# Stability analysis of nontrivial stationary solution and constant equilibrium point of reaction-diffusion neural networks with time delays

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**Abstract**

Firstly, the existence of asymptotically stable nontrivial stationary solution is derived by the comprehensive applications of Lyapunov-Razumikhin technique, design of state-dependent switching laws, a fixed point theorem, variational methods, and construction of compact operators on a convex set. This new theorem shows that the diffusion is a double-edged sword to the stability, refuting the views in previous literature that the greater the diffusion effect, the more stable the system will be. Next, a series of new theorems are presented one by one, which illustrates that the globally asymptotical stability of ordinary differential equations model for delayed neural networks may be locally stable in actual operation due to the inevitable diffusion. Besides, the non-zero constant equilibrium point is pointed out to be not the solution of delayed reaction diffusion system so that the stability of the non-zero constant equilibrium point of reaction diffusion system must lead to a contradiction. That is, non-zero constant equilibrium points are not in the phase plane of dynamic system. In addition, new theorems are further presented to prove that the small diffusion effect will cause the essential change of the phase plane structure of the dynamic behavior of the delayed neural networks, and thereby one equilibrium solution may become several stationary solutions, even infinitely many positive stationary solutions. Finally, a numerical example illustrates the feasibility of the proposed methods.

*Keywords:* reaction-diffusion; cellular neural networks; exponential stability; stationary solutions ; Saddle point theorem

## 1. Introduction

Firstly, we recalled the reason **why we need to study the stability of reaction-diffusion neural networks system**.

In 1988, inspired by cellular automata, Chua and Yang proposed a new neural network based on Hopfield network, i.e. cellular neural network (CNN), which is formed by a number of cells with the same structure after a well-organized combination ([22,23]). Each neuron in the network will automatically choose to connect with the nearest neuron. Because of its local connectivity, CNN is especially suitable for ultra large scale integrated circuit

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implementation. The characteristics of the above cellular neural network make it widely used in pattern recognition, image processing, signal processing and other fields. The main function of cellular neural network is to transform an input image into a corresponding output image. For example, the existing target motion direction detection, edge detection, and connected slice detection all use this function. In order to achieve these functions, the cellular neural network must be completely stable, that is, all output trajectories must converge to a stable equilibrium point. So the stability of cellular neural network has become a hot topic ([24-26]). As we all know, time delay may destroy the stability of the system and lead to oscillation, bifurcation, chaos and other phenomena, thus changing the characteristics of the system. In cellular neural networks, time delay is inevitable. For example, there are cell delay, transmission delay and synapse delay in biological neural network ([27]). **As pointed out in [28] that many pattern formation and wave propagation phenomena that appear in nature can be described by systems of coupled nonlinear differential equations, generally known as reaction-diffusion equations. These wave propagation phenomena are exhibited by systems belonging to very different scientific disciplines. Besides, the interactions arising from the space-distributed structure of the multilayer cellular neural networks can be seen as diffusion phenomenon([3, 29]).** Thereby, the reaction-diffusion effects cannot be neglected in both biological and man-made neural networks, especially when electrons are moving in non-even electromagnetic field. **Moreover, although the diffusion coefficients may be very small, the topological structure of the phase plane of the dynamic behavior of the following reaction-diffusion system (1.1) is likely to change substantially from a constant equilibrium point of the following system (1.3) to multiple stationary solutions of the reaction-diffusion system (1.1). Therefore, many global stability results of delayed neural networks in the form of ordinary differential equations may only be locally asymptotical stability criteria in real engineering (see Remark 13).** So, in this paper, we have to investigate the stability of reaction-diffusion cellular neural networks (delayed partial differential equation model), rather than that of ordinary differential equations model with time delays.

Next, we shall point out the fact that **the stability results in previous literature involved to delayed reaction-diffusion neural networks make it unnecessary to study the reaction-diffusion system (partial differential equations model), but only its corresponding ordinary differential equations model. What's the problem?**

For a long time, the stability of the reaction diffusion neural networks was investigated in many literatures[1-10], in which the stability of the constant equilibrium point was studied. For example, in [1], the following cellular neural networks with time-varying delays and reaction-diffusion terms was considered,

$$\begin{aligned}\frac{\partial y(t,x)}{\partial t} &= \sum_{q=1}^{m}\frac{\partial y(t,x)}{\partial x_q}(D_q\frac{\partial y(t,x)}{\partial x_q}) - Cy(t,x) + Ag(y(t,x)) \\ &\quad + Bg(y(t-\tau(t),x)) + J,\ (t,x)\in\mathbb{R}_+\times\Omega,\end{aligned} \tag{1.1}$$

Next, the authors of [1] defined the equilibrium point of the time-delayed reaction-diffusion system (1.1) as the constant vector $y^*$ satisfying

$$Cy^* = Ag(y^*) + Bg(y^*) + J. \tag{1.2}$$

Here, we have to say, the equilibrium point $y^*$ is also the equilibrium point of the following ordinary differential equations corresponding to the time-delayed partial differential equations (1.1),

$$\frac{dx(t)}{dt} = -Cx(t) + Ag(x(t)) + Bg(x(t-\tau(t))) + J, \; t \in \mathbb{R}_+, \tag{1.3}$$

Due to the Poincare inequality, we see, the diffusion items actually promote the stability of the reaction diffusion system (1.1). That is, we only need to study the ordinary differential equations (1.3) because the stability criteria of the ordinary differential equations (1.3) must make the system (1.1) stable. In other words, **the stability of the reaction-diffusion model does not need to be studied** because it is included in the stability of its corresponding ordinary differential equations model.

So we need to ask where the problem is? **The answer lies in the choice of the equilibrium point**.

In fact, the diffusions are inevitable in actual engineering. Thereby, whether the system (1.1) is close to stability or reaches stability, the diffusion phenomenon still exists, and hence the equilibrium point with with practical engineering significance of the system (1.1) should be related to the variable $x \in \Omega$, which reflects the characteristics of the diffusion system, rather than the constant equilibrium solution $y^*$ satisfying the condition (1.2). The constant equilibrium solution $y^*$ is actually independent of $x \in \Omega$, and is called the trivial equilibrium solution of the system (1.1). So, in this paper, we should select the nontrivial equilibrium point, that is, we should study the nontrivial stationary solution $y^*(x)$ with practical engineering significance. In fact, **not only in neural networks, but also some equilibrium points in other dynamical systems, such as the constant equilibrium points with zero values in the financial mathematical model of option pricing, are obviously trivial, because option prices cannot be zero in the real financial market. For another example, in the financial system, the equilibrium point with the negative interest rate is usually trivial, because most countries, such as China, have a positive interest rate when their economies reach stable ([11,12,30]). So we should pay more attention to the stability of the nontrivial stationary solutions of reaction-diffusion neural networks with time delays.**

Usually, a positive solution $y(x)(x \in \Omega)$ of the system (1.1) is not the solution of the constant equations (1.2). Generally speaking, the positive solution accords with many practical engineering meanings. **As far as we are concerned, the stability of the nontrivial stationary solution has never been investigated, which inspires us to write this paper.**

Motivated by some methods of [1-31], we investigate the stability of the nontrivial stationary solution of switched reaction-diffusion neural networks with time delays. This paper has the following innovations:

★ It is the first paper to study the stability of the nontrivial stationary solution of reaction-diffusion neural networks with time delays;

★ It is the first time that the existence of asymptotically stable nontrivial stationary solution is derived by the

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comprehensive applications of Lyapunov-Razumikhin technique, design of state-dependent switching laws, a fixed point theorem, variational methods, and construction of compact operators on a convex set.

★ For the first time, the stability criterion obtained in this paper shows the two sides of the diffusion phenomena in practical engineering. On the one hand, diffusion promotes the stability of the reaction-diffusion neural networks due to the Poincare inequility. On the other hand, diffusion causes the complexity of the dynamic behavior of the reaction-diffusion neural networks, which makes it more difficult to judge the stability, because the stability criterion of the nontrivial stationary solution is completely different from the stability criteria of the constant equilibrium point, which is more rigorous.

★ It is the first time to give the theorems and remarks to point out that we should study the reaction diffusion model for neural networks, rather than ordinary differential equation model.

★ It is the first time to give the theorems and remarks to point out that we should pay more attention to stability analysis of the nontrivial stationary solution of reaction diffusion neural networks in the future, rather than the stability of the constant equilibrium point.

★ It is the first time to study how the tiny diffusion causes the essential change of the phase plane structure of the dynamic behavior of the delayed neural networks.

## 2. System descriptions

Consider the following switched neural networks with time-varying delays and reaction-diffusion terms

$$\begin{cases} \dfrac{\partial y(t,x)}{\partial t} =D_\sigma \Delta y(t,x) - C_\sigma y(t,x) + A_\sigma g(y(t,x)) + B_\sigma g(y(t-\tau(t),x)) + J_\sigma, \quad (t,x)\in \mathbb{R}_+\times\Omega_\sigma, \\ y_i(t,x) =0,\ t\geqslant 0,\ x\in\partial\Omega_\sigma,\ \ i=1,2,\cdots,n, \end{cases} \tag{2.1}$$

where $\Omega_\sigma \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega_\sigma$, the state variable $y(t,x) = (y_1(t,x), y_2(t,x), \cdots, y_n(t,x))^T$ with $y_i$ representing state variable related to a neuron. $J_\sigma = (J_{\sigma 1},\cdots,J_{\sigma n})^T \in \mathbb{R}^n$ is the constant external input vector, and both $D_\sigma$ and $C_\sigma$ are positive definite diagonal matrices, in which $D_\sigma$ represents the diffusion coefficient matrix, and $C_\sigma$ represents the connection weight matrix of neural network. Besides, $A_\sigma$ and $B_\sigma$ both are the connection weight matrices of neural network. For each $x\in\Omega_\sigma$, $g(y(t,x)) = (g_1(y_1(t,x)),\cdots,g_n(y_n(t,x)))^T$ represents a time-dependent signal function vector. $\tau(t)$ represents the time delay required for signal transmission from neuron $j$ to neuron $i$, satisfying $0\leqslant\tau(t)\leqslant\tau$. Assumed that $y^\sigma(x) = (y_1^\sigma(x),\cdots,y_n^\sigma(x))^T$ is a nontrivial stationary solution of reaction-diffusion switched system (2.1), then $y^\sigma(x)$ satisfies two equations of the system (2.1), in addition,

$$-C_\sigma y^\sigma(x) + A_\sigma g(y^\sigma(x)) + B_\sigma g(y^\sigma(x)) + J_\sigma \not\equiv 0, \quad x\in\Omega_\sigma. \tag{2.2}$$

Of course, the sufficient condition should be given to ensure the existence of such nontrivial stationary solution.

Set $u(t,x) = y(t,x) - y^\sigma(x)$, then the system (2.1) is translated into the following system:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} =D_\sigma \Delta u(t,x) - C_\sigma u(t,x) + A_\sigma f(u(t,x)) + B_\sigma f(u(t-\tau(t),x)), \quad (t,x)\in \mathbb{R}_+\times\Omega_\sigma, \\ u_i(t,x) =0,\ t\geqslant 0,\ x\in\partial\Omega_\sigma,\ \ i=1,2,\cdots,n, \end{cases} \tag{2.3}$$

where $f(u(t,x)) = g(y(t,x)) - g(y^{\sigma}(x))$, $f(u(t-\tau(t),x)) = g(y(t-\tau(t),x)) - g(y^{\sigma}(x))$. Here, the nontrivial stationary solution $y^{\sigma}(x)$ of the system (2.1) corresponds to the null solution of the system (2.3).

Besides, we may equip the system (2.3) with the initial value:

$$u_i(s,x) = \phi_i(s,x), \ -\tau \leqslant s \leqslant 0, \ x \in \Omega_{\sigma}, \tag{2.4}$$

where $(\phi_1(s,x), \phi_2(s,x), \cdots, \phi_n(s,x))^T = \phi(s,x)$, and each $\phi_i(s,x)$ is bounded on $[-\tau, 0] \times \Omega_{\sigma}$.

Throughout this paper, we make the assumptions as follows,

(A1) There is a positive definite diagonal matrix $G = diag(G_1, G_2, \cdots, G_n)$ such that

$$|g_i(s) - g_i(t)| \leqslant G_i|s-t|, \quad \forall\, s, t \in \mathbb{R};$$

(A2) There are three positive real numbers $\lambda \in (0, \lambda_{\sigma 1})$, $c > 0$ and $\alpha \in (0,1)$ with $(\alpha+1)\lambda < \lambda_{\sigma 1}$ such that

$$0 \leqslant \Big[ -\Big(C_{\sigma} + (1+\alpha)\lambda I\Big)v + A_{\sigma} g(v) + B_{\sigma} g(v) + J_{\sigma}\Big] \leqslant cE, \quad \forall\, v \in \mathbb{R}^n$$

where $\lambda_{\sigma 1} > 0$ is the first positive eigenvalue of the Laplace operator $-\Delta$ on the Sobolev space $W_0^{1,2}(\Omega_{\sigma})$, $I$ is the identity matrix, and $E = (1, 1, \cdots, 1)^T \in \mathbb{R}^n$.

Define the switching law as follows,

**switching law** $\mathfrak{F}$: At each switching we determine the next mode according to the following minimum law :

$$\sigma(t) = \arg\min (y - y^{\sigma})^T\Big[\Big(-2\lambda_{\sigma 1} D_{\sigma} - 2C_{\sigma} + A_{\sigma}A_{\sigma}^T + B_{\sigma}B_{\sigma}^T + G^2 + e^{\gamma\tau} q G^2\Big) + \Psi\Big](y - y^{\sigma}). \tag{2.5}$$

($\mathfrak{F}_1$) Choose the initial mode $\sigma(t) = i_0$, if $(y(t_0, x) - y^{\sigma}(x)) \in \Upsilon_{i_0}$.

($\mathfrak{F}_2$) For each $t > t_0$, if $\sigma(t^-) = i$ and $(y - y^{\sigma}) \in \Upsilon_i$, keep $\sigma(t) = i$. On the other hand, if $\sigma(t^-) = i$ but $(y - y^{\sigma}) \notin \Upsilon_i$. i.e., hitting a switching surface, choose the next mode by applying (2.5) and begin to switch.

