Parametric Gevrey asymptotics in two complex time variables through truncated Laplace transforms

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Abstract

The work is devoted to the study of a family of linear initial value problems of partial differential equations in the complex domain, dealing with two complex time variables. The use of a truncated Laplace-like transformation in the construction of the analytic solution allows to overcome a small divisor phenomenon arising from the geometry of the problem and represents an alternative approach to the one proposed in a recent work [9] by the last two authors. The result leans on the application of a fixed point argument and the classical Ramis-Sibuya theorem.

Key words: asymptotic expansion, Borel-Laplace transform, Fourier transform, initial value problem, formal power series, nonlinear partial differential equation, singular perturbation. 2010 MSC: 35C10, 35C20.

1 Introduction

This work is devoted to the study of a family of singularly perturbed partial differential equations in the complex domain of the form

$$Q(\partial_z)u(t_1,t_2,z,\epsilon) = P(t_1^{k_1+1}\partial_{t_1}, t_2, \partial_{t_2}, \partial_z, z, \epsilon) + f(t_1,t_2,z,\epsilon), \quad (1)$$

under initial data $u(0,t_2,z,\epsilon) \equiv u(t_1,0,z,\epsilon) \equiv 0$, with $Q(X) \in \mathbb{C}[X]$ and $P(T_1,T_{21},T_{22},Z,z,\epsilon)$ being a polynomial in (T_1,T_{21},T_{22},Z) with holomorphic coefficients w.r.t. (z,ϵ) on $H_\beta \times D(0,\epsilon_0)$. Here, H_β and $D(0,\epsilon_0)$ stand for the horizontal strip $\{z \in \mathbb{C} : |\mathrm{Im}(z)| < \beta\}$ and the disc at the

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origin and radius ϵ_0, for some $\beta > 0, \epsilon_0 > 0$, respectively. The forcing term $f(t_1, t_2, z, \epsilon)$ is holomorphic on $\mathbb{C}^{\star} \times D(0, h') \times H_\beta \times \mathcal{E}$, for any open sector $\mathcal{E}$ centered at 0 and contained in $D(0, \epsilon_0)$, for some $h' > 0$, and remains close to a polynomial in t_1, analytic in t_2 on $D(0, h')$ and in z on H_β, as ϵ becomes close to the origin in $\mathbb{C}$. The variable ϵ acts as a small complex parameter. The concrete assumptions on the elements involved in the main problem (1) are to be described and analysed in detail throughout the work.

The study of a problem of such form is motivated by the recent research [9] of the second and third authors. The main aim in the preceeding work was related to the description of the asymptotic behavior of the analytic solutions, with respect to the perturbation parameter, near the origin, of singularly perturbed equations

$$Q(\partial_z)u(t_1, t_2, z, \epsilon) = \tilde{P}(t_1^{k_1+1}\partial_{t_1}, t_2^{k_2+1}\partial_{t_2}, \partial_z, z, \epsilon) + \tilde{f}(t_1, t_2, z, \epsilon), \tag{2}$$

with $\tilde{P}(T_1, T_2, Z, \epsilon)$ being a polynomial in $(T_1, T_2, Z, z, \epsilon)$ with holomorphic coefficients w.r.t. (z, ϵ) on $H_\beta \times D(0, \epsilon_0)$.

Two main novelties are considered here with respect to it. On the one hand, the irregular singular operators related to the second time variable stay rigid in (2), as a polynomial function of the operator $t_2^{k_2+1}\partial_{t_2}$. In the present study, the irregular operators in this variable fit a more general scheme within the problem, under certain technical assumptions (see (5) and (6)). This, at first sight slight, variation on the form of the main problem varies its underlying geometry radically. On the other hand, the appearance of different types of solutions observed in [9], known as inner and outer solutions, which describe boundary layer expansions do not appear in the present situation, since we study local solutions in time t_1, t_2 near the origin in the complex domain. It is worth mentioning that, despite the fact that the form of the main equation under study resembles that of [9], the nature of the singularities appearing in the problem require to appeal different approaches and apply novel techniques, to be briefly described below.

This work continues a line of research on the study of the asymptotic behavior of solutions of singularly perturbed PDEs in the complex domain, under the action of two time variables: dealing with a symmetric factorized (resp. asymmetric) leading term [8] (resp. [6]), the mentioned work [9], and the corresponding $q-$analog [10] in the framework of $q-$difference$-$differential equations.

The technique developed in the present work consists on searching for solutions of the main problem (see (8) for its precise expression) in the form of a Fourier, truncated Laplace and Laplace transform of certain function, for every fixed value of the perturbation parameter ϵ:

$$u(\boldsymbol{t}, z, \epsilon) := \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{L_{1,\epsilon}} \int_{L_2} \omega(u_1, u_2, m, \epsilon) \exp\left(-\left(\frac{u_1}{\epsilon t_1}\right)^{k_1} - \left(\frac{u_2}{\epsilon t_2}\right)^{k_2} \right) e^{izm} \frac{du_2}{u_2} \frac{du_1}{u_1}. \tag{3}$$

The integration path $L_{1,\epsilon}$ stands for the segment $[0, h_1(\epsilon)e^{\sqrt{-1}\theta_1}]$, for some holomorphic function $\epsilon \mapsto h_1(\epsilon)$ on the domain of definition of the perturbation parameter, approaching to infinity when ϵ tends to 0, and some $\theta_1 \in \mathbb{R}$ which does not depend on ϵ. The integration with respect to the path L_2 stands for a usual Laplace transform along certain half line $[0, \infty)e^{\sqrt{-1}d_2}$, for some $d_2 \in \mathbb{R}$, whereas the function ω belongs to certain Banach space which depends on the choice of the perturbation parameter. From this point, the main problem is replaced by an auxiliary convolution problem (see (17)) in the Borel plane with respect to the time variables (t_1, t_2). The precise knowledge on the geometry of the auxiliary problem is crucial in order to understand the location and control of the singularities (see Section 3). As a matter of fact, the singularities of the auxiliary problem are always located outside (but remain close to) a product of discs in the domain of the transformed time variables, say (τ_1, τ_2). The radius of the discs depend on ϵ tending to infinity regarding τ_1 and shrinking to the origin with respect to τ_2. In addition to this, one can choose narrow finite (resp. infinite) sectors with vertex at the origin with respect to τ_1 (resp. τ_2), valid for all the values of the pertubation parameter, which avoid the singular points. In other words, the function ω can be extended to the product of such sectors w.r.t. (τ_1, τ_2). Therefore, a small denominator problem regarding movable singularities to infinity and to zero at the same time (in each of the time variables) has to be analysed. This construction through a truncated Laplace transform is proposed in order that the solutions (3) remain close, as ϵ tends to 0, to a double usual Laplace transform in both variables t_1, t_2. For such a complete double Laplace representable solution, a direct analysis of the asymptotic behaviour w.r.t. ϵ is unfortunately not possible (as shown in our previous work [9]). However, such study turns achievable within the new approach regarding truncated Laplace transform solutions.

The use of truncated Laplace transform with respect to one of the time variables in the Borel plane is used successfully to control the growth of the solutions, via complex Banach spaces of functions not only subject to an exponential growth in the monomial variables, but also whose domain of definition depends on each value of ϵ (see Section 4). Given a finite family of finite sectors $\underline{\mathcal{E}} = (\mathcal{E}_p)_{0\le p\le \iota-1}$ which conform a good covering (see Definition 3), the first main result in the work, Theorem 1, states the existence of a solution of the main problem in the form (3) for every $0 \le p \le \iota - 1$, remaining holomorphic in a domain $\mathcal{T}_1 \times \mathcal{T}_2 \times H_\beta \times \mathcal{E}_p$, where $\mathcal{T}_1, \mathcal{T}_2$ are finite sectors with vertex at the origin. Moreover, the exponential decrease of the difference of two solutions associated to consecutive sectors in $\underline{\mathcal{E}}$ enables the application of the classical Ramis-Sibuya theorem (RS) in order to achieve the second main result of our study, namely the asymptotic relation of the analytic solutions and the formal solution of the main problem in powers of ϵ, with coefficients in some complex Banach space (see Theorem 3).

In recent years, several steps have been taken to contribute to the knowledge of the asymptotic behavior of analytic solutions of singularly perturbed partial differential equations in the complex domain. We first refer to the recent works [17, 18], by H. Yamazawa and M. Yoshino, and M. Yoshino resp. in which the parametric Borel summability of semilinear systems of PDEs is studied, first in the case of fuchsian operators, and second combining both irregular and fuchsian operators. We refer to [1, 14] as introductory texts on the classical theory of summability of formal solutions of differential equations in the complex domain.

The appearance of truncated Laplace transform is closely related to the classical theory of asymptotic approximation of analytic functions (examples of this situation is the classical proof of Ritt's Theorem for Gevrey asymptotics, see [1] Proposition 10, and also Lemma 1.3.2 in [14]). Truncated Laplace transform also appears as a recent object of study in the literature, related to differential operators [12, 13], but also from the numerical point of view [11]. The choice of an integration path for Laplace transform which depends on each fixed value of the perturbation parameter ϵ has been inspired from [3, 15].

Throughout the work, we use bold letters to indicate a vector of two variables: we write $\boldsymbol{\tau}$ for the pair (τ_1, τ_2), $\boldsymbol{u}$ for (u_1, u_2), $\boldsymbol{T}$ for (T_1, T_2), etc.

The paper is organized as follows.

In Section 2.1, we recall some properties on Fourier transform which allow to transform the main problem, stated in Section 2.2, in the form a convolution problem, described in Section 2.3. The geometry of the problem is an important matter in this work, which needs to be explained in detail. Section 3 is focused on this issue. The Banach spaces involved in the construction of the analytic solution of the auxiliary problem, and some of their main properties, are stated in Section 4. Such function is constructed in Section 5. The analytic solution of the main problem is obtained in Section 6 (Theorem 1), and the work concludes with the description of the parametric Gevrey asymptotic expansions of the analytic solution, obtained in Section 7 (Theorem 3).

2 Layout of the main and auxiliary problems

In this initial section, we describe in detail the main problem under study (8) (Section 2.2), and the conditions on the elements involved in it. The solution of this problem is reduced to a convolution auxiliary problem in the Borel plane (17) when inspecting solutions in the particular form of a triple Fourier, Laplace and truncated Laplace transform (see Section 2.3). We first give some words about inverse Fourier transform on certain Banach spaces which act on the transformation of the problem (Section 2.1).

2.1 Inverse Fourier transform on certain function spaces

The transformation of the main problem with respect to variable z requires recalling some basic facts about inverse Fourier transform when acting on certain Banach spaces of real functions of exponential decrease at infinity.

Definition 1 *Let $\beta,\mu \in \mathbb{R}$. We write $E_{(\beta,\mu)}$ for the set of all continuous functions $h : \mathbb{R} \to \mathbb{C}$ such that*

$$\|h(m)\|_{(\beta,\mu)} := \sup_{m\in\mathbb{R}} (1+|m|)^{\mu} \exp(\beta|m|)|h(m)| < \infty.$$

The pair $(E_{\beta,\mu}, \|\cdot\|_{(\beta,\mu)})$ is a Banach space.

The next result will be needed in our reasoning. We refer to [4], Proposition 7, for its proof.

Proposition 1 *Let $\beta > 0$ and $\mu > 1$. The inverse Fourier transform*

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(m)\exp(ixm)dm, \quad x \in \mathbb{R}$$

satisfies the following properties acting on every $f \in E_{(\beta,\mu)}$:

- *The function $\mathcal{F}^{-1}(f)$ is well defined in $\mathbb{R}$ and can be analytically extended to the set*

$$H_{\beta} := \{z \in \mathbb{C} : |Im(z)| < \beta\}. \tag{4}$$

- *Let $\phi(m) := imf(m)$. Then, $\phi \in E_{(\beta,\mu-1)}$ and $\partial_z \mathcal{F}^{-1}(f)(z) = \mathcal{F}^{-1}(\phi)(z)$ for $z \in H_{\beta}$.*

- *Let $g \in E_{(\beta,\mu)}$ and let $\psi(m) = \frac{1}{(2\pi)^{1/2}} f * g(m)$ be the convolution product of f and g, for all $m \in \mathbb{R}$. Then, $\psi \in E_{(\beta,\mu)}$ and it holds that*

$$\mathcal{F}^{-1}(f)(z)\mathcal{F}^{-1}(g)(z) = \mathcal{F}^{-1}(\psi)(z), \quad z \in H_{\beta}.$$

2.2 Statement of the main problem

Let k_1 and k_2 and $D_1, D_2 \geq 2$ be positive integers. Let δ_{ℓ_1} (resp. $\tilde{\delta}_{\ell_2}$) be a nonnegative integer for every $1 \leq \ell_1 \leq D_1$ (resp. every $1 \leq \ell_2 \leq D_2$). We also fix nonnegative integers $\Delta_{\ell_1,\ell_2}, d_{\ell_2}$ for all $1 \leq \ell_1 \leq D_1 - 1$ and $1 \leq \ell_2 \leq D_2 - 1$.

We assume that

$$\Delta_{D_1D_2} := k_1\delta_{D_1} + k_2\tilde{\delta}_{D_2}. \tag{5}$$

and

$$\Delta_{\ell_1\ell_2} > k_1\delta_{\ell_1} + \frac{k_2\tilde{\delta}_{D_2}\delta_{\ell_1}}{\delta_{D_1}}, \quad d_{\ell_2} > \tilde{\delta}_{\ell_2}(k_2+1), \quad \frac{\tilde{\delta}_{D_2}\delta_{l_1}}{\delta_{D_1}} \geq \tilde{\delta}_{l_2} + \frac{1}{k_2}, \tag{6}$$

for every $1 \leq \ell_1 \leq D_1 - 1$ and $1 \leq \ell_2 \leq D_2 - 1$.

Let $Q(X), R_{D_1D_2}(X)$ and $R_{\ell_1\ell_2}(X)$ for all $1 \leq \ell_1 \leq D_1 - 1$ and $1 \leq \ell_2 \leq D_2 - 1$ belong to $\mathbb{C}[X]$. We assume that

$$\deg(R_{D_1D_2}) \geq \deg(R_{\ell_1\ell_2}), \qquad R_{D_1D_2}(im) \neq 0 \tag{7}$$

for every $0 \leq \ell_1 \leq D_1 - 1$ and $0 \leq \ell_2 \leq D_2 - 1$, and all $m \in \mathbb{R}$.

Remark: In Section 3 we assume further geometric conditions on these polynomials. In particular, observe that condition (18) implies that $\deg(Q) \geq \deg(R_{D_1D_2})$.

We choose $\mu \in \mathbb{R}$ with

$$\mu > \max_{\substack{0\leq\ell_1\leq D_1-1 \\ 0\leq\ell_2\leq D_2-1}} \deg(R_{\ell_1\ell_2}) + 1.$$

The main aim in this work is to study the following initial value problem:

$$\begin{aligned} Q(\partial_z)u(t_1,t_2,z,\epsilon) &= \epsilon^{\Delta_{D_1D_2}}(t_1^{k_1+1}\partial_{t_1})^{\delta_{D_1}}(t_2^{k_2+1}\partial_{t_2})^{\tilde{\delta}_{D_2}}R_{D_1D_2}(\partial_z)u(t_1,t_2,z,\epsilon) \\ &+ \sum_{\substack{1\leq\ell_1\leq D_1-1 \\ 1\leq\ell_2\leq D_2-1}} \epsilon^{\Delta_{\ell_1\ell_2}}(t_1^{k_1+1}\partial_{t_1})^{\delta_{\ell_1}} t_2^{d_{\ell_2}}\partial_{t_2}^{\tilde{\delta}_{\ell_2}} c_{\ell_1\ell_2}(z,\epsilon)R_{\ell_1\ell_2}(\partial_z)u(t_1,t_2,z,\epsilon) + f(t_1,t_2,z,\epsilon), \end{aligned} \tag{8}$$

for the initial conditions $u(t_1,0,z,\epsilon) \equiv u(0,t_2,z,\epsilon) \equiv 0$. Let us describe the form of the elements involved in the problem.

