On third-order $\overline{h}$-Jacobsthal and third-order $\overline{h}$-Jacobsthal–Lucas sequences, and related quaternions

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Abstract. In this paper, inspired by recent articles of G. Anatriello and G. Vincenzi (see [1]), and G. Cerda-Morales (see [3, 4]), we will introduce the third-order $\overline{h}$-Jacobsthal and third-order $\overline{h}$-Jacobsthal–Lucas sequences and their associated quaternions. The new results that we have obtained extend most of those obtained in [4].

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1. Introduction

The quaternions are objects introduced by Hamilton order to attempt a definition of coordinate system different from cartesian one. Hamilton idea's had a great success by his contemporary scientist, and in particular it had a strong application in physics. Actually, after a period of shadow, they have been recently considered in different branches of mathematics (see for example [10] and reference therein), and many researches are devoted to them.

The set of real quaternions is denoted by $\mathbb{H}$ and a quaternion number appears in the form $q = q_0 + q_1 i + q_2 j + q_3 k$, where $q_l \in \mathbb{R}$ $(l = 0, 1, 2, 3)$. The basic rules is given by

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1.1}$$

For an introduction to quaternions theory see [11] (see also the introduction of [4] for basic rules).

In 1963, A. F. Horadam [9] had the idea to investigate special quaternion numbers. Precisely, the Fibonacci quaternions Q_n, that are quaternions of the form $Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$, where F_n is the n-th Fibonacci number. Following him other authors had investigated other kind of quaternions, and many interesting properties of Fibonacci and generalized Fibonacci quaternions had ben obtained (see [7, 8] and reference therein).

Recently, in [3] and [4] the author starting from third-order Jacobsthal sequence $\{J_n^{(3)}\}_{n\geq 0}$ and third-order Jacobsthal–Lucas sequence $\{j_n^{(3)}\}_{n\geq 0}$

$$\begin{cases} J_0^{(3)}=0,\ J_1^{(3)}=J_2^{(3)}=1 \\ J_n^{(3)}=J_{n-1}^{(3)}+J_{n-2}^{(3)}+2J_{n-3}^{(3)}, \end{cases} \quad \begin{cases} j_0^{(3)}=2,\ j_1^{(3)}=1,\ j_2^{(3)}=5 \\ j_n^{(3)}=j_{n-1}^{(3)}+j_{n-2}^{(3)}+2j_{n-3}^{(3)}, \end{cases} \tag{1.2}$$

considered the third-order Jacobsthal quaternions and third-order Jacobsthal–Lucas quaternions, obtaining properties and matrix representation of such numbers.

As the author had highlighted in [3], the numbers of Tribonacci type have many applications in distinct area of mathematics (see also [2] and reference therein). Therefore, it seems a natural question to investigate quaternions connected to either a Tribonacci-like sequence or more generally, to a whichever recursive sequence of third order; but it seems not easy to develop a general theory for quaternions connected to a whichever sequence too much different from Tribonacci sequence.

In this paper, in order to attempt to this kind of investigation, we restrict our attention to quaternions connected to generalizations of third-order Jacobsthal sequences and third-order Jacobsthal–Lucas sequence (see Eq. (1.4)).

Let h be a complex number, we will define third-order $\overline{h}$-Jacobsthal sequence $\{J_{h,n}^{(3)}\}_{n\geq 0}$ and third-order $\overline{h}$-Jacobsthal–Lucas sequence $\{j_{h,n}^{(3)}\}_{n\geq 0}$ the homogenous recursive sequences of third order with constant coefficients, whose characteristic polynomial is

$$x^3-(h-1)x^2-(h-1)x-h=(x-h)(x^2+x+1), \tag{1.3}$$

and initial conditions are 0, 1, 1 and 2, 1, 5 respectively:

$$\begin{aligned} &\begin{cases} J_{h,0}^{(3)}=0,\ J_{h,1}^{(3)}=1,\ J_{h,2}^{(3)}=h-1 \\ J_{h,n}^{(3)}=(h-1)J_{h,n-1}^{(3)}+(h-1)J_{h,n-2}^{(3)}+hJ_{h,n-3}^{(3)}, \end{cases} \\ &\begin{cases} j_{h,0}^{(3)}=2,\ j_{h,1}^{(3)}=h-1,\ j_{h,2}^{(3)}=h^2+1 \\ j_{h,n}^{(3)}=(h-1)j_{h,n-1}^{(3)}+(h-1)j_{h,n-2}^{(3)}+hj_{h,n-3}^{(3)}. \end{cases} \end{aligned} \tag{1.4}$$

Note that when $h=2$, we have the quoted third-order Jacobsthal sequence and third-order Jacobsthal–Lucas sequence. Taking the paper of G. Cerda-Morales as model we will extend most of the results showed in [3] to third-order $\overline{h}$-Jacobsthal quaternions and third-order $\overline{h}$-Jacobsthal–Lucas quaternions.

