac{W_{q, h_2}(\tau, \epsilon)}{h_2!} \right)$$

provided that

$$(113) \quad W_{n,j}(\tau, \epsilon) \equiv 0 \quad , \quad 0 \leq j \leq S - 1$$

that originates from our requirement (25) on the Cauchy data of our problem (24).

We now need to specify in which spaces of functions our sequence of functions $\tau \mapsto W_{n,\beta}(\tau, \epsilon)$ are going to live, provided that $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$. These Banach spaces have already been introduced in a former work of the author in [18]. Namely,

Definition 1 *Let S_d be an unbounded sector centered at 0 with bisecting direction $d \in \mathbb{R}$, $D(0, \rho)$ be a disc centered at 0 with radius $\rho > 0$ and $\sigma > 0$ be a fixed real number. For each integer*

$\beta \geq 0$, we set $F_{(\beta, \sigma, S_d, \rho)}$ as the vector space of holomorphic functions $v : S_d \cup D(0, \rho) \rightarrow \mathbb{C}$ such that the norm

$$\|v(\tau)\|_{(\beta, \sigma, S_d, \rho)} := \sup_{\tau \in S_d \cup D(0, \rho)} |v(\tau)| \frac{1 + |\tau|^2}{|\tau|} \exp(-\sigma r_b(\beta)|\tau|)$$

is finite, where $r_b(\beta)$ represents the partial Riemann series $r_b(\beta) := \sum_{n \geq 0}^{\beta} 1/(n+1)^b$, for some integer $b \geq 2$.

The next technical proposition turns to be essential in the discussion that will lead to the fact that the sequence of functions $\tau \mapsto W_{n, \beta}(\tau, \epsilon)$ actually belong to the space $F_{(\beta, \sigma, S_{d^-}, \rho)}$ for the direction d^- chosen as in Proposition 1 and for some small radius $\rho > 0$.

Proposition 4 Take a real number σ such that $\sigma > M$ for M given in (31). Select a radius $\rho > 0$ such that the disc $D(0, \rho)$ does not contain any element of the set \mathcal{R}_2 of the roots of the polynomial $P_2(\tau)$.

1) There exists a constant $P_{1,2} > 0$ (which relies on $\sigma, r_b(S), P_1, P_2$) such that

$$(114) \quad \|P_1(\tau)/P_2(\tau)\|_{(S, \sigma, S_{d^-}, \rho)} \leq P_{1,2}.$$

2) Let $n, \beta \geq 0$ be integers and $\mathbf{l} = (l_1, l_2, l_3) \in I$. Moreover let $0 \leq \beta_2 \leq \beta$, let $p, q \geq 0$ satisfy the relation $p + q = n$. Finally, let $0 \leq m_1 \leq l_2$. We make the assumption that the map $\tau \mapsto W_{q, \beta_2 + l_3}(\tau, \epsilon)$ belongs to the space $F_{(\beta_2 + l_3, \sigma, S_{d^-}, \rho)}$, for all $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$. Then, the estimates

$$(115) \quad \begin{aligned} & \|\omega_{l_1, p}(\tau, \epsilon) *_{\tau} \{\tau^{m_1} W_{q, \beta_2 + l_3}(\tau, \epsilon)\}\|_{(\beta + S, \sigma, S_{d^-}, \rho)} \\ & \leq K_{l_1} (L_{l_1})^p F m_1^{m_1} e^{-m_1} (\beta + S + 1)^{bm_1} \|W_{q, \beta_2 + l_3}(\tau, \epsilon)\|_{(\beta_2 + l_3, \sigma, S_{d^-}, \rho)} \end{aligned}$$

and

$$(116) \quad \begin{aligned} & \|a_{l_1, p}(\epsilon) \tau^{m_1} W_{q, \beta_2 + l_3}(\tau, \epsilon)\|_{(\beta + S, \sigma, S_{d^-}, \rho)} \\ & \leq A_{l_1} (B_{l_1})^p m_1^{m_1} e^{-m_1} (\beta + S + 1)^{bm_1} \|W_{q, \beta_2 + l_3}(\tau, \epsilon)\|_{(\beta_2 + l_3, \sigma, S_{d^-}, \rho)} \end{aligned}$$

hold where F is given by (84).

3) Let $n, \beta \geq 0$ be integers. Let $p, q \geq 0$ be integers with $p + q = n$ and $h_1, h_2 \geq 0$ integers with $h_1 + h_2 \leq \beta$. Take for granted that $\tau \mapsto W_{p, h_1}(\tau, \epsilon)$ belongs to $F_{(h_1, \sigma, S_{d^-}, \rho)}$ and that $\tau \mapsto W_{q, h_2}(\tau, \epsilon)$ belongs to $F_{(h_2, \sigma, S_{d^-}, \rho)}$. Then, the next bounds

$$(117) \quad \begin{aligned} & \|W_{p, h_1}(\tau, \epsilon) *_{\tau} W_{q, h_2}(\tau, \epsilon)\|_{(\beta + S, \sigma, S_{d^-}, \rho)} \\ & \leq F \|W_{p, h_1}(\tau, \epsilon)\|_{(h_1, \sigma, S_{d^-}, \rho)} \|W_{q, h_2}(\tau, \epsilon)\|_{(h_2, \sigma, S_{d^-}, \rho)} \end{aligned}$$

are valid for F displayed in (84).

Proof We turn to the first point 1). By construction, both sets $D(0, \rho)$ and S_{d^-} a properly chosen in a way that they avoid the roots \mathcal{R}_2 of the polynomial $P_2(\tau)$. We distinguish two cases. *First case* : assume that $\deg(P_1) < \deg(P_2)$. Then, we can get a constant $A_{1,2} > 0$ (which relies on P_1, P_2) with

$$(118) \quad \sup_{\tau \in S_{d^-} \cup D(0, \rho)} |P_1(\tau)/P_2(\tau)| \frac{1 + |\tau|^2}{|\tau|} \exp(-\sigma r_b(S)|\tau|) \leq \sup_{x \geq 0} A_{1,2} \exp(-\sigma r_b(S)x) = A_{1,2}.$$

Second case : assume that $\deg(P_1) \geq \deg(P_2)$. We need to recall the classical estimates for some real numbers $m_1 \geq 0$, $m_2 > 0$,

$$(119) \quad \sup_{x \geq 0} x^{m_1} \exp(-m_2 x) = (m_1/m_2)^{m_1} e^{-m_1}$$

with the convention that $0^0 = 1$. Then, by construction, two constants $\hat{A}_{1,2}, \hat{B}_{1,2} > 0$ can be singled out with

$$(120) \quad \begin{aligned} & \sup_{\tau \in S_{d^-} \cup D(0, \rho)} |P_1(\tau)/P_2(\tau)| \frac{1 + |\tau|^2}{|\tau|} \exp(-\sigma r_b(S)|\tau|) \\ & \leq \sup_{x \geq 0} (\hat{A}_{1,2} x^{\deg(P_1) - \deg(P_2) + 1} + \hat{B}_{1,2}) \exp(-\sigma r_b(S)x) \\ & \leq \hat{A}_{1,2} \left(\frac{\deg(P_1) - \deg(P_2) + 1}{\sigma r_b(S)} \right)^{\deg(P_1) - \deg(P_2) + 1} \exp(-(\deg(P_1) - \deg(P_2) + 1)) + \hat{B}_{1,2}. \end{aligned}$$

We now focus on the second point 2). From Proposition 3, one can choose two constants $K_{l_1}, L_{l_1} > 0$ and $M > 0$ introduced in (31) such that

$$(121) \quad |\omega_{l_1, p}(\tau, \epsilon)| \leq K_{l_1} (L_{l_1})^p \frac{|\tau|}{1 + |\tau|^2} \exp(M|\tau|)$$

holds for all $\tau \in \mathbb{C}$, provided that $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$. Besides, from our hypothesis, we know that

$$(122) \quad |W_{q, \beta_2 + l_3}(\tau, \epsilon)| \leq \|W_{q, \beta_2 + l_3}(\tau, \epsilon)\|_{(\beta_2 + l_3, \sigma, S_{d^-}, \rho)} \frac{|\tau|}{1 + |\tau|^2} \exp(\sigma r_b(\beta_2 + l_3)|\tau|)$$

provided that $\tau \in S_{d^-} \cup D(0, \rho)$ and $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$. Furthermore, since $r_b(\beta_2 + l_3) \geq 1$, we observe that $M \leq \sigma r_b(\beta_2 + l_3)$, and from the definition of F in (84), we deduce the next sequence of bounds