Here, $\Psi$ is a positive definite symmetric matrix with $\lambda_{\min}\Psi > 0$, and $\Upsilon_{\sigma}$ is defined as follows,

$$\Upsilon_{\sigma} = \Big\{ y \in \mathbb{R}^n \Big| (y - y^{\sigma})^T\Big(-2\lambda_{\sigma 1} D_{\sigma} - 2C_{\sigma} + A_{\sigma}A_{\sigma}^T + B_{\sigma}B_{\sigma}^T + G^2 + e^{\gamma\tau} q G^2 + \Psi\Big)(y - y^{\sigma}) < 0 \Big\},$$

where $\lambda_{\min}\Psi$ represents the minimum of all the eigenvalues of the symmetric matrix $\Psi > 0$.

**Definition 1.**([42]) Let $\psi$ be a real $C^1$ functional defined on a Banach space $X$. If any sequence $\{u_n\}$ in $X$ with $\psi(u_n) \to a$ and $\|\psi'(u_n)\| \to 0$ has a convergent subsequence, and this holds for every $a \in \mathbb{R}$, one says that $\psi$ satisfies the (PS) condition.

**Lemma 2.1.**([32]) Let $\mathfrak{Z}$ be a Banach space, and $\mathfrak{K}$ is a closed convex set. If $\mathfrak{T} : \mathfrak{K} \to \mathfrak{K}$ is a compact mapping such that for any $\varphi \in \mathfrak{K}$ with $\|\varphi\| = M$, the inequality $\varphi \neq r\mathfrak{T}(\varphi)$ holds for each $r \in [0,1]$, where $M$ is any given positive constant, then there exits at least a fixed point of $\mathfrak{T}$, say, $\varphi \in \mathfrak{K}$ with $\|\varphi\| \leqslant M$.

**Lemma 2.2.** ([42]) Let $H = H_1 \oplus H_2$ be a Banach space, and $H_1$ is a finite dimension subspace. If $\psi \in C^1(H, \mathbb{R})$, satisfying $\psi(0) = 0$, the (PS) condition. Besides, for some $\delta > 0$, the following conditions hold,

(P1) $\psi(u) \leqslant 0$ if $u \in H_1$ with $\|u\| \leqslant \delta$;

(P2) $\psi(u) \geqslant 0$ if $u \in H_2$ with $\|u\| \leqslant \delta$;

(P3) $\psi$ is bounded below, satisfying $\inf_H \psi < 0$,

then $\psi$ owns at least non-zero critical points.

## 3. Main results

**Theorem 3.1.** Suppose that the conditions (A1) and (A2) hold, then the system (2.1) possesses a positive bounded stationary solution $y^\sigma(x)$ for $x \in \Omega_\sigma$ with $y^\sigma|_{\partial\Omega_\sigma} = 0$. In addition, there is a sequence of nonnegative constants $\beta_\sigma$ $(\sigma = 1, 2, \cdots, N)$ with $\sum_{\sigma=1}^{N} \beta_\sigma = 1$ and $0 \leqslant \beta_\sigma \leqslant 1$ and positive constants $\gamma \in (0, \lambda_{\min}\Psi)$ and $q > 1$ such that

$$\sum_{\sigma=1}^{N} \beta_\sigma \Big( -2\lambda_{\sigma 1} D_\sigma - 2C_\sigma + A_\sigma A_\sigma^T + B_\sigma B_\sigma^T \Big) + G^2 + e^{\gamma\tau} q G^2 + \Psi < 0, \tag{3.1}$$

then the null solution of the switched delayed reaction-diffusion system (2.3) equipped with the initial value (2.4) is exponentially stable.

*Proof.* Firstly, we denote $\|u_i\| = \sqrt{\int_{\Omega_\sigma} |\nabla u_i|^2 dx}$, and $\|u\| = \sum_{i=1}^{n} \|u_i\|$. Besides, denote by $I$ the identity matrix. If the stationary solution of the system (2.1) exists, we may denote $y^\sigma(x)$.

Define the operator $\mathfrak{M} : [C(\overline{\Omega_\sigma})]^n \to [C(\overline{\Omega_\sigma})]^n$ as follows,

$$\mathfrak{M} = \begin{pmatrix} -\Delta - \lambda & 0 & 0 & \cdots & 0 \\ 0 & -\Delta - \lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -\Delta - \lambda \end{pmatrix}. \tag{3.2}$$

Due to $0 < \lambda < \lambda_{\sigma 1}$, the operator $\mathfrak{M}$ has the inverse operator $\mathfrak{M}^{-1}$ as follows,

$$\mathfrak{M}^{-1} = \begin{pmatrix} (-\Delta - \lambda)^{-1} & 0 & 0 & \cdots & 0 \\ 0 & (-\Delta - \lambda)^{-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & (-\Delta - \lambda)^{-1} \end{pmatrix}, \tag{3.3}$$

where $\mathfrak{M}^{-1} : [C(\overline{\Omega_\sigma})]^n \to [C(\overline{\Omega_\sigma})]^n$ is a linear compact positive operator ([13]). It is obvious that $\Big[ -\Big(C_\sigma + (1 + \alpha)\lambda I\Big) y^\sigma(x) + A_\sigma g(y^\sigma(x)) + B_\sigma g(y^\sigma(x)) + J_\sigma \Big]$ is continuous for all the variables $x, y_1^\sigma, \cdots, y_n^\sigma$. Define

$$\mathfrak{K} = \{\varphi(x) \in [C(\overline{\Omega_\sigma})]^n : \varphi(x) \geqslant 0,\ x \in \Omega;\ \varphi(x) = 0,\ x \in \partial\Omega\},$$

then $\mathfrak{K}$ is a positive cone, which must be a closed convex subset of $[C(\overline{\Omega_\sigma})]^n$. Define an operator $\mathfrak{T}:\mathfrak{K}\to\mathfrak{K}$ such that

$$\mathfrak{T}\varphi = \mathfrak{M}^{-1}\alpha\lambda\varphi + \mathfrak{M}^{-1}\Big[-\Big(C_\sigma + (1+\alpha)\lambda I\Big)\varphi + A_\sigma g(\varphi) + B_\sigma g(\varphi) + J_\sigma\Big]$$

Because $\mathfrak{M}^{-1}$ is the linear positive compact operator([13]), we can conclude from (A1) and (A2) that $\mathfrak{T}:\mathfrak{K}\to\mathfrak{K}$ is a positive compact operator.

Next, we claim that $\mathfrak{T}$ satisfies all the assumption conditions of Lemma 2.1, which implies that $\mathfrak{T}$ has at least a fixed point in $\mathfrak{K}$ .

Indeed, if it is not true, there must be $\{r_n\}\subset[0,1]$ and $\{\varphi_n\}\subset\mathfrak{K}$ with

$$\varphi_n = r_n\mathfrak{T}(\varphi_n) = r_n\mathfrak{M}^{-1}\alpha\lambda\varphi_n + r_n\mathfrak{M}^{-1}\Big[-\Big(C_\sigma + (1+\alpha)\lambda I\Big)\varphi_n + A_\sigma g(\varphi_n) + B_\sigma g(\varphi_n) + J_\sigma\Big] \tag{3.4}$$

and

$$\|\varphi_n\| = M_n \to +\infty, \qquad n\to+\infty.$$

The compactness of bounded closed sets in a finite dimensional space yields that there is a subsequence of $\{r_n\}$, say, $\{r_n\}$ such that $\lim_{n\to\infty} r_n = r_0$.

Let

$$\mathfrak{L}_n = \frac{\varphi_n}{\|\varphi_n\|},$$

then it is easy to conclude from (3.4) and (A2) that if $r_n\to r_0\in[0,1]$,

$$\mathfrak{L}_n \to \mathfrak{L}_0 \in \mathfrak{K}, \qquad \|\mathfrak{L}_0\| = 1. \tag{3.5}$$

In fact, (A2) yields

$$\frac{\Big[-\Big(C_\sigma + (1+\alpha)\lambda I\Big)\varphi + A_\sigma g(\varphi) + B_\sigma g(\varphi) + J_\sigma\Big]}{\|\varphi\|} \to 0\in\mathbb{R}^n, \qquad n\to\infty$$

Besides, since $\mathfrak{M}^{-1}\alpha\lambda I$ is a linear positive compact operator, and $\{\mathfrak{L}_n\}$ is a bounded set, we know that $\{r_n\mathfrak{M}^{-1}\alpha\lambda\mathfrak{L}_n\}$ is a sequential compact set. By means of the suitable choice of subsequence, we can get

$$r_n\mathfrak{M}^{-1}\alpha\lambda\mathfrak{L}_n \to \mathfrak{L}_0,$$

which together with (3.4) implies

$$\begin{aligned}\mathfrak{L}_n &= \frac{\varphi_n}{\|\varphi_n\|} = \frac{r_n\mathfrak{M}^{-1}\alpha\lambda\varphi_n}{\|\varphi_n\|} + \frac{r_n\mathfrak{M}^{-1}\Big[-\Big(C_\sigma + (1+\alpha)\lambda I\Big)\varphi_n + A_\sigma g(\varphi_n) + B_\sigma g(\varphi_n) + J_\sigma\Big]}{\|\varphi_n\|}\\ &\to \mathfrak{L}_0,\end{aligned}$$

hence $\|\mathfrak{L}_n\|\to 1 = \|\mathfrak{L}_0\|$ and

$$\mathfrak{L}_0(x) \equiv 0, \qquad x\in\Omega_\sigma. \tag{3.6}$$

In fact, $W_0^{1,2}(\Omega_\sigma)$ is a Banach space, and hence $\mathfrak{L}_n \to \mathfrak{L}_0$ yields $\mathfrak{L}_0(x)|_{\partial\Omega_\sigma} = 0$. Besides, $\mathfrak{M}\mathfrak{L}_0(x) = r_0\alpha\lambda\mathfrak{L}_0(x),\ \ x \in \Omega_\sigma$.

Denote $\mathfrak{L}_0(x) = (l_{01}(x), \cdots, l_{0n}(x))^T$. Poincare inequality and Dirichlet boundary value yields

$$\begin{aligned}\int_{\Omega_\sigma} |\nabla l_{0i}(x)|^2 dx =& (r_0\alpha\lambda + \lambda)\int_{\Omega_\sigma} l_{0i}^2(x)dx \\ \leqslant& \frac{\alpha\lambda + \lambda}{\lambda_{\sigma 1}} \int_{\Omega_\sigma} |\nabla l_{0i}(x)|^2 dx,\ \ \forall\, i,\end{aligned} \tag{3.7}$$

which together with $\mathfrak{L}_0(x) \in W_0^{1,2}(\Omega_\sigma)$ yields (3.6).

This contradicts $\|\mathfrak{L}_0\| = 1$, which means that there exists $y^\sigma \in \mathfrak{K}$ such that $y^\sigma = \mathfrak{T}y^\sigma$ with $\|y^\sigma\| \leqslant M$, and $y^\sigma$ is a bounded positive solution of the system (2.1).

Next, we consider the following Lyapunov functional :

$$V = \int_{\Omega_\sigma} [y(t,x) - y^\sigma(x)]^T [y(t,x) - y^\sigma(x)]dx = \int_{\Omega_\sigma} u^T(t,x)u(t,x)dx. \tag{3.8}$$

Set

$$U(t,u(t,x)) = \begin{cases} e^{\gamma t}\displaystyle\int_{\Omega_\sigma} u^T(t,x)u(t,x)dx, & t \geqslant 0, \\ \displaystyle\int_{\Omega_\sigma} u^T(t,x)u(t,x)dx, & t \in [-\tau, 0], \end{cases}$$

It is obvious that $U$ is continuous for $t \geqslant -\tau$. For $t \geqslant 0$ and $\gamma > 0$,

Now we claim that there is a positive constants $C_0 > 1$ and $K \in \mathbb{R}$ with $K > 1$ such that

$$U(t,u(t,x)) \leqslant KC_0\|\phi\|_\tau^2, \quad \forall\, t \geqslant 0, \tag{3.9}$$

where $\|\phi\|_\tau^2 = \sup\limits_{s\in[-\tau,0]} \int_{\Omega_\sigma} \phi^T(s,x)\phi(s,x)dx$.