Let $\epsilon_0 > 0$ and $\beta > 0$. For all $1 \leq \ell_1 \leq D_1 - 1$ and $1 \leq \ell_2 \leq D_2 - 1$, the term $c_{\ell_1\ell_2}(z,\epsilon)$ are holomorphic functions on $H_\beta \times D(0,\epsilon_0)$. We recall that H_β stands for the horizontal strip (4). The function $c_{\ell_1\ell_2}$ is defined by

$$c_{\ell_1\ell_2}(z,\epsilon) := \mathcal{F}^{-1}(m \mapsto C_{\ell_1\ell_2}(m,\epsilon))(z) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} C_{\ell_1\ell_2}(m,\epsilon)e^{izm}dm,$$

where $m \mapsto C_{\ell_1\ell_2}(m,\epsilon)$ is continuous for $m \in \mathbb{R}$ and is subject to uniform exponentially flat upper bounds with respect to $\epsilon \in D(0,\epsilon_0)$, i.e. there exists $C_c > 0$ such that

$$\sup_{\epsilon\in D(0,\epsilon_0)} |C_{\ell_1\ell_2}(m,\epsilon)| \leq \frac{C_c}{(1+|m|)^\mu}\exp(-\beta|m|), \quad m \in \mathbb{R}. \tag{9}$$

Observe that $m \mapsto C_{\ell_1\ell_2}(m,\epsilon)$ belongs to $E_{(\beta,\mu)}$ with

$$\sup_{\epsilon\in D(0,\epsilon_0)} \|C_{\ell_1\ell_2}(\cdot,\epsilon)\|_{(\beta,\mu)} \leq C_c,$$

for all $0 \leq \ell_1 \leq D_1 - 1$ and $0 \leq \ell_2 \leq D_2 - 1$.

The forcing term $f(t_1,t_2,z,\epsilon)$ is a holomorphic function in $\mathbb{C}^\star \times D(0,h') \times H_\beta \times \mathcal{E}$, for any given open sector $\mathcal{E}$ centered at 0, and contained in $D(0,\epsilon_0) \setminus \{0\}$, for some positive number h'.

The forcing term is constructed as follows. Let $N_1 \geq 0$ and $F_{n_1,n_2}(m,\epsilon) \in E_{(\beta,\mu)}$ under uniform bounds with respect to ϵ in the disc $D(0,\epsilon_0)$. More precisely, assume that

$$\sup_{\epsilon \in D(0,\epsilon_0)} ||F_{n_1 n_2}(m,\epsilon)||_{(\beta,\mu)} \leq K_0 \left(\frac{1}{T_0}\right)^{n_2}, \quad 0 \leq n_1 \leq N_1, n_2 \geq 0,$$

for some $K_0, T_0 > 0$. We consider

$$\psi(\boldsymbol{\tau},m,\epsilon) := \sum_{n_1=0}^{N_1} \sum_{n_2 \geq 0} F_{n_1 n_2}(m,\epsilon) k_1 \tau_1^{n_1} \frac{k_2 \tau_2^{n_2}}{\Gamma\left(\frac{n_2}{k_2}\right)},$$

which turns out to be a holomorphic function on $\mathbb{C}^2$ with respect to the first two variables, with coefficients in $E_{(\beta,\mu)}$. We write

$$F(\boldsymbol{T},z,\epsilon) = \sum_{n_1=0}^{N_1} \sum_{n_2 \geq 0} \mathcal{F}^{-1}(m \mapsto F_{n_1 n_2}(m,\epsilon)) T_1^{n_1} \gamma(\frac{n_1}{k_1}, \left(\frac{\kappa h_1(\epsilon) e^{\sqrt{-1}\theta_1}}{T_1}\right)^{k_1}) T_2^{n_2}, \tag{10}$$

where $\kappa h_1(\epsilon)$ is a holomorphic function on any open sector centered at 0 in the punctured disc $D(0,\epsilon_0) \setminus \{0\}$ (see (20)), θ_1 is a real number to be determined and $\gamma(n,z)$ stands for the incomplete Gamma function

$$\gamma(n,z) = \int_0^z u^{n-1} e^{-u} du,$$

which is an entire function w.r.t. z, when n is a fixed positive real number. Observe that the forcing term F depends in particular on the choice of θ_1.

The following property related to the lower incomplete Gamma function will be crucial in the construction of the auxiliary equation of the problem. Namely,

$$\int_0^{\kappa h_1(\epsilon) e^{\sqrt{-1}\theta_1}} u^{n-1} \exp(-(\frac{u}{T})^k) du = \frac{T^n}{k} \gamma(\frac{n}{k}, \left(\frac{\kappa h_1(\epsilon) \exp(\sqrt{-1}\theta_1)}{T}\right)^k). \tag{11}$$

We recall that the infinite Laplace transform satisfies

$$\int_0^{\infty} u^{n-1} \exp(-(\frac{u}{T})^k) du = \frac{T^n}{k} \Gamma\left(\frac{n}{k}\right), \tag{12}$$

for every positive natural numbers n, k.

This property will be used with respect to the second time variable, whereas a truncated Laplace transform depending on each value of the perturbation parameter near the origin is applied on the first variable. Both, (11) and (12) give rise to adequate algebraic properties which allow to reduce the main equation in the form of an auxiliary problem.

Regarding (10), F is holomorphic w.r.t. T_1 on $\mathbb{C}^\star$, T_2 on the disc $D(0,T_0/2)$ and on H_β w.r.t. z. Furthermore, according to (11), (12), we observe that

$$\gamma(\frac{n_1}{k_1}, (\frac{\kappa h_1(\epsilon) e^{\sqrt{-1}\theta_1}}{T_1})^k) \to \Gamma(\frac{n_1}{k_1})$$

as ϵ tends to 0, for (well chosen) fixed T_1. Therefore, F is getting closer to a polynomial in T_1 as ϵ tends to 0. The function f defined by

$$f(\boldsymbol{t},z,\epsilon) = F(\epsilon t_1, \epsilon t_2, z, \epsilon)$$

is holomorphic on $\mathbb{C}^\star \times D(0,h') \times H_\beta \times \mathcal{E}$, for any given open sector $\mathcal{E}$ centered at 0 and contained in $D(0,\epsilon_0) \setminus \{0\}$, with $h' > 0$ such that $0 < h'\epsilon_0 < T_0/2$. From the remark above, we check in particular that f becomes close to a polynomial in t_1 as ϵ becomes closer to the origin.

2.3 Auxiliary problems

We search for solutions of (8) in the form of an inverse Fourier transform

$$u(t_1, t_2, z, \epsilon) = \mathcal{F}^{-1}(m \mapsto U(\epsilon t_1, \epsilon t_2, z, \epsilon)).$$

The classical properties of inverse Fourier transform, together with (5), lead to an auxiliary functional equation satisfied by the expression $U(T_1, T_2, m, \epsilon)$, namely

$$\begin{aligned} Q(im)U(T_1,T_2,m,\epsilon) &= (T_1^{k_1+1}\partial_{T_1})^{\delta_{D_1}}(T_2^{k_2+1}\partial_{T_2})^{\tilde{\delta}_{D_2}} R_{D_1D_2}(im)U(T_1,T_2,m,\epsilon) \\ &+ \sum_{\substack{1\le \ell_1\le D_1-1\\ 1\le \ell_2\le D_2-1}} \epsilon^{\Delta_{\ell_1\ell_2}-k_1\delta_{\ell_1}-d_{\ell_2}+\tilde{\delta}_{\ell_2}}(T_1^{k_1+1}\partial_{T_1})^{\delta_{\ell_1}} T_2^{d_{\ell_2}} \partial_{T_2}^{\tilde{\delta}_{\ell_2}} \\ &\times \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} C_{\ell_1\ell_2}(m-m_1,\epsilon) R_{\ell_1\ell_2}(im_1) U(T_1,T_2,m_1,\epsilon)dm_1 + F(T_1,T_2,m,\epsilon). \end{aligned} \tag{13}$$

Let $0 < \kappa < 1$ and $\theta_1, d_2 \in \mathbb{R}$. Let $\epsilon \mapsto h_1(\epsilon)$ be a holomorphic function defined on the domain of definition of the perturbation parameter, to be detailed afterwards. For every fixed value of the perturbation parameter ϵ, we search for solutions of (13) written as the Laplace transform with respect to T_2 along direction d_2 and the truncated Laplace transform with respect to T_1 along direction $\theta_1 - \lambda k_2 \tilde{\delta}_{D_2} \arg(\epsilon)$ applied to a second auxiliary function. More precisely, we search for solutions of (13) of the form

$$U_{d_1d_2}(\boldsymbol{T}, m, \epsilon) = \int_{L_{d_1,\epsilon}} \int_{L_{d_2}} \omega(\boldsymbol{u}, m, \epsilon) \exp\left(-\left(\frac{u_1}{T_1}\right)^{k_1} - \left(\frac{u_2}{T_2}\right)^{k_2} \right) \frac{du_1}{u_1}\frac{du_2}{u_2}, \tag{14}$$

where $d_1 = d_1(\epsilon) := \arg(h_1(\epsilon)) + \theta_1$ and $L_{d_1,\epsilon}$ stands for the segment $[0, \kappa h_1(\epsilon) e^{\sqrt{-1}\theta_1}]$; L_{d_2} consists of the half-line with endpoint at the origin and direction d_2. The domain of definition of ω and $U_{d_1d_2}$ will be discussed in the subsequent sections.

Lemma 1 ((8.7), [16]) *For every $m, k \in \mathbb{N}$ one has*

$$t^{m(k+1)}\partial_t^m = (t^{k+1}\partial_t)^m + \sum_{1\le \ell\le m-1} A_{m\ell} t^{k(m-\ell)} (t^{k+1}\partial_t)^{\ell},$$

for some constants $A_{m\ell}$, $1 \le \ell \le m-1$.

The assumption (6) guarantees the existence of $d_{\ell_2 k_2} \in \mathbb{N}$ such that

$$d_{\ell_2} = \tilde{\delta}_{\ell_2}(k_2+1) + d_{\ell_2k_2}, \qquad 1 \le \ell_2 \le D_2 - 1. \tag{15}$$

The following result states a one-to-one correspondence between the solution of (13) and (17). Its proof, which is omitted, can be adapted with minor modifications from [6], Lemma 1.

Lemma 2 *Let $U_{d_1d_2}(T_1, T_2, m, \epsilon)$ be defined by (14). Then, it holds that*

$$T_j^{k_j+1}\partial_{T_j} U_{d_1d_2}(T_1,T_2,m,\epsilon) = \int_{L_{d_1,\epsilon}}\int_{L_{d_2}} (k_j u_j^{k_j})\omega(u_1,u_2,m,\epsilon) e^{-\left(\frac{u_1}{T_1}\right)^{k_1} - \left(\frac{u_2}{T_2}\right)^{k_2}} \frac{du_2}{u_2}\frac{du_1}{u_1}, \quad j = 1, 2.$$

$$\begin{aligned} T_2^{m_2} U_{d_1d_2}(T_1,T_2,m,\epsilon) &= \int_{L_{d_1,\epsilon}}\int_{L_{d_2}} \frac{u_2^{k_2}}{\Gamma\left(\frac{m_2}{k_2}\right)} \int_0^{u_2^{k_2}} (u_2^{k_2} - s_2)^{\frac{m_2}{k_2}-1} \omega(u_1, s_2^{1/k_2}, m, \epsilon) \frac{ds_2}{s_2} \\ &\times e^{-\left(\frac{u_1}{T_1}\right)^{k_1} - \left(\frac{u_2}{T_2}\right)^{k_2}} \frac{du_2}{u_2}\frac{du_1}{u_1}, \qquad m_2 \in \mathbb{N}. \end{aligned} \tag{16}$$

Lemma 1, (15) together with the shape of the solution in (14), the assumptions on the coefficients $c_{\ell_1\ell_2}$ and the forcing term f, and Lemma 2 entail ω being a solution of the following auxiliary convolution equation in the Borel plane:

$$\begin{aligned}(17)\quad &\left(Q(im) - R_{D_1D_2}(im)(k_1\tau_1^{k_1})^{\delta_{D_1}}(k_2\tau_2^{k_2})^{\tilde{\delta}_{D_2}}\right)\omega(\boldsymbol{\tau},m,\epsilon)\\ &= \sum_{\substack{1\le\ell_1\le D_1-1\\ 1\le\ell_2\le D_2-1}} \epsilon^{\Delta_{\ell_1\ell_2}-k_1\delta_{\ell_1}-d_{\ell_2}+\tilde{\delta}_{\ell_2}}(k_1\tau_1^{k_1})^{\delta_{\ell_1}}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}C_{\ell_1\ell_2}(m-m_1,\epsilon)R_{\ell_1\ell_2}(im_1)\\ &\times\left[\frac{\tau_2^{k_2}}{\Gamma\left(\frac{d_{\ell_2k_2}}{k_2}\right)}\int_0^{\tau_2^{k_2}}(\tau_2^{k_2}-s_2)^{\frac{d_{\ell_2k_2}}{k_2}-1}(k_2s_2)^{\tilde{\delta}_{\ell_2}}\omega(\tau_1,s_2^{1/k_2},m_1,\epsilon)\frac{ds_2}{s_2}dm_1\right.\\ &+\sum_{1\le p_2\le\tilde{\delta}_{\ell_2}-1}\frac{A_{\delta_{\ell_2p_2}}\tau_2^{k_2}}{\Gamma\left(\frac{d_{\ell_2k_2}+k_2(\tilde{\delta}_{\ell_2}-p_2)}{k_2}\right)}\int_0^{\tau_2^{k_2}}(\tau_2^{k_2}-s_2)^{\frac{d_{\ell_2k_2}+k_2(\tilde{\delta}_{\ell_2}-p_2)}{k_2}-1}(k_2s_2)^{p_2}\\ &\left.\times\omega(\tau_1,s_2^{1/k_2},m_1,\epsilon)\frac{ds_2}{s_2}dm_1\right]+\psi(\boldsymbol{\tau},m,\epsilon)\end{aligned}$$

So far, the solution is of symbolic nature. The geometry of the problem, detailed in the following section, together with Section 4 provide convergence and growth estimates of such solution.

3 On the geometry of the problem

In this section, we preserve the objects and assumptions detailed in Section 2.2 on the elements involved in the construction of the main problem under study (8), giving rise to the auxiliary problem (17). This section is devoted to the study of the geometry of the problem, which is crucial in the asymptotic approximation of the solution.

We define for every $m\in\mathbb{R}$ the polynomial

$$P_m(\boldsymbol{\tau}) = Q(im) - R_{D_1D_2}(im)(k_1\tau_1^{k_1})^{\delta_{D_1}}(k_2\tau_2^{k_2})^{\tilde{\delta}_{D_2}}.$$

In the case that $\tau_1\neq 0$ one can factorize P_m in the form

$$P_m(\boldsymbol{\tau}) = -R_{D_1D_2}(im)k_1^{\delta_{D_1}}k_2^{\tilde{\delta}_{D_2}}\tau_1^{k_1\delta_{D_1}}\prod_{\ell=0}^{k_2\tilde{\delta}_{D_2}-1}(\tau_2-q_\ell(\tau_1,m))$$

where $q_\ell(\tau_1,m)$ are the $k_2\tilde{\delta}_{D_2}$ roots of $Q(im)/(R_{D_1D_2}k_1^{\delta_{D_1}}k_2^{\tilde{\delta}_{D_2}}\tau_1^{k_1\delta_{D_1}})$ with respect to τ_2, i.e.

$$q_\ell(\tau_1,m) = \left(\frac{|Q(im)|}{|R_{D_1D_2}(im)|k_1^{\delta_{D_1}}k_2^{\tilde{\delta}_{D_2}}|\tau_1|^{k_1\delta_{D_1}}}\right)^{\frac{1}{k_2\tilde{\delta}_{D_2}}}\exp\left(\sqrt{-1}\left(\arg\left(\frac{Q(im)}{R_{D_1D_2}(im)\tau_1^{k_1\delta_{D_1}}}\right)\frac{1}{k_2\tilde{\delta}_{D_2}}+\frac{2\pi\ell}{k_2\tilde{\delta}_{D_2}}\right)\right)$$

for every $0\le\ell\le k_2\tilde{\delta}_{D_2}-1$.