$$(123) \quad \begin{aligned} & \left| \tau \int_0^\tau \omega_{l_1, p}(\tau - s, \epsilon) s^{m_1} W_{q, \beta_2 + l_3}(s, \epsilon) \frac{ds}{(\tau - s)s} \right| \\ & \leq |\tau| \int_0^{|\tau|} K_{l_1} (L_{l_1})^p \frac{1}{1 + (|\tau| - h)^2} \exp(M(|\tau| - h)) h^{m_1} \|W_{q, \beta_2 + l_3}(\tau, \epsilon)\|_{(\beta_2 + l_3, \sigma, S_{d^-}, \rho)} \\ & \quad \times \frac{1}{1 + h^2} \exp(\sigma r_b(\beta_2 + l_3)h) dh \leq K_{l_1} (L_{l_1})^p \|W_{q, \beta_2 + l_3}(\tau, \epsilon)\|_{(\beta_2 + l_3, \sigma, S_{d^-}, \rho)} \\ & \quad \times |\tau| \int_0^{|\tau|} \frac{1}{1 + (|\tau| - h)^2} \frac{1}{1 + h^2} \exp(\sigma r_b(\beta_2 + l_3)(|\tau| - h)) \exp(\sigma r_b(\beta_2 + l_3)h) h^{m_1} dh \\ & \leq K_{l_1} (L_{l_1})^p \|W_{q, \beta_2 + l_3}(\tau, \epsilon)\|_{(\beta_2 + l_3, \sigma, S_{d^-}, \rho)} F \frac{|\tau|}{1 + |\tau|^2} |\tau|^{m_1} \exp(\sigma r_b(\beta_2 + l_3)|\tau|) \\ & \leq \left[K_{l_1} (L_{l_1})^p \|W_{q, \beta_2 + l_3}(\tau, \epsilon)\|_{(\beta_2 + l_3, \sigma, S_{d^-}, \rho)} F \times |\tau|^{m_1} \exp((\sigma r_b(\beta_2 + l_3) - \sigma r_b(\beta + S))|\tau|) \right] \\ & \quad \times \frac{|\tau|}{1 + |\tau|^2} \exp(\sigma r_b(\beta + S)|\tau|) \end{aligned}$$

for all $\tau \in S_{d^-} \cup D(0, \rho)$ and $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$. On the other hand, by construction of $r_b(\beta)$, we observe that

$$(124) \quad r_b(\beta + S) - r_b(\beta_2 + l_3) = \sum_{n=\beta_2 + l_3 + 1}^{\beta + S} \frac{1}{(n+1)^b} \geq \frac{\beta + S - (\beta_2 + l_3)}{(\beta + S + 1)^b} = \frac{\beta_1 + S - l_3}{(\beta + S + 1)^b}$$

for $\beta_1 = \beta - \beta_2$. Then, according to the exponential bounds (119), we observe that

$$(125) \quad \begin{aligned} |\tau|^{m_1} \exp((\sigma r_b(\beta_2 + l_3) - \sigma r_b(\beta + S))|\tau|) &\leq |\tau|^{m_1} \exp\left(-\sigma \frac{\beta_1 + S - l_3}{(\beta + S + 1)^b} |\tau|\right) \\ &\leq \sup_{x \geq 0} x^{m_1} \exp\left(-\sigma \frac{\beta_1 + S - l_3}{(\beta + S + 1)^b} x\right) = m_1^{m_1} e^{-m_1} \frac{(\beta + S + 1)^{bm_1}}{(\beta_1 + S - l_3)^{m_1}} \leq m_1^{m_1} e^{-m_1} (\beta + S + 1)^{bm_1} \end{aligned}$$

for all $\tau \in \mathbb{C}$. Finally, gathering (123) and (125) gives raise to (115).

For the second inequality of 2), we recall the bounds (73) from Proposition 3,

$$(126) \quad \sup_{\epsilon \in D(0, \epsilon_0)} |a_{l_1, p}(\epsilon)| \leq A_{l_1}(B_{l_1})^p$$

and (122) from which the next estimates follow,

$$(127) \quad \begin{aligned} |a_{l_1, p}(\epsilon) \tau^{m_1} W_{q, \beta_2 + l_3}(\tau, \epsilon)| &\leq A_{l_1}(B_{l_1})^p |\tau|^{m_1} \|W_{q, \beta_2 + l_3}(\tau, \epsilon)\|_{(\beta_2 + l_3, \sigma, S_{d-}, \rho)} \\ &\times \frac{|\tau|}{1 + |\tau|^2} \exp(\sigma r_b(\beta_2 + l_3)|\tau|) \leq \left[A_{l_1}(B_{l_1})^p \|W_{q, \beta_2 + l_3}(\tau, \epsilon)\|_{(\beta_2 + l_3, \sigma, S_{d-}, \rho)} \right. \\ &\left. |\tau|^{m_1} \exp((\sigma r_b(\beta_2 + l_3) - \sigma r_b(\beta + S))|\tau|) \right] \times \frac{|\tau|}{1 + |\tau|^2} \exp(\sigma r_b(\beta + S)|\tau|) \end{aligned}$$

whenever $\tau \in S_{d-} \cup D(0, \rho)$ and $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$. Calling in again (125) yields the expected bounds (116).

At last, we discuss the third point 3). Owing to our assumption, we can control the functions W_{p, h_1} and W_{q, h_2} from above as follows

$$|W_{p, h_1}(\tau, \epsilon)| \leq \|W_{p, h_1}(\tau, \epsilon)\|_{(h_1, \sigma, S_{d-}, \rho)} \frac{|\tau|}{1 + |\tau|^2} \exp(\sigma r_b(h_1)|\tau|)$$

and

$$|W_{q, h_2}(\tau, \epsilon)| \leq \|W_{q, h_2}(\tau, \epsilon)\|_{(h_2, \sigma, S_{d-}, \rho)} \frac{|\tau|}{1 + |\tau|^2} \exp(\sigma r_b(h_2)|\tau|)$$

for $\tau \in S_{d-} \cup D(0, \rho)$ and $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$. Observing that $r_b(h_j) \leq r_b(\beta + S)$ for $j = 1, 2$, and bearing in mind the definition of the constant F in (84), we deduce estimates for the convolution product

$$(128) \quad \begin{aligned} &\left| \tau \int_0^\tau W_{p, h_1}(\tau - s, \epsilon) W_{q, h_2}(s, \epsilon) \frac{ds}{(\tau - s)s} \right| \\ &\leq |\tau| \int_0^{|\tau|} \|W_{p, h_1}(\tau, \epsilon)\|_{(h_1, \sigma, S_{d-}, \rho)} \frac{1}{1 + (|\tau| - h)^2} \exp(\sigma r_b(h_1)(|\tau| - h)) \\ &\quad \times \|W_{q, h_2}(\tau, \epsilon)\|_{(h_2, \sigma, S_{d-}, \rho)} \frac{1}{1 + h^2} \exp(\sigma r_b(h_2)h) dh \\ &\leq \left[F \|W_{p, h_1}(\tau, \epsilon)\|_{(h_1, \sigma, S_{d-}, \rho)} \|W_{q, h_2}(\tau, \epsilon)\|_{(h_2, \sigma, S_{d-}, \rho)} \right] \times \frac{|\tau|}{1 + |\tau|^2} \exp(\sigma r_b(\beta + S)|\tau|) \end{aligned}$$

provided that $\tau \in S_{d-} \cup D(0, \rho)$ and $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$. This leads to (117). \square

In the sequel, we define the next sequence of numbers

$$(129) \quad \mathbb{W}_{n, \beta} = \sup_{\epsilon \in D(0, \epsilon_0) \setminus \{0\}} \|W_{n, \beta}(\tau, \epsilon)\|_{(\beta, \sigma, S_{d-}, \rho)}$$

for all $n, \beta \geq 0$.

According to the recursion (110) together with the constraints (111) and taking into account the estimates of Proposition 4, we obtain the next inequalities for the sequence $\mathbb{W}_{0,\beta}$, $\beta \geq 0$,

$$(130) \quad \frac{\mathbb{W}_{0,\beta+S}}{\beta!} \leq \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1 \geq 1} \epsilon_0^{\Delta_1 - l_1} \left(\sum_{\beta_1 + \beta_2 = \beta} \left(K_{l_1} \frac{\mathbb{W}_{0,\beta_2+l_3}}{\beta_2!} F l_2^{l_2} e^{-l_2} (\beta + S + 1)^{bl_2} \right. \right. \\ \left. \left. + A_{l_1} \frac{\mathbb{W}_{0,\beta_2+l_3}}{\beta_2!} l_2^{l_2} e^{-l_2} (\beta + S + 1)^{bl_2} \right) \times \sup_{\epsilon \in D(0, \epsilon_0)} \frac{|d_{\mathbf{l}, \beta_1}(\epsilon)|}{\beta_1!} \right) \\ + \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1 = 0} \epsilon_0^{\Delta_1 - l_1} \left(\sum_{\beta_1 + \beta_2 = \beta} \sup_{\epsilon \in D(0, \epsilon_0)} \frac{|d_{\mathbf{l}, \beta_1}(\epsilon)|}{\beta_1!} \frac{\mathbb{W}_{0,\beta_2+l_3}}{\beta_2!} l_2^{l_2} e^{-l_2} (\beta + S + 1)^{bl_2} \right) \\ + \sum_{\beta_1 + \beta_2 = \beta} \sup_{\epsilon \in D(0, \epsilon_0)} \frac{|e_{\beta_1}(\epsilon)|}{\beta_1!} \times \left(\sum_{h_1 + h_2 = \beta_2} F \frac{\mathbb{W}_{0,h_1}}{h_1!} \frac{\mathbb{W}_{0,h_2}}{h_2!} \right) + P_{1,2} \delta_{0,\beta}$$

for $\delta_{0,0} = 1$ and $\delta_{0,\beta} = 0$ if $\beta \geq 1$, under the condition that