2. Basic Properties

In [6], the authors provided many basic identities for third-order Jacobsthal numbers, $\{J_n^{(3)}\}_{n\geq 0}$, and third-order Jacobsthal–Lucas numbers, $\{j_n^{(3)}\}_{n\geq 0}$ (see also [5] and reference therein):

$$3J_n^{(3)} + j_n^{(3)} = 2^{n+1}, \tag{2.1}$$

$$j_n^{(3)} - 3J_n^{(3)} = 2j_{n-3}^{(3)}, \tag{2.2}$$

$$J_{n+2}^{(3)} - 4J_n^{(3)} = \begin{cases} -2 & \text{if} \quad n \equiv 1 \pmod 3 \\ 1 & \text{if} \quad n \not\equiv 1 \pmod 3 \end{cases}, \tag{2.3}$$

$$j_n^{(3)} - 4J_n^{(3)} = \begin{cases} 2 & \text{if} \quad n \equiv 0 \pmod 3 \\ -3 & \text{if} \quad n \equiv 1 \pmod 3 \\ 1 & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}, \tag{2.4}$$

$$j_{n+1}^{(3)} + j_n^{(3)} = 3J_{n+2}^{(3)}, \tag{2.5}$$

$$j_n^{(3)} - J_{n+2}^{(3)} = \begin{cases} 1 & \text{if} \quad n \equiv 0 \pmod 3 \\ -1 & \text{if} \quad n \equiv 1 \pmod 3 \\ 0 & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}, \tag{2.6}$$

$$\left(j_{n-3}^{(3)}\right)^2 + 3J_n^{(3)} j_n^{(3)} = 4^n, \tag{2.7}$$

$$\sum_{k=0}^{n} J_k^{(3)} = \begin{cases} J_{n+1}^{(3)} & \text{if} \quad n \not\equiv 0 \pmod 3 \\ J_{n+1}^{(3)} - 1 & \text{if} \quad n \equiv 0 \pmod 3 \end{cases}, \tag{2.8}$$

$$\sum_{k=0}^{n} j_k^{(3)} = \begin{cases} j_{n+1}^{(3)} - 2 & \text{if} \quad n \not\equiv 0 \pmod 3 \\ j_{n+1}^{(3)} + 1 & \text{if} \quad n \equiv 0 \pmod 3 \end{cases} \tag{2.9}$$

and

$$\left(j_n^{(3)}\right)^2 - 9\left(J_n^{(3)}\right)^2 = 2^{n+2} j_{n-3}^{(3)}. \tag{2.10}$$

Using standard techniques for solving recurrence relations, the auxiliary equation, and its roots are given by

$$x^3 - x^2 - x - 2 = 0; \; x = 2, \text{ and } x = \frac{-1 \pm i\sqrt{3}}{2}.$$

Note that the latter two are the complex conjugate cube roots of unity. Call them ω_1 and ω_2, respectively. Thus the Binet formulas can be written as

$$J_n^{(3)} = \frac{2}{7} \cdot 2^n - \left(\frac{3 + 2i\sqrt{3}}{21}\right) \omega_1^n - \left(\frac{3 - 2i\sqrt{3}}{21}\right) \omega_2^n \tag{2.11}$$

and

$$j_{n}^{(3)}=\frac{8}{7}\cdot 2^{n}+\left(\frac{3+2i\sqrt{3}}{7}\right)\omega_{1}^{n}+\left(\frac{3-2i\sqrt{3}}{7}\right)\omega_{2}^{n}, \tag{2.12}$$

respectively.

Remark 2.1. We note that when $h=\omega_1$ or $h=\omega_2$, then the characteristic polynomial has the unique roots ω_1 and ω_2 (with multiplicity 2 one of them). In this case the sequences $\{J_{h,n}^{(3)}\}_{n\geq 0}$ and $\{j_{h,n}^{(3)}\}_{n\geq 0}$ are both geometric and their study can be considered trivial.

Let $h\neq\omega_1,\omega_2$ be a complex number. In order to extend similar properties to third-order $\overline{h}$-Jacobsthal and third-order $\overline{h}$-Jacobsthal–Lucas sequences we will provide the Binet's formula for these sequences.