Indeed, suppose this claim is not true, then there must be a $t \geqslant 0$ such that $U(t,u(t,x)) > KC_0\|\phi\|_\tau^2$. Obviously, (3.9) holds for $t \in [-\tau, 0]$, and hence there must exist $t^* > 0$ such that

$$U(t^*, u(t^*,x)) = KC_0\|\phi\|_\tau^2 \quad \text{and} \quad U(t,u(t,x)) \leqslant KC_0\|\phi\|_\tau^2,\ \forall\, t \in [0, t^*],$$

and hence

$$U(t^*, u(t^*,x)) = KC_0\|\phi\|_\tau^2 \quad \text{and} \quad U(t,u(t,x)) \leqslant KC_0\|\phi\|_\tau^2,\ \forall\, t \in [-\tau, t^*]. \tag{3.10}$$

Let $q > 1$, and due to $U(0,u(0,x)) < KC_0\|\phi\|_\tau^2 = U(t^*, u(t^*,x))$, there is $t^{**} \in [0, t^*]$ such that

$$\begin{cases} U(t^{**}, u(t^{**})) = \dfrac{1}{q}KC_0\|\phi\|_\tau^2 < KC_0\|\phi\|_\tau^2 = U(t^*, u(t^*))\,; \\ U(t^{**}, u(t^{**})) \leqslant U(t,u(t,x)) \leqslant U(t^*, u(t^*)) = KC_0\|\phi\|_\tau^2, \quad \forall\, t \in [t^{**}, t^*]. \end{cases} \tag{3.11}$$

It follows from (3.10), (3.11) and the definition of $U(t,u(t,x))$ that for $s \in [-\tau, 0]$ and $t \in [t^{**}, t^*]$,

$$
\begin{aligned}
&\int_{\Omega_\sigma} e^{\gamma s}[u^T(t+s)u(t+s)]dx \\
=&\begin{cases} e^{-\gamma t}U(t+s,u(t+s)), & t+s\geqslant 0 \\ e^{\gamma s}U(t+s,u(t+s)), & t+s\leqslant 0 \end{cases} \\
\leqslant& e^{-\gamma t}U(t+s,u(t+s)) \\
\leqslant& q\int_{\Omega_\sigma}[u^T(t,x)u(t,x)]dx,
\end{aligned} \tag{3.12}
$$

which yields that for any $s\in[-\tau,0]$,

$$
\int_{\Omega_\sigma}[u^T(t-\tau(t))u(t-\tau(t))]dx \leqslant e^{\gamma\tau}q\int_{\Omega_\sigma}u^T(t,x)u(t,x)dx, \quad t\in[t^{**},t^*]. \tag{3.13}
$$

On the other hand, the condition (A1) yields

$$
\begin{aligned}
u^TA_\sigma f(u)+f^T(u)A_\sigma^Tu =&(A_\sigma^Tu)^Tf(u)+f^T(u)(A_\sigma^Tu) \\
\leqslant& u^T(A_\sigma A_\sigma^T)u+u^TG^2u,
\end{aligned}
$$

and

$$
u^TB_\sigma f(u(t-\tau(x),x))+f^T(u(t-\tau(x),x))B_\sigma^Tu \leqslant u^T(B_\sigma B_\sigma^T)u+u^T(t-\tau(x),x)G^2u(t-\tau(x),x).
$$

Now, we calculate the derivative $\frac{dV}{dt}$ alongside with the trajectories of the system (2.1) or (2.3) as follows,

$$
\begin{aligned}
\frac{dV}{dt} =&2\int_{\Omega_\sigma}u^T(t,x)\Big[D_\sigma\Delta u(t,x)-C_\sigma u(t,x)+A_\sigma f(u(t,x))+B_\sigma f(u(t-\tau(t),x))\Big]dx \\
\leqslant& \int_{\Omega_\sigma}u^T(t,x)\Big(-2\lambda_1D_\sigma-2C_\sigma+A_\sigma A_\sigma^T+B_\sigma B_\sigma^T+G^2\Big)u(t,x)dx \\
&+\int_{\Omega_\sigma}u^T(t-\tau(x),x)G^2u(t-\tau(x),x)dx,
\end{aligned}
$$

which together with (3.13) implies that

$$
\frac{dV}{dt}\leqslant\int_{\Omega_\sigma}u^T(t,x)\Big(-2\lambda_1D_\sigma-2C_\sigma+A_\sigma A_\sigma^T+B_\sigma B_\sigma^T+G^2+e^{\gamma\tau}qG^2\Big)u(t,x)dx,\ t\in[t^{**},t^*]. \tag{3.14}
$$

For any given $t\geqslant t_0$, according to the switching law $\mathfrak{F}$, when $\sigma(t^-)=i$ and $u(t,x)\in\Upsilon_i$, then keep $\sigma(t)=i$, and we can conclude that

$$
\begin{aligned}
\frac{dV}{dt}\leqslant&\int_{\Omega_\sigma}u^T(t,x)\Big(-2\lambda_1D_\sigma-2C_\sigma+A_\sigma A_\sigma^T+B_\sigma B_\sigma^T+G^2+e^{\gamma\tau}qG^2\Big)u(t,x)dx \\
\leqslant&-\lambda_{\min}\Psi V(t,u(t,x)),\ t\in[t^{**},t^*].
\end{aligned} \tag{3.15}
$$

When $\sigma(t^-)=i$ and $u(t,x)\notin\Upsilon_i$, which means that the trajectory hits a switching surface. On the other hand, it is not difficult to deduce from (3.1) that $\bigcup_{i=1}^{N}\Upsilon_i=R^n\setminus\{0\}$, which together with the minimum law (2.5) yields (3.15), too.

Thereby, it follows from the definition of $U(t,u(t,x))$ that

$$
\frac{dU}{dt}=(\gamma-\lambda_{\min}\Psi)U(t,u(t,x)\leqslant 0,\ t\in[t^{**},t^*],
$$

which derives that $U(t^*, u(t^*)) \leqslant U(t^{**}, u(t^{**}))$. This contradicts (3.11). So we have prove the claim (3.9), which means

$$e^{\gamma t}\int_{\Omega_\sigma} u^T(t,x)u(t,x)dx \leqslant KC_0\|\phi\|_\tau^2, \quad \forall\, t \geqslant 0,$$

or

$$\|u\|^2_{L^2(\Omega_\sigma)} \leqslant KC_0\|\phi\|_\tau^2 e^{-\gamma t}, \quad \forall\, t \geqslant 0,$$

which implies that the switched delayed reaction-diffusion system (2.3) equipped with the initial value (2.4) is exponentially stable. □

**Remark 1.** Under the assumption conditions of Theorem 3.1, the positive bounded stationary solution of the switched delayed reaction-diffusion system (2.1) is exponentially stable, too. Particularly, in the case of $\sigma(t) \equiv i_0$ or $N = 1$, the system (2.1) is the common delayed reaction-diffusion system without any switches. Theorem 3.1 includes the exponential stability in the classical sense for the positive bounded stationary solution of the following common delayed reaction-diffusion system:

$$\begin{cases} \dfrac{\partial y(t,x)}{\partial t} = D\Delta y(t,x) - Cy(t,x) + Ag(y(t,x)) + Bg(y(t-\tau(t),x)) + J, \quad (t,x) \in \mathbb{R}_+ \times \Omega, \\ y_i(t,x) = 0,\ t \geqslant 0,\ x \in \partial\Omega,\ i = 1, 2, \cdots, n. \end{cases} \tag{3.16}$$

**Remark 2.** Generally speaking, the positive bounded solution is the nontrivial stationary solution in the practical engineering sense. Besides, the following system of algebraic equations is composed of $n$ equations about $n$ unknown variables $Y_1, Y_2, \cdots, Y_n$ :

$$-C_\sigma Y + A_\sigma g(Y) + B_\sigma g(Y) + J_\sigma = 0, \quad Y_i \in \mathbb{R},\ i = 1, 2\cdots, n, \tag{3.17}$$

where $Y = (Y_1, Y_2, \cdots, Y_n)^T$. In most cases, $J_\sigma \neq 0$ means that the zero solution is not the solution of the equations (3.16). Moreover, we have to point out the fact that except the zero solution, other constant equilibrium points are not the solutions of the system (2.1), even in the Sobolev space $W_0^{1,2}(\Omega_\sigma)$. Thus, in Theorem 3.1, we select a positive bounded stationary solution as the reasonable nontrivial equilibrium solution to be investigated.

**Remark 3.** For the first time, Theorem 3.1 shows the two sides of the diffusion phenomena in practical engineering. In fact, the item $-2\lambda_{\sigma 1}D_\sigma$ in the condition (3.1) illuminates that the poincare inequality makes diffusion promote the stability of the reaction-diffusion neural networks (2.1). On the other hand, the condition (A2) explains why diffusion makes it more difficult to judge the stability. Indeed, the condition (A2) is rather harsh, and if this condition is not satisfied, we can not judge the stability of the positive bounded stationary solution $y^*(x)$ for the reaction-diffusion cellular neural networks (2.1) with time delays. Moreover, if $y(t,x) \equiv y(t)$ in the reaction-diffusion system (2.1), then the corresponding ordinary differential equations model of the system (2.1) is follows as,

$$\frac{\partial y(t)}{\partial t} = -C_\sigma y(t) + A_\sigma g(y(t)) + B_\sigma g(y(t-\tau(t))) + J_\sigma, \quad t \geqslant 0. \tag{3.18}$$

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It is obvious that the constant equilibrium point $y^*$ of the system (3.18) satisfies the algebraic equations (3.17). And obviously the stability of he constant equilibrium point $y^*$ of the system (3.18) do not need the harsh condition (A2) at all. This illuminates that there are big differences between the stability criterion of the nontrivial stationary solution $y^*(x)$ and that of the constant equilibrium point $y^*$ . However, in previous related literature [1,2,15,16], for example, in [1], due to the poincare inequality or its other forms, there is completely similar between the stability criterion of the system (1.1) and that of its corresponding ordinary differential equations model (1.3) because the equilibrium point satisfying (1.2) is just the constant equilibrium point. That is, [1, Theorem 1] only shows one side of the diffusion phenomena which promotes the stability of reaction-diffusion neural networks. So are those of previous related literature [2-10] and the references therein (or, see, e.g. Theorem 3.2, Remark 7-8). Here, the constant equilibrium point includes the case of the null solution in previous literature (see, e.g., [3, 8-10]). For example, in [8, theorem 4.1], the null solution of reaction-diffusion neural networks is only the transformation of constant equilibrium point of its corresponding reaction-diffusion neural networks.

**Remark 4.** Particularly in the case of $\sigma(t) \equiv i_0$ or $N = 1$, the switched reaction-diffusion system is the common reaction diffusion neural networks (3.16) without any switches. Theorem 3.1 includes the classical exponential stability criterion in this case. In Theorem 3.1, the inevitable diffusions in real engineering are conducive to the stability of the system, but also makes the dynamic behavior of the system more complex and difficult to judge its stability. This explains the reason why we need to study the stability of reaction-diffusion neural networks system.

**Remark 5.** To show the fact that the constant equilibrium point of the delayed reaction-diffusion models for all neural networks (not only the cellular neural networks) is trivial, which implies that there are completely similar stability criteria between the constant equilibrium point of reaction-diffusion neural networks (partial differential equations) and that of its corresponding ordinary differential equations, we may consider the stability of the constant equilibrium point of the following delayed reaction-diffusion Cohen-Grossberg neural networks which is the partial differential equations model studied in [2]:

$$\begin{cases} \dfrac{\partial u_i(t,x)}{\partial t} =r_i\Delta u_i(t,x) - a_i(u_i(t,x))\Big[b_i(u_i(t,x)) - \sum\limits_{j=1}^{n} c_{ij}f_j(u_j(t,x)) - \sum\limits_{j=1}^{n} d_{ij}g_j(u_j(t-\tau_j(t),x)) + I_i\Big], \quad t \geqslant 0,\ t \neq t_k, \\ u_i(t^+,x) =m_i u_i(t^-,x) + \sum\limits_{j=1}^{n} n_{ij}h_j(u_j(t^- - \tau_j(t),x)), \quad t = t_k,\ 0 \leqslant \tau_j(t) \leqslant \tau_j,\ \forall\, j, \\ u_i(t,x) =0, \quad t \geqslant 0,\ x \in \partial\Omega,\ i = 1,2,\cdots,n, \\ u_i(s,x) =\phi_i(s,x), \quad -\tau \leqslant s \leqslant 0,\ \tau = \max\limits_{1\leqslant j\leqslant n} \tau_j, \end{cases} \tag{3.19}$$

where all the variables, coefficients and functions are defined in [2], which are different from those of our Theorem 3.1. We always assume $u_i(t_k^+,x) = u_i(t_k,x)$. Below, we will give two completely similar criteria for the stability of the constant equilibrium point of the system (3.19) and its corresponding ordinary differential equations:

$$\begin{cases} \dfrac{du_i(t)}{dt} = -a_i(u_i(t))\Big[b_i(u_i(t)) - \sum_{j=1}^{n} c_{ij}f_j(u_j(t)) - \sum_{j=1}^{n} d_{ij}g_j(u_j(t-\tau_j(t))) + I_i\Big], \quad t \geqslant 0,\ t \neq t_k, \\ u_i(t^+) = m_i u_i(t^-) + \sum_{j=1}^{n} n_{ij}h_j(u_j(t^- - \tau_j(t))), \quad t = t_k, \\ u_i(s) = \phi_i(s), \quad -\tau \leqslant s \leqslant 0,\ \tau = \max_{1\leqslant j\leqslant n} \tau_j, \quad i = 1, 2, \cdots, n. \end{cases} \tag{3.20}$$

For the convenience of readers, we may copy the assumption conditions in the document [2] as follows,

(H1) Each function $a_i(u)$ is bounded, positive and continuous, i.e., there exist two positive diagonal matrices $\underline{A} = diag(\underline{A}_1, \underline{A}_2, \cdots, \underline{A}_n)$ and $\underline{A} = diag(\overline{A}_1, \overline{A}_2, \cdots, \overline{A}_n)$ such that

$$0 < \underline{A}_i \leqslant a_i(u) \leqslant \overline{A}_i, \quad \forall\, u \in \mathbb{R},\ \forall\, i.$$

(H2) Each function $b_i(u)$ is monotone increasing, i.e., there exist a positive diagonal matrix $B = diag(B_1, B_2, \cdots, B_n)$ such that

$$\frac{b_i(u) - b_i(v)}{u - v} \geqslant B_i, \quad \forall\, u, v(u \neq v) \in \mathbb{R},\ \forall\, i.$$

(H3) There exist three positive diagonal matrices $F = diag(F_1, F_2, \cdots, F_n)$, $G = diag(G_1, G_2, \cdots, G_n)$ and $H = diag(H_1, H_2, \cdots, H_n)$ such that

$$0 \leqslant \frac{f_i(u) - f_i(v)}{u - v} \leqslant F_i, \quad 0 \leqslant \frac{g_i(u) - g_i(v)}{u - v} \leqslant G_i, \quad 0 \leqslant \frac{h_i(u) - h_i(v)}{u - v} \leqslant H_i, \qquad \forall\, u, v(u \neq v) \in \mathbb{R},\ \forall\, i.$$

It is obvious that the system (3.19) and its corresponding ordinary differential equations (3.20) own the same constant equilibrium point $u^* = (u_1^*, \cdots, u_n^*)^T \in \mathbb{R}^n$ if $u^* = (u_1^*, \cdots, u_n^*)^T$ satisfies

$$\begin{cases} b_i(u_i^*) - \sum_{j=1}^{n} c_{ij}f_j(u_j^*) \sum_{j=1}^{n} d_{ij}g_j(u_j^*) + I_i = 0, \\ (m_i - 1)u_i(t^*) + \sum_{j=1}^{n} n_{ij}h_j(u_j^*) = 0, \end{cases} \quad \forall\, i, j = 1, 2, \cdots, n. \tag{3.21}$$