$$(131) \quad \mathbb{W}_{0,j} = \sup_{\epsilon \in D(0, \epsilon_0) \setminus \{0\}} \|Q_j(\tau)\|_{(j, \sigma, S_{d-}, \rho)} \quad , \quad 0 \leq j \leq S-1$$

which are, by construction, finite positive numbers. Besides, owing to the recursion (112) subjected to the conditions (113), with the help of the binomial expansion

$$(\tau + q \epsilon^{\frac{\alpha+1}{2}})^{l_2} = \sum_{m_1 + m_2 = l_2} \frac{l_2!}{m_1! m_2!} \tau^{m_1} \epsilon^{\frac{\alpha+1}{2} m_2} q^{m_2}$$

for $0 \leq q \leq n$, the bounds of proposition 4 allows us to get inequalities for the whole sequence $\mathbb{W}_{n,\beta}$, for any $n \geq 1$, all $\beta \geq 0$. Namely,

$$(132) \quad \frac{\mathbb{W}_{n,\beta+S}}{\beta!} \leq \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1 \geq 1} \epsilon_0^{\Delta_1 - l_1} \left(\sum_{\beta_1 + \beta_2 = \beta} \left(\sum_{p+q=n} \left[\sum_{m_1 + m_2 = l_2} \frac{l_2!}{m_1! m_2!} \epsilon_0^{\frac{\alpha+1}{2} m_2} q^{m_2} \right. \right. \right. \\ \left. \left. \left. \times K_{l_1} (L_{l_1})^p \frac{\mathbb{W}_{q,\beta_2+l_3}}{\beta_2!} F m_1^{m_1} e^{-m_1} (\beta + S + 1)^{bm_1} + \sum_{m_1 + m_2 = l_2} \frac{l_2!}{m_1! m_2!} \epsilon_0^{\frac{\alpha+1}{2} m_2} q^{m_2} \right. \right. \right. \\ \left. \left. \left. \times A_{l_1} (B_{l_1})^p \frac{\mathbb{W}_{q,\beta_2+l_3}}{\beta_2!} m_1^{m_1} e^{-m_1} (\beta + S + 1)^{bm_1} \right] \right) \times \sup_{\epsilon \in D(0, \epsilon_0)} \frac{|d_{\mathbf{l}, \beta_1}(\epsilon)|}{\beta_1!} \right) \\ + \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1 = 0} \epsilon_0^{\Delta_1 - l_1} \left(\sum_{\beta_1 + \beta_2 = \beta} \sup_{\epsilon \in D(0, \epsilon_0)} \frac{|d_{\mathbf{l}, \beta_1}(\epsilon)|}{\beta_1!} \times \left[\sum_{m_1 + m_2 = l_2} \frac{l_2!}{m_1! m_2!} \epsilon_0^{\frac{\alpha+1}{2} m_2} n^{m_2} \right. \right. \\ \left. \left. \times \frac{\mathbb{W}_{n,\beta_2+l_3}}{\beta_2!} m_1^{m_1} e^{-m_1} (\beta + S + 1)^{bm_1} \right] \right) \\ + \sum_{\beta_1 + \beta_2 = \beta} \sup_{\epsilon \in D(0, \epsilon_0)} \frac{|e_{\beta_1}(\epsilon)|}{\beta_1!} \times \left[\sum_{p+q=n} \sum_{h_1 + h_2 = \beta_2} \frac{\mathbb{W}_{p,h_1}}{h_1!} \frac{\mathbb{W}_{q,h_2}}{h_2!} F \right]$$

under the additional condition that

$$(133) \quad \mathbb{W}_{n,j} = 0 \quad , \quad 0 \leq j \leq S-1 \quad , \quad n \geq 1.$$

At this point of the proof, we plan to apply a *majorant series method* in order to be able to provide upper bounds for the whole sequence $\mathbb{W}_{n,\beta}$, for any integers $n, \beta \geq 0$. Indeed, let us introduce a sequence of positive numbers $\mathbf{W}_{n,\beta}$ for integers $n, \beta \geq 0$ which are submitted to the next recursive relations.

For $n = 0$, the sequence $\mathbf{W}_{0,\beta}$, $\beta \geq 0$ fulfills

$$(134) \quad \frac{\mathbf{W}_{0,\beta+S}}{\beta!} = \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1 \geq 1} \epsilon_0^{\Delta_{\mathbf{l}} - l_1} \left(\sum_{\beta_1 + \beta_2 = \beta} \left(K_{l_1} \frac{\mathbf{W}_{0,\beta_2+l_3}}{\beta_2!} F l_2^{l_2} e^{-l_2} (\beta + S + 1)^{bl_2} \right. \right. \\ \left. \left. + A_{l_1} \frac{\mathbf{W}_{0,\beta_2+l_3}}{\beta_2!} l_2^{l_2} e^{-l_2} (\beta + S + 1)^{bl_2} \right) \times \sup_{\epsilon \in D(0, \epsilon_0)} \frac{|d_{\mathbf{l}, \beta_1}(\epsilon)|}{\beta_1!} \right) \\ + \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1 = 0} \epsilon_0^{\Delta_{\mathbf{l}} - l_1} \left(\sum_{\beta_1 + \beta_2 = \beta} \sup_{\epsilon \in D(0, \epsilon_0)} \frac{|d_{\mathbf{l}, \beta_1}(\epsilon)|}{\beta_1!} \frac{\mathbf{W}_{0,\beta_2+l_3}}{\beta_2!} l_2^{l_2} e^{-l_2} (\beta + S + 1)^{bl_2} \right) \\ + \sum_{\beta_1 + \beta_2 = \beta} \sup_{\epsilon \in D(0, \epsilon_0)} \frac{|e_{\beta_1}(\epsilon)|}{\beta_1!} \times \left(\sum_{h_1 + h_2 = \beta_2} F \frac{\mathbf{W}_{0,h_1}}{h_1!} \frac{\mathbf{W}_{0,h_2}}{h_2!} \right) + P_{1,2} \delta_{0,\beta}$$

for $\delta_{0,0} = 1$ and $\delta_{0,\beta} = 0$ if $\beta \geq 1$, strained to

$$(135) \quad \mathbf{W}_{0,j} = \sup_{\epsilon \in D(0, \epsilon_0) \setminus \{0\}} \|Q_j(\tau)\|_{(j, \sigma, S_{d-}, \rho)} , \quad 0 \leq j \leq S-1.$$

For any integer $n \geq 1$, the sequence $\mathbf{W}_{n,\beta}$ obeys the next rule

$$(136) \quad \frac{\mathbf{W}_{n,\beta+S}}{\beta!} = \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1 \geq 1} \epsilon_0^{\Delta_{\mathbf{l}} - l_1} \left(\sum_{\beta_1 + \beta_2 = \beta} \left(\sum_{p+q=n} \left[\sum_{m_1 + m_2 = l_2} \frac{l_2!}{m_1! m_2!} \epsilon_0^{\frac{\alpha+1}{2} m_2} q^{m_2} \right. \right. \right. \\ \times K_{l_1} (L_{l_1})^p \frac{\mathbf{W}_{q,\beta_2+l_3}}{\beta_2!} F m_1^{m_1} e^{-m_1} (\beta + S + 1)^{bm_1} + \sum_{m_1 + m_2 = l_2} \frac{l_2!}{m_1! m_2!} \epsilon_0^{\frac{\alpha+1}{2} m_2} q^{m_2} \\ \times A_{l_1} (B_{l_1})^p \frac{\mathbf{W}_{q,\beta_2+l_3}}{\beta_2!} m_1^{m_1} e^{-m_1} (\beta + S + 1)^{bm_1} \left. \right] \times \sup_{\epsilon \in D(0, \epsilon_0)} \frac{|d_{\mathbf{l}, \beta_1}(\epsilon)|}{\beta_1!} \right) \\ + \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1 = 0} \epsilon_0^{\Delta_{\mathbf{l}} - l_1} \left(\sum_{\beta_1 + \beta_2 = \beta} \sup_{\epsilon \in D(0, \epsilon_0)} \frac{|d_{\mathbf{l}, \beta_1}(\epsilon)|}{\beta_1!} \times \left[\sum_{m_1 + m_2 = l_2} \frac{l_2!}{m_1! m_2!} \epsilon_0^{\frac{\alpha+1}{2} m_2} n^{m_2} \right. \right. \\ \times \frac{\mathbf{W}_{n,\beta_2+l_3}}{\beta_2!} m_1^{m_1} e^{-m_1} (\beta + S + 1)^{bm_1} \left. \right] \\ + \sum_{\beta_1 + \beta_2 = \beta} \sup_{\epsilon \in D(0, \epsilon_0)} \frac{|e_{\beta_1}(\epsilon)|}{\beta_1!} \times \left[\sum_{p+q=n} \sum_{h_1 + h_2 = \beta_2} \frac{\mathbf{W}_{p,h_1}}{h_1!} \frac{\mathbf{W}_{q,h_2}}{h_2!} F \right]$$

for the given vanishing data

$$(137) \quad \mathbf{W}_{n,j} = 0 , \quad 0 \leq j \leq S-1 , \quad n \geq 1.$$

We can check by induction the important fact that

$$(138) \quad \mathbb{W}_{n,\beta} \leq \mathbf{W}_{n,\beta}$$

for any integers $n, \beta \geq 0$. We build up the generating series

$$(139) \quad \mathbf{W}(X, Z) = \sum_{n \geq 0, \beta \geq 0} \frac{\mathbf{W}_{n, \beta}}{\beta!} X^n Z^\beta.$$