We assume that the polynomials Q and $R_{D_1D_2}$ satisfy that

$$(18)\qquad \frac{Q(im)}{R_{D_1D_2}(im)}\in S_{Q,R_{D_1D_2}},$$

where $S_{Q,R_{D_1D_2}}$ stands for an unbounded sector

$$S_{Q,R_{D_1D_2}} = \{z \in \mathbb{C} : |z| \geq \rho_{Q,R_{D_1D_2}}, |\arg(z) - d_{Q,R_{D_1D_2}}| \leq \eta_{Q,R_{D_1D_2}}\},$$

for some small $\eta_{Q,R_{D_1D_2}} > 0$, large $\rho_{Q,R_{D_1D_2}}, > 0$, and some $d_{Q,R_{D_1D_2}} \in \mathbb{R}$ to be determined.

Let λ be a real number which satisfies that

$$0 < \lambda < \frac{1}{k_1\delta_{D_1}}. \tag{19}$$

Let $\mathcal{E}$ be a sector with vertex at the origin which is contained in the disc $D(0, \epsilon_0)$ and for every $\epsilon \in \mathcal{E}$ we define

$$h_1(\epsilon) := \left(\frac{\rho_{Q,R_{D_1D_2}}}{k_1^{\delta_{D_1}} k_2^{\tilde{\delta}_{D_2}}}\right)^{\frac{1}{k_1\delta_{D_1}}} \frac{1}{\epsilon^{r_{11}}}, \tag{20}$$

and the quantities

$$r_1(\epsilon) := \left(\frac{\rho_{Q,R_{D_1D_2}}}{k_1^{\delta_{D_1}} k_2^{\tilde{\delta}_{D_2}}}\right)^{\frac{1}{k_1\delta_{D_1}}} \frac{1}{|\epsilon|^{r_{11}}} \quad \text{and } r_2(\epsilon) = \frac{1}{2}|\epsilon|^{r_{22}}, \tag{21}$$

with $r_{11} := \lambda k_2\tilde{\delta}_{D_2}$, and $r_{22} := \lambda k_1\delta_{D_1}$.

The next result summarizes the main properties of the geometric construction above, which will be used to state the asymptotic behavior of the solutions of the main problem.

Lemma 3 *Let $m \in \mathbb{R}$ and $\epsilon \in \mathcal{E}$. The following statements hold:*

- $\{\boldsymbol{\tau} \in \mathbb{C}^2 : P_m(\boldsymbol{\tau}) = 0\} \cap (D(0, r_1(\epsilon)) \times D(0, 2r_2(\epsilon))) = \emptyset$.
- *Provided that $\lambda > 0$ is small enough, for any couple of directions (θ_1, d_2) which satisfy that*

$$d_2 \neq \Big(d - k_1\delta_{D_1}(\lambda k_2\tilde{\delta}_{D_2} arg(\epsilon) - \theta_1)\Big)\frac{1}{k_2\tilde{\delta}_{D_2}} + \frac{2\pi\ell}{k_2\tilde{\delta}_{D_2}},$$

  for all $0 \leq \ell \leq k_2\tilde{\delta}_{D_2} - 1$, where $d \in (d_{Q,R_{D_1D_2}} - \eta_{Q,R_{D_1D_2}}, d_{Q,R_{D_1D_2}} + \eta_{Q,R_{D_1D_2}})$, all $\epsilon \in \mathcal{E}$, there exist an unbounded sector S_{d_2} with bisecting direction d_2 and small opening, and a sector $S_{d_1,\epsilon}$, with

$$S_{d_1,\epsilon} = \{z \in \mathbb{C}^\star : 0 < |z| < \kappa r_1(\epsilon), |arg(z) - d_1| < \tilde{\delta}_1\},$$

  where $d_1 = d_1(\epsilon) = \lambda k_2\tilde{\delta}_{D_2} arg(\epsilon) - \theta_1$ such that

$$\{\boldsymbol{\tau} \in \mathbb{C}^2 : P_m(\boldsymbol{\tau}) = 0\} \cap (\overline{S_{d_1,\epsilon}} \times \overline{S_{d_2}}) = \emptyset,$$

  for all $\epsilon \in \mathcal{E}$.
- *Let $S_{d_1,\epsilon}$ and S_{d_2} be as above. We put*

$$\Omega_1(\epsilon) := S_{d_1,\epsilon}, \quad \text{and} \quad \Omega_2(\epsilon) := D(0, r_2(\epsilon)) \cup S_{d_2}. \tag{22}$$

  Then, there exists $C_P > 0$ which does not depend on $\epsilon \in \mathcal{E}$ such that

$$|P_m(\boldsymbol{\tau})| \geq C_P|R_{D_1D_2}(im)|(1 + |\tau_1|^{k_1\delta_{D_1}}|\tau_2|^{k_2\tilde{\delta}_{D_2}}), \tag{23}$$

  for every $m \in \mathbb{R}$, $\boldsymbol{\tau} \in \overline{\Omega_1(\epsilon)} \times \Omega_2(\epsilon)$.

Proof

Let $\boldsymbol{\tau} = (\tau_1, \tau_2) \in \mathbb{C}^2$ such that $P_m(\boldsymbol{\tau}) = 0$. One has that $\tau_2 = q_\ell(\tau_1, m)$, for some $0 \le \ell \le k_2\tilde{\delta}_{D_2} - 1$. In the case that $|q_\ell(\tau_1, m)| \le 2r_2(\epsilon)$, from the definition of P_m and (18) we derive that

$$|\tau_1| \ge \left(\frac{\rho_{Q,R_{D_1D_2}}}{k_1^{\delta_{D_1}} k_2^{\tilde{\delta}_{D_2}}} \right)^{\frac{1}{k_1\delta_{D_1}}} \frac{1}{(2r_2(\epsilon))^{\frac{k_2\tilde{\delta}_{D_2}}{k_1\delta_{D_1}}}} = r_1(\epsilon).$$

The first statements follows from here.

The second statement is a direct consequence of the fact that for all $\tau_1 \in \mathbb{C}^\star$ and $m \in \mathbb{R}$ one has

$$\arg(q_\ell(\tau_1, m)) = \left[\arg\left(\frac{Q(im)}{R_{D_1D_2}(im)} \right) - k_1\delta_{D_1}\arg(\tau_1) \right] \frac{1}{k_2\tilde{\delta}_{D_2}} + \frac{2\pi\ell}{k_2\tilde{\delta}_{D_2}}, \tag{24}$$

for every $0 \le \ell \le k_2\tilde{\delta}_{D_2} - 1$. Regarding the construction of $S_{d_1,\epsilon}$ we have that for all $\tau_1 \in \overline{S_{d_1,\epsilon}}$ it holds that

$$-\tilde{\delta}_1 + \lambda k_2\tilde{\delta}_{D_2}\arg(\epsilon) + \theta_1 < \arg(\tau_1) < \tilde{\delta}_1 + \lambda k_2\tilde{\delta}_{D_2}\arg(\epsilon) + \theta_1.$$

The pair (θ_1, d_2) can be chosen accordingly, provided that $\lambda, \eta_{Q,R_{D_1D_2}}, \tilde{\delta}_1 > 0$ are small enough.

In order to give proof to the third item, we first give estimates on $|\tau_2/q_\ell(\tau_1, m)|$ for any $0 \le \ell \le k_2\tilde{\delta}_{D_2} - 1$ and $m \in \mathbb{R}$. First, assume that $\tau_1 \in D(0, r_1(\epsilon))$ and $\tau_2 \in D(0, r_2(\epsilon))$. Then, it holds that

$$\left| \frac{\tau_2}{q_\ell(\tau_1, m)} \right| \le \frac{1}{2}|\epsilon|^{\lambda k_1\delta_{D_1}} \left(\frac{|R_{D_1D_2}(im)| k_1^{\delta_{D_1}} k_2^{\tilde{\delta}_{D_2}} |\tau_1|^{k_1\delta_{D_1}}}{|Q(im)|} \right)^{\frac{1}{k_2\tilde{\delta}_{D_2}}} \tag{25}$$

$$\le \frac{1}{2}|\epsilon|^{\lambda k_1\delta_{D_1}} \left(\frac{k_1^{\delta_{D_1}} k_2^{\tilde{\delta}_{D_2}} r_1(\epsilon)^{k_1\delta_{D_1}}}{\rho_{Q,R_{D_1D_2}}} \right)^{\frac{1}{k_2\tilde{\delta}_{D_2}}} \le \frac{1}{2}.$$

The previous estimates yield $\mathrm{dist}(q_\ell(\tau_1, m)/\tau_2, 1) \ge \frac{1}{2}$. Moreover, the choice made for $S_{d_1,\epsilon}$ can be made in order to guarantee the existence of a positive constant M_2 such that $\mathrm{dist}(q_\ell(\tau_1, m)/\tau_2, 1) \ge M_2$ for every $\tau_1 \in \overline{\Omega_1(\epsilon)}$, $\tau_2 \in S_{d_2}$ and $m \in \mathbb{R}$. One gets from the previous argument that $|q_\ell(\tau_1, m)/\tau_2 - 1| \ge \min\{M_2, 1/2\}$, for every $m \in \mathbb{R}$ and $\boldsymbol{\tau} \in (\overline{\Omega_1(\epsilon)} \times \Omega_2(\epsilon))$. This entails the existence of a constant $c_1 > 0$ such that

$$|q_\ell(\tau_1, m) - \tau_2| \ge c_1|\tau_2|, \qquad |q_\ell(\tau_1, m) - \tau_2| \ge c_1|q_\ell(\tau_1, m)| \ge \frac{c_2}{|\tau_1|^{\frac{k_1\delta_{D_1}}{k_2\tilde{\delta}_{D_2}}}},$$

where $c_2 = c_1 \left(\frac{\rho_{Q,R_{D_1D_2}}}{k_1^{\delta_{D_1}}} \right)^{\frac{1}{k_2\tilde{\delta}_{D_2}}} \frac{1}{k_2^{1/k_2}}$. The previous estimates yield

$$|q_\ell(\tau_1, m) - \tau_2| \ge \frac{c_1}{2} \left(|\tau_2| + \frac{c_2}{c_1} \frac{1}{|\tau_1|^{\frac{k_1\delta_{D_1}}{k_2\tilde{\delta}_{D_2}}}} \right),$$

and from the factorization of P_m,

$$|P_m(\boldsymbol{\tau})| \geq |R_{D_1D_2}(im)| k_1^{\delta_{D_1}} k_2^{\tilde{\delta}_{D_2}} |\tau_1|^{k_1\delta_{D_1}} \left(\frac{c_1}{2}(|\tau_2| + \frac{c_2}{c_1} \frac{1}{|\tau_1|^{\frac{k_1\delta_{D_1}}{k_2\tilde{\delta}_{D_2}}}}) \right)^{k_2\tilde{\delta}_{D_2}}. \tag{26}$$

From (26), we conclude the existence of $\tilde{c} > 0$ such that

$$|P_m(\boldsymbol{\tau})| \geq \tilde{c}|R_{D_1D_2}(im)| \left(|\tau_1|^{\frac{k_1\delta_{D_1}}{k_2\tilde{\delta}_{D_2}}} |\tau_2| + \frac{c_2}{c_1} \right)^{k_2\tilde{\delta}_{D_2}},$$

for all $m \in \mathbb{R}$ and $\boldsymbol{\tau} \in (\overline{\Omega_1(\epsilon)} \times \Omega_2(\epsilon))$.

It only rests to prove that

$$\left(|\tau_1|^{\frac{k_1\delta_{D_1}}{k_2\tilde{\delta}_{D_2}}} |\tau_2| + \frac{c_2}{c_1} \right)^{k_2\tilde{\delta}_{D_2}} \geq C_P(1 + |\tau_1|^{k_1\delta_{D_1}}|\tau_2|^{k_2\tilde{\delta}_{D_2}}), \tag{27}$$

for some constant $C_P > 0$, all $m \in \mathbb{R}$ and $\boldsymbol{\tau} \in (\overline{\Omega_1(\epsilon)} \times \Omega_2(\epsilon))$.

Usual estimates guarantee that

$$\left(|\tau_1|^{\frac{k_1\delta_{D_1}}{k_2\tilde{\delta}_{D_2}}} |\tau_2| + \frac{c_2}{c_1} \right)^{k_2\tilde{\delta}_{D_2}} = \left(\frac{c_2}{c_1}\right)^{k_2\tilde{\delta}_{D_2}} \left(c_3|\tau_1|^{\frac{k_1\delta_{D_1}}{k_2\tilde{\delta}_{D_2}}} |\tau_2| + 1 \right)^{k_2\tilde{\delta}_{D_2}} \geq c_4 \left(|\tau_1|^{\frac{k_1\delta_{D_1}}{k_2\tilde{\delta}_{D_2}}} |\tau_2| + 1 \right)^{k_2\tilde{\delta}_{D_2}}$$

with $c_3 = c_1/c_2$, and some $c_4 > 0$. Taking into account that

$$\lim_{x\to 0^+} \frac{(1+x)^m}{1+x^m} = \lim_{x\to\infty} \frac{(1+x)^m}{1+x^m} = 1, \quad m > 0,$$

we get the existence of $c_5 > 0$ such that

$$\left(|\tau_1|^{\frac{k_1\delta_{D_1}}{k_2\tilde{\delta}_{D_2}}} |\tau_2| + 1 \right)^{k_2\tilde{\delta}_{D_2}} \geq c_5(1 + |\tau_1|^{k_1\delta_{D_1}}|\tau_2|^{k_2\tilde{\delta}_{D_2}}),$$

which concludes the proof. □

4 Banach spaces of functions with exponential growth

In this section, we recall the definition and main properties of certain Banach spaces previously used by the authors in [4], and adapted to the several variable case in [6, 8]. The dependence of the domains of definition involved in the norm with respect to the values of the perturbation parameter has previously been consider in [5].

Let $\mathcal{E}$ be a sector of finite radius in the complex plane. For every $\epsilon \in \mathcal{E}$ we consider the following two domains: a finite sector $\Omega_1(\epsilon)$ with vertex at the origin, bisecting direction d_1 which depends on ϵ, and radius $r_1(\epsilon)$; and the union of an infinite sector S_{d_2} with vertex at the origin, fixed bisecting direction d_2 and positive opening which do not depend on ϵ together with the disc $D(0, r_2(\epsilon))$ for some $r_2(\epsilon) > 0$, say $\Omega_2(\epsilon)$, i.e. $\Omega_2(\epsilon) = S_{d_2} \cup D(0, r_2(\epsilon))$.

In the following we write $\boldsymbol{d} = (d_1, d_2)$.

Definition 2 *Let $\nu_1,\nu_2,\beta,\mu>0$ and let k_1,k_2 be positive integers. We write $\boldsymbol{k}=(k_1,k_2)$ and $\boldsymbol{\nu}=(\nu_1,\nu_2)$. For every $\epsilon\in\mathcal{E}$, $F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$ stands for the vector space of continuous functions $(\boldsymbol{\tau},m)\mapsto h(\boldsymbol{\tau},m)$ defined on $\overline{\Omega_1(\epsilon)}\times\overline{\Omega_2(\epsilon)}\times\mathbb{R}$ which are holomorphic with respect to the first two variables on $\Omega_1(\epsilon)\times\Omega_2(\epsilon)$, and satisfy that*

$$\|h(\boldsymbol{\tau},m)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} = \sup_{\substack{\boldsymbol{\tau}\in\Omega_1(\epsilon)\times\Omega_2(\epsilon)\\ m\in\mathbb{R}}}(1+|m|)^{\mu}\frac{1+\left|\frac{\tau_1}{\epsilon}\right|^{2k_1}}{\left|\frac{\tau_1}{\epsilon}\right|}\frac{1+\left|\frac{\tau_2}{\epsilon}\right|^{2k_2}}{\left|\frac{\tau_2}{\epsilon}\right|}\exp\left(\beta|m|-\nu_1\left|\frac{\tau_1}{\epsilon}\right|^{k_1}-\nu_2\left|\frac{\tau_2}{\epsilon}\right|^{k_2}\right)|h(\boldsymbol{\tau},m)| \tag{28}$$

is finite. The pair $(F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)},\|\cdot\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)})$ is a complex Banach space.

In the rest of this section, we fix the values of $\nu_1,\nu_2,\beta,\mu>0$ and the positive integers k_1,k_2. We write $\boldsymbol{\nu}=(\nu_1,\nu_2)$, and $\boldsymbol{k}=(k_1,k_2)$.