**Theorem 3.2.** Under assumptions (H1)-(H3), if the following conditions hold:

($\tilde{C}1$) there exists a positive diagonal matrix $P > 0$ such that

$$\widetilde{\Psi} = \begin{pmatrix} -2P\underline{A}B + F^2 & P\overline{A}|C| & P\overline{A}|D| \\ |C^T|\overline{A}P & -I & 0 \\ |D^T|\overline{A}P & 0 & -I \end{pmatrix} < 0,$$

where $R = diag(r_1, r_2, \cdots, r_n)$, $|C| = (|c_{ij}|)_{n\times n}$, $|D| = (|d_{ij}|)_{n\times n}$, $I = diag(1, 1, \cdots, 1)$;

($\tilde{C}2$) $\tilde{a} = \frac{\lambda_{\min}\widetilde{\Phi}}{\lambda_{\max}P} > \frac{\lambda_{\max}G^2}{\lambda_{\min}P} = b \geqslant 0$, where

$$\widetilde{\Phi} = 2P\underline{A}B - P\overline{A}|C||C^T|\overline{A}P - P\overline{A}|D||D^T|\overline{A}P - F^2 > 0\,;$$

(C3) there exists a constant $\delta$ such that $\delta > \ln(\rho e^{\lambda\tau})/\delta\tau$, where $\lambda > 0$ is the unique solution of the equation $\lambda = a - be^{\lambda\tau}$, and $\rho = \max\{1, \frac{2\lambda_{\max}(PMP)}{\lambda_{\min}P} + \frac{2\lambda_{\max}(HN^TPNH)}{\lambda_{\min}P}e^{\lambda\tau}\}$, $M = diag(m_1, m_2, \cdots, m_n)$, $N = (n_{ij})_{n\times n}$, $H = diag(H_1, \cdots, H_n)$;

then we have the conclusions:

(1) the constant equilibrium point $u^*$ of system (3.20) is globally exponentially stable with convergence rate $\frac{1}{2}(\lambda - \frac{\ln(\rho e^{\lambda\tau})}{\delta\tau})$;

(2) the globally exponential stability of the equilibrium point $u^*$ of system (3.20) directly yields that the equilibrium point $u^*$ of system (3.19) is also globally exponentially stable with the same convergence rate $\frac{1}{2}(\lambda - \frac{\ln(\rho e^{\lambda\tau})}{\delta\tau})$, which implies that the diffusion promotes the stability, or the diffusion is not harmful to the stability.

*Proof.* Firstly, we may prove the first conclusion of Theorem 3.1 involved in the system (3.20).

Next, for any given $i$, let $y_i(t) = u_i(t) - u_i^*$, where $u^*$ is the constant equilibrium point of the system (3.20). Then the system (3.20) can be transformed into

$$\begin{cases} \frac{dy_i(t)}{dt} = -\tilde{a}_i(y_i(t))\Big[\tilde{b}_i(y_i(t)) - \sum_{j=1}^n c_{ij}\tilde{f}_j(y_j(t)) - \sum_{j=1}^n d_{ij}\tilde{g}_j(y_j(t-\tau_j(t)))\Big], & t \geqslant 0, t \neq t_k, \\ y_i(t^+) = m_i y_i(t^-) + \sum_{j=1}^n n_{ij}\tilde{h}_j(y_j(t^- - \tau_j(t))), & t = t_k, \\ y_i(s) = \phi_i(s) - u_i^*(s), \quad -\tau \leqslant s \leqslant 0, \quad \tau = \max_{1\leqslant j\leqslant n} \tau_j, & i = 1, 2, \cdots, n. \end{cases} \tag{3.22}$$

where $\tilde{a}_i(y_i(t)) = a_i(y_i(t) + u_i^*)$, $\tilde{b}_i(y_i(t)) = b_i(y_i(t) + u_i^*) - b_i(u_i^*)$, $\tilde{f}_j(y_j(t)) = f_j(y_j(t) + u_j^*) - f_j(u_j^*)$, $\tilde{g}_j(y_j(t)) = g_j(y_j(t) + u_j^*) - g_j(u_j^*)$, $\tilde{h}_j(y_j(t)) = h_j(y_j(t) + u_j^*) - h_j(u_j^*)$ for all $i, j = 1, 2, \cdots, n$.

Similarly as the proof of [2, Theorem 3.1], we may set up the Lyapunov function as follows,

$$\widetilde{V}(t) = Y^T(t)PY(t) = |Y^T(t)|P|Y(t)|,$$

where $Y(t) = (y_1(t), \cdots, y_n(t))^T$, $P = diag(p_1, p_2, \cdots, p_n)$.

For the case of $t \neq t_k$, we compute the Dini derivative of $\widetilde{V}(t)$ alongside with the trajectories of (3.22),

$$\begin{aligned} D^+\widetilde{V}(t) =& -2Y^T(t)P\widetilde{A}(Y(t))\widetilde{B}(Y(t)) + 2Y^T(t)P\widetilde{A}(Y(t))C\widetilde{F}(Y(t)) + 2Y^T(t)P\widetilde{A}(Y(t))D\widetilde{G}(Y(t-\tau(t))) \\ \leqslant& -|Y^T(t)|\widetilde{\Phi}|Y(t)| + Y^T(t-\tau(t))G^2Y(t-\tau(t)) \\ \leqslant& -\tilde{a}\widetilde{V}(t) + b[\widetilde{V}(t)]_\tau, \end{aligned}$$

where $\widetilde{A}(Y(t)) = diag(\tilde{a}_1(y_1(t)), \cdots, \tilde{a}_n(y_n(t)))$, $\widetilde{B}(Y(t)) = (\tilde{b}_1(y_1(t)), \cdots, \tilde{b}_n(y_n(t)))^T$, $\widetilde{F}(Y(t)) = (\tilde{f}_1(y_1(t)), \cdots, \tilde{f}_n(y_n(t)))^T$, $\widetilde{G}(Y(t)) = (\tilde{g}_1(y_1(t)), \cdots, \tilde{g}_n(y_n(t)))^T$, $\widetilde{G}(Y(t-\tau(t))) = (\tilde{g}_1(y_1(t-\tau_1(t))), \cdots, \tilde{g}_n(y_n(t-\tau_n(t))))^T$.

When $t = t_k$, using the similar methods in the proof of [2, Theorem 3.1] results in that

$$\widetilde{V}(t_k) = Y^T(t_k)PY(t_k) \leqslant 2\frac{\lambda_{\max}(PMP)}{\lambda_{\min}P}\widetilde{V}(t_k^-) + 2\frac{\lambda_{\max}(HN^TPNH)}{\lambda_{\min}P}[\widetilde{V}(t_k^-)]_\tau.$$

Now it follows from $(\widetilde{C}1)$, $(\widetilde{C}2)$, (C3) and [2, Lemma 2.2] that

$$\widetilde{V}(t) \leqslant \rho[\widetilde{V}(0)]_\tau e^{-(\lambda - \frac{\ln(\rho e^{\lambda\tau})}{\delta\tau})t}, \quad t \geqslant 0,$$

or

$$\sqrt{(u(t)-u^*)^T(u(t)-u^*)} = \sqrt{Y^T(t)Y(t)} \leqslant \sqrt{\frac{\rho\lambda_{\max}P}{\lambda_{\min}P}}\sqrt{[Y^T(0)Y(0)]_\tau}\, e^{-\frac{1}{2}(\lambda-\frac{\ln(\rho e^{\lambda\tau})}{\delta\tau})t}, \quad t \geqslant 0,$$

which has proved that the equilibrium point $u^*$ of system (3.20) is globally exponentially stable with convergence rate $\frac{1}{2}(\lambda - \frac{\ln(\rho e^{\lambda\tau})}{\delta\tau})$.

Finally, we shall prove the second conclusion of Theorem 3.2.

Indeed, let $Y(t,x) = u(t,x) - u^*$, where the constant equilibrium point $u^* = (u_1^*, \cdots, u_n^*)^T$ is a constant vector satisfying (3.21) for all $x \in \Omega$ with $u^*|_{\partial\Omega} = 0$, then the system (3.19) can be transformed into

$$\begin{cases} \frac{\partial y_i(t,x)}{\partial t} = r_i\Delta y_i(t,x) - \tilde{a}_i(y_i(t,x))\Big[\tilde{b}_i(y_i(t,x)) - \sum_{j=1}^n c_{ij}\tilde{f}_j(y_j(t,x)) - \sum_{j=1}^n d_{ij}\tilde{g}_j(y_j(t-\tau_j(t),x))\Big], & t \geqslant 0,\ t \neq t_k, \\ y_i(t^+,x) = m_i y_i(t^-,x) + \sum_{j=1}^n n_{ij}\tilde{h}_j(y_j(t^- - \tau_j(t),x)), & t = t_k,\ 0 \leqslant \tau_j(t) \leqslant \tau_j,\ \forall\, j, \\ y_i(t,x) = 0, & t \geqslant 0,\ x \in \partial\Omega,\ i = 1,2,\cdots,n, \\ y_i(s,x) = \phi_i(s,x) - u_i^*, & -\tau \leqslant s \leqslant 0,\ \tau = \max_{1\leqslant j\leqslant n}\tau_j, \end{cases} \tag{3.23}$$

where $y_i, \tilde{a}_i, \tilde{b}_i, \tilde{f}_j, \tilde{g}_j$ and $\tilde{h}_i$ all are defined as those of [2].

$$\int_\Omega |\nabla u_i(t,x)|^2 dx \geqslant \lambda_1 \int_\Omega u_i^2(t,x)dx$$

On the other hand, the Poincare inequality and the Dirichlet zero boundary value yields

$$\begin{aligned} \int_\Omega Y^T(t,x)PR\Delta Y(t,x)dx &= -\int_\Omega \sum_{i=1}^n p_i r_i \sum_{j=1}^m (\frac{\partial y_i}{\partial x_j})^2 dx \\ &\leqslant -\lambda_1 \int_\Omega \sum_{i=1}^n p_i r_i y_i^2(t,x)dx \leqslant 0, \end{aligned} \tag{3.24}$$

where $\lambda_1$ is the smallest positive eigenvalue of the following eigenvalue problem:

$$\begin{cases} -\Delta\varphi(x) = \lambda\varphi(x), & x \in \Omega \subset \mathbb{R}^m, \\ \varphi(x) = 0, & x \in \partial\Omega. \end{cases}$$

Constructing the Lyapunov functional as follows,

$$\mathcal{V}(t) = \int_\Omega Y^T(t,x)PY(t,x)dx = \int_\Omega |Y^T(t,x)|P|Y(t,x)|dx$$

For the case of $t \neq t_k$, we compute the Dini derivative of $\mathcal{V}(t)$ alongside with the trajectories of (3.23),

$$\begin{aligned} D^+\mathcal{V}(t) &= \int_\Omega \Big[-2Y^T(t,x)P\widetilde{A}(Y(t,x))\widetilde{B}(Y(t,x)) + 2Y^T(t,x)P\widetilde{A}(Y(t,x))C\widetilde{F}(Y(t,x)) + 2Y^T(t,x)P\widetilde{A}(Y(t,x))D\widetilde{G}(Y(t-\tau(t),x))\Big]dx \\ &\leqslant \int_\Omega \Big[-|Y^T(t,x)|\widetilde{\Phi}|Y(t,x)| + Y^T(t-\tau(t),x)G^2Y(t-\tau(t),x)\Big]dx \\ &\leqslant \int_\Omega \Big[-\tilde{a}\mathcal{V}(t,x) + b[\mathcal{V}(t,x)]_\tau\Big]dx, \end{aligned}$$

where $\widetilde{A}, \widetilde{B}, \widetilde{F}, \widetilde{G}$ all are defined as those of [2]. Completely similar as the proof of the first conclusion of Theorem 3.2, we can also obtain

$$\mathcal{V}(t_k) = \int_{\Omega} Y^T(t_k, x)PY(t_k, x)dx \leqslant 2\frac{\lambda_{\max}(PMP)}{\lambda_{\min}P}\mathcal{V}(t_k^-) + 2\frac{\lambda_{\max}(HN^TPNH)}{\lambda_{\min}P}[\mathcal{V}(t_k^-)]_\tau ,$$

and

$$\mathcal{V}(t) \leqslant \rho[\mathcal{V}(0)]_\tau e^{-(\lambda - \frac{\ln(\rho e^{\lambda\tau})}{\delta\tau})t}, \quad t \geqslant 0,$$

or

$$\sqrt{\int_{\Omega}(u(t,x)-u^*)^T(u(t,x)-u^*)dx} \leqslant \sqrt{\frac{\rho\lambda_{\max}P}{\lambda_{\min}P}}\sqrt{[\int_{\Omega}Y^T(0,x)Y(0,x)dx]_\tau}\ e^{-\frac{1}{2}(\lambda - \frac{\ln(\rho e^{\lambda\tau})}{\delta\tau})t}, \quad t \geqslant 0,$$

which has proved that the equilibrium point $u^*$ of system (3.20) is globally exponentially stable with convergence rate $\frac{1}{2}(\lambda - \frac{\ln(\rho e^{\lambda\tau})}{\delta\tau})$.

In fact, due to (3.24), the globally exponential stability of the equilibrium point $u^*$ of system (3.20) directly yields that the equilibrium point $u^*$ of system (3.19) is also globally exponentially stable with the same convergence rate $\frac{1}{2}(\lambda - \frac{\ln(\rho e^{\lambda\tau})}{\delta\tau})$, which implies that the diffusion promotes the stability. The proof is completed.

□

**Remark 6.** Due to (3.24), we can know from two conclusions of Theorem 3.2 that the stability criterion of the constant equilibrium point of delayed reaction-diffusion neural networks (3.19) is same as that of its corresponding ordinary differential equations (3.20). So we only need to study the ordinary differential equations model if we only study the stability of the constant equilibrium point. In the other word, if if we only study the stability of the constant equilibrium point, the reaction-diffusion partial differential equations model is unnecessary. What's the problem? So, in this paper, we should investigate stability of the nontrivial stationary solution of neural networks, rather than that of the constant equilibrium point. Moreover, under Dirichlet zero boundary value, we have to point out that except the zero solution, other constant equilibrium points are not the solutions of reaction diffusion neural networks, even in the Sobolev space $W_0^{1,2}(\Omega)$.