Our next intention is to show that these series solve a Cauchy-Kowalevski type PDE, see (145), (146). Indeed, we define the next convergent series

$$(140) \quad \mathbf{G}_{1, l_1}(X) = \sum_{n \geq 0} K_{l_1}(L_{l_1})^n X^n, \quad \mathbf{G}_{2, l_1}(X) = \sum_{n \geq 0} A_{l_1}(B_{l_1})^n X^n$$

and

$$(141) \quad \mathbf{D}_1(Z) = \sum_{\beta \geq 0} \sup_{\epsilon \in D(0, \epsilon_0)} \frac{|d_{1, \beta}(\epsilon)|}{\beta!} Z^\beta, \quad \mathbf{E}(Z) = \sum_{\beta \geq 0} \sup_{\epsilon \in D(0, \epsilon_0)} \frac{|e_\beta(\epsilon)|}{\beta!} Z^\beta$$

for $\mathbf{l} = (l_1, l_2, l_3) \in I$. We now briefly describe the action of basic differential operators on $\mathbf{W}(X, Z)$, namely

$$(142) \quad \partial_z^l \mathbf{W}(X, Z) = \sum_{n \geq 0, \beta \geq 0} \frac{\mathbf{W}_{n, \beta+l}}{\beta!} X^n Z^\beta, \quad (Z \partial_Z)^{m_1} \mathbf{W}(X, Z) = \sum_{n \geq 0, \beta \geq 0} \beta^{m_1} \frac{\mathbf{W}_{n, \beta}}{\beta!} X^n Z^\beta,$$

$$(X \partial_X)^{m_2} \mathbf{W}(X, Z) = \sum_{n \geq 0, \beta \geq 0} n^{m_2} \frac{\mathbf{W}_{n, \beta}}{\beta!} X^n Z^\beta$$

for integers $l, m_1, m_2 \geq 1$ and the product by convergent series

$$\mathbf{G}(X) = \sum_{n \geq 0} \mathbf{G}_n X^n, \quad \mathbf{D}(Z) = \sum_{\beta \geq 0} \frac{\mathbf{D}_\beta}{\beta!} Z^\beta$$

with \mathbf{W} which yields the expansions

$$(143) \quad \mathbf{G}(X) \mathbf{W}(X, Z) = \sum_{n \geq 0, \beta \geq 0} \left(\sum_{p+q=n} \mathbf{G}_p \frac{\mathbf{W}_{q, \beta}}{\beta!} \right) X^n Z^\beta,$$

$$\mathbf{D}(Z) \mathbf{W}(X, Z) = \sum_{n \geq 0, \beta \geq 0} \left(\sum_{\beta_1+\beta_2=\beta} \frac{\mathbf{D}_{\beta_1}}{\beta_1!} \frac{\mathbf{W}_{n, \beta_2}}{\beta_2!} \right) X^n Z^\beta$$

together with

$$(144) \quad \mathbf{W}(X, Z) \mathbf{W}(X, Z) = \sum_{n \geq 0, \beta \geq 0} \left(\sum_{p+q=n} \sum_{h_1+h_2=\beta} \frac{\mathbf{W}_{p, h_1}}{h_1!} \frac{\mathbf{W}_{q, h_2}}{h_2!} \right) X^n Z^\beta.$$

From the recursions (134) and (136) under the Cauchy data (135) and (137), with the help of the above identities (142), (143) and (144) we obtain that $\mathbf{W}(X, Z)$ solves the next Cauchy

problem

$$\begin{aligned}
 (145) \quad \partial_Z^S \mathbf{W}(X, Z) = & \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1 \geq 1} \epsilon_0^{\Delta_{\mathbf{l}} - l_1} \left[\sum_{m_1+m_2=l_2} \frac{l_2!}{m_1!m_2!} \epsilon_0^{\frac{\alpha+1}{2}m_2} F m_1^{m_1} e^{-m_1} \right. \\
 & \times (Z\partial_Z + S + 1)^{bm_1} (\mathbf{G}_{1, l_1}(X) \times (X\partial_X)^{m_2} \partial_Z^{l_3} \mathbf{W}(X, Z) \times \mathbf{D}_{\mathbf{l}}(Z)) + \sum_{m_1+m_2=l_2} \frac{l_2!}{m_1!m_2!} \epsilon_0^{\frac{\alpha+1}{2}m_2} m_1^{m_1} e^{-m_1} \\
 & \left. \times (Z\partial_Z + S + 1)^{bm_1} (\mathbf{G}_{2, l_1}(X) \times (X\partial_X)^{m_2} \partial_Z^{l_3} \mathbf{W}(X, Z) \times \mathbf{D}_{\mathbf{l}}(Z)) \right] \\
 & + \sum_{\mathbf{l}=(l_1, l_2, l_3) \in I, l_1=0} \epsilon_0^{\Delta_{\mathbf{l}} - l_1} \left(\sum_{m_1+m_2=l_2} \frac{l_2!}{m_1!m_2!} \epsilon_0^{\frac{\alpha+1}{2}m_2} m_1^{m_1} e^{-m_1} (Z\partial_Z + S + 1)^{bm_1} (\mathbf{D}_{\mathbf{l}}(Z) \right. \\
 & \left. \times (X\partial_X)^{m_2} \partial_Z^{l_3} \mathbf{W}(X, Z)) \right) + \mathbf{E}(Z) F(\mathbf{W}(X, Z))^2 + P_{1,2}
 \end{aligned}$$

for given constant Cauchy data

$$(146) \quad (\partial_Z^j \mathbf{W})(X, 0) \equiv \mathbf{W}_{0,j} \quad , \quad 0 \leq j \leq S - 1.$$

We now need to call upon the classical Cauchy-Kowalevski theorem (see [12], Chapter 1 for a reference), outlined below.

Theorem CK Consider a non-linear partial differential equation of the form

$$(a) \quad \partial_x^\beta u(t, x) = F(t, x, (\partial_t^{\alpha_0} \partial_x^{\alpha_1} u(t, x))_{(\alpha_0, \alpha_1) \in \mathcal{I}})$$

for some integer $\beta \geq 1$, where $\mathcal{I} = \{(\alpha_0, \alpha_1) \in \mathbb{N}^2 : \alpha_0 + \alpha_1 \leq \beta, \alpha_1 < \beta\}$, and F is analytic in the variables t, x in a neighborhood of the origin in \mathbb{C}^2 and polynomial in its other arguments, with Cauchy data

$$(b) \quad (\partial_x^k u)(t, 0) = \varphi_k(t) \quad , \quad 0 \leq k \leq \beta - 1$$

for $0 \leq k \leq \beta - 1$, where φ_k are analytic functions in a neighborhood of the origin in \mathbb{C} . Then, the problem (a), (b) has a unique solution u that is analytic in a neighborhood of the origin in \mathbb{C}^2 .

Under the hypothesis that (10) hold, we observe that the above theorem applies. Hence, the problem (145), (146) has a unique analytic solution on some polydisc $D(0, X_0) \times D(0, Z_0)$ near 0 in \mathbb{C}^2 for some $X_0, Z_0 > 0$. As a result, the constructed formal series $\mathbf{W}(X, Z)$ is actually convergent on the domain $D(0, X_0) \times D(0, Z_0)$. This means that one can find constants $\mathbf{W} > 0$, $X_1, Z_1 > 0$ such that

$$(147) \quad 0 \leq \mathbf{W}_{n,\beta} \leq \mathbf{W}(X_1)^n (Z_1)^\beta \beta!$$

for all $n, \beta \geq 0$.

In particular, from the lower bounds (138) and the definition of the sequence $\mathbb{W}_{n,\beta}$ in (129), we deduce the next upper bounds for the set of functions $W_{n,\beta}(\tau, \epsilon)$, $n, \beta \geq 0$ that solve the recursive relations (110), (112) for Cauchy data (111), (113).

Proposition 5 One can single out constants $\mathbf{W}, X_1, Z_1 > 0$ with

$$(148) \quad |W_{n,\beta}(\tau, \epsilon)| \leq \mathbf{W}(X_1)^n (Z_1)^\beta \beta! \frac{|\tau|}{1 + |\tau|^2} \exp(\sigma r_b(\beta)|\tau|)$$

whenever $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$, for all $\tau \in S_{d^-} \cup D(0, \rho)$, for a direction d^- (that relies on ϵ) constructed as in Proposition 1 and a small radius ρ chosen in a way that the domain $S_{d^-} \cup D(0, \rho)$ avoids the set of the roots \mathcal{R}_2 of the polynomial $P_2(\tau)$.

As a consequence of the latter proposition, the formal power series $z \mapsto W_n(\tau, z, \epsilon)$ defined in (102) turns out to be an analytic function on the disc $D(0, 1/(2Z_1))$. More precisely, for all $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$, all integers $n \geq 0$, the map $(\tau, z) \mapsto W_n(\tau, z, \epsilon)$ is holomorphic on the product $(S_{d-} \cup D(0, \rho)) \times D(0, 1/(2Z_1))$ and suffers the next bounds

$$(149) \quad |W_n(\tau, z, \epsilon)| \leq 2\mathbf{W}(X_1)^n \frac{|\tau|}{1 + |\tau|^2} \exp(\sigma\zeta(b)|\tau|)$$

where $\zeta(b) = \sum_{n \geq 0} 1/(n+1)^b$, provided that $\tau \in S_{d-} \cup D(0, \rho)$, $z \in D(0, 1/(2Z_1))$. Furthermore, for $n = 0$, $W_0(\tau, z, \epsilon)$ solves (100) and for $n \geq 1$, $W_n(\tau, z, \epsilon)$ fulfills (101) on the above domain.