The first result follows directly from the definition of the norm of the Banach space in Definition 2.

Lemma 4 *Let $\epsilon\in\mathcal{E}$, and let $(\boldsymbol{\tau},m)\mapsto a(\boldsymbol{\tau},m)$ be a bounded continuous function on $\overline{\Omega_1(\epsilon)}\times\overline{\Omega_2(\epsilon)}$. Then it holds that*

$$\|a(\boldsymbol{\tau},m)h(\boldsymbol{\tau},m)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}\le M_a\,\|h(\boldsymbol{\tau},m)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)},$$

for every $h(\boldsymbol{\tau},m)\in F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$, where $M_a:=\sup_{\boldsymbol{\tau}\in(\overline{\Omega_1(\epsilon)}\times\overline{\Omega_2(\epsilon)})}|a(\boldsymbol{\tau},m)|$. Moreover, if $\boldsymbol{\tau}\mapsto a(\boldsymbol{\tau},m)$ is a holomorphic function, then $a(\boldsymbol{\tau},m)h(\boldsymbol{\tau},m)$ belongs to $F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$.

Some parts of the proof of the following result can be adapted from that of Proposition 2 in [4]. We decided to include it completely for the sake of completeness and a self-contained work.

Lemma 5 *Let $\epsilon\in\mathcal{E}$. Let $\boldsymbol{\sigma}=(\sigma_1,\sigma_2)\in(0,\infty)^2$, $\tilde{\sigma}_1<\sigma_1$, and let $a_{\boldsymbol{\sigma},\boldsymbol{k}}$ be a holomorphic function in $\Omega_1(\epsilon)\times\Omega_2(\epsilon)$, continuous up to $\overline{\Omega_1(\epsilon)}\times\overline{\Omega_2(\epsilon)}$, such that*

$$|a_{\boldsymbol{\sigma},\boldsymbol{k}}(\boldsymbol{\tau})|\le\frac{1}{1+|\tau_1|^{k_1\sigma_1}|\tau_2|^{k_2\sigma_2}},$$

for $\boldsymbol{\tau}\in(\overline{\Omega_1(\epsilon)}\times\overline{\Omega_2(\epsilon)})$. Assume that $\sigma_3,\sigma_4>0$ with

$$\sigma_3=\frac{\chi}{k_2}-1,\quad\textit{and}\quad\frac{\sigma_2\tilde{\sigma}_1}{\sigma_1}-1\ge\sigma_4+\frac{1}{k_2}, \tag{29}$$

for some positive integer χ. Then, there exists $C_1>0$, depending on $\boldsymbol{k},\nu_2,\tilde{\sigma}_2,\boldsymbol{\sigma}$, such that

$$\left\|a_{\boldsymbol{\sigma},\boldsymbol{k}}(\boldsymbol{\tau})\tau_1^{k_1\tilde{\sigma}_1}\tau_2^{k_2}\int_0^{\tau_2^{k_2}}(\tau_2^{k_2}-s_2)^{\sigma_3}s_2^{\sigma_4}f(\tau_1,s_2^{1/k_2},m)ds_2\right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \le C_1|\epsilon|^{k_2(\sigma_3+\sigma_4+2)}\max\{|\epsilon|^{-\frac{k_2\sigma_2\tilde{\sigma}_1}{\sigma_1}},|\epsilon|^{-r_{11}k_1\tilde{\sigma}_1}\}\,\|f(\boldsymbol{\tau},m)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}, \tag{30}$$

for every $f\in F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$.

Proof Let $f \in F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$. It holds that

$$\left\| a_{\boldsymbol{\sigma},\boldsymbol{k}}(\boldsymbol{\tau})\tau_1^{k_1\tilde{\sigma}_1}\tau_2^{k_2}\int_0^{\tau_2^{k_2}}(\tau_2^{k_2}-s_2)^{\sigma_3}s_2^{\sigma_4}f(\tau_1,s_2^{1/k_2},m)ds_2\right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$$
$$=\sup_{\boldsymbol{\tau}\in\Omega_1(\epsilon)\times\Omega_2(\epsilon) m\in\mathbb{R}}(1+|m|)^{\mu}\frac{1+\left|\frac{\tau_2}{\epsilon}\right|^{2k_2}}{\left|\frac{\tau_2}{\epsilon}\right|}\exp\left(\beta|m|-\nu_2\left|\frac{\tau_2}{\epsilon}\right|^{k_2}\right)\left|a_{\boldsymbol{\sigma},\boldsymbol{k}}(\boldsymbol{\tau})\tau_1^{k_1\tilde{\sigma}_1}\tau_2^{k_2}\right.$$
$$\int_0^{\tau_2^{k_2}}\{(1+|m|)^{\mu}\frac{1+\left|\frac{\tau_1}{\epsilon}\right|^{2k_1}}{\left|\frac{\tau_1}{\epsilon}\right|}\frac{1+\frac{|s_2|^2}{|\epsilon|^{2k_2}}}{\frac{|s_2|^{1/k_2}}{|\epsilon|}}\exp(\beta|m|-\nu_1\left|\frac{\tau_1}{\epsilon}\right|^{k_1}-\nu_2\frac{|s_2|}{|\epsilon|^{k_2}})f(\tau_1,s_2^{1/k_2},m)\}$$
$$\left.\times\{\exp(-\nu_2\frac{|\tau_2^{k_2}-s_2|}{|\epsilon|^{k_2}})\frac{1+\frac{|\tau_2^{k_2}-s_2|^2}{|\epsilon|^{2k_2}}}{\frac{|\tau_2^{k_2}-s_2|^{1/k_2}}{|\epsilon|}}(\tau_2^{k_2}-s_2)^{\chi/k_2}\mathcal{B}(\tau_2,s_2,m,\epsilon)\}\right|ds_2,$$

with

$$\mathcal{B}(\tau_2,s_2,m,\epsilon)=e^{-\beta|m|}\frac{1}{(1+|m|)^{\mu}}\exp\left(\nu_2\frac{|\tau_2^{k_2}-s_2|}{|\epsilon|^{k_2}}\right)\frac{|s_2|^{1/k_2}}{|\epsilon|}\frac{|\tau_2^{k_2}-s_2|^{1/k_2}}{|\epsilon|}$$
$$\times\left((1+\frac{|s_2|^2}{|\epsilon|^{2k_2}})(1+\frac{|\tau_2^{k_2}-s_2|^2}{|\epsilon|^{2k_2}})|\tau_2^{k_2}-s_2|\right)^{-1}s_2^{\sigma_4}.$$

Therefore, one has

$$\left\| a_{\boldsymbol{\sigma},\boldsymbol{k}}(\boldsymbol{\tau})\tau_1^{k_1\tilde{\sigma}_1}\tau_2^{k_2}\int_0^{\tau_2^{k_2}}(\tau_2^{k_2}-s_2)^{\sigma_3}s_2^{\sigma_4}f(\tau_1,s_2^{1/k_2},m)ds_2\right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}\le C_{2.2}(\epsilon)C_{2.3}(\epsilon)\left\|f(\boldsymbol{\tau},m)\right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)},$$

with

$$C_{2.2}(\epsilon)=\sup_{x\ge 0}\left(\exp\left(-\nu_2\frac{x}{|\epsilon|^{k_2}}\right)\frac{1+\frac{x^2}{|\epsilon|^{2k_2}}}{\frac{x^{1/k_2}}{|\epsilon|}}x^{\chi/k_2}\right),$$

and

$$C_{2.3}(\epsilon)=\sup_{\boldsymbol{\tau}\in\Omega_1(\epsilon)\times\Omega_2(\epsilon)}\frac{1+\left|\frac{\tau_2}{\epsilon}\right|^{2k_2}}{\left|\frac{\tau_2}{\epsilon}\right|}\frac{|\tau_1|^{k_1\tilde{\sigma}_1}|\tau_2|^{k_2}}{1+|\tau_1|^{k_1\sigma_1}|\tau_2|^{k_2\sigma_2}}$$
$$\times\int_0^{|\tau_2|^{k_2}}\frac{h_2^{1/k_2}}{|\epsilon|}\frac{(|\tau_2|^{k_2}-h_2)^{1/k_2}}{|\epsilon|}\left((1+\frac{h_2^2}{|\epsilon|^{2k_2}})(1+\frac{(|\tau_2|^{k_2}-h_2)^2}{|\epsilon|^{2k_2}})(|\tau_2|^{k_2}-h_2)\right)^{-1}h_2^{\sigma_4}dh_2. \tag{31}$$

The classical estimates

$$\sup_{x\ge 0}x^{m_1}\exp(-m_2x)=\left(\frac{m_1}{m_2}\right)^{m_1}\exp(-m_1),$$

for $m_1 \ge 0$ and $m_2 > 0$, yield

$$C_{2.2}(\epsilon)\le|\epsilon|^{\chi}\left(\left(\frac{\chi-1}{k_2\nu_2}\right)^{\frac{\chi-1}{k_2}}\exp(-\frac{\chi-1}{k_2})+\left(\frac{2+\frac{\chi-1}{k_2}}{\nu_2}\right)^{2+\frac{\chi-1}{k_2}}\exp(-2-\frac{\chi-1}{k_2})\right).$$

At this point, we provide upper bounds for $C_{2.3}(\epsilon)$. Let $g(y) = \frac{ay}{1+cy^b}$, for some $a,c>0$ and $b>1$. The function g attains its maximum at $y_0 = (c(b-1))^{-1/b}$. We apply this result to the case $a = |\tau_2|^{k_2}$, $b = \sigma_1/\tilde{\sigma}_1$ and $c = |\tau_2|^{k_2\sigma_2}$ to arrive at

$$\frac{|\tau_1|^{k_1\tilde{\sigma}_1}|\tau_2|^{k_2}}{1+|\tau_1|^{k_1\sigma_1}|\tau_2|^{k_2\sigma_2}} \leq \frac{C(\sigma_1,\tilde{\sigma}_1)}{|\tau_2|^{k_2(\frac{\sigma_2\tilde{\sigma}_1}{\sigma_1}-1)}}, \tag{32}$$

for some $C(\sigma_1,\tilde{\sigma}_1)>0$. We plug the bound (32) into (31) and make the change of variable $h = |\epsilon|^{k_2}h'$ at the integral in $C_{2.3}(\epsilon)$ to arrive at

$$\begin{aligned}(33)\quad C_{23}(\epsilon) \leq C_{24} \sup_{\tau_2\in\Omega_2(\epsilon)} \frac{1+\left|\frac{\tau_2}{\epsilon}\right|^{2k_2}}{\left|\frac{\tau_2}{\epsilon}\right|} \frac{1}{|\epsilon|^{k_2(\frac{\sigma_2\tilde{\sigma}_1}{\sigma_1}-1)}\left|\frac{\tau_2}{\epsilon}\right|^{k_2(\frac{\sigma_2\tilde{\sigma}_1}{\sigma_1}-1)}}\\ \times \int_0^{\left|\frac{\tau_2}{\epsilon}\right|^{k_2}} (h')^{1/k_2}\left(\left|\frac{\tau_2}{\epsilon}\right|^{k_2}-h'\right)^{1/k_2} \frac{1}{1+(h')^2}\frac{1}{1+(\left|\frac{\tau_2}{\epsilon}\right|^{k_2}-h')^2}\left(\left|\frac{\tau_2}{\epsilon}\right|^{k_2}-h'\right)^{-1}|\epsilon|^{k_2\sigma_4}(h')^{\sigma_4}dh'\end{aligned}$$

for some $C_{24}>0$ only depending on $\tilde{\sigma}_1,\sigma_1$.

Let $x_0>0$. The previous sup. particularized for those $\tau_2\in\Omega_2(\epsilon)$ such that $x := |\tau_2/\epsilon|^{k_2} > x_0$ reads as

$$\begin{aligned}(34)\quad & C_{24}|\epsilon|^{k_2(1-\frac{\sigma_2\tilde{\sigma}_1}{\sigma_1}+\sigma_4)} \sup_{x>x_0} \frac{1+x^2}{x^{1/k_2}}\frac{1}{x^{\frac{\sigma_2\tilde{\sigma}_1}{\sigma_1}-1}}\\ &\times \int_0^x (h')^{1/k_2}(x-h')^{1/k_2}\frac{1}{1+(h')^2}\frac{1}{1+(x-h')^2}(x-h')^{-1}(h')^{\sigma_4}dh'\\ \leq\; & C_{24}|\epsilon|^{k_2(1-\frac{\sigma_2\tilde{\sigma}_1}{\sigma_1}+\sigma_4)} \sup_{x>x_0}(1+x^2)\frac{1}{x^{\frac{\sigma_2\tilde{\sigma}_1}{\sigma_1}-1}}\int_0^x \frac{1}{(x-h')^{1-\frac{1}{k_2}}}\frac{1}{1+(h')^2}\frac{1}{1+(x-h')^2}(h')^{\sigma_4}dh',\end{aligned}$$

Let

$$\Delta(x) = (1+x^2)\frac{1}{x^{\frac{\sigma_2\tilde{\sigma}_1}{\sigma_1}-1}}\int_0^x \frac{1}{(x-h')^{1-\frac{1}{k_2}}}\frac{1}{1+(h')^2}\frac{1}{1+(x-h')^2}(h')^{\sigma_4}dh'.$$

For $x>x_0$, one can perform the change of variable $h'=xu$ in the integral of $\Delta(x)$, in order to get that

$$\Delta(x) = (1+x^2)x^{\sigma_4+\frac{1}{k_2}+1-\frac{\sigma_2\tilde{\sigma}_1}{\sigma_1}}F_{k_2}(x),$$

where

$$F_{k_2}(x) = \int_0^1 \frac{u^{\sigma_4}}{(1+x^2u^2)(1+x^2(1-u)^2)(1-u)^{1-\frac{1}{k_2}}}du.$$

A partial fraction decomposition allows to write $F_{k_2}(x) = F_{1,k_2}(x)+F_{2,k_2}(x)$, where

$$F_{1,k_2}(x) = \frac{1}{4+x^2}\int_0^1 \frac{(2u+1)u^{\sigma_4}}{(1+x^2u^2)(1-u)^{1-\frac{1}{k_2}}}du,$$

$$F_{2,k_2}(x) = \frac{1}{4+x^2}\int_0^1 \frac{(3-2u)u^{\sigma_4}}{(1+x^2(1-u)^2)(1-u)^{1-\frac{1}{k_2}}}du.$$

We observe that
$$F_{1,k_2}(x) \leq \frac{\mathcal{F}_{1,k_2}}{4+x^2}, \quad F_{2,k_2}(x) \leq \frac{\mathcal{F}_{2,k_2}}{4+x^2},$$
for some positive constants $\mathcal{F}_{1,k_2}, \mathcal{F}_{2,k_2}$, valid for all $x \geq x_0$. Under the second assumption in (29), we obtain that $\sup_{x>x_0} \Delta(x)$ is upper bounded by a constant. We conclude that the expression in (34) is upper bounded by

$$C_{25}|\epsilon|^{k_2(1-\frac{\sigma_2\tilde{\sigma}_1}{\sigma_1}+\sigma_4)}, \tag{35}$$

for some $C_{25} > 0$.