Lemma 2.2 (Binet's Formula). *Let $h\neq\omega_1,\omega_2$ be a complex number, and let $\{J_{h,n}^{(3)}\}_{n\geq 0}$ be the third-order $\overline{h}$-Jacobsthal sequence and $\{j_{h,n}^{(3)}\}_{n\geq 0}$ be the third-order $\overline{h}$-Jacobsthal–Lucas sequence. Then the Binet's formulas reads:*

$$J_{h,n}^{(3)}=\frac{1}{\sigma_h}\left[h^{n+1}-\left(\frac{h-\omega_2}{\omega_1-\omega_2}\right)\omega_1^{n+1}+\left(\frac{h-\omega_1}{\omega_1-\omega_2}\right)\omega_2^{n+1}\right] \tag{2.13}$$

and

$$j_{h,n}^{(3)}=\frac{1}{\sigma_h}\left[(h^2+h+2)h^{n}+(h+1)\left(\left(\frac{h-\omega_2}{\omega_1-\omega_2}\right)\omega_1^{n+1}-\left(\frac{h-\omega_1}{\omega_1-\omega_2}\right)\omega_2^{n+1}\right)\right], \tag{2.14}$$

where $\sigma_h=h^2+h+1$.

Proof. The characteristic polynomial of both the sequences $\{J_{h,n}^{(3)}\}_{n\geq 0}$ and $\{j_{h,n}^{(3)}\}_{n\geq 0}$ is $(x-h)(x^2+x+1)$, and by hypothesis it has three distinct roots, namely ω_1, ω_2 and h. It follows that there exist suitable a_1, a_2, a_3, b_1, b_2, $b_3\in\mathbb{C}$ such that the Binet's formula for $\{J_{h,n}^{(3)}\}_{n\geq 0}$ and $\{j_{h,n}^{(3)}\}_{n\geq 0}$ is of the type:

$$J_{h,n}^{(3)}=a_1h^n+a_2\omega_1^n+a_3\omega_2^n,\ j_{h,n}^{(3)}=b_1h^n+b_2\omega_1^n+b_3\omega_2^n.$$

In particular, using the initial conditions we have the relations:

$$\begin{cases} J_{h,0}^{(3)}=a_1h^0+a_2\omega_1^0+a_3\omega_2^0=0\\ J_{h,1}^{(3)}=a_1h^1+a_2\omega_1^1+a_3\omega_2^1=1\\ J_{h,2}^{(3)}=a_1h^2+a_2\omega_1^2+a_3\omega_2^2=h-1\end{cases}$$

and

$$\begin{cases} j_{h,0}^{(3)}=b_1h^0+b_2\omega_1^0+b_3\omega_2^0=2\\ j_{h,1}^{(3)}=b_1h^1+b_2\omega_1^1+b_3\omega_2^1=h-1\\ j_{h,2}^{(3)}=b_1h^2+b_2\omega_1^2+b_3\omega_2^2=h^2+1\end{cases}.$$

Thus we have the following systems

$$\begin{cases} a_1 + a_2 + a_3 = 0 \\ a_1 h + a_2\omega_1 + a_3\omega_2 = 1 \\ a_1 h^2 + a_2\omega_1^2 + a_3\omega_2^2 = h - 1 \end{cases} \quad \text{and} \quad \begin{cases} b_1 + b_2 + b_3 = 2 \\ b_1 h + b_2\omega_1 + b_3\omega_2 = h - 1 \\ b_1 h^2 + b_2\omega_1^2 + b_3\omega_2^2 = h^2 + 1 \end{cases},$$

whose solutions are:

$$a_1 = \frac{h}{h^2+h+1},\ a_2 = -\frac{\omega_2}{(h-\omega_1)(\omega_1-\omega_2)},\ a_3 = \frac{\omega_1}{(h-\omega_2)(\omega_1-\omega_2)}$$

and

$$b_1 = \frac{h^2+h+2}{h^2+h+1},\ b_2 = \frac{\omega_2(h+1)}{(h-\omega_1)(\omega_1-\omega_2)},\ b_3 = -\frac{\omega_1(h+1)}{(h-\omega_2)(\omega_1-\omega_2)}.$$

□

Remark 2.3. We highlight that when $h = 2$ the above Binet's formulas (2.13) and (2.14) give the Binet's formula of the third-order Jacobsthal sequence $\{J_n^{(3)}\}_{n\geq 0}$:

$$J_0^{(3)} = 0, J_1^{(3)} = 1, J_2^{(3)} = 1, J_3^{(3)} = 2, J_4^{(3)} = 5, J_5^{(3)} = 9, \cdots$$

and third-order Jacobsthal–Lucas sequence $\{j_n^{(3)}\}_{n\geq 0}$:

$$j_0^{(3)} = 2, j_1^{(3)} = 1, j_2^{(3)} = 5, j_3^{(3)} = 10, j_4^{(3)} = 17, j_5^{(3)} = 37, \cdots.$$