4 Construction of inner analytic solutions to the main problem and their parametric asymptotic expansions

We build up a set of genuine solutions to our initial problems (11) and (23) subjected to (12). We label these solutions as *inner solutions* in the terminology of the so-called boundary layer expansions since their domain of holomorphy in time turns out to depend on the parameter ϵ and comes close to the origin when ϵ tends to 0. This family of solutions is based on the following definition of so-called good covering in \mathbb{C}^* .

Definition 2 Let $\iota \geq 2$ be an integer. We consider a finite set $\underline{\mathcal{E}} = \{\mathcal{E}_p\}_{0 \leq p \leq \iota-1}$ where \mathcal{E}_p stand for open sectors with vertex at 0 such that $\mathcal{E}_p \subset D(0, \epsilon_0)$ which fulfills the next three assumptions:

- i) $\mathcal{E}_p \cap \mathcal{E}_{p+1} \neq \emptyset$ for all $0 \leq p \leq \iota - 1$ (with the convention that $\mathcal{E}_\iota = \mathcal{E}_0$).
- ii) $\mathcal{E}_{p_1} \cap \mathcal{E}_{p_2} \cap \mathcal{E}_{p_3} = \emptyset$ for any distinct integers $0 \leq p_j \leq \iota - 1$, $j = 1, 2, 3$.
- iii) The union of all the sectors \mathcal{E}_p covers a punctured disc centered at 0 in \mathbb{C} .

Then, the set $\underline{\mathcal{E}}$ is called a good covering in \mathbb{C}^* .

In the forthcoming definition, we describe the notion of admissible set of sectors relatively to a good covering.

Definition 3 We consider a good covering $\underline{\mathcal{E}} = \{\mathcal{E}_p\}_{0 \leq p \leq \iota-1}$ in \mathbb{C}^* and a set of unbounded sectors $S_{\mathfrak{d}_p}$, $0 \leq p \leq \iota - 1$ with bisecting direction $\mathfrak{d}_p \in \mathbb{R}$ that fulfill the next two properties:

- 1) The next inclusion

$$(150) \quad S_{\mathfrak{d}_p} \subset \bigcap_{\epsilon \in \mathcal{E}_p} S_{d-}$$

holds.

2) There exists $\Delta > 0$ such that for all $\epsilon \in \mathcal{E}_p$, one can choose a direction $\gamma_p \in \mathbb{R}$ (that may depend on ϵ) with $L_{\gamma_p} = \mathbb{R}_+ \exp(\sqrt{-1}\gamma_p) \subset S_{\mathfrak{d}_p} \cup \{0\}$ for which

$$(151) \quad \cos(\gamma_p - \arg(\epsilon^{\frac{\alpha+1}{2}}) + \pi) < -\Delta$$

holds.

In that case, the family of sectors $\mathcal{S} = \{S_{\mathfrak{d}_p}\}_{0 \leq p \leq \iota-1}$ is called admissible relatively to $\underline{\mathcal{E}}$.

We now discuss the feasibility of such a construction. Indeed, for a given sector \mathcal{E}_p , we construct the sector

$$S_{\mathfrak{d}_p} = \bigcup_{\epsilon \in \mathcal{E}_p} L_{\arg(\epsilon^{(\alpha+1)/2}) + \Delta_p^-}$$

where

$$L_{\arg(\epsilon^{(\alpha+1)/2})+\Delta_p^-} = (0, +\infty) \exp(\sqrt{-1}(\arg(\epsilon^{(\alpha+1)/2}) + \Delta_p^-))$$

for some positive number $\Delta_p^- > 0$ (which is taken less than $\pi/2$ and relies on p) and for each $\epsilon \in \mathcal{E}_p$, we choose a sector $S_{\arg(\epsilon^{(\alpha+1)/2})+\Delta_p^-}$ with bisecting direction $\arg(\epsilon^{(\alpha+1)/2}) + \Delta_p^-$ and with large enough opening such that

$$S_{\mathfrak{d}_p} \subset S_{\arg(\epsilon^{(\alpha+1)/2})+\Delta_p^-}$$

Furthermore, we adjust $\Delta_p^- > 0$ in a way that the union

$$\bigcup_{\epsilon \in \mathcal{E}_p} S_{\arg(\epsilon^{(\alpha+1)/2})+\Delta_p^-}$$

avoids the set of the roots \mathcal{R}_2 of the polynomial $P_2(\tau)$. Such an arrangement is achievable if the opening of \mathcal{E}_p is taken small enough. Therefore the first property 1) follows.

For the second point 2), let $\epsilon \in \mathcal{E}_p$ and set

$$\gamma_p = \arg(\epsilon^{(\alpha+1)/2}) + \Delta_p^-.$$

Observe in particular that $L_{\gamma_p} = \mathbb{R}_+ e^{\sqrt{-1}\gamma_p} \subset S_{\mathfrak{d}_p} \cup \{0\}$. By construction, we get that

$$\cos(\gamma_p - \arg(\epsilon^{(\alpha+1)/2}) + \pi) = \cos(\Delta_p^- + \pi) < -\Delta$$

for some $\Delta > 0$.

In the first main statement of the work, we construct a family of actual holomorphic solutions to our main problems (11) and (23) under the Cauchy data (12). These solutions are defined on the sectors of a good covering $\underline{\mathcal{E}} = \{\mathcal{E}_p\}_{0 \leq p \leq \iota-1}$ in \mathbb{C}^* w.r.t the perturbation parameter ϵ . We control the difference between neighboring solutions on the intersection of sectors $\mathcal{E}_p \cap \mathcal{E}_{p+1}$ where exponentially flat estimates are witnessed.

Theorem 1 Assume that the requirement (10) on the shape of the equation (11) holds. We fix a good covering $\underline{\mathcal{E}} = \{\mathcal{E}_p\}_{0 \leq p \leq \iota-1}$ in \mathbb{C}^* together with an admissible set of sectors $\mathcal{S} = \{S_{\mathfrak{d}_p}\}_{0 \leq p \leq \iota-1}$ relatively to $\underline{\mathcal{E}}$.

Then, for all $\epsilon \in \mathcal{E}_p$, one can exhibit a solution $(t, z) \mapsto u_p(t, z, \epsilon)$ of the main problems (11) and (23) submitted to the Cauchy data (12), that remains bounded holomorphic on a domain $\mathcal{T}_\epsilon \times D(0, r)$ provided that $r, \epsilon_0 > 0$ are taken small enough. This solution is represented as an exponential transseries expansion which contains infinitely many special Laplace transforms

$$(152) \quad u_p(t, z, \epsilon) = \sum_{n \geq 0} \left(\int_{L_{\gamma_p}} W_n(\tau, z, \epsilon) \exp(\tau a(\epsilon t, \epsilon)) \frac{d\tau}{\tau} \right) \exp\left(n\epsilon^{\frac{\alpha+1}{2}} a(\epsilon t, \epsilon)\right)$$

where $W_n(\tau, z, \epsilon)$ is the sequence of functions disclosed at the end of Section 3, for $\epsilon \in \mathcal{E}_p$.

Furthermore, the functions

$$(x, z, \epsilon) \mapsto u_p(\epsilon^{\frac{\alpha-1}{2}} x, z, \epsilon)$$

are bounded holomorphic on the domain $\chi_1 \times D(0, r) \times \mathcal{E}_p$, for $0 \leq p \leq \iota-1$. These functions suffer the next bounds: There exist constants $K_p, M_p > 0$ such that

$$(153) \quad \sup_{x \in \chi_1, z \in D(0, r)} \left| u_{p+1}(\epsilon^{\frac{\alpha-1}{2}} x, z, \epsilon) - u_p(\epsilon^{\frac{\alpha-1}{2}} x, z, \epsilon) \right| \leq K_p \exp\left(-M_p/|\epsilon|^{\frac{\alpha+1}{2}}\right)$$

for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, for $0 \leq p \leq \iota-1$ (where by convention $u_\iota = u_0$).

Proof In view of the feature (150) for the admissible set \mathcal{S} , we remind the reader the construction we have reached at the end of Section 3. Namely, we have singled out a sequence of functions $W_n(\tau, z, \epsilon)$, $n \geq 0$ satisfying the following property:

Let $0 \leq p \leq \iota - 1$ and $n \geq 0$ an integer, for all $\epsilon \in \mathcal{E}_p$, the map $(\tau, z) \mapsto W_n(\tau, z, \epsilon)$ is holomorphic on the product $(S_{\mathfrak{d}_p} \cup D(0, \rho)) \times D(0, r)$, provided that $0 < r \leq 1/(2Z_1)$ and is subjected to the bounds (149) whenever $\tau \in S_{\mathfrak{d}_p} \cup D(0, \rho)$ and $z \in D(0, r)$.