It only rests to provide upper bounds for $C_{2.3}(\epsilon)$ regarding the set of $\tau_2 \in \Omega_2(\epsilon)$ such that $0 \leq x \leq x_0$. We observe that

$$\frac{|\tau_1|^{k_1\tilde{\sigma}_1}|\tau_2|^{k_2}}{1+|\tau_1|^{k_1\sigma_1}|\tau_2|^{k_2\sigma_2}} \leq |\tau_1|^{k_1\tilde{\sigma}_1}|\tau_2|^{k_2}. \tag{36}$$

We plug this last expression into (31) to arrive at

$$\sup_{\substack{(\tau_1,\tau_2)\in\Omega_1(\epsilon)\times\Omega_2(\epsilon)\\ |\tau_2/\epsilon|^{k_2}\leq x_0}} \frac{1+\left|\frac{\tau_2}{\epsilon}\right|^{2k_2}}{\left|\frac{\tau_2}{\epsilon}\right|}|\tau_1|^{k_1\tilde{\sigma}_1}|\tau_2|^{k_2} \tag{37}$$
$$\times \int_0^{\left|\frac{\tau_2}{\epsilon}\right|^{k_2}} (h')^{1/k_2}(\left|\frac{\tau_2}{\epsilon}\right|^{k_2} - h')^{1/k_2}\frac{1}{1+(h')^2}\frac{1}{1+(\left|\frac{\tau_2}{\epsilon}\right|^{k_2}-h')^2}\left(\left|\frac{\tau_2}{\epsilon}\right|^{k_2}-h'\right)^{-1}|\epsilon|^{k_2\sigma_4}(h')^{\sigma_4}dh'$$
$$\leq C_{26}|\epsilon|^{-r_{11}k_1\tilde{\sigma}_1}\sup_{\tau_2\in\Omega_2(\epsilon),|\tau_2/\epsilon|^{k_2}\leq x_0}\frac{1+\left|\frac{\tau_2}{\epsilon}\right|^{2k_2}}{\left|\frac{\tau_2}{\epsilon}\right|}|\epsilon|^{k_2}\left|\frac{\tau_2}{\epsilon}\right|^{k_2}$$
$$\times \int_0^{\left|\frac{\tau_2}{\epsilon}\right|^{k_2}} (h')^{1/k_2}(\left|\frac{\tau_2}{\epsilon}\right|^{k_2} - h')^{1/k_2}\frac{1}{1+(h')^2}\frac{1}{1+(\left|\frac{\tau_2}{\epsilon}\right|^{k_2}-h')^2}\left(\left|\frac{\tau_2}{\epsilon}\right|^{k_2}-h'\right)^{-1}|\epsilon|^{k_2\sigma_4}(h')^{\sigma_4}dh'$$
$$\leq C_{27}|\epsilon|^{-r_{11}k_1\tilde{\sigma}_1+k_2(1+\sigma_4)}\sup_{0\leq x\leq x_0}\frac{1+x^2}{x^{1/k_2}}x\int_0^x (h')^{1/k_2}\frac{1}{1+(h')^2}\frac{1}{1+(x-h')^2}\frac{1}{(x-h')^{1-\frac{1}{k_2}}}(h')^{\sigma_4}dh'$$
$$\leq C_{28}|\epsilon|^{-r_{11}k_1\tilde{\sigma}_1+k_2(1+\sigma_4)}\sup_{0\leq x\leq x_0}(1+x^2)x\int_0^x \frac{1}{1+(h')^2}\frac{1}{1+(x-h')^2}\frac{1}{(x-h')^{1-\frac{1}{k_2}}}(h')^{\sigma_4}dh'$$
$$\leq C_{29}|\epsilon|^{-r_{11}k_1\tilde{\sigma}_1+k_2(1+\sigma_4)},$$

for some $C_{26}, C_{27}, C_{28}, C_{29} > 0$. We conclude that the left-hand side in (37) is upper bounded by

$$C_{29}|\epsilon|^{-r_{11}k_1\tilde{\sigma}_1+k_2(1+\sigma_4)}. \tag{38}$$

In view of (35) and (38), we derive that

$$C_{23}(\epsilon) \leq \sup\left\{C_{25}|\epsilon|^{k_2(1-\frac{\sigma_2\tilde{\sigma}_1}{\sigma_1}+\sigma_4)}, C_{29}|\epsilon|^{-r_{11}k_1\tilde{\sigma}_1+k_2(1+\sigma_4)}\right\},$$

which concludes the result. □

The proof of the following result can be reproduced under minor adjustments from that of Proposition 2, [7].

Lemma 6 *Let $P_1, P_2 \in \mathbb{C}[X]$ such that*

$$deg(P_1) \geq deg(P_2), \qquad P_1(im) \neq 0 \quad \text{for all } m \in \mathbb{R},$$

and let $\mu > deg(P_2)+1$. For every $f \in E_{(\beta,\mu)}$ and $g \in F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$, then one has that the function

$$\Phi(\boldsymbol{\tau}, m) := \frac{1}{P_1(im)} \int_{-\infty}^{\infty} f(m-m_1)P_2(im_1)g(\boldsymbol{\tau}, m_1)dm_1,$$

belongs to $F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$ and it holds that

$$\|\Phi(\boldsymbol{\tau}, m)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \leq C_2 \|f(m)\|_{(\beta,\mu)} \|g(\boldsymbol{\tau}, m)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)},$$

for some constant $C_2 > 0$.

5 Solution of an auxiliary problem

In this section, we preserve the elements and assumptions made on the main problem under study described in Section 2.2. More precisely, we assume the conditions (5), (6), (7) are satisfied by the parameters and elements involved. The coefficients $c_{\ell_1\ell_2}(z,\epsilon)$ and the forcing term $f(\boldsymbol{t}, z, \epsilon)$ are constructed accordingly. We also assume the geometry of the problem is set in accordance with the assumptions made in Section 3 (see condition (18)), and preserve the values of $r_1(\epsilon), r_2(\epsilon)$ for each $\epsilon \in \mathcal{E}$ and $\lambda > 0$ (see (19) and (21)).

We provide a solution of the auxiliary problem (17) by means of a fixed point method in the Banach spaces introduced in Section 4.

Let $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$ and let d_1, d_2 be chosen as described in Section 3. We define the operator

$$(39) \quad \mathcal{H}_\epsilon(\omega(\boldsymbol{\tau}, m)) = \sum_{\substack{1\leq \ell_1 \leq D_1-1 \\ 1 \leq \ell_2 \leq D_2-1}} \frac{\epsilon^{\Delta_{\ell_1\ell_2} - k_1\delta_{\ell_1} - d_{\ell_2} + \tilde{\delta}_{\ell_2}} (k_1 \tau_1^{k_1})^{\delta_{\ell_1}}}{P_m(\boldsymbol{\tau})} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} C_{\ell_1\ell_2}(m-m_1,\epsilon) R_{\ell_1\ell_2}(im_1)$$
$$\times \left[\frac{\tau_2^{k_2}}{\Gamma\left(\frac{d_{\ell_2}k_2}{k_2}\right)} \int_0^{\tau_2^{k_2}} (\tau_2^{k_2} - s_2)^{\frac{d_{\ell_2}k_2}{k_2} - 1} (k_2 s_2)^{\tilde{\delta}_{\ell_2}} \omega(\tau_1, s_2^{1/k_2}, m_1, \epsilon) \frac{ds_2}{s_2} dm_1 \right.$$
$$+ \sum_{1 \leq p_2 \leq \tilde{\delta}_{\ell_2} - 1} \frac{A_{\delta_{\ell_2}p_2} \tau_2^{k_2}}{\Gamma\left(\frac{d_{\ell_2}k_2 + k_2(\tilde{\delta}_{\ell_2} - p_2)}{k_2}\right)} \int_0^{\tau_2^{k_2}} (\tau_2^{k_2} - s_2)^{\frac{d_{\ell_2}k_2 + k_2(\tilde{\delta}_{\ell_2} - p_2)}{k_2} - 1} (k_2 s_2)^{p_2}$$
$$\left. \times \omega(\tau_1, s_2^{1/k_2}, m_1, \epsilon) \frac{ds_2}{s_2} dm_1 \right] + \frac{1}{P_m(\boldsymbol{\tau})} \psi(\boldsymbol{\tau}, m, \epsilon)$$

We consider the Banach space of Definition 2, when fixing the domains described in (22), in accordance with the geometric analysis of the problem, in Section 3.

Proposition 2 *Under the assumptions adopted in this section, for every $\varpi > 0$ there exist $\xi_\psi, \epsilon_0 > 0$ such that if*

$$\|\psi(\boldsymbol{\tau}, m, \epsilon)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \leq \xi_\psi,$$

then the operator $\mathcal{H}_\epsilon$ admits a unique fixed point $\omega^{\boldsymbol{d}}_{\boldsymbol{k}}(\boldsymbol{\tau}, m, \epsilon) \in F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$ which satisfies that $\|\omega^{\boldsymbol{d}}_{\boldsymbol{k}}(\boldsymbol{\tau}, m, \epsilon)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \leq \varpi$.

Proof Let $\omega \in F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$. For every $0 \le \ell_1 \le D_1 - 1$ and $0 \le \ell_2 \le D_2 - 1$ we define

$$L_{1\ell_1\ell_2} := \frac{\tau_1^{k_1\delta_{\ell_1}}\tau_2^{k_2}}{P_m(\boldsymbol{\tau})}\int_{-\infty}^{\infty} C_{\ell_1\ell_2}(m-m_1,\epsilon)R_{\ell_1\ell_2}(im_1) \times \int_0^{\tau_2^{k_2}} (\tau_2^{k_2}-s_2)^{\frac{d_{\ell_2 k_2}}{k_2}-1} s_2^{\tilde{\delta}_{\ell_2}-1}\omega(\tau_1, s_2^{1/k_2}, m_1,\epsilon)ds_2dm_1. \tag{40}$$

Taking into account the assumptions in (6) and (19), one can apply Lemmas 4–6 to arrive at

$$\|L_{1\ell_1\ell_2}\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \le C_1C_2\|C_{\ell_1\ell_2}\|_{(\beta,\mu)}|\epsilon|^{d_{\ell_2 k_2}+k_2\tilde{\delta}_{\ell_2}-\frac{k_2\tilde{\delta}_{D_2}\delta_{\ell_1}}{\delta_{D_1}}}\|\omega(\boldsymbol{\tau},m,\epsilon)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}. \tag{41}$$

We put

$$L_{2\ell_1\ell_2} := \frac{\tau_1^{k_1\delta_{\ell_1}}\tau_2^{k_2}}{P_m(\boldsymbol{\tau})}\int_{-\infty}^{\infty} C_{\ell_1\ell_2}(m-m_1,\epsilon)R_{\ell_1\ell_2}(im_1)\int_0^{\tau_2^{k_2}} (\tau_2^{k_2}-s_2)^{\frac{d_{\ell_2 k_2}+k_2(\tilde{\delta}_{\ell_2}-p_2)}{k_2}-1} s_2^{p_2-1} \omega(\tau_1, s_2^{1/k_2}, m_1,\epsilon)ds_2dm_1. \tag{42}$$

An analogous argument as before leads to

$$\|L_{2\ell_1\ell_2}\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \le C_1C_2\|C_{\ell_1\ell_2}\|_{(\beta,\mu)}|\epsilon|^{d_{\ell_2 k_2}+k_2\tilde{\delta}_{\ell_2}-\frac{k_2\tilde{\delta}_{D_2}\delta_{\ell_1}}{\delta_{D_1}}}\|\omega(\boldsymbol{\tau},m,\epsilon)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \tag{43}$$

Finally, the definition of the forcing term and Lemma 4 lead to

$$\begin{aligned}\left\|\frac{1}{P_m(\boldsymbol{\tau})}\psi(\boldsymbol{\tau},m,\epsilon)\right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} &\le \frac{1}{C_P}\sup_{m\in\mathbb{R}}\frac{1}{|R_{D_1D_2}(im)|}\|\psi(\boldsymbol{\tau},m,\epsilon)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}\\ &\le \frac{1}{C_P}\sup_{m\in\mathbb{R}}\frac{1}{|R_{D_1D_2}(im)|}\xi_\psi.\end{aligned} \tag{44}$$

In view of (41), (43) and (44) we get that

$$\|\mathcal{H}_\epsilon(\omega(\boldsymbol{\tau},m))\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \le \sum_{\substack{1\le\ell_1\le D_1-1\\1\le\ell_2\le D_2-1}}|\epsilon|^{\Delta}\frac{k_1^{\delta_{\ell_1}}C_1C_2\|C_{\ell_1\ell_2}\|_{(\beta,\mu)}}{(2\pi)^{1/2}}\left[\frac{k_2^{\tilde{\delta}_{\ell_2}}}{\Gamma\left(\frac{d_{\ell_2k_2}}{k_2}\right)} + \sum_{1\le p_2\le\tilde{\delta}_{\ell_2}-1}\frac{|A_{\delta_{\ell_2}p_2}|k_2^{p_2}}{\Gamma\left(\frac{d_{\ell_2k_2}+k_2(\tilde{\delta}_{\ell_2}-p_2)}{k_2}\right)}\right]\|\omega(\boldsymbol{\tau},m,\epsilon)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} + \frac{\xi_\psi}{C_P}\sup_{m\in\mathbb{R}}\frac{1}{|R_{D_1D_2}(im)|}, \tag{45}$$

where $\Delta = \Delta_{\ell_1\ell_2} - k_1\delta_{\ell_1} - \frac{k_2\tilde{\delta}_{D_2}\delta_{\ell_1}}{\delta_{D_1}} > 0$, in view of (6). Let $\varpi > 0$, and assume that $\omega(\boldsymbol{\tau},m) \in F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$ with $\|\omega(\boldsymbol{\tau},m,\epsilon)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} < \varpi$. Any choice of small enough $\xi_\psi, \epsilon_0 > 0$ which satisfies

$$\sum_{\substack{1\le\ell_1\le D_1-1\\1\le\ell_2\le D_2-1}}\epsilon_0^{\Delta}\frac{k_1^{\delta_{\ell_1}}C_1C_2\|C_{\ell_1\ell_2}\|_{(\beta,\mu)}}{(2\pi)^{1/2}}\left[\frac{k_2^{\tilde{\delta}_{\ell_2}}}{\Gamma\left(\frac{d_{\ell_2k_2}}{k_2}\right)} + \sum_{1\le p_2\le\tilde{\delta}_{\ell_2}-1}\frac{|A_{\delta_{\ell_2}p_2}|k_2^{p_2}}{\Gamma\left(\frac{d_{\ell_2k_2}+k_2(\tilde{\delta}_{\ell_2}-p_2)}{k_2}\right)}\right]\varpi + \frac{\xi_\psi}{C_P}\sup_{m\in\mathbb{R}}\frac{1}{|R_{D_1D_2}(im)|} \le \varpi \tag{46}$$

leads to

$$\left\| \mathcal{H}_{\epsilon}(\omega(\boldsymbol{\tau},m)) \right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \leq \varpi. \tag{47}$$

Let $\varpi > 0$ and $\omega_1, \omega_2 \in F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$ with $\|\omega_j(\boldsymbol{\tau},m,\epsilon)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} < \varpi$ for $j = 1,2$. Then, analogous arguments as above entail

$$\begin{aligned}\left\| \mathcal{H}_{\epsilon}(\omega_1(\boldsymbol{\tau},m)) - \mathcal{H}_{\epsilon}(\omega_2(\boldsymbol{\tau},m)) \right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \leq \sum_{\substack{1\le \ell_1 \le D_1-1\\ 1\le \ell_2\le D_2-1}} |\epsilon|^{\Delta} \frac{k_1^{\delta_{\ell_1}} C_1 C_2 \left\| C_{\ell_1\ell_2}\right\|_{(\beta,\mu)}}{(2\pi)^{1/2}} \left[\frac{k_2^{\tilde{\delta}_{\ell_2}}}{\Gamma\left(\frac{d_{\ell_2 k_2}}{k_2}\right)} \right. \\ \left. + \sum_{1\le p_2\le \tilde{\delta}_{\ell_2}-1} \frac{|A_{\delta_{\ell_2 p_2}}| k_2^{p_2}}{\Gamma\left(\frac{d_{\ell_2k_2}+k_2(\tilde{\delta}_{\ell_2}-p_2)}{k_2}\right)} \right] \left\| \omega_1(\boldsymbol{\tau},m,\epsilon) - \omega_2(\boldsymbol{\tau},m,\epsilon)\right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}. \end{aligned} \tag{48}$$

Let $\epsilon_0 > 0$ such that

$$\sum_{\substack{1\le \ell_1 \le D_1-1\\ 1\le \ell_2\le D_2-1}} \epsilon_0^{\Delta} \frac{k_1^{\delta_{\ell_1}} C_1 C_2 \left\| C_{\ell_1\ell_2}\right\|_{(\beta,\mu)}}{(2\pi)^{1/2}} \left[\frac{k_2^{\tilde{\delta}_{\ell_2}}}{\Gamma\left(\frac{d_{\ell_2 k_2}}{k_2}\right)} + \sum_{1\le p_2\le \tilde{\delta}_{\ell_2}-1} \frac{|A_{\delta_{\ell_2 p_2}}| k_2^{p_2}}{\Gamma\left(\frac{d_{\ell_2k_2}+k_2(\tilde{\delta}_{\ell_2}-p_2)}{k_2}\right)} \right] < \frac{1}{2}.$$

Then, it holds that

$$\left\| \mathcal{H}_{\epsilon}(\omega_1(\boldsymbol{\tau},m)) - \mathcal{H}_{\epsilon}(\omega_2(\boldsymbol{\tau},m)) \right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \leq \frac{1}{2} \left\| \omega_1(\boldsymbol{\tau},m,\epsilon) - \omega_2(\boldsymbol{\tau},m,\epsilon)\right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}. \tag{49}$$

In view of (47) and (49), we get that the operator $\mathcal{H}_{\epsilon}$, restricted to $\overline{B}(0,\varpi) \subseteq F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$ turns out to be a contractive map in the complete metric space $\overline{B}(0,\varpi) \subseteq F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$ for the distante $d(x,y) = \|\cdot\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$. The classical contractive mapping theorem guarantees the existence of a unique fixed point, say $\omega^{\boldsymbol{d}}_{\boldsymbol{k}}(\boldsymbol{\tau},m,\epsilon) \in F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$ with $\left\|\omega^{\boldsymbol{d}}_{\boldsymbol{k}}(\boldsymbol{\tau},m,\epsilon)\right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} < \varpi$.