Precisely, the Binet's formulas for $J_n^{(3)}$ and $j_n^{(3)}$ are:

$$J_n^{(3)} = J_{2,n}^{(3)} = \frac{1}{7}\left[2^{n+1} + \left(\frac{2\omega_2+3}{\omega_1-\omega_2}\right)\omega_1^n - \left(\frac{2\omega_1+3}{\omega_1-\omega_2}\right)\omega_2^n\right] \tag{2.15}$$

and

$$j_n^{(3)} = j_{2,n}^{(3)} = \frac{1}{7}\left[2^{n+3} - 3\left(\frac{2\omega_2+3}{\omega_1-\omega_2}\right)\omega_1^n + 3\left(\frac{2\omega_1+3}{\omega_1-\omega_2}\right)\omega_2^n\right]. \tag{2.16}$$

(see [6, Eq. (3.1)])

For simplicity of notation, we define

$$Z_{h,n} = \left(\frac{h-\omega_2}{\omega_1-\omega_2}\right)\omega_1^{n+1} - \left(\frac{h-\omega_1}{\omega_1-\omega_2}\right)\omega_2^{n+1}. \tag{2.17}$$

Next result show that each term of $\{Z_{h,n}\}_{n\geq 0}$ has one more representation. It will be useful for proving condition of the Lemma 2.5.

Corollary 2.4. *Let $h \neq \omega_1, \omega_2$ be a complex number, and let $\{Z_{h,n}\}_{n\geq 0}$ as in Eq. (2.17). Then the following identities hold:*

$$Z_{h,n} + Z_{h,n+1} + Z_{h,n+2} = 0, \tag{2.18}$$

$$Z_{h,n} = \begin{cases} h & \text{if} \quad n \equiv 0 \pmod 3 \\ -(h+1) & \text{if} \quad n \equiv 1 \pmod 3 \\ 1 & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}. \tag{2.19}$$

Proof. (2.18): Using the Eq. (2.17), we obtain

$$\begin{aligned} Z_{h,n}+Z_{h,n+1}+Z_{h,n+2} &= \left(\frac{h-\omega_2}{\omega_1-\omega_2}\right)\omega_1^{n+1}(1+\omega_1+\omega_1^2) \\ &\quad -\left(\frac{h-\omega_1}{\omega_1-\omega_2}\right)\omega_2^{n+1}(1+\omega_2+\omega_2^2) \\ &= 0. \end{aligned}$$

(2.19): Using the Eq. (2.17), $\omega_1\omega_2=1$ and $\omega_1+\omega_2=-1$. Then,

$$\begin{aligned} Z_{h,n} &= \left(\frac{h-\omega_2}{\omega_1-\omega_2}\right)\omega_1^{n+1}-\left(\frac{h-\omega_1}{\omega_1-\omega_2}\right)\omega_2^{n+1} \\ &= h\left(\frac{\omega_1^{n+1}-\omega_2^{n+1}}{\omega_1-\omega_2}\right)-\left(\frac{\omega_1^{n}-\omega_2^{n}}{\omega_1-\omega_2}\right) \\ &= \begin{cases} h & \text{if} \quad n\equiv 0 \pmod 3 \\ -(h+1) & \text{if} \quad n\equiv 1 \pmod 3 \\ 1 & \text{if} \quad n\equiv 2 \pmod 3 \end{cases}. \end{aligned}$$

□

Then, using Corollary 2.4 we can write

$$J_{h,n}^{(3)}=\frac{1}{h^2+h+1}\left[h^{n+1}-Z_{h,n}\right] \tag{2.20}$$

and

$$j_{h,n}^{(3)}=\frac{1}{h^2+h+1}\left[(h^2+h+2)h^n+(h+1)Z_{h,n}\right]. \tag{2.21}$$

The next Lemma shows that similar properties to (2.1)–(2.10) also hold for third-order $\overline{h}$-Jacobsthal and third-order $\overline{h}$-Jacobsthal–Lucas sequences:

Lemma 2.5. *Let $h\neq\omega_1,\omega_2$ be a complex number, and let $\{J_{h,n}^{(3)}\}_{n\geq 0}$ be the third-order $\overline{h}$-Jacobsthal sequence and $\{j_{h,n}^{(3)}\}_{n\geq 0}$ be the $\overline{h}$-Jacobsthal–Lucas sequence. Then the following identities hold:*

$$(h+1)J_{h,n}^{(3)}+j_{h,n}^{(3)}=2h^n, \tag{2.22}$$

$$J_{h,n+2}^{(3)}-h^2J_{h,n}^{(3)}=\begin{cases} h-1 & \textit{if} \quad n\equiv 0 \pmod 3 \\ -h & \textit{if} \quad n\equiv 1 \pmod 3 \\ 1 & \textit{if} \quad n\equiv 2 \pmod 3 \end{cases}, \tag{2.23}$$