For each $0 \leq p \leq \iota - 1$, we define the function

$$U_p(T, z, \epsilon) = \sum_{n \geq 0} \left(\int_{L_{\gamma_p}} W_n(\tau, z, \epsilon) \exp(\tau a(T, \epsilon)) \frac{d\tau}{\tau} \right) \exp \left(n\epsilon^{\frac{\alpha+1}{2}} a(T, \epsilon) \right)$$

where the integration halfline L_{γ_p} is chosen accordingly to Definition 3. We now show that when $\epsilon \in \mathcal{E}_p$, the map $(T, z) \mapsto U_p(T, z, \epsilon)$ is well defined on the domain $\epsilon\mathcal{T}_\epsilon \times D(0, r)$ whenever the outer radius $r_2 > 0$ of χ_0 in (26) is taken small enough. Indeed, owing to the factorization (28), the next bounds hold

$$(154) \quad |U_p(T, z, \epsilon)| \leq \sum_{n \geq 0} \int_0^{+\infty} 2\mathbf{W}(X_1)^n \frac{1}{1+s^2} \exp(\sigma\zeta(b)s) \\ \times \exp \left(\frac{s}{2|\epsilon|^{\frac{\alpha+1}{2}}} \left| \log \left| \frac{x-1}{x+1} \right| \right| |1+a_{\text{in}}(x)| \times \cos \left(\gamma_p - \arg(\epsilon^{\frac{\alpha+1}{2}}) + \pi + \arg(1+a_{\text{in}}(x)) \right) \right) ds \\ \times \exp \left(\frac{n}{2} \left| \log \left| \frac{x-1}{x+1} \right| \right| |1+a_{\text{in}}(x)| \times \cos \left(\pi + \arg(1+a_{\text{in}}(x)) \right) \right)$$

for $\epsilon \in \mathcal{E}_p$ and $(T, z) \in \epsilon\mathcal{T}_\epsilon \times D(0, r)$.

Bearing in mind (29), the lower bounds

$$(155) \quad |1+a_{\text{in}}(x)| \geq 1 - \delta$$

hold for $x \in \chi_1$, whenever $\delta > 0$ is small enough and from (30), we deduce that

$$(156) \quad \left| \log \left| \frac{x-1}{x+1} \right| \right| > \log \left(\frac{2-r_2}{r_2} \right)$$

provided that $r_2 > 0$ is small enough, when $x \in \chi_1$. Besides, owing to the constraint (151), we deduce that

$$(157) \quad \cos \left(\gamma_p - \arg(\epsilon^{\frac{\alpha+1}{2}}) + \pi + \arg(1+a_{\text{in}}(x)) \right) < -\Delta$$

for all $x \in \chi_1$ whenever $r_2 > 0$ is taken small enough. Similarly, we can get a constant $\Delta_1 > 0$ for which

$$(158) \quad \cos \left(\pi + \arg(1+a_{\text{in}}(x)) \right) < -\Delta_1$$

when x belongs to χ_1 and $r_2 > 0$ is chosen close to 0.

Now, we choose $r_2 > 0$ in the vicinity of 0 in a way that both inequalities

$$(159) \quad 0 < X_1 \exp \left(-\frac{1}{2} \log \left(\frac{2-r_2}{r_2} \right) (1-\delta)\Delta_1 \right) \leq 1/2$$

and

$$(160) \quad \sigma\zeta(b) - \frac{1}{2\epsilon_0^{\frac{\alpha+1}{2}}} \log \left(\frac{2-r_2}{r_2} \right) (1-\delta)\Delta \leq -1/2$$

hold. Applying the lower bounds (155), (156) together with (157), (158) to (154) under the additional restrictions (159), (160), we deduce that

$$|U_p(T, z, \epsilon)| \leq 2\mathbf{W} \int_0^{+\infty} e^{-s/2} ds \times \sum_{n \geq 0} (1/2)^n = 8\mathbf{W}$$

for $\epsilon \in \mathcal{E}_p$ and $(T, z) \in \epsilon\mathcal{T}_\epsilon \times D(0, r)$. Furthermore, from the bounds given above, for all $\epsilon \in \mathcal{E}_p$, the series defining $U_p(T, z, \epsilon)$ converge uniformly on the domain $\epsilon\mathcal{T}_\epsilon \times D(0, r)$. Therefore, the map $(T, z) \mapsto U_p(T, z, \epsilon)$ is holomorphic on the domain $\epsilon\mathcal{T}_\epsilon \times D(0, r)$. Since the function $W_0(\tau, z, \epsilon)$ solves the convolution equation (100) and $W_n(\tau, z, \epsilon)$ satisfies (101) for all $n \geq 1$, the sequence of formal computations (95), (96), (97), (98), (99) are now justified and once performed they show that, for all $\epsilon \in \mathcal{E}_p$, the map $(T, z) \mapsto U_p(T, z, \epsilon)$ solves the Cauchy problem (24), (25) on the domain $\epsilon\mathcal{T}_\epsilon \times D(0, r)$.

At last, for $0 \leq p \leq \iota - 1$, we set forth

$$u_p(t, z, \epsilon) = U_p(\epsilon t, z, \epsilon) = \sum_{n \geq 0} \left(\int_{L_{\gamma_p}} W_n(\tau, z, \epsilon) \exp(\tau a(\epsilon t, \epsilon)) \frac{d\tau}{\tau} \right) \exp\left(n\epsilon^{\frac{\alpha+1}{2}} a(\epsilon t, \epsilon)\right).$$

According to the construction above, for each $0 \leq p \leq \iota - 1$, the function $(t, z) \mapsto u_p(t, z, \epsilon)$ is bounded, holomorphic and solves the main Cauchy problem (11), (12) (and hence the singularly perturbed Cauchy problem (23), (12)) on the domain $\mathcal{T}_\epsilon \times D(0, r)$. Furthermore, the maps

$$(x, z, \epsilon) \mapsto u_p(\epsilon^{\frac{\alpha-1}{2}} x, z, \epsilon)$$

represent bounded holomorphic functions on the domain $\chi_1 \times D(0, r) \times \mathcal{E}_p$, for all $0 \leq p \leq \iota - 1$.

In the second part of the proof, we discuss the bounds (153). Let $0 \leq p \leq \iota - 1$. For all integers $n \geq 0$, the partial maps

$$\tau \mapsto W_n(\tau, z, \epsilon) \exp(\tau a(\epsilon t, \epsilon)) / \tau$$

are bounded and holomorphic on the disc $D(0, \rho)$, provided that $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$ and $z \in D(0, r)$. By Cauchy's theorem, its integral must vanish along the next oriented path depicted as the union of

- a) the segment joining 0 and $(\rho/2) \exp(\sqrt{-1}\gamma_{p+1})$,
- b) the arc of circle with radius $\rho/2$ which relates the two points $(\rho/2) \exp(\sqrt{-1}\gamma_{p+1})$ and $(\rho/2) \exp(\sqrt{-1}\gamma_p)$,
- c) the segment which attaches $(\rho/2) \exp(\sqrt{-1}\gamma_p)$ and 0.

As a result, we can split the difference $u_{p+1} - u_p$ into a sum of three exponential transseries. Namely,

$$\begin{aligned}
 (161) \quad & u_{p+1}(\epsilon^{\frac{\alpha-1}{2}} x, z, \epsilon) - u_p(\epsilon^{\frac{\alpha-1}{2}} x, z, \epsilon) \\
 &= \sum_{n \geq 0} \left(\int_{L_{\rho/2, \gamma_{p+1}}} W_n(\tau, z, \epsilon) \exp(\tau a(\epsilon^{\frac{\alpha+1}{2}} x, \epsilon)) \frac{d\tau}{\tau} \right) \exp\left(n\epsilon^{\frac{\alpha+1}{2}} a(\epsilon^{\frac{\alpha+1}{2}} x, \epsilon)\right) \\
 &\quad - \sum_{n \geq 0} \left(\int_{L_{\rho/2, \gamma_p}} W_n(\tau, z, \epsilon) \exp(\tau a(\epsilon^{\frac{\alpha+1}{2}} x, \epsilon)) \frac{d\tau}{\tau} \right) \exp\left(n\epsilon^{\frac{\alpha+1}{2}} a(\epsilon^{\frac{\alpha+1}{2}} x, \epsilon)\right) \\
 &\quad + \sum_{n \geq 0} \left(\int_{C_{\rho/2, \gamma_p, \gamma_{p+1}}} W_n(\tau, z, \epsilon) \exp(\tau a(\epsilon^{\frac{\alpha+1}{2}} x, \epsilon)) \frac{d\tau}{\tau} \right) \exp\left(n\epsilon^{\frac{\alpha+1}{2}} a(\epsilon^{\frac{\alpha+1}{2}} x, \epsilon)\right)
 \end{aligned}$$

where

$$L_{\rho/2, \gamma_j} = [\rho/2, +\infty) \exp(\sqrt{-1}\gamma_j) , \quad j = p, p+1$$

are halflines in directions γ_j at the distance $\rho/2$ from the origin and $C_{\rho/2, \gamma_p, \gamma_{p+1}}$ represents an arc of circle with radius $\rho/2$ that joins the above two lines.