□

As a result, and regarding the proof of the previous result, one attains the following statement.

Corollary 1 *Under the assumptions made in Proposition 2, the function $\omega^{\boldsymbol{d}}_{\boldsymbol{k}}(\boldsymbol{\tau},m,\epsilon)$ is a solution of the auxiliary equation (17). Moreover, for every $\epsilon \in \mathcal{E}$, it satisfies that*

$$|\omega^{\boldsymbol{d}}_{\boldsymbol{k}}(\boldsymbol{\tau},m,\epsilon)| \leq C_{\omega} \frac{1}{(1+|m|)^{\mu}} \frac{\left|\frac{\tau_1}{\epsilon}\right|}{1+\left|\frac{\tau_1}{\epsilon}\right|^{2k_1}} \frac{\left|\frac{\tau_2}{\epsilon}\right|}{1+\left|\frac{\tau_2}{\epsilon}\right|^{2k_2}} \exp\left(-\beta|m| + \nu_1 \left|\frac{\tau_1}{\epsilon}\right|^{k_1} + \nu_2 \left|\frac{\tau_2}{\epsilon}\right|^{k_2}\right), \tag{50}$$

for every $\boldsymbol{\tau} \in \Omega_1(\epsilon)\times\Omega_2(\epsilon)$ and $m \in \mathbb{R}$. The constant $C_{\omega} > 0$ can be chosen uniformly for all $\epsilon \in \mathcal{E}$.

6 Analytic solutions of the main problem

The main aim in this section is to provide analytic solutions of (8) for each of the elements of a family of sectors with respect to the perturbation parameter in the form of a truncated Laplace, Laplace and Fourier transforms. We first fix the geometric elements in this construction.

Definition 3 *Let ι be an integer number, $\iota \geq 2$. Let $\underline{\mathcal{E}} := (\mathcal{E}_p)_{0\leq p\leq \iota-1}$, where $\mathcal{E}_p$ stands for a finite open sector with vertex at the origin, radius smaller than ϵ_0. We assume the intersection of three different elements in $\underline{\mathcal{E}}$ is empty, and $\bigcup_{0\leq p\leq \iota-1}\mathcal{E}_p = \mathcal{U}\setminus\{0\}$, for some neighborhood of the origin $\mathcal{U}\subseteq\mathbb{C}$. For the sake of simplicity, we arrange the sectors in order that nonempty intersections of sectors in $\underline{\mathcal{E}}$ correspond to consecutive indices in the ring of integers modulo ι. Under this configuration, we say that $\underline{\mathcal{E}}$ describes a good covering in $\mathbb{C}^\star$.*

Definition 4 *Let ι be an integer number, $\iota \geq 2$, and let $\underline{\mathcal{E}} := (\mathcal{E}_p)_{0\leq p\leq \iota-1}$ be a good covering in $\mathbb{C}^\star$. Let $\mathcal{T}_j$ be an open sector with vertex at the origin in $\mathbb{C}$ and finite radius $r_{\mathcal{T}_j} > 0$, for $j = 1, 2$. For all $0 \leq p \leq \iota-1$ we consider two bounded sectors $S_{\mathfrak{d}_{j,p}}$ of bisecting direction $\mathfrak{d}_{j,p}$, and small opening.*

In the following statements, we identify the indices $p = \iota$ and $p = 0$.

We say that the set

$$\{\mathcal{T}_1, \mathcal{T}_2, \underline{\mathcal{E}}, (S_{\mathfrak{d}_{1,p}})_{0\leq p\leq\iota-1}, (S_{\mathfrak{d}_{2,p}})_{0\leq p\leq\iota-1}\} \tag{51}$$

is admissible if there exists $\delta > 0$ such that for $j = 1, 2$ one has

$$k_j(\xi_j - arg(\epsilon t_j)) \in \left(-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta\right), \tag{52}$$

for every $0 \leq p \leq \iota-1$, $\epsilon\in\mathcal{E}_p$, $t_j \in \mathcal{T}_j$ and $\xi_j \in \mathbb{R}$ (which may depend on t_j and ϵ) such that $e^{\xi_j\sqrt{-1}} \in S_{\mathfrak{d}_{j,p}}$. The directions $\mathfrak{d}_{j,p}$ are given by $\mathfrak{d}_{2,p} := d_2 \in \mathbb{R}$ and $\mathfrak{d}_{1,p} := d_1$ according to the choice made on the directions d_2 and $d_1 = d_1(\epsilon)$ in Lemma 3.

Let $\iota \geq 2$ be an integer number. Let $\underline{\mathcal{E}} = (\mathcal{E}_p)_{0\leq p\leq\iota-1}$ be a good covering and consider an admissible set $\{\mathcal{T}_1, \mathcal{T}_2, \underline{\mathcal{E}}, (S_{\mathfrak{d}_{1,p}})_{0\leq p\leq\iota-1}, (S_{\mathfrak{d}_{2,p}})_{0\leq p\leq\iota-1}\}$, which is associated to the good covering $\underline{\mathcal{E}}$. We briefly discuss the feasibility of such a construction. Indeed, let $0 \leq p \leq \iota-1$ be fixed. We can first choose the direction $\nu_{1,p}$ (related to a fixed direction θ_1 depending on p) such that (52) holds for $j = 1$. Then, select the direction $\nu_{2,p} = d_2$ in order that (52) holds for $j = 2$ together with the condition stated in the second item of Lemma 3.

Let $0 \leq p \leq \iota-1$. For each $0 \leq p \leq \iota-1$, we consider the main problem under study (8) under the assumptions (5)-(7), and departing from the coefficients $c_{\ell_1\ell_2}(z,\epsilon)$ and the forcing term $f(\boldsymbol{t}, z, \epsilon)$ defined in Section 2.2. In virtue of the geometry of the problem described in Section 3 and Corollary 1, in particular the assumption of condition (18) and the choice of λ and $r_j(\epsilon)$ for $j = 1, 2$ in (19) and (21) resp., one has that for every $\epsilon\in\mathcal{E}_p$ there exist a vector of directions $\boldsymbol{d}_p = (d_{p,1}(\epsilon), d_{p,2})$, a bounded sector with vertex at the origin $S_{d_{p,1},\epsilon}$ and bisecting direction $d_{p,1}$, with $\overline{S_{d_{p,1},\epsilon}} \subseteq D(0, r_1(\epsilon))$ and an infinite sector $S_{d_{p,2}}$ of bisecting direction $d_{p,2}$ such that the problem (17) admits a solution, say $\omega_{\boldsymbol{k}}^{\boldsymbol{d}_p}(\boldsymbol{\tau}, m, \epsilon)$.

Let us write $\Omega_{p,1}(\epsilon) := S_{d_{p,1},\epsilon}$ and $\Omega_2(\epsilon) := D(0, r_2(\epsilon)) \cup S_{d_{p,2}}$.

In view of Corollary 1, one has that for every $\epsilon\in\mathcal{E}_p$, the function $(\boldsymbol{\tau}, m) \mapsto \omega_{\boldsymbol{k}}^{\boldsymbol{d}_p}(\boldsymbol{\tau}, m, \epsilon)$ is continuous on $\overline{\Omega_{p,1}(\epsilon)} \times \overline{\Omega_{p,2}(\epsilon)} \times \mathbb{R}$, holomorphic with respect to the first two variables on $\Omega_{p,1}(\epsilon) \times \Omega_{p,2}(\epsilon)$ which satisfies that

$$|\omega_{\boldsymbol{k}}^{\boldsymbol{d}_p}(\boldsymbol{\tau}, m, \epsilon)| \leq C_{\omega_{\boldsymbol{k}}^{\boldsymbol{d}_p}} \frac{1}{(1+|m|)^\mu} \frac{\left|\frac{\tau_1}{\epsilon}\right|}{1+\left|\frac{\tau_1}{\epsilon}\right|^{2k_1}} \frac{\left|\frac{\tau_2}{\epsilon}\right|}{1+\left|\frac{\tau_2}{\epsilon}\right|^{2k_2}} \exp\left(-\beta|m| + \nu_1\left|\frac{\tau_1}{\epsilon}\right|^{k_1} + \nu_2\left|\frac{\tau_2}{\epsilon}\right|^{k_2}\right), \tag{53}$$

for every $\boldsymbol{\tau}\in\Omega_{p,1}(\epsilon)\times\Omega_{p,2}(\epsilon)$ and $m\in\mathbb{R}$. The constant $C_{\omega_{\boldsymbol{k}}^{\boldsymbol{d}_p}}$ can be uniformly chosen for all $\epsilon\in\mathcal{E}_p$.

The application of a Fourier, Laplace and truncated Laplace transforms to the function $\omega_{\boldsymbol{k}}^{\boldsymbol{d}_p}(\boldsymbol{\tau},m,\epsilon)$ leads to a solution of the main problem under study: for every $0 \le p \le \iota - 1$ and $\epsilon \in \mathcal{E}_p$, we define the function $u_p(\boldsymbol{t},z,\epsilon)$ by

$$(54)\qquad \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{L_{d_{p,1},\epsilon}} \int_{L_{d_{p,2}}} \omega_{\boldsymbol{k}}^{\boldsymbol{d}_p}(\boldsymbol{u},m,\epsilon) \exp\left(- \left(\frac{u_1}{\epsilon t_1}\right)^{k_1} - \left(\frac{u_2}{\epsilon t_2}\right)^{k_2} \right) e^{izm} \frac{du_2}{u_2}\frac{du_1}{u_1} dm.$$

The integration path $L_{d_{p,1},\epsilon}$ stands for the segment $[0, \kappa h_1(\epsilon) e^{\sqrt{-1}\theta_1}]$ (see Lemma 3 and (20)), and $L_{d_{p,2}}$ stands for a usual Laplace transform along the half line $[0,\infty) e^{\sqrt{-1}d_2}$.

We observe that the choice of the admissible set, compatible with the good covering, together with the bounds in (53) guarantee that $(\boldsymbol{t},z) \mapsto u_p(\boldsymbol{t},z,\epsilon)$ is holomorphic on the domain $(\mathcal{T}_1 \cap D(0,h')) \times (\mathcal{T}_2 \cap D(0,h')) \times H_{\beta'}$, for $0 < \beta' < \beta$ and some $h' > 0$. We recall that H_β stands for the horizontal strip

$$H_\beta = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \beta\}.$$

Indeed, the construction of $\omega_{\boldsymbol{k}}^{\boldsymbol{d}_p}(\boldsymbol{\tau},m,\epsilon)$ and the definition of $u_p(\boldsymbol{t},z,\epsilon)$ in (54) allow to affirm that the function

$$(55)\qquad (\boldsymbol{t},z,\epsilon) \mapsto u_p(\boldsymbol{t},z,\epsilon)$$

is holomorphic on the domain $(\mathcal{T}_1 \cap D(0,h')) \times (\mathcal{T}_2 \cap D(0,h')) \times H_{\beta'} \times \mathcal{E}_p$, for every $0 \le p \le \iota - 1$.

The properties of Fourier transform (see Section 2.1) and Laplace transform (see Lemma 2), together with the definition of the elements involved in the main equation guarantee that (55) represents a solution of the main problem (8).

From now on, we refer to consecutive solutions of (8) to solutions associated to consecutive sectors in the corresponding good covering, which have nonempty intersection.

The next property on the difference of two consecutive solutions will be crucial in order to provide the asymptotic behavior of the solution at 0 regarding the perturbation parameter.

Theorem 1 *Let $\underline{\mathcal{E}} = (\mathcal{E}_p)_{0\le p\le \iota-1}$ be a good covering and consider an admissible set (51) associated to $\underline{\mathcal{E}}$. For every $0 \le p \le \iota - 1$, the function $u_p(\boldsymbol{t},z,\epsilon)$ in (54) is a holomorphic solution of (8) defined in $(\mathcal{T}_1 \cap D(0,h')) \times (\mathcal{T}_2 \cap D(0,h')) \times H_{\beta'} \times \mathcal{E}_p$ for some $h' > 0$ and all $0 < \beta' < \beta$.*

Moreover, there exist $K, M > 0$ such that for every $0 \le p \le \iota - 1$, one has

$$(56)\qquad \sup_{\boldsymbol{t} \in (\mathcal{T}_1 \cap D(0,h')) \times (\mathcal{T}_2 \cap D(0,h')), z \in H_{\beta'}} |u_{p+1}(\boldsymbol{t},z,\epsilon) - u_p(\boldsymbol{t},z,\epsilon)| \le K \exp\left(-\frac{M}{|\epsilon|^\alpha}\right),$$

for every $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, with

$$(57)\qquad \alpha = \min\{k_2(1 - \lambda k_1 \delta_{D_1}), k_1(1 + \lambda k_2 \tilde{\delta}_{D_2})\}.$$

Proof The first part of the proof is guaranteed from the construction of the function $u_p(\boldsymbol{t},z,\epsilon)$ for every $0 \le p \le \iota - 1$.

Let $0 \le p \le \iota - 1$. For every $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$ we distinguish different situations depending on the relative position of the directions $d_{p,1}, d_{p+1,1}$, and $d_{p,2}, d_{p+1,2}$.

Let $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$ and assume that $L_{d_{p,1},\epsilon}$ can be transformed into $L_{d_{p+1,1},\epsilon}$ by a path deformation and the same holds for $L_{d_{p,2}}$ and $L_{d_{p+1,2}}$ without meeting any $(\tau_1,\tau_2) \in D(0,r_1(\epsilon)) \times (\mathbb{C} \setminus D(0,r_2(\epsilon)))$ with $P_m(\tau_1,\tau_2) = 0$ for $m \in \mathbb{R}$, i.e. the movable singularities in $D(0,r_1(\epsilon)) \times (\mathbb{C} \setminus D(0,r_2(\epsilon)))$ fall apart from the arguments between $d_{p,1}$ and $d_{p+1,1}$ with respect to the first component, nor between $d_{p,2}$ and $d_{p+1,2}$ with respect to the second component. Whenever this configuration holds, Cauchy theorem ensures that $u_p(\boldsymbol{t},z,\epsilon) \equiv u_{p+1}(\boldsymbol{t},z,\epsilon)$ for all $(\boldsymbol{t},z) \in (\mathcal{T}_1 \cap D(0,h')) \times (\mathcal{T}_2 \cap D(0,h')) \times H_{\beta'}$. The same argument can be applied to all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$ concluding that the sectors $\mathcal{E}_p$ and $\mathcal{E}_{p+1}$ can merge in the configuration of the good covering.

It is worth mentioning that the following cases state three equivalence classes regarding each element in the good covering. A continuity argument yields that for all $0 \le p \le \iota - 1$, if there exists $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$ such that one of the following mutually excluded cases holds for such ϵ, then the same case holds for every element in $\mathcal{E}_p \cap \mathcal{E}_{p+1}$.

Case 1: Assume that $L_{d_{p,1},\epsilon} \equiv L_{d_{p+1,1},\epsilon}$ and $L_{d_{p,2}}$ differs from $L_{d_{p+1,2}}$. This situation occurs in case that the first component of every singularity in the Borel plane does not fall between the directions $d_{p,1}$ and $d_{p+1,1}$ but at least the second component of one singular point in the Borel plane occurs within angles between $d_{p,2}$ and $d_{p+1,2}$.