$$(h+1)J_{h,n+2}^{(3)}-j_{h,n+1}^{(3)}-j_{h,n}^{(3)}=(h-2)(h+1)h^n, \tag{2.24}$$

$$j_{h,n}^{(3)}-J_{h,n+2}^{(3)}+(h-2)h^n=\begin{cases} 1 & \textit{if} \quad n\equiv 0 \pmod 3 \\ -1 & \textit{if} \quad n\equiv 1 \pmod 3 \\ 0 & \textit{if} \quad n\equiv 2 \pmod 3 \end{cases}, \tag{2.25}$$

$$j_{h,n}^{(3)}-h^2J_{h,n}^{(3)}+(h-2)h^n=Z_{h,n}, \tag{2.26}$$

$$\left(j_{h,n}^{(3)}\right)^2 - (h+1)^2\left(J_{h,n}^{(3)}\right)^2 = 4h^n\left(j_{h,n}^{(3)} - h^n\right), \tag{2.27}$$

$$J_{h,n}^{(3)} + J_{h,n+1}^{(3)} + J_{h,n+2}^{(3)} = h^{n+1}, \tag{2.28}$$

$$\sum_{l=0}^{n} J_{h,l}^{(3)} = \frac{1}{3(h-1)}\left[J_{h,n+2}^{(3)} - (h-2)J_{h,n+1}^{(3)} + hJ_{h,n}^{(3)} - 1\right],\ h \neq 1. \tag{2.29}$$

Proof. By Eqs. (2.18), (2.19), (2.20) and (2.21), we have
(2.22):

$$\begin{aligned}(h+1)J_{h,n}^{(3)} + j_{h,n}^{(3)} &= \frac{1}{h^2+h+1}\left[(h+1)h^{n+1} - (h+1)Z_{h,n}\right]\\ &\quad + \frac{1}{h^2+h+1}\left[(h^2+h+2)h^n + (h+1)Z_{h,n}\right]\\ &= \frac{1}{h^2+h+1}\left[2(h^2+h+1)h^n\right] = 2h^n.\end{aligned}$$

(2.23):

$$\begin{aligned}J_{h,n+2}^{(3)} - h^2 J_{h,n}^{(3)} &= \frac{1}{h^2+h+1}\left[h^{n+3} - Z_{h,n+2}\right]\\ &\quad - \frac{1}{h^2+h+1}\left[h^2\cdot h^n + h^2 Z_{h,n}\right]\\ &= \frac{1}{h^2+h+1}\left[Z_{h,n+1} + (h^2+1)Z_{h,n}\right]\\ &= \begin{cases} h-1 & \text{if} \quad n \equiv 0 \pmod 3\\ -h & \text{if} \quad n \equiv 1 \pmod 3\\ 1 & \text{if} \quad n \equiv 2 \pmod 3\end{cases}.\end{aligned}$$

(2.24):

$$\begin{aligned}&(h+1)J_{h,n+2}^{(3)} - j_{h,n+1}^{(3)} - j_{h,n}^{(3)}\\ &\quad= \frac{1}{h^2+h+1}\left[(h+1)h^{n+3} - (h+1)Z_{h,n+2}\right]\\ &\quad\quad - \frac{1}{h^2+h+1}\left[(h^2+h+2)h^{n+1} + (h+1)Z_{h,n+1}\right]\\ &\quad\quad - \frac{1}{h^2+h+1}\left[(h^2+h+2)h^n + (h+1)Z_{h,n}\right]\\ &\quad= \frac{1}{h^2+h+1}\left[(h-2)(h+1)(h^2+h+1)\right]h^n\\ &\quad= (h-2)(h+1)h^n.\end{aligned}$$

(2.25):

$$\begin{aligned}
j_{h,n}^{(3)} - J_{h,n+2}^{(3)} + (h-2)h^n \\
&= \frac{1}{h^2+h+1}\left[(h^2+h+2)h^n + (h+1)Z_{h,n}\right] \\
&\quad - \frac{1}{h^2+h+1}\left[h^{n+3} - Z_{h,n+2}\right] \\
&\quad + \frac{1}{h^2+h+1}\left[(h^2+h+1)(h-2)\right]h^n \\
&= \frac{1}{h^2+h+1}\left[(h+1)Z_{h,n} + Z_{h,n+2}\right] \\
&= \begin{cases} 1 & \text{if} \quad n\equiv 0 \pmod 3 \\ -1 & \text{if} \quad n\equiv 1 \pmod 3 \\ 0 & \text{if} \quad n\equiv 2 \pmod 3 \end{cases}.
\end{aligned}$$