We evaluate the first sum

$$I_1 = \left| \sum_{n \geq 0} \left(\int_{L_{\rho/2, \gamma_{p+1}}} W_n(\tau, z, \epsilon) \exp(\tau a(\epsilon^{\frac{\alpha+1}{2}} x, \epsilon)) \frac{d\tau}{\tau} \right) \exp\left(n\epsilon^{\frac{\alpha+1}{2}} a(\epsilon^{\frac{\alpha+1}{2}} x, \epsilon)\right) \right|.$$

Bearing in mind the bounds (149) and the factorization (28), we get that

$$(162) \quad I_1 \leq \sum_{n \geq 0} \int_{\rho/2}^{+\infty} 2\mathbf{W}(X_1)^n \frac{1}{1+s^2} \exp(\sigma\zeta(b)s) \\ \times \exp\left(\frac{s}{2|\epsilon|^{\frac{\alpha+1}{2}}} \left| \log \left| \frac{x-1}{x+1} \right| \right| |1+a_{\text{in}}(x)| \times \cos\left(\gamma_{p+1} - \arg(\epsilon^{\frac{\alpha+1}{2}}) + \pi + \arg(1+a_{\text{in}}(x))\right)\right) ds \\ \times \exp\left(\frac{n}{2} \left| \log \left| \frac{x-1}{x+1} \right| \right| |1+a_{\text{in}}(x)| \times \cos\left(\pi + \arg(1+a_{\text{in}}(x))\right)\right)$$

for $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, $x \in \chi_1$ and $z \in D(0, r)$. The lower bounds (155), (156) and angles estimates (157) (for γ_p replaced by γ_{p+1}), (158) yield that

$$(163) \quad I_1 \leq \sum_{n \geq 0} \int_{\rho/2}^{+\infty} 2\mathbf{W}(X_1)^n \exp(\sigma\zeta(b)s) \times \exp\left(-\frac{s}{2|\epsilon|^{\frac{\alpha+1}{2}}} \log\left(\frac{2-r_2}{r_2}\right)(1-\delta)\Delta\right) ds \\ \times \exp\left(-\frac{n}{2} \log\left(\frac{2-r_2}{r_2}\right)(1-\delta)\Delta_1\right)$$

for $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, $x \in \chi_1$, $z \in D(0, r)$. We select a positive real number $\delta_1 > 0$ such that

$$(164) \quad \sigma\zeta(b)\epsilon_0^{\frac{\alpha+1}{2}} - \frac{1}{2} \log\left(\frac{2-r_2}{r_2}\right)(1-\delta)\Delta \leq -\delta_1$$

holds. Notice that such a $\delta_1 > 0$ is ensured to exist if $r_2 > 0$ is taken close enough to 0. On the other hand, we take for granted that (159) is verified. Then, we obtain

$$(165) \quad I_1 \leq 2\mathbf{W} \sum_{n \geq 0} (1/2)^n \times \int_{\rho/2}^{+\infty} \exp\left(-\frac{\delta_1}{|\epsilon|^{\frac{\alpha+1}{2}}} s\right) ds = 4\mathbf{W} \frac{|\epsilon|^{\frac{\alpha+1}{2}}}{\delta_1} \exp\left(-\frac{\delta_1}{|\epsilon|^{\frac{\alpha+1}{2}}} \frac{\rho}{2}\right)$$

provided that $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, $x \in \chi_1$, $z \in D(0, r)$.

In a similar manner, we can reach upper bounds for the second piece of the decomposition (161)

$$I_2 = \left| \sum_{n \geq 0} \left(\int_{L_{\rho/2, \gamma_p}} W_n(\tau, z, \epsilon) \exp(\tau a(\epsilon^{\frac{\alpha+1}{2}} x, \epsilon)) \frac{d\tau}{\tau} \right) \exp\left(n\epsilon^{\frac{\alpha+1}{2}} a(\epsilon^{\frac{\alpha+1}{2}} x, \epsilon)\right) \right|.$$

Namely,

$$(166) \quad I_2 \leq 4\mathbf{W} \frac{|\epsilon|^{\frac{\alpha+1}{2}}}{\delta_1} \exp\left(-\frac{\delta_1}{|\epsilon|^{\frac{\alpha+1}{2}}} \frac{\rho}{2}\right)$$

for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, $x \in \chi_1$, $z \in D(0, r)$ and $\delta_1 > 0$ chosen as above in (164).

For our last task, we handle the transseries integrated along an arc of circle

$$I_3 = \left| \sum_{n \geq 0} \left(\int_{C_{\rho/2, \gamma_p, \gamma_{p+1}}} W_n(\tau, z, \epsilon) \exp(\tau a(\epsilon^{\frac{\alpha+1}{2}} x, \epsilon)) \frac{d\tau}{\tau} \right) \exp\left(n \epsilon^{\frac{\alpha+1}{2}} a(\epsilon^{\frac{\alpha+1}{2}} x, \epsilon)\right) \right|.$$

Again the bounds (149) and the factorization (28) give rise to

$$\begin{aligned} I_3 \leq \sum_{n \geq 0} & \left| \int_{\gamma_p}^{\gamma_{p+1}} 2\mathbf{W}(X_1)^n \frac{\rho/2}{1 + (\rho/2)^2} \exp(\sigma\zeta(b)\rho/2) \right. \\ & \times \exp\left(\frac{\rho/2}{2|\epsilon|^{\frac{\alpha+1}{2}}} \left| \log \left| \frac{x-1}{x+1} \right| \right| |1 + a_{\text{in}}(x)| \cos\left(\theta - \arg(\epsilon^{\frac{\alpha+1}{2}}) + \pi + \arg(1 + a_{\text{in}}(x))\right)\right) d\theta \left. \right| \\ & \times \exp\left(\frac{n}{2} \left| \log \left| \frac{x-1}{x+1} \right| \right| |1 + a_{\text{in}}(x)| \cos(\pi + \arg(1 + a_{\text{in}}(x)))\right). \end{aligned}$$

Since the estimates (157) hold true for both angles γ_p and γ_{p+1} , one checks that

$$(167) \quad \cos(\theta - \arg(\epsilon^{\frac{\alpha+1}{2}}) + \pi + \arg(1 + a_{\text{in}}(x))) < -\Delta$$

is valid for all $\theta \in (\gamma_p, \gamma_{p+1})$ if $\gamma_p < \gamma_{p+1}$ or $\theta \in (\gamma_{p+1}, \gamma_p)$ if $\gamma_{p+1} > \gamma_p$. Furthermore, taking into account the lower bounds (155), (156) together with the angle inequality (158), we come out with

$$\begin{aligned} I_3 \leq \sum_{n \geq 0} & |\gamma_p - \gamma_{p+1}| 2\mathbf{W}(X_1)^n \frac{\rho/2}{1 + (\rho/2)^2} \exp(\sigma\zeta(b)\rho/2) \\ & \times \exp\left(-\frac{\rho/2}{2|\epsilon|^{\frac{\alpha+1}{2}}} \log\left(\frac{2-r_2}{r_2}\right)(1-\delta)\Delta\right) \times \exp\left(-\frac{n}{2} \log\left(\frac{2-r_2}{r_2}\right)(1-\delta)\Delta_1\right) \end{aligned}$$

provided that $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, $x \in \chi_1$, $z \in D(0, r)$. We select $\delta_1 > 0$ as in (164) and we assume that (159) to be valid. Then,

$$\begin{aligned} (168) \quad I_3 \leq 2\mathbf{W} & |\gamma_p - \gamma_{p+1}| \frac{\rho/2}{1 + (\rho/2)^2} \exp\left(-\frac{\delta_1 \rho}{2|\epsilon|^{\frac{\alpha+1}{2}}}\right) \sum_{n \geq 0} (1/2)^n \\ & = 4\mathbf{W} |\gamma_p - \gamma_{p+1}| \frac{\rho/2}{1 + (\rho/2)^2} \exp\left(-\frac{\delta_1 \rho}{2|\epsilon|^{\frac{\alpha+1}{2}}}\right) \end{aligned}$$

whenever $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, $x \in \chi_1$, $z \in D(0, r)$.

In conclusion, the collection of bounds (165), (166), (168) applied to the decomposition (161) give rise to the expected exponentially flat estimates (153). \square

In the second central result of the paper, we show that the holomorphic inner solutions to (11) and (23) under the Cauchy data (12) obtained in Theorem 1 share a common asymptotic expansion relatively to ϵ on the sectors \mathcal{E}_p that turns out to be of Gevrey type.

Theorem 2 *Assume that the foregoing constraints listed in Theorem 1 hold. Let us denote $\mathcal{O}_b(\chi_1 \times D(0, r))$ the Banach space of \mathbb{C} -valued bounded holomorphic functions on $\chi_1 \times D(0, r)$ endowed with the sup norm.*

Then, for all $0 \leq p \leq \iota - 1$, the holomorphic and bounded functions $\epsilon \mapsto ((x, z) \mapsto u_p(\epsilon^{\frac{\alpha-1}{2}} x, z, \epsilon))$ built up in Theorem 1 and viewed as maps from \mathcal{E}_p into $\mathcal{O}_b(\chi_1 \times D(0, r))$, admit a formal power series

$$\hat{I}(\epsilon) = \sum_{k \geq 0} I_k \epsilon^k \in \mathcal{O}_b(\chi_1 \times D(0, r))[[\epsilon]]$$

as Gevrey asymptotic expansion of order $\frac{2}{\alpha+1}$. This means that for all $0 \leq p \leq \iota - 1$, one can find constants $A_p, B_p > 0$ such that

$$(169) \quad \sup_{x \in \chi_1, z \in D(0, r)} \left| u_p(\epsilon^{\frac{\alpha-1}{2}} x, z, \epsilon) - \sum_{k=0}^{n-1} I_k \epsilon^k \right| \leq A_p (B_p)^n \Gamma(1 + \frac{2}{\alpha+1} n) |\epsilon|^n$$

for all $n \geq 1$, whenever $\epsilon \in \mathcal{E}_p$.