Then, one has

$$u_{p+1}(\boldsymbol{t},z,\epsilon) - u_p(\boldsymbol{t},z,\epsilon) = I_{11} - I_{12},$$

where

$$I_{11} := \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{L_{d_{p,1},\epsilon}} \int_{L_{d_{p+1,2}}} \omega_{\boldsymbol{k}}^{\boldsymbol{d}_{p+1}}(\boldsymbol{u},m,\epsilon) \exp\left(-\left(\frac{u_1}{\epsilon t_1}\right)^{k_1} - \left(\frac{u_2}{\epsilon t_2}\right)^{k_2} \right) e^{izm} \frac{du_2}{u_2}\frac{du_1}{u_1} dm,$$

$$I_{12} := \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{L_{d_{p,1},\epsilon}} \int_{L_{d_{p,2}}} \omega_{\boldsymbol{k}}^{\boldsymbol{d}_{p}}(\boldsymbol{u},m,\epsilon) \exp\left(-\left(\frac{u_1}{\epsilon t_1}\right)^{k_1} - \left(\frac{u_2}{\epsilon t_2}\right)^{k_2} \right) e^{izm} \frac{du_2}{u_2}\frac{du_1}{u_1} dm,$$

for every $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$. Taking into account the first statement in Lemma 3, the functions $\omega_{\boldsymbol{k}}^{\boldsymbol{d}_p}(\boldsymbol{\tau},m,\epsilon)$ and $\omega_{\boldsymbol{k}}^{\boldsymbol{d}_{p+1}}(\boldsymbol{\tau},m,\epsilon)$ define a common function , say $\omega_{\boldsymbol{k}}(\boldsymbol{\tau},m,\epsilon)$ in $D(0,r_1(\epsilon)) \times D(0,2r_2(\epsilon))$ with respect to the first two variables. This entails that a deformation of the integration path in the second time variable can be performed in the previous difference in order to obtain after the application of Cauchy theorem that for all $\tau_1 \in L_{d_{p,1},\epsilon}$ and $m \in \mathbb{R}$

$$\int_{L_{d_{p+1,2}}} \omega_{\boldsymbol{k}}^{\boldsymbol{d}_{p+1}}(\boldsymbol{u},m,\epsilon) \exp\left(-\left(\frac{u_2}{\epsilon t_2}\right)^{k_2} \right) \frac{du_2}{u_2} - \int_{L_{d_{p,2}}} \omega_{\boldsymbol{k}}^{\boldsymbol{d}_{p}}(\boldsymbol{u},m,\epsilon) \exp\left(-\left(\frac{u_2}{\epsilon t_2}\right)^{k_2} \right) \frac{du_2}{u_2}$$

can be expressed in the form

(58)
$$\int_{L_{d_{p+1,2},r_2(\epsilon)}} \omega_{\boldsymbol{k}}^{\boldsymbol{d}_{p+1}}(\boldsymbol{u},m,\epsilon) \exp\left(-\left(\frac{u_2}{\epsilon t_2}\right)^{k_2} \right) \frac{du_2}{u_2} + \int_{C_{p,p+1,r_2(\epsilon)}} \omega_{\boldsymbol{k}}(\boldsymbol{u},m,\epsilon) \exp\left(-\left(\frac{u_2}{\epsilon t_2}\right)^{k_2} \right) \frac{du_2}{u_2}$$
$$- \int_{L_{d_{p,2},r_2(\epsilon)}} \omega_{\boldsymbol{k}}^{\boldsymbol{d}_{p}}(\boldsymbol{u},m,\epsilon) \exp\left(-\left(\frac{u_2}{\epsilon t_2}\right)^{k_2} \right) \frac{du_2}{u_2} = I_{13} + I_{14} - I_{15},$$

where $L_{d_{p,2},r_2(\epsilon)} = [r_2(\epsilon),\infty)e^{\sqrt{-1}d_{p,2}}$, $L_{d_{p+1,2},r_2(\epsilon)} = [r_2(\epsilon),\infty)e^{\sqrt{-1}d_{p+1,2}}$ and $C_{p,p+1,r_2(\epsilon)}$ stands for the arc of circle centered at 0 and radius $r_2(\epsilon)$ which connects the points $r_2(\epsilon)e^{\sqrt{-1}d_{p,2}}$ and $r_2(\epsilon)e^{\sqrt{-1}d_{p+1,2}}$. Taking into account (53) and by construction of the solutions, the direction $d_{p+1,2}$ (depending on ϵt_2) is such that there exists $\delta_1 > 0$ with $\cos(k_2(d_{p+1,2} - \arg(\epsilon t_2))) \ge \delta_1 > 0$

for every $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$ and every $\boldsymbol{t} \in (\mathcal{T}_1 \cap D(0,h')) \cap (\mathcal{T}_2 \cap D(0,h'))$. This entails that

$$(59) \quad |I_{13}| \le C_{\omega_{\boldsymbol{k}}^{\boldsymbol{d}_{p+1}}} \exp(-\beta|m|) \frac{1}{(1+|m|)^{\mu}} \frac{\left|\frac{u_1}{\epsilon}\right|}{1+\left|\frac{u_1}{\epsilon}\right|^{2k_1}} \exp\left(\nu_1 \left|\frac{u_1}{\epsilon}\right|^{k_1}\right) \int_{r_2(\epsilon)}^{\infty} \frac{\frac{s_2}{|\epsilon|}}{1+\left(\frac{s_2}{|\epsilon|}\right)^{2k_2}} \exp\left(\left(\frac{s_2}{|\epsilon|}\right)^{k_2}\left(\nu_2 - \frac{\cos(k_2(d_{p+1,2}-\arg(\epsilon t_2)))}{|t_2|^{k_2}}\right)\right)\frac{ds_2}{s_2}$$
$$\le C_{\omega_{\boldsymbol{k}}^{\boldsymbol{d}_{p+1}}} \exp(-\beta|m|) \frac{1}{(1+|m|)^{\mu}} \frac{\left|\frac{u_1}{\epsilon}\right|}{1+\left|\frac{u_1}{\epsilon}\right|^{2k_1}} \exp\left(\nu_1 \left|\frac{u_1}{\epsilon}\right|^{k_1}\right) \int_{r_2(\epsilon)}^{\infty} \frac{s_2}{|\epsilon|} \exp\left(\left(\frac{s_2}{|\epsilon|}\right)^{k_2}\left(\nu_2 - \frac{\delta_1}{|t_2|^{k_2}}\right)\right)\frac{ds_2}{s_2}.$$

We choose $0 < h' < (\delta_1/\nu_2)^{1/k_2}$, to get that the previous expression is upper bounded by

$$(60) \qquad C_{\omega_{\boldsymbol{k}}^{\boldsymbol{d}_{p+1}}} \exp(-\beta|m|) \frac{1}{(1+|m|)^{\mu}} \frac{\left|\frac{u_1}{\epsilon}\right|}{1+\left|\frac{u_1}{\epsilon}\right|^{2k_1}} \exp\left(\nu_1 \left|\frac{u_1}{\epsilon}\right|^{k_1}\right) \exp\left(-\frac{C_{21}}{|\epsilon|^{k_2(1-\lambda k_1 \delta_{D_1})}}\right),$$

for some $C_{21} > 0$. The expression I_{15} is upper estimated following analogous arguments. We consider I_{14}, and apply (53) to analogous argument as above arriving at

$$(61) \quad |I_{14}| \le C_{\omega_{\boldsymbol{k}}} \exp(-\beta|m|) \frac{1}{(1+|m|)^{\mu}} \frac{\left|\frac{u_1}{\epsilon}\right|}{1+\left|\frac{u_1}{\epsilon}\right|^{2k_1}} \exp\left(\nu_1 \left|\frac{u_1}{\epsilon}\right|^{k_1}\right) \times \frac{\frac{r_2(\epsilon)}{|\epsilon|}}{1+\left(\frac{r_2(\epsilon)}{|\epsilon|}\right)^{2k_2}} \int_{d_{p,2}}^{d_{p+1,2}} \exp\left(\left(\frac{r_2(\epsilon)}{|\epsilon|}\right)^{k_2}\left(\nu_2 - \frac{\cos(k_2(\theta-\arg(\epsilon t_2)))}{|t_2|^{k_2}}\right)\right) d\theta$$
$$\le C_{\omega_{\boldsymbol{k}},2} \exp(-\beta|m|) \frac{1}{(1+|m|)^{\mu}} \frac{\left|\frac{u_1}{\epsilon}\right|}{1+\left|\frac{u_1}{\epsilon}\right|^{2k_1}} \exp\left(\nu_1 \left|\frac{u_1}{\epsilon}\right|^{k_1}\right) \exp\left(-\frac{C_{21}}{|\epsilon|^{k_2(1-\lambda k_1 \delta_{D_1})}}\right),$$

for some $C_{\omega_{\boldsymbol{k}},2} > 0$.

In view of (60) and (61), and regarding (52) we get that

$$(62) \quad |u_{p+1}(\boldsymbol{t},z,\epsilon) - u_p(\boldsymbol{t},z,\epsilon)| \le C_{\omega_{\boldsymbol{k}}^{\boldsymbol{d}_p},3} \frac{1}{(2\pi)^{1/2}} \left(\int_{-\infty}^{\infty} \exp((|\mathrm{Im}(z)|-\beta)|m|) \frac{1}{(1+|m|)^{\mu}} dm\right) \times \left(\int_0^{\kappa r_1(\epsilon)} \frac{\frac{s_1}{|\epsilon|}}{1+\left(\frac{s_1}{|\epsilon|}\right)^{2k_1}} \exp\left(\left(\frac{s_1}{|\epsilon|}\right)^{k_1}\left(\nu_1 - \frac{\delta_1}{|t_1|^{k_1}}\right)\right)\frac{ds_1}{s_1}\right) \exp\left(-\frac{C_{21}}{|\epsilon|^{k_2(1-\lambda k_1 \delta_{D_1})}}\right)$$

for some $C_{\omega_{\boldsymbol{k}}^{\boldsymbol{d}_p},3} > 0$. This is valid for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, $\boldsymbol{t} \in ((\mathcal{T}_1 \cap D(0,h')) \times (\mathcal{T}_2 \cap D(0,h')))$ and $z \in H_{\beta'}$. We point out that

$$(63) \qquad \int_{-\infty}^{\infty} \exp((|\mathrm{Im}(z)|-\beta)|m|) \frac{1}{(1+|m|)^{\mu}} dm < \infty, \qquad z \in H_{\beta'}.$$

Finally, observe that the change of variable $s_1 = |\epsilon| s$ and usual estimates yield

(64)
$$\int_0^{\kappa r_1(\epsilon)} \frac{\frac{s_1}{|\epsilon|}}{1+\left(\frac{s_1}{|\epsilon|}\right)^{2k_1}} \exp\left(\left(\frac{s_1}{|\epsilon|}\right)^{k_1}\left(\nu_1 - \frac{\delta_1}{|t_1|^{k_1}}\right)\right)\frac{ds_1}{s_1} \le \int_0^{\infty} \frac{1}{1+s^{2k_1}} \exp(-As^{k_1}) ds < \infty,$$

for some $A > 0$. We conclude that

$$|u_{p+1}(\boldsymbol{t},z,\epsilon) - u_p(\boldsymbol{t},z,\epsilon)| \leq C_{\omega_{\boldsymbol{k}}^{\boldsymbol{d}_p},4} \exp\left(-\frac{C_{21}}{|\epsilon|^{k_2(1-\lambda k_1 \delta_{D_1})}}\right),$$

for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, $\boldsymbol{t} \in ((\mathcal{T}_1 \cap D(0,h')) \times (\mathcal{T}_2 \cap D(0,h')))$ and $z \in H_{\beta'}$.

Case 2: Assume that $L_{d_{p,2}} \equiv L_{d_{p+1,2}}$ and $L_{d_{p,1},\epsilon}$ differs from $L_{d_{p+1,1},\epsilon}$. We only provide details on the steps which differ from the proof of Case 1. We have

$$u_{p+1}(\boldsymbol{t},z,\epsilon) - u_p(\boldsymbol{t},z,\epsilon) = I_{21} - I_{22},$$

where

$$I_{21} := \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{L_{d_{p+1,1},\epsilon}} \int_{L_{d_{p,2}}} \omega_{\boldsymbol{k}}^{\boldsymbol{d}_{p+1}}(\boldsymbol{u},m,\epsilon) \exp\left(-\left(\frac{u_1}{\epsilon t_1}\right)^{k_1} - \left(\frac{u_2}{\epsilon t_2}\right)^{k_2}\right) e^{izm} \frac{du_2}{u_2}\frac{du_1}{u_1} dm,$$

$$I_{22} := \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{L_{d_{p,1},\epsilon}} \int_{L_{d_{p,2}}} \omega_{\boldsymbol{k}}^{\boldsymbol{d}_{p}}(\boldsymbol{u},m,\epsilon) \exp\left(-\left(\frac{u_1}{\epsilon t_1}\right)^{k_1} - \left(\frac{u_2}{\epsilon t_2}\right)^{k_2}\right) e^{izm} \frac{du_2}{u_2}\frac{du_1}{u_1} dm,$$

for every $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$.

We split the integration path on the second time variable into $L_{d_{p,2},[0,r_2(\epsilon)]} := [0, r_2(\epsilon)]e^{\sqrt{-1}d_{p,2}}$ and $L_{d_{p,2},[r_2(\epsilon),\infty)} := [r_2(\epsilon),\infty)e^{\sqrt{-1}d_{p,2}}$. The first statement of Lemma 3 and Cauchy theorem allow to write

$$I_{21} - I_{22} = I_{23} - I_{24} + I_{25},$$

where

$$I_{23} := \int_{-\infty}^{\infty} \int_{L_{d_{p+1,1},\epsilon}} \int_{L_{d_{p,2},[r_2(\epsilon),\infty)}} \Delta_{p+1}(\boldsymbol{u},\epsilon,\boldsymbol{t}) du_2 du_1 dm,$$

$$I_{24} := \int_{-\infty}^{\infty} \int_{L_{d_{p,1},\epsilon}} \int_{L_{d_{p,2},[r_2(\epsilon),\infty)}} \Delta_{p}(\boldsymbol{u},\epsilon,\boldsymbol{t}) du_2 du_1 dm,$$

$$I_{25} := \int_{-\infty}^{\infty} \int_{C_{p,p+1,\kappa r_1(\epsilon)}} \int_{L_{d_{p,2},[0,r_2(\epsilon)]}} \Delta(\boldsymbol{u},\epsilon,\boldsymbol{t}) du_2 du_1 dm,$$

where $C_{p,p+1,\kappa r_1(\epsilon)}$ is the arc of circle centered at the origin, radius $\kappa r_1(\epsilon)$ connecting the points $\kappa r_1(\epsilon)e^{\sqrt{-1}d_{p,1}}$ and $\kappa r_1(\epsilon)e^{\sqrt{-1}d_{p+1,1}}$. Here, we have used the notation

$$\Delta_j = \frac{1}{(2\pi)^{1/2}} \omega_{\boldsymbol{k}}^{\boldsymbol{d}_j}(\boldsymbol{u},m,\epsilon) \exp\left(-\left(\frac{u_1}{\epsilon t_1}\right)^{k_1} - \left(\frac{u_2}{\epsilon t_2}\right)^{k_2}\right) e^{izm} \frac{1}{u_1 u_2}, \qquad j \in \{p, p+1\},$$

and $\Delta = \Delta_p = \Delta_{p+1}$ whenever both functions coincide. In practice, this last consideration holds if $|\tau_1| < r_1(\epsilon)$ and $|\tau_2| < 2r_2(\epsilon)$ as it follows from the first statement in Lemma 3.

The estimates for I_{23} coincide with those for I_{13}, together with the bounds provided after (62) to get that

$$|I_{23}| \leq C_{\omega_{\boldsymbol{k}}^{\boldsymbol{d}_p},4} \exp\left(-\frac{C_{21}}{|\epsilon|^{k_2(1-\lambda k_1 \delta_{D_1})}}\right),$$

for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, $\boldsymbol{t} \in ((\mathcal{T}_1 \cap D(0,h')) \times (\mathcal{T}_2 \cap D(0,h')))$ and $z \in H_{\beta'}$.