**Remark 7.** Obviously, [2, Theorem 3.1] is the direct corollary of our Theorem 3.2 due to the Poincare inequality. In fact, the condition (C1) of [2, Theorem 3.1] is as follows,

$$\Psi = \widetilde{\Psi} + \begin{pmatrix} -2lPR & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} < 0 \tag{3.25}$$

and the condition (C2) of [2, Theorem 3.1] is as follows,

$$a = \frac{\lambda_{\min}\Phi}{\lambda_{\max}P} \geqslant \tilde{a} = \frac{\lambda_{\min}\widetilde{\Phi}}{\lambda_{\max}P} > \frac{\lambda_{\max}G^2}{\lambda_{\min}P} = b \geqslant 0, \tag{3.26}$$

where

$$\Phi = 2lPR + \widetilde{\Phi} > 0. \tag{3.27}$$

Therefore, our Theorem 3.2 yields that all the conditions of [2, Theorem 3.1] are satisfied, which implies that [2, Theorem 3.1] is the direct corollary of our Theorem 3.2.

**Remark 8.** [2, Lemma 2.1] is an approximation of the Poincare inequality. In (3.25)-(3.27), the Poincare inequality play the key role, which illuminates the fact that the diffusion promotes stability. However, [2, Theorem 3.1] and our Theorem 3.2 only point out that diffusion contributes to stability, and Theorem 3.2 does not reflect that diffusion makes the dynamic behavior of the system more complex and it is more difficult to judge the stability of the system. So we have reason to believe that the constant equilibrium point is indeed trivial, and only our Theorem 3.1 involved in the stability of the nontrivial stationary solution points out the two sides of the diffusion phenomena in reality engineering (see Remark 3 for details). So we should pay more attention to stability analysis of the nontrivial stationary solution of reaction diffusion neural networks in the future, rather than the stability of the constant equilibrium point.

**Remark 9.** Particularly let $a_i(u_i) \equiv 1$ and $b_i(u_i) = b_i u_i$ with $b_i \in \mathbb{R}$ in the delayed reaction-diffusion Cohen-Grossberg neural networks (3.19), then the Cohen-Grossberg neural networks (3.19) is reduced to the following cellular neural networks

$$\begin{cases} \frac{\partial u_i(t,x)}{\partial t} = r_i \Delta u_i(t,x) - b_i u_i(t,x) + \sum_{j=1}^{n} c_{ij} f_j(u_j(t,x)) + \sum_{j=1}^{n} d_{ij} g_j(u_j(t-\tau_j(t),x)) - I_i, \quad t \geqslant 0,\ t \neq t_k, \\ u_i(t^+,x) = m_i u_i(t^-,x) + \sum_{j=1}^{n} n_{ij} h_j(u_j(t^- - \tau_j(t),x)), \quad t = t_k,\ 0 \leqslant \tau_j(t) \leqslant \tau_j,\ \forall\, j, \\ u_i(t,x) = 0, \quad t \geqslant 0,\ x \in \partial\Omega,\ i = 1,2,\cdots,n, \\ u_i(s,x) = \phi_i(s,x), \quad -\tau \leqslant s \leqslant 0,\ \tau = \max_{1 \leqslant j \leqslant n} \tau_j. \end{cases} \tag{3.28}$$

So the conclusions of Theorem 3.2 include the case of cellular neural networks.

**Remark 10.** Below, we shall employ a simple example to show that although the diffusion coefficients may be very small, the topological structure of the phase plane of the dynamic behavior of the following reaction-diffusion system is likely to change substantially from a constant equilibrium point of the following system (1.3) to multiple stationary solutions of the reaction-diffusion system. Therefore, many global stability results of delayed neural networks in the form of ordinary differential equations may only be locally asymptotical stability criteria in real engineering. So, in this paper, we have to investigate the stability of reaction-diffusion neural networks with time delays.

**[31, Theorem 1]**. Suppose that there exists a positive constant $\bar{l}_i$ such that

$$|f_i(u) - f(v)| \leqslant \bar{l}_i |u - v|, \quad u, v \in \mathbb{R},\ i = 1,2,\cdots,n, \tag{3.29}$$

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then the constant equilibrium point of the following system (cellular neural networks):

$$\frac{dx_i}{dt} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j(t))) + J_i, \qquad i = 1,2,\cdots,n. \tag{3.30}$$

is globally exponentially stable if there exists a nonsingular matrix $N$ such that

$$-\mu_p(-NCN^{-1}) - \bar{l}\|NA\|_p\|N^{-1}\|_p > \bar{l}\|NB\|_p\|N^{-1}\|_p > 0, \tag{3.31}$$

where $\bar{l} = \max_{1\leqslant i\leqslant n} \bar{l}_i$ and $p = 1, 2, \infty$.

**Remark 11.** To give a simple proof of the idea of Remark 10, we may consider the following cellular neural networks in the case of $n = 2$,

$$\begin{cases} \dfrac{dx_1(t)}{dt} = -c_1x_1(t) + a_{11}f_1(x_1(t)) + a_{12}f_2(x_2(t)) + b_{11}f_1(x_1(t-\tau_1(t))) + b_{12}f_2(x_2(t-\tau_2(t))) + J_1 \\ \dfrac{dx_2(t)}{dt} = -c_2x_2(t) + a_{21}f_1(x_1(t)) + a_{22}f_2(x_2(t)) + b_{21}f_1(x_1(t-\tau_1(t))) + b_{22}f_2(x_2(t-\tau_2(t))) + J_2, \end{cases} \tag{3.32}$$

where $J_1 = 0.6$, $J_2 = 0.3$, $f_j(x_j) = j(0.05x_j - \frac{0.3}{j})$ for $j = 1, 2$, and $\bar{l}_j = 0.05j$, then $\bar{l} = 0.1$, and hence the condition (3.29) holds. Besides, let $N = diag(1, 1)$, and

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1.8 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.2 & 0.2 \\ 0.4 & 0.4 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 0.05 & 0.05 \\ 0.1 & 0.1 \end{pmatrix}, \tag{3.33}$$

$$\begin{cases} \dfrac{dx_1}{dt} = -x_1 + 0.2(0.05x_1 - k) + 0.2(0.1x_2 - k) + 0.05(0.05x_1(t-\tau_1) - k) + 0.05(0.1x_2(t-\tau_2) - k) + J_1 \\ \dfrac{dx_2}{dt} = -1.8x_2 + 0.4(0.05x_1 - k) + 0.4(0.1x_2 - k) + 0.1(0.05x_1(t-\tau_1) - k) + 0.1(0.1x_2(t-\tau_2) - k) + J_2, \end{cases} \Rightarrow \begin{cases} k = 0.5J_1 \\ k = J_2 \end{cases} \tag{3.32*}$$

$$\Rightarrow \begin{cases} k = 0.5J_1, \quad J_1 = 0.6 \\ k = J_2 = 0.3 \end{cases} \Rightarrow \begin{cases} \dfrac{dx_1}{dt} = -x_1 + 0.2(0.05x_1) + 0.2(0.1x_2) + 0.05(0.05x_1(t-\tau_1)) + 0.05(0.1x_2(t-\tau_2)) \\ \dfrac{dx_2}{dt} = -1.8x_2 + 0.4(0.05x_1) + 0.4(0.1x_2) + 0.1(0.05x_1(t-\tau_1)) + 0.1(0.1x_2(t-\tau_2)), \end{cases} \tag{3.32**}$$

Let $p = 1$ and $N = I = diag(1, 1)$, then

$$-\mu_1(-NCN^{-1}) - \bar{l}\|NA\|_1\|N^{-1}\|_1 = 1.8 - 0.1 \times 0.6 = 1.74 > 0.1 \times 0.15 = \bar{l}\|NB\|_1\|N^{-1}\|_1 > 0,$$

and hence the condition (3.31) holds. Therefore, according to [Theorem 3.1], the constant equilibrium point $x^* = (x_1^*, x_2^*)^T$ of the cellular neural networks (3.32) is globally exponentially stable, where the constant vector $x^* = (x_1^*, x_2^*)^T$ is the unique of the following equations

$$\begin{cases} 0 = -c_1x_1 + a_{11}f_1(x_1) + a_{12}f_2(x_2) + b_{11}f_1(x_1) + b_{12}f_2(x_2) + J_1 \\ 0 = -c_2x_2 + a_{21}f_1(x_1) + a_{22}f_2(x_2) + b_{21}f_1(x_1) + b_{22}f_2(x_2) + J_2. \end{cases} \tag{3.34}$$

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Direct computation on the equations (3.34) results in the fact that the constant vector $x^* = (x_1^*, x_2^*)^T = (0,0)^T$. So we have concluded the following true proposition due to [31, Theorem 1]:

**Proposition 1.** In the delayed reaction diffusion cellular neural networks (3.32), let $J_1 = 0.6$, $J_2 = 0.3$, $f_j(u_j) = j(0.05u_j - \frac{0.3}{j})$ for $j = 1,2$, and other data are defined in (3.33), then there is the unique constant equilibrium point $x^* = (x_1^*, x_2^*)^T = (0,0)^T$ is globally exponentially stable.

Below, we shall prove that although diffusion coefficients ($D_1 = 0.003, D_2 = 0.006$) both are very small, the following reaction diffusion cellular neural networks

$$\begin{cases} \frac{\partial u_1}{\partial t} = D_1\Delta u_1(t,x) - c_1u_1(t,x) + a_{11}f_1(u_1(t,x)) + a_{12}f_2(u_2(t,x)) + b_{11}f_1(u_1(t-\tau_1(t),x)) + b_{12}f_2(u_2(t-\tau_2(t),x)) + J_1,\ t \geqslant 0,\ x \in \Omega, \\ \frac{\partial u_2}{\partial t} = D_2\Delta u_2(t,x) - c_2u_2(t,x) + a_{21}f_1(u_1(t,x)) + a_{22}f_2(u_2(t,x)) + b_{21}f_1(u_1(t-\tau_1(t),x)) + b_{22}f_2(u_2(t-\tau_2(t),x)) + J_2,\ t \geqslant 0,\ x \in \Omega, \\ u_i(t,x) = 0, \quad x \in \partial\Omega,\ i = 1,2, \end{cases} \tag{3.35}$$

owns at least two equilibrium points, where the corresponding ordinary differential equations model of the system (3.35) is the system (3.32) under the same data in Remark 11. This will illuminate that the globally asymptotical stability of the ordinary differential equations model (3.32) in Proposition 1 or [31, Theorem 1] may be the locally asymptotical stability as best possible in reality engineering because diffusion is inevitable in practice.

**Theorem 3.3.** In the delayed reaction diffusion cellular neural networks (3.35), let $D_1 = 0.003, D_2 = 0.006$, $J_1 = 0.6$, $J_2 = 0.3$, $f_j(u_j) = j(0.05u_j - \frac{0.3}{j})$ for $j = 1,2$, $\tau = 0.5$, $\Omega = (0,1)\times(0,1)$, and other data are defined in (3.33), then there are at least two equilibrium solutions (a constant zero solution $(0,0)^T$ and another stable positive stationary solution $u(x)$, $x \in \Omega$ with $u|_{\partial\Omega} = 0$) for the delayed reaction diffusion system (3.35).

*Proof.* By means of direct calculations, we know the system (3.35) is just the following system

$$\begin{cases} \frac{\partial u_1(t,x)}{\partial t} = 0.003\Delta u_1(t,x) - 0.99u_1 + 0.02u_2(t,x) + 0.0025u_1(t-\tau_1(t),x) + 0.005u_2(t-\tau_2(t),x)),\ t \geqslant 0,\ x \in \Omega, \\ \frac{\partial u_2(t,x)}{\partial t} = 0.006\Delta u_2(t,x) + 0.02u_1(t,x) - 1.76u_2(t,x) + 0.005u_1(t-\tau_1(t),x)) + 0.01u_2(t-\tau_2(t),x),\ t \geqslant 0,\ x \in \Omega, \\ u_i(t,x) = 0, \quad x \in \partial\Omega,\ i = 1,2, \end{cases} \tag{3.36}$$

or

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = D\Delta u(t,x) - Cu(t,x) + Af(u(t,x)) + Bf(u(t-\tau(t),x)) + J, \quad (t,x) \in \mathbb{R}_+ \times \Omega, \\ u_i(t,x) = 0,\ t \geqslant 0,\ x \in \partial\Omega,\ i = 1,2,\cdots,n, \end{cases} \tag{3.37}$$

where $J_1 = 0.6$, $J_2 = 0.3$, $f_j(x_j) = j(0.05x_j - \frac{0.3}{j})$ for $j = 1,2$, and matrices $C, A, B$ all are defined in (3.33). Besides,

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} = \begin{pmatrix} 0.003 & 0 \\ 0 & 0.006 \end{pmatrix},$$

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and the stationary solution $u(x)$ of the system (3.36) or (3.37) must satisfy

$$\begin{cases} 0.003\Delta u_1 - 0.99u_1 + 0.02u_2 + 0.0025u_1(x) + 0.005u_2(x) = 0, \ x \in \Omega, \\ 0.006\Delta u_2 + 0.02u_1 - 1.76u_2 + 0.005u_1(x) + 0.01u_2(x) = 0, \ x \in \Omega, \\ u_i(x) = 0, \quad x \in \partial\Omega, \ i = 1, 2, \end{cases}$$

It is obvious from (3.36) that the zero solution is a constant equilibrium point of the system (3.36) or (3.37).

On the other hand, each $f_i$ is a Lipschitz continuous, and hence the condition (A1) of Theorem 3.1 is satisfied. From $\Omega = (0,1) \times (0,1)$ and Remark 18, we can compute that $\lambda_1 = 19.7392$. Set $\lambda = 0.0001 \in (0, \lambda_1)$, $c = 10000 > 0$ and $\alpha = 0.0001 \in (0,1)$ with $(\alpha+1)\lambda < \lambda_1$. Now direct computation results in

$$0 \leqslant \Big[ -\Big(C + (1+\alpha)\lambda I\Big)v + Ag(v) + Bg(v) + J \Big] \leqslant cE, \quad \forall\, v \in \mathbb{R}^2,$$

where $I = diag(1,1)$ and $E = (1,1)^T \in \mathbb{R}^2$. And hence the condition (A2) holds. In addition, let $q = 1.0001 > 1$, $\Psi = 0.0001I$, $\gamma = 0.00005$, and $L = diag(\bar{l}_1, \bar{l}_2) = (0.05, 0.1)$, we can employ Matlab software to compute, verifying the following inequality:

$$\Big( -2\lambda_1 D - 2C + AA^T + BB^T \Big) + L^2 + e^{\gamma\tau} qG^2 + \Psi < 0,$$

which implies that the condition (3.1) holds. Now, according to our Theorem 3.1, there is a stable positive stationary solution $u(x)$, $x \in \Omega$ with $u|_{\partial\Omega} = 0$) for the delayed reaction diffusion system (3.35). And the proof is completed.

