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On new inequalities involving circular, inverse circular, inverse hyperbolic and exponential functions

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Abstract

In this paper, we provide alternative proofs to some results proposed in the article *"New inequalities involving circular, inverse circular, hyperbolic, inverse hyperbolic and exponential functions"* authored by Yogesh J. Bagul.

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1 Introductory remarks

In the year 2018, Yogesh J. Bagul[2] proposed following theorems.

Theorem 1.1. *([2, Theorem 1]) If $x \in (0,1)$, then*

$$e^{-ax^2} < \frac{x}{\sin^{-1} x} < e^{-x^2/6}$$

with the best possible constants $a \approx 0.451583$ and $1/6$.

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Theorem 1.2. *([2, Theorem 2]) If $x \in (0, \infty)$, then*

$$e^{-x^2/3} < \frac{\tan^{\text{-}1} x}{x} < 1.$$

Theorem 1.3. *([2, Theorem 3]) If $x \in (0, 1)$, then*

$$e^{-ax^2} < \frac{x}{\tan x} < e^{-x^2/3}$$

with the best possible constants $a \approx 0.443023$ and $1/3$.

Theorem 1.4. *([2, Theorem 4]) If $x \in (0, 1)$, then*

$$e^{-x^2/6} < \frac{\sinh^{\text{-}1} x}{x} < e^{-bx^2}$$

with the best possible constants $1/6$ and $b \approx 0.126274$.

To prove above theorems, the author of paper[2] used contradictory methods. After careful reading, we found that the contradictions at the end of each proof are not clear and convincing. Thus, the proofs presented in [2] can not be considered flawless. We give alternative and pellucid proof of each theorem listed above and this is the main aim of this paper. While providing alternative proofs, we also genaralize Theorems 1.3 and 1.4.

In the whole paper, it is to be noted that the superscript "-" for circular and hyperbolic functions is used for their inverses.

2 Main Results

We need following lemma for our alternative proofs.

Lemma 2.1. *([l'Hôpital's rule of monotonicity] [1]) Let f, g be two real valued functions which are continuous on $[a, b]$ and differentiable on (a, b), where $-\infty < a < b < \infty$ and $g'(x) \neq 0$, for $\forall x \in (a, b)$. Let,*

$$A(x) = \frac{f(x) - f(a)}{g(x) - g(a)} \; and \, B(x) = \frac{f(x) - f(b)}{g(x) - g(b)}.$$

Then
I) $A(x)$ and $B(x)$ are increasing on (a, b) if f'/g' is increasing on (a, b) and
II) $A(x)$ and $B(x)$ are decreasing on (a, b) if f'/g' is decreasing on (a, b).
The strictness of the monotonicity of $A(x)$ and $B(x)$ depends on the strictness of monotonicity of f'/g'.

In addition to this, we also prove two lemmas that are useful for our proofs.

Lemma 2.2. *The function* $\phi(x) = \dfrac{x - \sqrt{1-x^2}\sin^{-1} x}{x^2 \sin^{-1} x}$ *is positive increasing on* $(0, 1)$.

Proof. Consider

$$\phi(x) = \frac{x - \sqrt{1-x^2}\sin^{-1} x}{x^2 \sin^{-1} x} = \frac{\phi_1(x)}{\phi_2(x)},$$

where $\phi_1(x) = x - \sqrt{1-x^2}\sin^{-1} x$ and $\phi_2(x) = x^2 \sin^{-1} x$ with $\phi_1(0) = 0$ and $\phi_2(0) = 0$. By differentiating with respect to x

$$\frac{\phi_1'(x)}{\phi_2'(x)} = \frac{x \sin^{-1} x}{x^2 + 2x\sqrt{1-x^2}\sin^{-1} x} = \frac{1}{\dfrac{x}{\sin^{-1} x} + 2\sqrt{1-x^2}}$$