Proof The proof leans on a cohomological criterion for the existence of Gevrey asymptotic expansions of order $1/k$, for real numbers $k > 1/2$, for suitable families of sectorial holomorphic functions known as Ramis-Sibuya theorem in the literature, see [1], p.121 or [14] Lemma XI-2-6. Here we need a Banach valued version of this result that can be stated as follows.

Theorem (Ramis-Sibuya) We set $(\mathbb{F}, \|\cdot\|_{\mathbb{F}})$ as a Banach space over \mathbb{C} and consider a good covering $\{\mathcal{E}_p\}_{0 \leq p \leq \iota-1}$ in \mathbb{C}^* . For all $0 \leq p \leq \iota - 1$, G_p stands for a holomorphic function from \mathcal{E}_p into the Banach space $(\mathbb{F}, \|\cdot\|_{\mathbb{F}})$ and let the cocycle $\Theta_p(\epsilon) = G_{p+1}(\epsilon) - G_p(\epsilon)$ be a holomorphic function from the sector $Z_p = \mathcal{E}_{p+1} \cap \mathcal{E}_p$ into \mathbb{E} (under the convention that $\mathcal{E}_\iota = \mathcal{E}_0$ and $G_\iota = G_0$). We ask for the following requirements.

- 1) The functions $G_p(\epsilon)$ are bounded on \mathcal{E}_p , for all $0 \leq p \leq \iota - 1$.
- 2) The functions $\Theta_p(\epsilon)$ suffer exponential flatness of order k on Z_p , for all $0 \leq p \leq \iota - 1$. Namely, there exist constants $C_p, A_p > 0$ such that

$$\|\Theta_p(\epsilon)\|_{\mathbb{F}} \leq C_p e^{-A_p/|\epsilon|^k}$$

for all $\epsilon \in Z_p$, all $0 \leq p \leq \iota - 1$.

Then, for all $0 \leq p \leq \iota - 1$, the functions $G_p(\epsilon)$ share a common formal power series $\hat{I}(\epsilon) = \sum_{k \geq 0} I_k \epsilon^k$ where the coefficients I_k belong to \mathbb{F} , as Gevrey asymptotic expansion of order $1/k$ on \mathcal{E}_p . In other words, constants $A_p, B_p > 0$ can be selected with

$$\|G_p(\epsilon) - \sum_{k=0}^{n-1} I_k \epsilon^k\|_{\mathbb{F}} \leq A_p (B_p)^n \Gamma(1 + \frac{n}{k}) |\epsilon|^n$$

for all $n \geq 1$, provided that $\epsilon \in \mathcal{E}_p$.

We apply the above theorem to the set of functions

$$G_p(\epsilon) := (x, z) \mapsto u_p(\epsilon^{\frac{\alpha-1}{2}} x, z, \epsilon)$$

for $0 \leq p \leq \iota - 1$, which represent holomorphic and bounded functions from \mathcal{E}_p into the Banach space $\mathbb{F} = \mathcal{O}_b(\chi_1 \times D(0, r))$ equipped with the sup norm over $\chi_1 \times D(0, r)$. Furthermore, the bounds (153) allow the cocycle $\Theta_p(\epsilon) = G_{p+1}(\epsilon) - G_p(\epsilon)$ to fulfill the constraint **2**) overhead. As a result, we deduce the existence of a formal power series $\hat{I}(\epsilon)$ that match the statement of Theorem 2. \square

5 Conclusion and perspectives

In this work we have considered Cauchy problems which are conjointly singularly perturbed and possess confluent fuchsian singularities relying on a single perturbation parameter. We constructed genuine solutions expressed as exponential transseries expansions which are shown to appear naturally from the polynomial structure of the coefficients in the equations involved.

It turns out that the nature of these special solutions and the extraction of their parametric asymptotic behaviour impose rather strong constraints on the shape of the equations under study in this work, namely the reduction to an equation of Kowalevski type displayed in (11). Concerning that point we may provide some explanatory comments that may help to investigate future lines of research in this topic. Indeed, departing from the singularly perturbed equation (23) under less restrictive conditions on the differential operator \mathcal{P} would lead to corresponding sets of convolution equations similar to (101) that ought to be solved relatively to τ (and ϵ on a good covering \mathcal{E}) on a domain where the expression $P_2(\tau + n\epsilon^{(\alpha+1)/2})$ is not equal to zero. Here P_2 is the polynomial introduced in Section 2. However, for $\tau = 0$, the quantity $P_2(n\epsilon^{(\alpha+1)/2})$ must vanish for at least some integer $n \geq 1$ and some value of the parameter ϵ near 0. As a consequence, the solution $W_n(\tau, z, \epsilon)$ to the convolution equation may not be defined at $\tau = 0$ and the solutions to (23) would not be constructible by means of Laplace transforms with special kernel. Hence, relaxing those constraints would need a new framework and novel ideas. It is worthwhile noting that strong restrictions on the shape of the coefficients are also asked in related works on confluence problems, see the references in our previous contribution [19].

At last, we expect that our approach can be adapted to other related problems, for instance in the context of q -difference or difference equations.

Competing interests

The author declares that there is no conflict of interest regarding the publication of this article.

References

- [1] W. Balser, *Formal power series and linear systems of meromorphic ordinary differential equations*. Universitext. Springer-Verlag, New York, 2000. xviii+299 pp.
- [2] W. Balser, M. Yoshino, *Integrability of Hamiltonian systems and transseries expansions*. Math. Z. 268 (2011), no. 1-2, 257–280.
- [3] B. Braaksma, R. Kuik, *Resurgence relations for classes of differential and difference equations*. Ann. Fac. Sci. Toulouse Math. (6) 13 (2004), no. 4, 479–492.
- [4] B. Braaksma, R. Kuik, *Asymptotics and singularities for a class of difference equations*. Analyzable functions and applications, 113135, Contemp. Math., 373, Amer. Math. Soc., Providence, RI, 2005.
- [5] O. Costin, *On Borel summation and Stokes phenomena for rank-1 nonlinear systems of ordinary differential equations*. Duke Math. J. 93 (1998), no. 2, 289–344.
- [6] O. Costin, *Asymptotics and Borel summability*. Chapman and Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 141. CRC Press, Boca Raton, FL, 2009. xiv+250 pp. ISBN: 978-1-4200-7031-6.
- [7] O. Costin, R. Costin, *On the formation of singularities of solutions of nonlinear differential systems in antistokes directions*. Invent. Math. 145 (2001), no. 3, 425–485.

- [8] O. Costin, R. Costin, M. Huang, *A direct method to find Stokes multipliers in closed form for P1 and more general integrable systems*. Trans. Amer. Math. Soc. 368 (2016), no. 11, 7579–7621.
- [9] O. Costin, M. Huang, F. Fauvet, *Global behavior of solutions of nonlinear ODEs: first order equations*. Int. Math. Res. Not. IMRN 2012, no. 21, 4830–4857.
- [10] O. Costin, S. Tanveer, *Complex singularity analysis for a nonlinear PDE*. Comm. Partial Differential Equations 31 (2006), no. 4-6, 593–637.
- [11] O. Costin, S. Tanveer, *Existence and uniqueness for a class of nonlinear higher-order partial differential equations in the complex plane*. Comm. Pure Appl. Math. 53 (2000), no. 9, 1092–1117.
- [12] G. Folland, *Introduction to partial differential equations*. Second edition. Princeton University Press, Princeton, NJ, 1995. xii+324 pp.
- [13] A. Fruchard, R. Schäfke, *On the parametric resurgence for a certain singularly perturbed linear differential equation of second order*. Asymptotics in dynamics, geometry and PDEs; generalized Borel summation. Vol. II, 213–243, CRM Series, 12, Ed. Norm., Pisa, 2011.
- [14] P. Hsieh, Y. Sibuya, *Basic theory of ordinary differential equations*. Universitext. Springer-Verlag, New York, 1999.
- [15] S. Lang, *Complex analysis*. Fourth edition. Graduate Texts in Mathematics, 103. Springer-Verlag, New York, 1999. xiv+485 pp.
- [16] A. Lastra, S. Malek, *On parametric multisummable formal solutions to some nonlinear initial value Cauchy problems*, Advances in Difference Equations 2015, 2015:200.
- [17] A. Lastra, S. Malek, *On parametric Gevrey asymptotics for some nonlinear initial value Cauchy problems*. J. Differential Equations 259 (2015), no. 10, 5220–5270.
- [18] S. Malek, *On the summability of formal solutions for doubly singular nonlinear partial differential equations*. J. Dyn. Control Syst. 18 (2012), no. 1, 45–82.
- [19] S. Malek, *On boundary layer expansions for a singularly perturbed problem with confluent fuchsian singularities*, Mathematics 2020, 8(2), 189.