□

**Remark 12. There has always been a problem (see,e.g. Remark 8): whether is it the case in the literature([1-10]) that the greater the diffusion, the more stable the system will be ?** Now, our Theorem 3.3 illuminates that the globally asymptotical stability of the ordinary differential equations model (3.32) in Proposition 1 or [31, Theorem 1] may be the locally asymptotical stability at best possible in reality engineering because diffusion is inevitable in practice, and **our Theorem 3.3 has verified that the positive stationary solution is stable due to our Theorem 3.1, which implies that the constant equilibrium point (zero solution) must be not globally stability in reaction diffusion system (3.35).** That is, the diffusion is a double-edged sword for the stability of delayed neural networks in reality engineering, which is reflected in our Theorem 3.1.

In our Theorem 3.3, we have verified the idea of Remark 10 via the cellular neural networks in the case of $n = 2$. Next, we shall verified it via the cellular neural networks in the case of $n = 1$.

Consider the following cellular neural networks (3.37) in the case of $n = 1$:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = D\Delta u(t,x) - Cu(t,x) + Af(u(t,x)) + Bf(u(t-\tau(t),x)) + J, \quad (t,x) \in \mathbb{R}_+ \times \Omega, \ \Omega \subset \mathbb{R}, \\ u(t,x) = 0, \ t \geqslant 0, \ x \in \partial\Omega, \\ u(s,x) = \xi(s,x) \text{ is bounded in } [-\tau, 0] \times (0,1). \end{cases} \tag{3.38}$$

and its corresponding ordinary differential equation model:

$$\begin{cases} \frac{dx(t)}{dt} = -Cx(t) + Af(x(t)) + Bf(x(t-\tau(t))) + J, \quad t \geqslant 0, \\ x(s) = \xi(s) \text{ is bounded in } [-\tau, 0]. \end{cases} \tag{3.39}$$

Similarly, [31, Theorem 1] yields the following proposition:

**Proposition 2.** In the delayed system (3.39), let $f(x) = 0.05(x-6)$, $C = 1.8$, $A = 0.2$, $B = 0.1$, $J = 1.09$, then there is the unique constant equilibrium point $x^* = \frac{1000}{1785} = \frac{200}{357}$ is globally exponentially stable.

*Proof.* Direct computation derives that $x = \frac{1000}{1785} = \frac{200}{357}$ is the constant equilibrium point of the system (3.39). Obviously, $f$ is Lipschitz continuous function with Lipschitz constant $\bar{l} = 0.05$ , and the condition (3.29) of [31, Theorem 1] holds.

Let $p = 1$ and $N = 1$, then

$$-\mu_1(-NCN^{-1}) - \bar{l}\|NA\|_1\|N^{-1}\|_1 = 1.8 - 0.05 \times 0.2 > 0.05 \times 0.1 = \bar{l}\|NB\|_1\|N^{-1}\|_1 > 0,$$

and hence the condition (3.31) holds. According to [31, Theorem 1], there is the unique constant equilibrium point $x^* = \frac{1000}{1785}$ is globally exponentially stable.

□

**Theorem 3.4.** In the system (3.38), set $\Omega = (0,1) \subset \mathbb{R}$, $D = 0.001$, $C = 1.8$, $A = 0.2$, $B = 0.1$, $J = 0.09$, $f(u) = 0.05(u-6)$, then there are at least two equilibrium solution for the system (3.38), including a nontrivial stationary solution and a constant solution $u^*(x) \equiv \frac{1000}{1785} = \frac{200}{357}$ for all $x \in (0,1)$ with $u^*(0) = u^*(1) = 0$.

*Proof.* Firstly, by direct computation, the system (3.38) becomes the following system:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = 0.001\Delta u(t,x) - 1.8u(t,x) + 0.01u(t,x) + 0.005u(t-\tau(t),x) + 1, \quad t \geqslant 0,\ x \in \Omega = (0,1), \\ u(t,0) = u(t,1) = 0, \\ u(s,x) = \xi(s,x) \text{ is bounded in } [-\tau,0] \times (0,1). \end{cases} \tag{3.40}$$

and its stationary solutions satisfy the following equation:

$$\begin{cases} \frac{d^2u(x)}{dx^2} = 1785u(x) - 1000, \quad x \in \Omega = (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{3.41}$$

We may firstly solve the following homogeneous equation

$$\frac{d^2u(x)}{dx^2} = 1785u(x)$$

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and obtain

$$u(x) = m_1 e^{\sqrt{1785}x} + m_2 e^{-\sqrt{1785}x}, \qquad m_1, m_2 \in \mathbb{R}.$$

It is obvious that $u = \frac{200}{357}$ is a solution of the first equation of (3.41).

Hence, the general solution of the first equation of (3.41) can be given as follows,

$$u(x) = m_1 e^{\sqrt{1785}x} + m_2 e^{-\sqrt{1785}x} + \frac{200}{357}, \qquad m_1, m_2 \in \mathbb{R}.$$

Moreover, the Dirichlet boundary value yields

$$\begin{cases} 0 = u(0) = m_1 + m_2 + \dfrac{200}{357}, \\ 0 = u(1) = m_1 e^{\sqrt{1785}} + m_2 e^{-\sqrt{1785}} + \dfrac{200}{357}, \end{cases}$$

which implies that

$$\begin{cases} m_1 = \dfrac{200(e^{-\sqrt{1785}} - 1)}{357(e^{\sqrt{1785}} - e^{-\sqrt{1785}})} \\ m_2 = -\dfrac{200(e^{\sqrt{1785}} - 1)}{357(e^{\sqrt{1785}} - e^{-\sqrt{1785}})} \end{cases}$$

Therefore, besides a constant equilibrium point $u^*(x) \equiv \frac{1000}{1785} = \frac{200}{357}$ for all $x \in (0, 1)$ with $u^*(0) = u^*(1) = 0$, the system (3.38) also has a nontrivial stationary solution

$$u(x) = \frac{200(e^{-\sqrt{1785}} - 1)}{357(e^{\sqrt{1785}} - e^{-\sqrt{1785}})} e^{\sqrt{1785}x} - \frac{200(e^{\sqrt{1785}} - 1)}{357(e^{\sqrt{1785}} - e^{-\sqrt{1785}})} e^{-\sqrt{1785}x} + \frac{200}{357}, \qquad \forall\, x \in [0, 1].$$

□

**Remark 13.** From Theorem 3.4, we know, although the diffusion coefficient $D = 0.001$ is very small, the topological structure of the phase plane of the dynamic behavior of the reaction-diffusion system is likely to change substantially from a constant equilibrium point of its corresponding ordinary differential equation model (see Proposition 2 or [31, Theorem 1] ) to multiple stationary solutions of the reaction-diffusion system. Therefore, many global stability results of delayed neural networks in the form of ordinary differential equations may only be locally asymptotical stability criteria in real engineering. This means that we should study the reaction diffusion model for neural networks, rather than ordinary differential equation model.

**Remark 14.** Besides [31, Theorem 1], there are many other literature involved in the globally asymptotical stability of the unique constant equilibrium point of delayed ordinary differential equations models for cellular neural networks, Cohen-Grossberg neural networks, Bidirectional Associative Memory (BAM) neural networks, and so on. For example, [34, Theorem 2.1], [34, Theorem 2.2], [34, Theorem 3.1], [35, Theorem 1], [35, Theorem 2], [36, Lemma 1], [36, Theorem 1-5], [37, Theorem 3.1], [38, Theorem 3.1], [38, Theorem 3.3], [39, Theorem 1], [39, Theorem 2] and so on. Generally, under the common Lipschitz condition and some other conditions, the unique existence and global asymptotic stability of the constant equilibrium point of ordinary differential equations model for neural networks are given in many previous related literature ([31,34-39] and the references therein). Limited to the length of the article, we can't point out one by one.

**Theorem 3.5.** Suppose that all the assumptions of Theorem 3.4 holds, then the constant equilibrium point $u^*$ of the reaction diffusion system (3.38) is globally exponentially stable, where $u^* = u^*(x) \equiv \frac{1000}{1785} = \frac{200}{357}$ for all $x \in (0,1)$ with $u^*(0) = u^*(1) = 0$.

*Proof.* Firstly, by direct computation, the system (3.38) becomes the following system:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} =0.001\Delta u(t,x) - 1.8u(t,x) + 0.01u(t,x) + 0.005u(t-\tau(t),x) + 1, \quad t \geqslant 0,\ x \in \Omega = (0,1), \\ u(t,0) =u(t,1) = 0, \\ u(s,x) =\xi(s,x) \text{ is bounded in } [-\tau,0] \times (0,1), \end{cases}$$

which is equivalent to the following system via the transformation $y(t,x) = u(t,x) - u^*$:

$$\begin{cases} \dfrac{\partial y(t,x)}{\partial t} =0.001\Delta y(t,x) - 1.8y(t,x) + 0.01y(t,x) + 0.005y(t-\tau(t),x), \quad t \geqslant 0,\ x \in \Omega = (0,1), \\ y(t,0) =y(t,1) = 0, \\ y(s,x) =\eta(s,x) \text{ is bounded in } [-\tau,0] \times (0,1), \end{cases} \tag{3.42}$$

Consider the following Lyapunov functional:

$$V(t,y(t,x)) = \int_\Omega y^2(t,x)dx,$$

then the derivative $\frac{dV}{dt}$ alongside with the trajectories of the system (3.42) yields

$$\begin{aligned} \frac{dV(t,y(t,x))}{dt} &= \int_\Omega 2y(t,x)\Big(0.001\Delta y(t,x) - 1.8y(t,x) + 0.01y(t,x) + 0.005y(t-\tau(t),x)\Big)dx \\ &\leqslant \int_\Omega \Big(-0.002\lambda_1 - 3.58\Big)y^2(t,x)dx + 0.005\int_\Omega (y^2(t,x) + y^2(t-\tau(t),x))dx \\ &= -(0.002\lambda_1 + 3.58 - 0.005)\int_\Omega y^2(t,x)dx + 0.005\int_\Omega y^2(t-\tau(t),x)dx \\ &= -(0.002\pi^2 + 3.575)\int_\Omega y^2(t,x)dx + 0.005\int_\Omega y^2(t-\tau(t),x)dx \\ &= -aV(t,y(t,x)) + bV(t,y(t-\tau(t),x)), \end{aligned}$$

where $a = 0.002\pi^2 + 3.575$, $b = 0.005$, satisfying $a > b > 0$. By employing the methods in the proof of [40, Theorem 3], we can similarly derive that the zero solution of the system (3.43) is globally exponentially stable with the convergence rate $\frac{1}{2}$, where $\lambda > 0$ is the unique solution of the equation $\lambda + a + be^{\lambda\tau} = 0$. That is, the constant equilibrium point $u^*$ of the reaction diffusion system (3.38) is globally exponentially stable, where $u^* = u^*(x) \equiv \frac{1000}{1785} = \frac{200}{357}$ for all $x \in (0,1)$ with $u^*(0) = u^*(1) = 0$.

□

**Remark 15.** Theorem 3.5 fully illustrates that the constant equilibrium point should not be investigated in in practical engineering because diffusion inevitably exists in the real engineering whether the reaction diffusion system is close to stability or reaches stable. Theorem 3.5 shows that the constant equilibrium point $u^*$ is globally asymptotically stable, which implies that the equilibrium point $u^*$ must be the unique equilibrium solution, where $u^* = u^*(x) \equiv \frac{1000}{1785} = \frac{200}{357}$ for all $x \in (0,1)$ with $u^*(0) = u^*(1) = 0$. **and hence there should not exist any other equilibrium solutions for the reaction diffusion system (3.43), which contradicts Theorem 3.4. In Theorem 3.4, we compute another stationary solution for the system (3.43).** So we should study the nontrivial solutions, for the diffusion is a double-edged sword. That is , it is not the case in the literature([1-10]) that the greater the diffusion, the more stable the system will be (see Remark 7).

Usually, if the activation function is strictly monotonous, the number the equilibrium points of the ordinary differential equation model for cellular neural networks will be not more than three in the case of $n = 1$. Particularly in many literature ([31,34-39]), under the Lipschitz conditions on activation functions, the equilibrium point is always unique (see,e.g. [31, Theorem 1]). Now we consider whether the small diffusion makes the equilibrium point become infinitely many nontrivial stationary solutions of reaction diffusion system under both strict monotonicity and Lipschitz continuity assumptions on the activation function. We thank that it may be possible. To show it conveniently, we may assume that time delay is very small so that it can be ignored. Now, consider the following reaction diffusion system,

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} =D\Delta u(t,x) - Cu(t,x) + Af(u(t,x)) + J, \quad t \geqslant 0,\ x \in \Omega = (0,1), \\ \qquad =0.1\Delta u(t,x) - 2u(t,x) + (2 + \frac{\pi^2}{10})f(u(t,x)) + 0, \quad t \geqslant 0,\ x \in \Omega = (0,1), \\ u(t,0) =0 = u(t,1), \\ u(0,x) =\xi(0,x), \quad \forall\, x \in (0,1). \end{cases} \tag{3.43}$$

At first, we recall that the first positive eigenvalue of the following eigenvalue problem is $\lambda_1 = \pi^2 = 9.8696$ (see Remark 18)

$$\begin{cases} -u''(x) =\lambda u(x), x \in \Omega = (0,1) \\ u(0) =0 = u(1) \end{cases}$$

and then the second eigenvalue is $\lambda_2 > \pi^2$ (see,e.g. [33]).