Now, clearly $\dfrac{x}{\sin^{-1} x}$ is decreasing in $(0, 1)$ (or see the proof of [2, Thm. 1]). Again $\sqrt{1-x^2}$ is positive decreasing in $(0, 1)$, implying that $\phi_1'(x)/\phi_2'(x)$ is increasing in $(0, 1)$. By Lemma 2.1, $\phi(x)$ is also increasing in $(0, 1)$. The positivity of $\phi(x)$ follows from the fact that $\phi_1'(x) > 0$ in $(0, 1)$. □

Lemma 2.3. *The function* $\psi(x) = \dfrac{x - \sqrt{1+x^2}\sinh^{-1} x}{x^2 \sinh^{-1} x}$ *is negative increasing on* $(0, \infty)$.

Proof. Consider,

$$\psi(x) = \frac{x - \sqrt{1+x^2}\sinh^{-1} x}{x^2 \sinh^{-1} x} = \frac{\psi_1(x)}{\psi_2(x)},$$

where $\psi_1(x) = x - \sqrt{1+x^2}\sinh^{-1} x$ and $\psi_2(x) = x^2 \sinh^{-1} x$ with $\psi_1(0) = 0$ and $\psi_2(0) = 0$.

After differentiating,

$$\frac{\psi_1'(x)}{\psi_2'(x)} = \frac{-x \sinh^{-1} x}{x^2 + 2x\sqrt{1+x^2}\sinh^{-1} x} = \frac{-1}{\dfrac{x}{\sinh^{-1} x} + 2\sqrt{1+x^2}}.$$

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Clearly $\dfrac{x}{\sinh^{-1} x}$ is strictly increasing on $(0,\infty)$(or see proof of [2, Thm. 4]) and $\sqrt{1+x^2}$ is also strictly increasing on $(0,\infty)$. Therefore, $\psi_1'(x)/\psi_2'(x)$ is strictly increasing on $(0,\infty)$. By Lemma 2.1, $\psi(x)$ is strictly increasing on $(0,\infty)$. The negativity of $\psi(x)$ follows from the fact that $\psi_1'(x) < 0$. □

We now give alternative proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1:
Set

$$f(x) = \frac{\log(\sin^{-1} x/x)}{x^2} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = \log(\sin^{-1} x/x)$ and $f_2(x) = x^2$ with $f_1(0^+) = 0$ and $f_2(0) = 0$. After differentiation, we get

$$\frac{f_1'(x)}{f_2'(x)} = \frac{1}{2}\frac{1}{\sqrt{1-x^2}}\frac{x-\sqrt{1-x^2}\sin^{-1} x}{x^2 \sin^{-1} x} = \frac{1}{2} f_3(x)\phi(x)$$

where $f_3(x) = \dfrac{1}{\sqrt{1-x^2}}$ is positive increasing on $(0,1)$ and $\phi(x)$ is also positive increasing on $(0,1)$ by Lemma 2.2. Therefore, $f_1'(x)/f_2'(x)$ is positive increasing on $(0,1)$. By Lemma 2.1, $f(x)$ is strictly increasing on $(0,1)$.
So,

$$f(0^+) < f(x) < f(1).$$

Lastly, $f(0^+) = 1/6$ and $f(1) = \log(\pi/2) \approx 0.451583$ prove the theorem. □

Proof of Theorem 2:
Set

$$g(x) = \frac{\log(x/\tan^{-1} x)}{x^2} = \frac{g_1(x)}{g_2(x)},$$

where $g_1(x) = \log(x/\tan^{-1} x)$ and $g_2(x) = x^2$ with $g_1(0^+) = 0$ and $g_2(0) = 0$. Differentiation yields

$$\frac{g_1'(x)}{g_2'(x)} = \frac{(1+x^2)\tan^{-1} x - x}{2x^2(1+x^2)\tan^{-1} x} = \frac{g_3(x)}{g_4(x)}$$

and

$$\frac{g_3'(x)}{g_4'(x)} = \frac{x\tan^{-1} x}{[2x(1+x^2)\tan^{-1} x + 2x^4\tan^{-1} x + 2x^2]} = \frac{1}{2[(1+x^2)+x^3+\frac{x}{\tan^{-1} x}]}$$