(2.26):

$$\begin{aligned}
j_{h,n}^{(3)} - h^2 J_{h,n}^{(3)} + (h-2)h^n &= \frac{1}{h^2+h+1}\left[(h^2+h+2)h^n + (h+1)Z_{h,n}\right] \\
&\quad - \frac{1}{h^2+h+1}\left[h^2\cdot h^{n+1} - h^2 Z_{h,n}\right] \\
&\quad + \frac{1}{h^2+h+1}\left[(h-2)(h^2+h+1)\right]h^n \\
&= Z_{h,n}.
\end{aligned}$$

(2.27):

$$\begin{aligned}
\left(j_{h,n}^{(3)}\right)^2 - (h+1)^2\left(J_{h,n}^{(3)}\right)^2 &= \left(j_{h,n}^{(3)} - (h+1)J_{h,n}^{(3)}\right)\left(j_{h,n}^{(3)} + (h+1)J_{h,n}^{(3)}\right) \\
&= 2\left(j_{h,n}^{(3)} - h^n\right)\cdot 2h^n \\
&= 4h^n\left(j_{h,n}^{(3)} - h^n\right).
\end{aligned}$$

(2.28):

$$\begin{aligned}
J_{h,n}^{(3)} + J_{h,n+1}^{(3)} + J_{h,n+2}^{(3)} &= \frac{1}{h^2+h+1}\left[h^{n+1} - Z_{h,n}\right] \\
&\quad + \frac{1}{h^2+h+1}\left[h^{n+2} - Z_{h,n+1}\right] \\
&\quad + \frac{1}{h^2+h+1}\left[h^{n+3} - Z_{h,n+2}\right] \\
&= \frac{1}{h^2+h+1}\left[h^{n+1}(h^2+h+1)\right] \\
&= h^{n+1}.
\end{aligned}$$

(2.29):

$$\begin{aligned}\sum_{l=0}^{n} J_{h,l}^{(3)} &= h + (h-1)\sum_{l=3}^{n} J_{h,l-1}^{(3)} + (h-1)\sum_{l=3}^{n} J_{h,l-2}^{(3)} + h\sum_{l=3}^{n} J_{h,l-3}^{(3)} \\ &= (3h-2)\sum_{l=0}^{n} J_{h,l}^{(3)} + 1 - (3h-2)J_{h,n}^{(3)} - (2h-1)J_{h,n-1}^{(3)} - hJ_{h,n-2}^{(3)}.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\sum_{l=0}^{n} J_{h,l}^{(3)} &= \frac{1}{3(h-1)}\left[(3h-2)J_{h,n}^{(3)} + (2h-1)J_{h,n-1}^{(3)} + hJ_{h,n-2}^{(3)} - 1\right] \\ &= \frac{1}{3(h-1)}\left[J_{h,n+1}^{(3)} + (2h-1)J_{h,n}^{(3)} + hJ_{h,n-1}^{(3)} - 1\right] \\ &= \frac{1}{3(h-1)}\left[J_{h,n+2}^{(3)} - (h-2)J_{h,n+1}^{(3)} + hJ_{h,n}^{(3)} - 1\right].\end{aligned}$$

□

3. The third-order $\overline{h}$-Jacobsthal and third-order $\overline{h}$-Jacobsthal–Lucas Quaternions

Let h be a real number. Following [3], we define the n-th third-order $\overline{h}$-Jacobsthal quaternion $\{JQ_{h,n}^{(3)}\}_{n\geq 0}$ and the n-th third-order $\overline{h}$-Jacobsthal–Lucas quaternion $\{jQ_{h,n}^{(3)}\}_{n\geq 0}$ as

$$\begin{aligned}JQ_{h,n}^{(3)} &= J_{h,n}^{(3)} + iJ_{h,n+1}^{(3)} + jJ_{h,n+2}^{(3)} + kJ_{h,n+3}^{(3)}, \\ jQ_{h,n}^{(3)} &= j_{h,n}^{(3)} + ij_{h,n+1}^{(3)} + jj_{h,n+2}^{(3)} + kj_{h,n+3}^{(3)}.\end{aligned} \tag{3.1}$$

Example. If we put $h = 2$ in the Eqns. (3.1), we have the third-order Jacobstal quaternions sequence $\{JQ_{n}^{(3)}\}_{n\geq 0}$ and the third-order Jacobstal-Lucas quaternions sequence $\{jQ_{n}^{(3)}\}_{n\geq 0}$ studied in [3]:

$$\begin{aligned}JQ_{0}^{(3)} &= i + j + 2k \\ JQ_{1}^{(3)} &= 1 + i + 2j + 5k \\ JQ_{2}^{(3)} &= 1 + 2i + 5j + 9k \\ JQ_{3}^{(3)} &= 2 + 5i + 9j + 18k \\ &\vdots\end{aligned}$$

The following result extends [3], next theorem just replacing $h = 2$.