Now we denote by $\mu_1, \mu_2$ the first positive eigenvalue and the second positive eigenvalue of the following eigenvalue problem :

$$\begin{cases} -u''(x) + 20u(x) =\lambda u(x), x \in \Omega = (0,1) \\ u(0) =0 = u(1) \end{cases} \tag{3.44}$$

then $\mu_1 = 20 + \lambda_1 = 20 + \pi^2$, and $\mu_2 > \mu_1$ ([33]).

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Let

$$f(u) = \begin{cases} 3u^{\frac{1}{3}} + 2, & u \leqslant -1; \\ u, & u \in [-1, 1]; \\ 3u^{\frac{1}{3}} - 2, & u \geqslant 1. \end{cases} \tag{3.45}$$

then $f(0) = 0$, and

$$f'(u) = \frac{df(u)}{du} = \begin{cases} u^{-\frac{2}{3}}, & u \leqslant -1; \\ 1, & u \in [-1, 1]; \\ u^{-\frac{2}{3}}, & u \geqslant 1. \end{cases} \tag{3.46}$$

It is obvious that $f(\varpi)$ is continuous for all $\varpi \in \mathbb{R}$, and $f(\varpi)$ is differentiable for all $\varpi \in \mathbb{R}$. In addition, $|f'(\varpi)| \leqslant 1$ for all $\varpi \in \mathbb{R}$. Obviously, $f'(\varpi) > 0$ and hence $f(\varpi)$ is a strictly monotonic function in $\mathbb{R}$, and

$$|f(\varpi) - f(v)| = |f'(\theta)| \cdot |\varpi - v| \leqslant |\varpi - v|, \quad \forall\ \varpi, v \in \mathbb{R}, \tag{3.47}$$

where $\theta \in [\varpi, v]$ or $\theta \in [v, \varpi]$.

$$\begin{aligned} F(u) &= \int_0^u f(v)dv \\ &= \begin{cases} \int_0^{-1} f(v)dv + \int_{-1}^u f(v)dv = \frac{9}{4}u^{\frac{4}{3}} + 2u + \frac{1}{4},, & u \leqslant -1; \\ \frac{1}{2}u^2, & u \in [-1, 1]; \\ \int_0^1 f(v)dv + \int_1^u f(v)dv = \frac{9}{4}u^{\frac{4}{3}} - 2u + \frac{1}{4}, & u \geqslant 1. \end{cases} \end{aligned} \tag{3.48}$$

It is obvious that

$$\lim_{|u| \to +\infty} \frac{F(u)}{u^2} = 0. \tag{3.49}$$

**Theorem 3.6** In the reaction diffusion system (3.43), $C = 2, A = 1, B = \frac{\pi^2}{20}, J = 0$, and the diffusion coefficient $D = 0.1$, and $f$ is defined in (3.45), which is strictly monotonous and Lipschitz continuous, then there are infinitely many positive stationary solutions, infinitely many negative stationary solutions and zero solution, or there are at least three equilibrium solutions including two non-zero stationary solutions and a asymptotically stable zero solution, for the reaction diffusion system (3.43).

*Proof.* Firstly, we shall prove that the zero solution $u = 0$ is the constant equilibrium point of the system (3.43), which is asymptotically stable.

Indeed, we consider the following Lyapunov functional

$$V(t) = \int_\Omega u^2(t, x)dx,$$

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and calculate the derivative $\frac{dV}{dt}$ along the trajectories of the system (3.43) as follows,

$$\begin{aligned}
\frac{dV(t)}{dt} &= \int_{\Omega} 2u(t,x)\Big(0.1\Delta u(t,x) - 2u(t,x) + (2+\frac{\pi^2}{10})f(u(t,x))\Big)dx \\
&\leqslant \int_{\Omega}\Big[\Big(-2\lambda_1 \times 0.1 - 4\Big)u^2(t,x) + 2u(t,x)(2+\frac{\pi^2}{10})f(u(t,x))\Big]dx \\
&= \int_{|u|\leqslant 1}\Big[\Big(-0.2\pi^2 - 4\Big)u^2(t,x) + 2u(t,x)(2+\frac{\pi^2}{10})u(t,x)\Big]dx \\
&\quad + \int_{u\leqslant -1}\Big[\Big(-0.2\pi^2 - 4\Big)u^2(t,x) + 2u(t,x)(2+\frac{\pi^2}{10})(3u^{\frac{1}{3}}(t,x)+2)\Big]dx \\
&\quad + \int_{u\geqslant 1}\Big[\Big(-0.2\pi^2 - 4\Big)u^2(t,x) + 2u(t,x)(2+\frac{\pi^2}{10})(3u^{\frac{1}{3}}(t,x)-2)\Big]dx \\
&= 0 - \Big(0.2\pi^2+4\Big)\int_{u\leqslant -1}\Big(u^2(t,x) - 3u^{\frac{4}{3}}(t,x) - 2u(t,x)\Big)dx \\
&\quad - \Big(0.2\pi^2+4\Big)\int_{u\geqslant 1}\Big(u^2(t,x) - 3u^{\frac{4}{3}}(t,x) + 2u(t,x)\Big)dx \\
&\leqslant -\Big(0.2\pi^2+4\Big)\int_{u\leqslant -1}\Big((-1)^2 - 3(-1)^{\frac{4}{3}} - 2(-1)\Big)dx \\
&\quad - \Big(0.2\pi^2+4\Big)\int_{u\geqslant 1}\Big(1^2 - 3\times 1^{\frac{4}{3}} + 2\times 1\Big)dx \\
&\leqslant 0,
\end{aligned}$$

which together with the definition of the functional $V$ implies that the zero solution $u = 0$ is asymptotically stable, where $\lambda_1 = \pi^2$.

Next, we shall prove the other conclusions in Theorem 3.6.

In fact, the stationary solution $u(x)$ of the system (3.43) must satisfy

$$\begin{cases} 0 = 0.1u''(x) - 2u(x) + 2f(u(x)) + \frac{\pi^2}{10}f(u(x)) + 0, & t \geqslant 0,\ x \in \Omega = (0,1), \\ u(0) = 0 = u(1), \end{cases}$$

or

$$\begin{cases} -u''(x) + 20u(x) = (20+\pi^2)f(u(x)), & t \geqslant 0,\ x \in \Omega = (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{3.50}$$

For any $u, v \in W_0^{1,2}(0,1)$, we define the inner product as follows,

$$\langle u, v\rangle = \int_{(0,1)} (u'(x)v'(x) + 20u(x)v(x))ds,$$

and then the induced norm $\|u\| = \sqrt{\int_{(0,1)}(|u'(x)|^2 + 20u^2(x))dx}$. For convenience, we denote by $H = \{u \in W_0^{1,2}(0,1) : \|u\| < +\infty\}$ the normed space. Then the corresponding functional of the equation (3.50) is as follows,

$$\psi(u) = \frac{1}{2}\|u\|^2 - (20+\pi^2)\int_{(0,1)} F(u)dx = \frac{1}{2}\|u\|^2 - \mu_1\int_{(0,1)} F(u)dx.$$

Obviously, $\psi \in C^1(H, \mathbb{R})$. And each critical point of the functional $\psi$ is a stationary solution of the system (3.47).

Next, we claim that $\psi(u)$ is bounded below. In fact, if it is not true, there must exist a sequence $\{u_n\}_{n=1}^{\infty} \subset H$ such that

$$\psi(u_n) \to -\infty, \quad \text{as } \|u_n\| \to \infty,$$

which implies that $\psi \leqslant C$, where $C > 0$ is a constant. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\{v_n\}$ is bounded in the Hilbert space $H$, and hence there is $v \in H$ such that $v_n$ converges weakly to $v$ in the space $H$, $v_n \to v$ in $L^p(0,1)$ with $p \geqslant 2$, and $v_n(x) \to v(x)$, a.e. in $(0,1)$.

On the other hand, the condition (3.49) yields that there exists $C_1 > 0$ big enough such that $2F(u) < \frac{1}{2}u^2$ for all $|u| \geqslant C_1$. So the definition of $F$ yields

$$\begin{aligned}\frac{C}{\|u_n\|^2} \geqslant & \frac{\psi(u_n)}{\|u_n\|^2} = \frac{\|u_n\|^2 - \int_{(0,1)} \mu_1 u_n^2 dx - \int_{(0,1)} [2\mu_1 F(u_n) - \mu_1 u_n^2] dx}{2\|u_n\|^2} \geqslant \frac{\|u_n\|^2 - \int_{(0,1)} \mu_1 u_n^2 dx - \int_{|u|\leqslant C_1} [2\mu_1 F(u_n) - \mu_1 u_n^2] dx}{2\|u_n\|^2}\\ \geqslant & \frac{\|u_n\|^2 - \int_{(0,1)} \mu_1 u_n^2 dx - C_2}{2\|u_n\|^2},\end{aligned}$$

which together with $v_n \to v$ in $L^2(0,1)$ and the weak lower semi-continuity of norm implies

$$\limsup_{n\to\infty} \|v_n\|^2 \leqslant \mu_1 \int_{(0,1)} v^2 dx \leqslant \|v\|^2 \leqslant \liminf_{n\to\infty} \|v_n\|^2,$$

which implies $v_n \to v$ in $H$, $\|v\| = 1$ and $\mu_1 \int_{(0,1)} v^2 dx = \|v\|^2$. This shows that $v$ is the eigenfunction in the eigensubspace with one dimension of the eigenvalue $\mu_1$, and hence $|v| > 0$ . Besides, $v_n = \frac{u_n}{\|u_n\|} \to v$ means $|u_n(x)| \to +\infty\, a.e. x \in (0,1)$. In addition,

$$\begin{aligned} C > & \psi(u_n) = [\frac{1}{2}(\|u_n\|^2 - \mu_1 \int_{(0,1)} u_n^2(x) dx] - \frac{1}{2} \int_{(0,1)} [2\mu_1 F(u_n) - u_n^2(x)] dx \\ \geqslant & -\frac{1}{2}\mu_1 \int_{(0,1)} [2F(u_n) - u_n^2(x)] dx \to +\infty, \end{aligned}$$

which implies that $\psi$ is bounded below.

The poincare inequality derives

$$\int_{(0,1)} |u'|^2 dx \leqslant \|u\|^2 = \int_{(0,1)} (|u'(x)|^2 + 20u^2(x)) dx \leqslant (1 + \frac{20}{\lambda_1}) \int_{(0,1)} |u'|^2 dx,$$

and then

$$\sqrt{\int_{(0,1)} |u'|^2 dx} \leqslant \|u\| \leqslant \sqrt{(1 + \frac{20}{\lambda_1})} \sqrt{\int_{(0,1)} |u'|^2 dx}, \tag{3.51}$$

which implies the norm $\|\cdot\|$ is equivalent to the norm of [41, Theorem 6]. Hence, to prove that the functional $\psi$ satisfies the $(PS)$ condition (see [41, Definition 3]), we only need to prove the boundedness of $\psi$ (see the proof of [41, Theorem 6]). Moreover, owing to the boundedness of $\{\psi(u_n)\}$, it is not difficult to prove by the application of reduction to absurdity that $\{u_n\}$ must be bounded in $H$, similarly as the above-mentioned method in proving that $\psi$ is bounded below.

On the one hand, if $\inf_H \psi \geqslant 0$, we claim that there are infinitely many positive stationary solutions and infinitely many negative stationary solutions for the reaction diffusion system (3.43) .

Indeed, Due to the orthogonal decomposition of Sobolev space $W_0^{1,2}(\Omega)$ ([43-45]), we may let $H = E(\mu_1) \oplus E(\mu_1)^\perp$, where $E(\mu_k)$ represents the eigenfunction space of $\mu_k$, and $E(\mu_1)^\perp = E(\mu_2) \oplus E(\mu_3) \oplus \cdots$. Obviously, $\psi$ satisfies (P1). In fact, if $u \in E(\mu_1)$ with $\|u\| \leqslant \delta$, the equivalence of norms in each finite dimensional space yields

$$\|u\| \leqslant \delta \Rightarrow \int_{0,1} |u(x)|dx < \delta_1 \Rightarrow |u(x)| < 1, \text{ a.e. } x \in (0,1).$$

where the positive number $\delta$ is small enough, and so is $\delta_1$. And then

$$\begin{aligned} \psi(u) =& \frac{1}{2}\|u\|^2 - \mu_1 \int_{(0,1)} F(u)dx \\ =& \frac{1}{2}\|u\|^2 - \mu_1 \int_{(0,1)} \frac{u^2}{2}dx = 0 \leqslant 0, \quad u \in E(\mu_1),\ \|u\| \leqslant \delta. \end{aligned} \tag{3.52}$$

which together with $\inf_H \psi \geqslant 0$ implies that all $u \in E(\mu_1)$ with $\|u\| \leqslant \delta$ are the stationary solutions of the reaction diffusion system (3.43). Moreover, $E(\mu_1)$ implies that there are infinitely many positive stationary solutions and infinitely many negative stationary solutions for the reaction diffusion system (3.43).

On the other hand, if $\inf_H \psi < 0$, we see, the condition (P3) holds. Then we claim that at least three equilibrium solutions including two non-zero stationary solutions and a asymptotically stable zero solution, for the reaction diffusion system (3.43) if $\inf_H \psi < 0$.

In fact, (3.52) implies that (P1) holds. In order to apply Lemma 2.2, we only need to verify the condition (P2) of Lemma 2.2. Next, for $u \in E(\mu_1)^\perp$, let $u = v + z$, where $v \in E(\mu_2)$, $z \in E(\mu_3) \oplus E(\mu_4) \oplus \cdots$. Then we get

$$\begin{aligned} \psi(u) =& \frac{1}{2}\|u\|^2 - \mu_1 \int_{(0,1)} F(u)dx \geqslant \frac{\mu_2}{2}\left(\int_{(0,1)} v^2dx + \int_{(0,1)} z^2dx\right) + \frac{1}{2}\left(1 - \frac{\mu_2}{\mu_3}\right)\|z\|^2 - \mu_1 \int_{(0,1)} F(u)dx \\ =& \left[\frac{1}{2}\mu_2 \int_{(0,1)} u^2dx - \mu_1 \int_{(0,1)} F(u)dx\right] + \frac{1}{2}\left(1 - \frac{\mu_2}{\mu_3}\right)\|z\|^2. \end{aligned}$$

Due to (3.48), there exists $\delta \in (0,1)$ such that $2\mu_1 F(u) = \mu_1 u^2 \leqslant \mu_2 u^2$ if $|u| \leqslant \delta$. Moreover, for this $\delta$, there exists correspondingly $\delta_2 > 0$ such that for $u \in E(\mu_2)$ with $\|u\| \leqslant \delta_2$, we get $|u(x)| \leqslant \frac{\delta}{2}$, a.e. $x \in (0,1)$ in view of the equivalence of norms in finite dimensional space.