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Since, $\dfrac{x}{\tan^{-1} x}$ is clearly increasing on $(0, \infty)$ (or see proof of [2, Thm. 2]), $g_3'(x)/g_4'(x)$ is strictly decreasing on $(0, \infty)$. By Lemma 2.1, $g(x)$ is also decreasing on $(0, \infty)$.
Hence,

$$g(0^+) > g(x) > g(\infty^-).$$

Therefore $g(0^+) = 1/3$ and $g(\infty^-) = 0$ give the desired result. $\square$

We slightly expand the interval of values of x and restate Theorem 1.3.

Theorem 2.4. *Let $x \in (0, \alpha)$ where $\alpha \in (0, \pi/2)$. Then*

$$e^{-ax^2} < \frac{x}{\tan x} < e^{-x^2/3} \tag{2.1}$$

with the best possible constants $a = \dfrac{\log(\tan \alpha/\alpha)}{\alpha^2}$ and $1/3$.

Proof. Set

$$h(x) = \frac{\log(\tan x/x)}{x^2} = \frac{h_1(x)}{h_2(x)},$$

where $h_1(x) = \log(\tan x/x)$ and $h_2(x) = x^2$ with $h_1(0^+) = 0$ and $h_2(0) = 0$. Then,

$$\frac{h_1'(x)}{h_2'(x)} = \frac{x\sec^2 x - \tan x}{2x^2 \tan x} = \frac{h_3(x)}{h_4(x)},$$

where $h_3(x) = x\sec^2 x - \tan x$ and $h_4(x) = 2x^2 \tan x$ with $h_3(0) = 0$ and $h_4(0) = 0$. Differentiation yields

$$\frac{h_3'(x)}{h_4'(x)} = \frac{1}{2\cos^2 x - x/\tan x},$$

which is strictly increasing in $(0, \infty)$ as $\cos x$ and $x/\tan x$ are decreasing in $(0, \pi/2)$. By Lemma 2.1, $h(x)$ is also strictly increasing in $(0, \infty)$. Consequently,

$$h(0^+) < h(x) < h(\infty).$$

Moreover, $h(0^+) = 1/3$ and $h(\alpha) = \dfrac{\log(\tan \alpha/\alpha)}{\alpha^2}$. This proves the result. $\square$

Note: The Theorem 2.4 was also proved by another method by Branko Malesevic *et al.* in [3].

We can similarly restate Theorem 1.4 as follows.

Theorem 2.5. *Let* $x \in (0,\infty)$. *Then,*

$$e^{-x^2/6} < \frac{\sinh^{-1} x}{x} < 1 \tag{2.2}$$

with the best possible constants $1/6$ *and* 0.

Proof. Set

$$k(x) = \frac{\log(\sinh^{-1} x/x)}{x^2} = \frac{k_1(x)}{k_2(x)},$$

where $k_1(x) = \log(\sinh^{-1} x/x)$ and $k_2(x) = x^2$ with $k_1(0^+) = 0$ and $k_2(0) = 0$. On differentiating,

$$\frac{k_1'(x)}{k_2'(x)} = \frac{1}{2\sqrt{1+x^2}} \frac{x - \sqrt{1+x^2}\sinh^{-1} x}{x^2 \sinh^{-1} x} = \frac{1}{2} k_3(x)\psi(x),$$

where $k_3(x) = \dfrac{1}{\sqrt{1+x^2}}$ is clearly positive decreasing on $(0,\infty)$ and by Lemma 2.3, $\psi(x)$ is negative increasing on $(0,\infty)$, implying that $k_1'(x)/k_2'(x)$ is strictly increasing on $(0,\infty)$. Consequently, by Lemma 2.1, $k(x)$ is increasing on $(0,\infty)$. Hence,

$$k(0^+) < k(x) < k(\infty^-).$$

Therefore, $k(0^+) = -1/6$ and $k(\infty^-) = 0$ finish the proof. $\square$

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