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Theorem 3.1. *Let h be a real number, and let $\{JQ^{(3)}_{h,n}\}_{n\geq 0}$ be the third-order $\overline{h}$-Jacobsthal and $\{jQ^{(3)}_{h,n}\}_{n\geq 0}$ be the third-order $\overline{h}$-Jacobsthal –Lucas quaternions sequence. Then, for every positive integer n, we have:*

$$JQ^{(3)}_{h,n}+JQ^{(3)}_{h,n+1}+JQ^{(3)}_{h,n+1}=h^{n+1}(1+hi+h^2j+h^3k), \tag{3.2}$$

$$(h+1)JQ^{(3)}_{h,n}+jQ^{(3)}_{h,n}=2h^{n}(1+hi+h^2j+h^3k), \tag{3.3}$$

$$JQ^{(3)}_{h,n+2}-h^2JQ^{(3)}_{h,n}=\begin{cases} h-1-hi+j+(h-1)k & \textit{if} \quad n\equiv 0 \pmod 3\\ -h+i+(h-1)j-hk & \textit{if} \quad n\equiv 1 \pmod 3\\ 1+(h-1)i-hj+k & \textit{if} \quad n\equiv 2 \pmod 3\end{cases}, \tag{3.4}$$

$$(h+1)JQ^{(3)}_{h,n+2}-jQ^{(3)}_{h,n+1}-jQ^{(3)}_{h,n}=(h-2)(h+1)h^{n}(1+hi+h^2j+h^3k). \tag{3.5}$$

Proof. (3.2): By definition, we have

$$\begin{aligned} JQ^{(3)}_{h,n}+JQ^{(3)}_{h,n+1}+JQ^{(3)}_{h,n+1} &= J^{(3)}_{h,n}+J^{(3)}_{h,n+1}+J^{(3)}_{h,n+1}\\ &\quad +i(J^{(3)}_{h,n+1}+J^{(3)}_{h,n+2}+J^{(3)}_{h,n+3})\\ &\quad +j(J^{(3)}_{h,n+2}+J^{(3)}_{h,n+3}+J^{(3)}_{h,n+4})\\ &\quad +k(J^{(3)}_{h,n+3}+J^{(3)}_{h,n+4}+J^{(3)}_{h,n+5}). \end{aligned}$$

Using the property (2.28) in Lemma 2.5, we have

$$\begin{aligned} JQ^{(3)}_{h,n}+JQ^{(3)}_{h,n+1}+JQ^{(3)}_{h,n+1} &= h^{n+1}+ih^{n+2}+jh^{n+3}+kh^{n+4}\\ &= h^{n+1}(1+hi+h^2j+h^3k). \end{aligned}$$

(3.3):

$$\begin{aligned} (h+1)JQ^{(3)}_{h,n}+jQ^{(3)}_{h,n} &= (h+1)J^{(3)}_{h,n}+j^{(3)}_{h,n}\\ &\quad +i((h+1)J^{(3)}_{h,n+1}+j^{(3)}_{h,n+1})\\ &\quad +j((h+1)J^{(3)}_{h,n+2}+j^{(3)}_{h,n+2})\\ &\quad +k((h+1)J^{(3)}_{h,n+3}+j^{(3)}_{h,n+3}). \end{aligned}$$

Using the property (2.22) in Lemma 2.5, we have

$$\begin{aligned} (h+1)JQ^{(3)}_{h,n}+jQ^{(3)}_{h,n} &= 2h^{n}+2ih^{n+1}+2jh^{n+2}+2kh^{n+3}\\ &= 2h^{n}(1+hi+h^2j+h^3k). \end{aligned}$$

(3.3):

$$\begin{aligned} JQ^{(3)}_{h,n+2} - h^2 JQ^{(3)}_{h,n} &= J^{(3)}_{h,n+2} - h^2 J^{(3)}_{h,n} \\ &+ i(J^{(3)}_{h,n+3} - h^2 J^{(3)}_{h,n+1}) \\ &+ j(J^{(3)}_{h,n+4} - h^2 J^{(3)}_{h,n+2}) \\ &+ k(J^{(3)}_{h,n+5} - h^2 J^{(3)}_{h,n+3}). \end{aligned}$$

Using the property (2.23) in Lemma 2.5 and $n \equiv 0 (\mathrm{mod}\ 3)$, we have

$$JQ^{(3)}_{h,n+2} - h^2 JQ^{(3)}_{h,n} = h - 1 + i(-h) + j(1) + k(h-1).$$

Finally, we obtain

$$JQ^{(3)}_{h,n+2} - h^2 JQ^{(3)}_{h,n} = \begin{cases} h - 1 - hi + j + (h-1)k & \text{if} \quad n \equiv 0 \pmod 3 \\ -h + i + (h-1)j - hk & \text{if} \quad n \equiv 1 \pmod 3 \\ 1 + (h-1)i - hj + k & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}.$$