Define

$$\Omega_1 = \{x \in (0,1) : |u(x)| \leqslant \delta\}, \qquad \Omega_2 = \{x \in (0,1) : |u(x)| > \delta\}.$$

Due to the orthogonal decomposition of the Sobolev space $W_0^{1,2}(0,1)$ and $u = v + z$, we see, $\|u\| \leqslant \delta_2 \Rightarrow \|v\| \leqslant \delta_2$, which implies

$$|v(x)| \leqslant \frac{\delta}{2} \leqslant \frac{1}{2}|u(x)|, \qquad |z(x)| \geqslant |u(x)| - |v(x)| \geqslant \frac{1}{2}|u(x)|, \text{ a.e. } x \in \Omega_2.$$

Besides, (3.48) yields that there exists $C_* > 0$ such that

$$\left|\frac{1}{2}\mu_2 u^2 dx - \mu_1 F(u)\right| \leqslant C_*|u|^3 \leqslant 2C_*|z|^3, \ x \in \Omega_2.$$

So we can see it from the orthogonal decomposition of the Sobolev space $W_0^{1,2}(0,1)$ and the Sobolev embedding theorem that for all $u \in E(\mu_1)^\perp$, there is $C_0 > 0$ big enough such that

$$\begin{aligned}\psi(u) =&[\frac{1}{2}\mu_2\int_{\Omega_1} u^2dx - \mu_1\int_{\Omega_1} F(u)dx] + \frac{1}{2}(1-\frac{\mu_2}{\mu_3})\|z\|^2 + [\frac{1}{2}\mu_2\int_{\Omega_2} u^2dx - \mu_1\int_{\Omega_2} F(u)dx]\\ \geqslant&\frac{1}{2}(1-\frac{\mu_2}{\mu_3})\|z\|^2 - 2C_*\int_{\Omega_2}|z|^3dx\\ \geqslant&\frac{1}{2}(1-\frac{\mu_2}{\mu_3})\|z\|^2 - C_0\|z\|^3,\end{aligned}$$

which implies that $\psi(u) \geqslant 0$ for all $u \in E(\mu_1)^{\perp}$ with $\|u\| \leqslant \delta$ by assuming that $\delta > 0$ is small enough. Now, according to Lemma 2.2, we have proved the claim. And the proof is completed.

□

**Remark 16.** Under strict monotonicity and Lipschitz condition on activation function, there are at most three constant equilibrium points in the reaction diffusion system of Theorem 3.6, in which only the zero solution is one of the solutions of the reaction diffusion system, and other constants are not the solutions of the system at all. Even if one of the conclusions of theorem 3.6 is two stationary solutions and one zero solution, these two station solutions are not non-zero constant equilibrium points at all. In other works, the number of the equilibrium solutions changes from three to five. Besides, Theorem 3.6 illuminates, it is possible that the small diffusion makes one equilibrium point become infinitely many nontrivial stationary solutions of the reaction diffusion system.

## 4. Numerical example

**Example 4.1.** Consider the following switched financial system with $N = 3$,

$$\begin{cases}\frac{\partial y(t,x)}{\partial t} =D_1\Delta y(t,x) - C_1 y(t,x) + A_1 g(y(t,x)) + B_1 g(y(t-\tau(t),x)) + J_1, \quad (t,x)\in\mathbb{R}_+\times\Omega_1,\\ y_i(t,x) =0,\ t\geqslant 0,\ x\in\partial\Omega_1,\ i=1,2,\end{cases} \tag{4.1a}$$

$$\begin{cases}\frac{\partial y(t,x)}{\partial t} =D_2\Delta y(t,x) - C_2 y(t,x) + A_2 g(y(t,x)) + B_2 g(y(t-\tau(t),x)) + J_2, \quad (t,x)\in\mathbb{R}_+\times\Omega_2,\\ y_i(t,x) =0,\ t\geqslant 0,\ x\in\partial\Omega_2,\ i=1,2,\end{cases} \tag{4.1b}$$

and

$$\begin{cases}\frac{\partial y(t,x)}{\partial t} =D_3\Delta y(t,x) - C_3 y(t,x) + A_3 g(y(t,x)) + B_3 g(y(t-\tau(t),x)) + J_3, \quad (t,x)\in\mathbb{R}_+\times\Omega_3,\\ y_i(t,x) =0,\ t\geqslant 0,\ x\in\partial\Omega_3,\ i=1,2,\end{cases} \tag{4.1c}$$

or the following corresponding homogeneous equations:

$$\begin{cases}\frac{\partial u(t,x)}{\partial t} =D_\sigma\Delta u(t,x) - C_\sigma u(t,x) + A_\sigma f(u(t,x)) + B_\sigma f(u(t-\tau(t),x)), \quad (t,x)\in\mathbb{R}_+\times\Omega_\sigma,\\ u_i(t,x) =0,\ t\geqslant 0,\ x\in\partial\Omega_\sigma,\ i=1,2; \qquad \sigma\in\{1,2,3\}\end{cases} \tag{4.2}$$

equipped with the initial value:

$$u_i(s,x) = \phi_i(s,x),\ -\tau \leqslant s \leqslant 0,\ x \in \Omega_\sigma, \qquad \sigma \in \{1,2,3\} \tag{4.3}$$

where $\Omega_1 = [0,10]\times[0,10]$, $\Omega_2 = [0,15]\times[0,15]$, $\Omega_3 = [0,20]\times[0,20]$, $\lambda_{11} = 0.1974$, $\lambda_{21} = 0.0877$, $\lambda_{31} = 0.0493$(see Remark 5).

Set $c = 1000$, $\lambda = 0.01$, $\alpha = 0.1$, $I = diag(1,1)$, $D_1 = diag(0.01, 0.015)$, $D_2 = diag(0.015, 0.02)$, $D_3 = diag(0.01, 0.015)$, $C_1 = diag(0.46, 0.4)$, $C_2 = diag(0.44, 0.43)$, $C_3 = diag(0.48, 0.44)$, $A_1 = diag(0.48, 0.5) + (1+\alpha)\lambda I$, $A_2 = diag(0.47, 0.43) + (1+\alpha)\lambda I$, $A_3 = diag(0.5, 0.4) + (1+\alpha)\lambda I$, $B_1 = diag(0.44, 0.3) + (1+\alpha)\lambda I$, $B_2 = diag(0.41, 0.43) + (1+\alpha)\lambda I$, $B_3 = diag(0.46, 0.48) + (1+\alpha)\lambda I$.

Set

$$D_1 = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.015 \end{pmatrix},\ D_2 = \begin{pmatrix} 0.015 & 0 \\ 0 & 0.02 \end{pmatrix},\ D_3 = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.015 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0.46 & 0 \\ 0 & 0.4 \end{pmatrix},\ C_2 = \begin{pmatrix} 0.44 & 0 \\ 0 & 0.43 \end{pmatrix},\ C_3 = \begin{pmatrix} 0.48 & 0 \\ 0 & 0.44 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0.491 & 0 \\ 0 & 0.511 \end{pmatrix},\ A_2 = \begin{pmatrix} 0.491 & 0 \\ 0 & 0.441 \end{pmatrix},\ A_3 = \begin{pmatrix} 0.511 & 0 \\ 0 & 0.481 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0.451 & 0 \\ 0 & 0.311 \end{pmatrix},\ B_2 = \begin{pmatrix} 0.411 & 0 \\ 0 & 0.441 \end{pmatrix},\ B_3 = \begin{pmatrix} 0.471 & 0 \\ 0 & 0.421 \end{pmatrix},$$

Let $g_i(y_i) = \frac{39+2y_i+0.000001\sin y_i}{4}$, and then $G = diag(0.51, 0.51)$. Set $|J_\sigma| \leqslant 3(1,1)^T$, $\sigma = 1,2,3$, then the direct calculation can verify that both conditions (A1) and (A2) hold. Let $\tau = 0.5, \gamma = 0.0001$, then employing computer LMI toolbox to solve the inequality (3.1) derives the following feasible data:

$$\beta_1 = 0.4179,\ \beta_2 = 0.3315,\ \beta_3 = 0.2506,$$

then the switched delayed reaction-diffusion system (2.3) equipped with the initial value (2.4) is exponentially stable due to Theorem 3.1.

Due to the flexibility of $J_\sigma$, we can easily select some $J_\sigma$ such that there is no positive solutions for the algebraic equations (3.17), and hence the positive bounded stationary solution of the system (4.1) is its nontrivial stationary solution, which is exponentially stable.

**Remark 17.** In Example 4.1, the switched delayed reaction-diffusion system (4.2) equipped with the initial value (4.3) is exponential stability with the convergence rate $\frac{\gamma}{2}$. Obviously, the larger the upper bound $\tau$ of time delay, the slower the convergence speed $\gamma$.

**Remark 18.** (see,e.g.,[14-16]) The smallest positive eigenvalue of $-\Delta_p$ in $W_0^{1,p}(0,T)$ is

$$\lambda_1 = \left(\frac{2}{T}\int_0^{(p-1)^{\frac{1}{p}}} \frac{dt}{\left(1 - \frac{t^p}{p-1}\right)^{\frac{1}{p}}}\right)^p.$$

If $\Omega = \{(x_1, x_2)^T : 0 < x_1 < \alpha,\ 0 < x_2 < \beta\} \subset \mathbb{R}^2$ and $W_0^{1,p}(\Omega)$ with $p = 2$, the first eigenvalue $\lambda_1 = (\frac{\pi}{\alpha})^2 + (\frac{\pi}{\beta})^2$.

Besides, there is the following approximate substitution of Poincare inequality lemma :

**Remark 19.** Let $\Omega$ be a cube $|x_i| < l_i (i = 1, 2, \cdots, n)$ and let $\mu(x)$ be a real-valued function belonging to $C^1(\Omega)$ which vanish on the boundary $\partial\Omega$ of $\Omega$, i.e., $\mu(x)|_{\partial\Omega} = 0$, then

$$\int_\Omega \mu^2(x)dx \leqslant l_i^2 \int_\Omega |\frac{\partial \mu}{\partial x_i}|^2 dx.$$

## 5. Conclusions and further considerations

By constructing a compact operator on a convex set, the author makes up for the loss of compactness in infinite dimensional space. Using a fixed point theorem, variational methods and Lyapunov functional method results in the existence positive bounded stationary solution, which is exponentially stable. Finally, a numerical example is presented to illuminate the effectiveness of the proposed methods. It is worth mentioning that the newly-obtained stability criterion illustrates that the inevitable diffusions in real engineering are conducive to the stability of the system, but also makes the dynamic behavior of the system more complex and difficult to judge its stability. Of course, some idea and methods of related literature ([1-51]) inspires our current work. Many innovations make the stability criterion (Theorem 3.1) and other results of this paper novel and new (see Remark 1-16 and Theorem 3.2-3.5 for details). Now the main theorems and contents are listed as follows,

♣ For the first time, our Theorem 3.1 gives the existence of asymptotically stable positive stationary solution of the reaction diffusion cellular neural networks with time delays.

♣ Our Theorem 3.2 points out that the reaction diffusion Cohen-Grossberg neural networks and its corresponding ordinary differential equations model have the same unique constant equilibrium point which is globally asymptotically stable under the same assumptions. This means that it seems unnecessary to study the reaction-diffusion system.

♣ [31, Theorem 1] or Proposition 1 pointed out that the zero solution is the unique equilibrium point of the ordinary differential equations model for the cellular neural networks with time delays. However, applying our Theorem 3.1 to prove our Theorem 3.3 results in that its corresponding reaction diffusion cellular neural networks owns two equilibrium solutions, including the zero solution and the asymptotical stable positive stationary solution, which implies that the globally asymptotical stability of [31, Theorem 1] or Proposition 1 may be locally asymptotically stable due to the ineluctable diffusion in actual operation, which implies that the reaction diffusion model should be studied, rather than the ordinary differential equations model.

♣ Our Theorem 3.4 points out that there exist two equilibrium solutions, including a constant equilibrium point $u^*(x) \equiv \frac{1000}{1785} = \frac{200}{357}$ and another nontrivial stationary solution, for the reaction diffusion system. However, our Theorem 3.5 shows that the constant equilibrium point $u^*(x) \equiv \frac{1000}{1785} = \frac{200}{357}$ is globally asymptotically stable, which means that the constant equilibrium point is the unique equilibrium solution of the reaction diffusion system. This contradicts the conclusion of theorem 3.4, and this contradiction implies that the constant equilibrium points, except the zero solution, should not be investigated. Only the nontrivial stationary solutions should be studied in reaction diffusion system. In fact, all the non-zero constant equilibrium points are not solutions of reaction diffusion systems, even in the Sobolev space $W_0^{1,2}(\Omega)$. That is, non-zero constant equilibrium points are not in the phase plane of dynamic system.

♣ Under Lipschitz condition on the activation function, Theorem 3.6 illuminates that it is possible that the small diffusion makes one equilibrium point become infinitely many nontrivial stationary solutions of the reaction diffusion system (see Remark 16).

Finally, we do not know how to give the sufficient and necessary condition of the stability of the positive bounded stationary solution for reaction-diffusion cellular neural networks though our Theorem 3.1 presents a sufficient condition on the stability. This is an interesting problem.

Besides, under Lipschitz condition on activation function, Theorem 3.6 illuminates, it is possible that the small diffusion makes the equilibrium point become infinitely many nontrivial stationary solutions of reaction diffusion system. But in Theorem 3.6, we have not prove that the small diffusion must make the equilibrium point become infinitely many positive stationary solutions of reaction diffusion system, because Theorem 3.6 owns another case of three equilibrium points. How to improve the conclusion of our Theorem 3.6? or how to construct a new example to directly prove that the small diffusion makes the equilibrium point become infinitely many positive stationary solutions of reaction diffusion system? This is another interesting problem.

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