(3.4): By definition, we have

$$\begin{aligned} (h+1)JQ^{(3)}_{h,n+2} - jQ^{(3)}_{h,n+1} - jQ^{(3)}_{h,n} &= (h+1)J^{(3)}_{h,n+2} - j^{(3)}_{h,n+1} - j^{(3)}_{h,n} \\ &+ i((h+1)J^{(3)}_{h,n+3} - j^{(3)}_{h,n+2} - j^{(3)}_{h,n+1}) \\ &+ j((h+1)J^{(3)}_{h,n+4} - j^{(3)}_{h,n+3} - j^{(3)}_{h,n+2}) \\ &+ k((h+1)J^{(3)}_{h,n+5} - j^{(3)}_{h,n+4} - j^{(3)}_{h,n+3}). \end{aligned}$$

Using the property (2.24) in Lemma 2.5, we have

$$\begin{aligned} (h+1)JQ^{(3)}_{h,n+2} - jQ^{(3)}_{h,n+1} - jQ^{(3)}_{h,n} &= (h-2)(h+1)h^n \\ &+ i((h-2)(h+1)h^{n+1}) \\ &+ j((h-2)(h+1)h^{n+2}) \\ &+ k((h-2)(h+1)h^{n+3}) \\ &= (h-2)(h+1)h^n(1 + hi + h^2 j + h^3 k). \end{aligned}$$

Then, the result is obtained. □

In addition, some formulae involving sums of terms of the third-order $\overline{h}$-Jacobsthal quaternion sequence will be provided in the following theorem.

Theorem 3.2. *Let $h \in \mathbb{R} - \{1\}$. For a natural numbers m, n, with $n \geq m$, if $JQ^{(3)}_{h,n}$ is the n-th third-order $\overline{h}$-Jacobsthal quaternion, then the following identities are true:*

$$\sum_{l=0}^{n} JQ^{(3)}_{h,l} = \frac{1}{3(h-1)} \left\{ \begin{array}{c} JQ^{(3)}_{h,n+1} + (2h-1)JQ^{(3)}_{h,n} + hJQ^{(3)}_{h,n-1} \\ -JQ^{(3)}_{h,2} + (2h-3)JQ^{(3)}_{h,1} + (h-2)JQ^{(3)}_{h,0} \end{array} \right\}. \quad (3.6)$$

Proof. Using Eqs. (1.4) and (3.1), we obtain

$$\begin{aligned}\sum_{l=m}^{n} JQ_{h,l}^{(3)} &= JQ_{h,m}^{(3)} + JQ_{h,m+1}^{(3)} + JQ_{h,m+2}^{(3)} + \sum_{l=m+3}^{n} JQ_{h,l}^{(3)} \\ &= JQ_{h,m}^{(3)} + JQ_{h,m+1}^{(3)} + JQ_{h,m+2}^{(3)} \\ &\quad + (h-1)\sum_{l=m+2}^{n-1} JQ_{h,l}^{(3)} + (h-1)\sum_{l=m+1}^{n-2} JQ_{h,l}^{(3)} + h\sum_{l=m}^{n-3} JQ_{h,l}^{(3)} \\ &= (3h-2)\sum_{l=m}^{n} JQ_{h,l}^{(3)} \\ &\quad + JQ_{h,m+2}^{(3)} - (2h-3)JQ_{h,m+1}^{(3)} - (h-2)JQ_{h,m}^{(3)} \\ &\quad - (3h-2)JQ_{h,n}^{(3)} - (2h-1)JQ_{h,n-1}^{(3)} - hJQ_{h,n-2}^{(3)} \\ &= (3h-2)\sum_{l=m}^{n} JQ_{h,l}^{(3)} \\ &\quad + JQ_{h,m+2}^{(3)} - (2h-3)JQ_{h,m+1}^{(3)} - (h-2)JQ_{h,m}^{(3)} \\ &\quad - JQ_{h,n+1}^{(3)} - (2h-1)JQ_{h,n}^{(3)} - hJQ_{h,n-1}^{(3)}.\end{aligned}$$

Then, the result in Eq. (3.6) is completed if $m = 0$. □

4. Conclusion

Sequences of quaternions have been studied over several years, including the well-known Tribonacci quaternion sequence [4] and, consequently, on the third-order Jacobsthal quaternion sequence [3]. In this paper we have also contributed for the study of third-order $\overline{h}$-Jacobsthal quaternion, deducing some formulae for the sums of such numbers, presenting their Binet-style formula. It is our intention to continue the study of this type of sequences, exploring some their applications in the science domain. For example, a new type of sequences in the octonion algebra with the use of these numbers and their combinatorial properties.

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