The expression I_{24} can be handled analogously. We finally provide upper bounds for $|I_{25}|$, which can be estimated via (53) and the choice made in (52) by

$$
(65) \quad |I_{25}| \leq C_{\omega_{\boldsymbol{k}}^{\boldsymbol{d}_p},4} \left(\int_{-\infty}^{\infty} \exp((|\mathrm{Im}(z)| - \beta)|m|) \frac{1}{(1+|m|)^{\mu}} dm \right)
\times \left(\int_{d_{p,1}}^{d_{p+1,1}} \frac{\frac{\kappa r_1(\epsilon)}{|\epsilon|}}{1 + \left(\frac{\kappa r_1(\epsilon)}{|\epsilon|}\right)^{2k_1}} \exp\left(\left(\frac{\kappa r_1(\epsilon)}{|\epsilon|}\right)^{k_1} \left(\nu_1 - \frac{\cos(k_1(\theta - \arg(\epsilon t_1)))}{|t_1|^{k_1}}\right)\right) d\theta \right)
\times \left(\int_0^{r_2(\epsilon)} \frac{\frac{s_2}{|\epsilon|}}{1 + \left(\frac{s_2}{|\epsilon|}\right)^{2k_2}} \exp\left(\left(\frac{s_2}{|\epsilon|}\right)^{k_2} \left(\nu_2 - \frac{\delta_1}{|t_2|^{k_2}}\right)\right) \frac{ds_2}{s_2} \right) = C_{\omega_{\boldsymbol{k}}^{\boldsymbol{d}_p},4} I_{26} I_{27} I_{28},
$$

for some $C_{\omega_{\boldsymbol{k}}^{\boldsymbol{d}_p},4} > 0$. I_{26} (resp. I_{28}) is upper bounded by a constant, see (63) (resp. a symmetric situation to that in (64)). We also have

$$
(66) \quad |I_{27}| \leq (d_{p+1,1} - d_{p,1}) \left(\sup_{x \geq 0} \frac{x}{1 + x^{2k_1}} \right) \exp\left(\left(\frac{\kappa r_1(\epsilon)}{|\epsilon|}\right)^{k_1} \left(\nu_1 - \frac{\delta_1}{|t_1|^{k_1}}\right)\right)
\leq (d_{p+1,1} - d_{p,1}) \left(\sup_{x \geq 0} \frac{x}{1 + x^{2k_1}} \right) \exp\left(-\frac{C_{22}}{|\epsilon|^{k_1(1+\lambda k_2 \tilde{\delta}_{D_2})}} \right)
$$

for $0 < h' < (\delta_1/\nu_1)^{1/k_1}$, and some $C_{22} > 0$. This entails the existence of $C_{\omega_{\boldsymbol{k}}^{\boldsymbol{d}_p},5}, C_{23} > 0$ such that

$$
|u_{p+1}(\boldsymbol{t}, z, \epsilon) - u_p(\boldsymbol{t}, z, \epsilon)| \leq C_{\omega_{\boldsymbol{k}}^{\boldsymbol{d}_p},5} \exp\left(-\frac{C_{23}}{|\epsilon|^{\alpha}} \right),
$$

for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, $\boldsymbol{t} \in ((\mathcal{T}_1 \cap D(0,h')) \times (\mathcal{T}_2 \cap D(0,h')))$ and $z \in H_{\beta'}$, with α defined in (57).

Case 3: Assume that $L_{d_{p,2}}$ does not coincide with $L_{d_{p+1,2}}$ and $L_{d_{p,1},\epsilon}$ differs from $L_{d_{p+1,1},\epsilon}$. For a more compact writing, we will only display the integration paths in which the integrals involved are subdivided. Each of them can be reduced to the situation in case 1 or case 2 above. In the following steps, we preserve the notation for Δ, Δ_p and Δ_{p+1}, and consider

$$
u_{p+1}(\boldsymbol{t}, z, \epsilon) - u_p(\boldsymbol{t}, z, \epsilon) = I_{31} - I_{32},
$$

where

$$
I_{31} := \int_{-\infty}^{\infty} \int_{L_{d_{p+1,1},\epsilon}} \int_{L_{d_{p+1,2}}} \Delta_{p+1} \frac{du_2}{u_2} \frac{du_1}{u_1} dm, \qquad I_{32} := \int_{-\infty}^{\infty} \int_{L_{d_{p,1},\epsilon}} \int_{L_{d_{p,2}}} \Delta_p \frac{du_2}{u_2} \frac{du_1}{u_1} dm,
$$

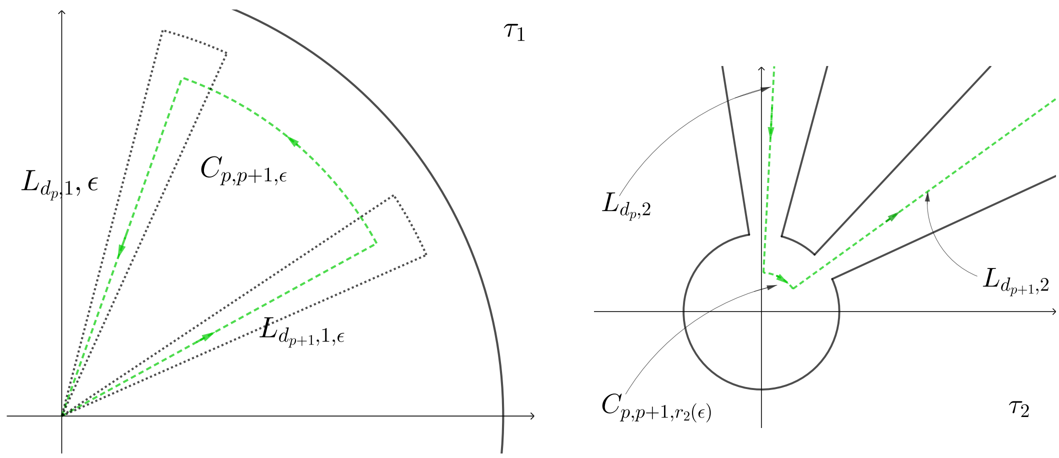

Figure 1: Deformation of the paths involved in the proof of Theorem 1

for every $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$. We deform and split the integration paths to obtain that $I_{31} - I_{32}$ equals

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{L_{d_{p+1},1,\epsilon}}\int_{L_{d_{p+1},2,[0,r_2(\epsilon)]}} \Delta du_2du_1dm + \int_{-\infty}^{\infty}\int_{L_{d_{p+1},1,\epsilon}}\int_{L_{d_{p+1},2,[r_2(\epsilon),\infty)}} \Delta_{p+1}du_2du_1dm\\
&- \int_{-\infty}^{\infty}\int_{L_{d_p,1,\epsilon}}\int_{L_{d_p,2,[0,r_2(\epsilon)]}} \Delta du_2du_1dm - \int_{-\infty}^{\infty}\int_{L_{d_p,1,\epsilon}}\int_{L_{d_p,2,[r_2(\epsilon),\infty)}} \Delta_p du_2du_1dm\\
&= \int_{-\infty}^{\infty}\int_{L_{d_{p+1},1,\epsilon}}\int_{L_{d_{p+1},2,[0,r_2(\epsilon)]}} \Delta du_2du_1dm + \int_{-\infty}^{\infty}\int_{L_{d_{p+1},1,\epsilon}}\int_{L_{d_{p+1},2,[r_2(\epsilon),\infty)}} \Delta_{p+1}du_2du_1dm\\
&- \int_{-\infty}^{\infty}\int_{L_{d_p,1,\epsilon}}\int_{L_{d_{p+1},2,[0,r_2(\epsilon)]}} \Delta du_2du_1dm + \int_{-\infty}^{\infty}\int_{L_{d_p,1,\epsilon}}\int_{C_{p,p+1,r_2(\epsilon)}} \Delta du_2du_1dm\\
&- \int_{-\infty}^{\infty}\int_{L_{d_p,1,\epsilon}}\int_{L_{d_p,2,[r_2(\epsilon),\infty)}} \Delta_p du_2du_1dm\\
&= \int_{-\infty}^{\infty}\int_{C_{p,p+1,\kappa r_1(\epsilon)}}\int_{L_{d_{p+1},2,[0,r_2(\epsilon)]}} \Delta du_2du_1dm + \int_{-\infty}^{\infty}\int_{L_{d_{p+1},1,\epsilon}}\int_{L_{d_{p+1},2,[r_2(\epsilon),\infty)}} \Delta_{p+1}du_2du_1dm\\
&+ \int_{-\infty}^{\infty}\int_{L_{d_p,1,\epsilon}}\int_{C_{p,p+1,r_2(\epsilon)}} \Delta du_2du_1dm - \int_{-\infty}^{\infty}\int_{L_{d_p,1,\epsilon}}\int_{L_{d_p,2,[r_2(\epsilon),\infty)}} \Delta_p du_2du_1dm\\
&= I_{33} + I_{34} + I_{35} - I_{36}.
\end{aligned}
$$

In the previous expression, we have extended in a natural manner the notation adopted for the integration paths in Case 1 and Case 2. Analogous bounds as those stated for the integral I_{25} (resp. I_{23}) are also valid for I_{33} (resp. I_{34}), in Case 2. For the expression I_{35} (resp. I_{36}) one can consider the estimates used to study I_{14} (resp. I_{13}), involved in Case 1. We conclude the existence of $C_{\omega_{\boldsymbol{k}},6}, C_{24} > 0$ such that

$$
|u_{p+1}(\boldsymbol{t}, z, \epsilon) - u_p(\boldsymbol{t}, z, \epsilon)| \leq C_{\omega_{\boldsymbol{k}},6} \exp\left(-\frac{C_{24}}{|\epsilon|^{\alpha}}\right),
$$

for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, $\boldsymbol{t} \in ((\mathcal{T}_1 \cap D(0, h')) \times (\mathcal{T}_2 \cap D(0, h')))$ and $z \in H_{\beta'}$, with α defined in (57). Figure 1 illustrates the deformation of the paths involved in the procedure.

□

7 Parametric Gevrey asymptotic expansions of the analytic solutions

In this section, we analyse the asymptotic behavior of the analytic solutions of the main problem (8) obtained in the previous section, regarding the perturbation parameter approaching the origin. The classical criterion for $k-$summability of formal power series with coefficients in a Banach space, known as Ramis-Sibuya Theorem (see [1], p.121, or Lemma XI–2–6 in [2]) will be used to describe the Gevrey asymptotic approximation of the solution.

The assumptions made in Section 2.2 and construction of the elements related to the main problem under study (8) are maintained in this section.

We first give some words on this classical summability theory for the sake of completeness.

7.1 $k-$summable formal power series and Ramis-Sibuya Theorem

Let $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ be a complex Banach space.

Definition 5 *Let $k \geq 1$ be an integer number. A formal power series $\hat{f}(\epsilon) = \sum_{n\geq 0} f_n\epsilon^n \in \mathbb{E}[[\epsilon]]$ is $k-$summable with respect to ϵ along direction $d \in \mathbb{R}$ if there exists a bounded holomorphic function f defined in a finite sector V_d of bisecting direction d and opening larger than π/k, and with values in $\mathbb{E}$, which admits $\hat{f}$ as its Gevrey asymptotic expansion of order $1/k$ on V_d, i.e. for every proper subsector V_1 of V_d, there exist $K, M > 0$ such that*

$$\left\| f(\epsilon) - \sum_{n=0}^{N-1} f_n\epsilon^n \right\|_{\mathbb{E}} \leq KM^N\Gamma\left(\frac{N}{k}+1\right)|\epsilon|^N,$$

for every integer $N \geq 1$ and $\epsilon \in V_1$. Watson's lemma guarantees uniqueness of such function, known as the $k-$sum of the formal power series.

Theorem 2 (RS) *Let $\iota \geq 2$ and let $(\mathcal{E}_p)_{0\leq p\leq \iota-1}$ be a good covering in $\mathbb{C}^\star$. For every $0 \leq p \leq \iota - 1$ we consider a holomorphic function $G_p : \mathcal{E}_p \to \mathbb{E}$, and define the function $\Theta_p(\epsilon) := G_{p+1}(\epsilon) - G_p(\epsilon)$ holomorphic in $Z_p := \mathcal{E}_p \cap \mathcal{E}_{p+1}$. We assume the following statements hold:*

- *G_p is a bounded function for $\epsilon \in Z_p$, $\epsilon \to 0$ for all $0 \leq p \leq \iota - 1$.*
- *Θ_p is an exponentially flat function of order k in Z_p for all $0 \leq p \leq \iota - 1$, i.e. there exist $K, M > 0$ such that $\|\Theta_p(\epsilon)\|_{\mathbb{E}} \leq K \exp\left(-\frac{M}{|\epsilon|^k}\right)$, valid for all $\epsilon \in Z_p$, and each $0 \leq p \leq \iota-1$.*

Then, each of the functions $G_p(\epsilon)$, for $0 \leq p \leq \iota - 1$ admits a common formal power series $\hat{G}(\epsilon) \in \mathbb{E}[[\epsilon]]$ as Gevrey asymptotic expansion of order $1/k$ on $\mathcal{E}_p$. In addition to this, if the opening of $\mathcal{E}_{p_0}$ is larger than π/k for some $0 \leq p_0 \leq \iota - 1$, then $G_{p_0}(\epsilon)$ is unique, being the $k-$sum of $\hat{G}(\epsilon)$ on $\mathcal{E}_{p_0}$.

7.2 Asymptotic behavior of the solutions of (8) in the perturbation parameter

We are in conditions to describe the asymptotic behavior of the analytic solutions of the main problem under study (8) with respect to the perturbation parameter, at the origin.

For this purpose, we consider a good covering $\underline{\mathcal{E}} = (\mathcal{E}_p)_{0\leq p\leq \iota-1}$, for some integer number $\iota \geq 2$. We also fix an admissible set $\{\mathcal{T}_1, \mathcal{T}_2, \underline{\mathcal{E}}, (S_{\mathfrak{d}_{1,p}})_{0\leq p\leq \iota-1}, (S_{\mathfrak{d}_{2,p}})_{0\leq p\leq \iota-1}\}$, which is associated to the good covering $\underline{\mathcal{E}}$, in accordance with the geometry of the problem (see Section 3) for each $0 \leq p \leq \iota - 1$, as described in Section 6.

Let $(u_p)_{0 \leq p \leq \iota-1}$ be the set of analytic solutions of (8), determined in Theorem 1. We recall that for every $0 \leq p \leq \iota - 1$, the function $(\boldsymbol{t}, z, \epsilon) \mapsto u_p(\boldsymbol{t}, z, \epsilon)$ is a holomorphic function in $\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'} \times \mathcal{E}_p$, for all $0 < \beta' < \beta$.

Let $\mathbb{E}$ be the Banach space of holomorphic and bounded functions on the domain $(\mathcal{T}_1 \cap D(0, h')) \times (\mathcal{T}_2 \cap D(0, h')) \times H_{\beta'}$, endowed with the norm of the supremum.

Theorem 3 *There exists a formal power series*

$$\hat{u}(\boldsymbol{t}, z, \epsilon) = \sum_{m \geq 0} H_m(\boldsymbol{t}, z) \frac{\epsilon^m}{m!} \in \mathbb{E}[[\epsilon]], \tag{67}$$

solution of (8), such that for every $0 \leq p \leq \iota - 1$, the function $\epsilon \mapsto u_p(\boldsymbol{t}, z, \epsilon)$ constructed in (54) admits $\epsilon \mapsto \hat{u}(\boldsymbol{t}, z, \epsilon)$ as its Gevrey asymptotic expansion of order $1/\alpha$, as $\epsilon \to 0$ with $\epsilon \in \mathcal{E}_p$ regarding them as functions and formal power series with coefficients in $\mathbb{E}$. Here, α is defined by (57). More precisely, there exist $C, M > 0$ such that

$$\sup_{\boldsymbol{t} \in ((\mathcal{T}_1 \cap D(0,h')) \times (\mathcal{T}_2 \cap D(0,h'))), z \in H_{\beta'}} \left| u_p(\boldsymbol{t}, z, \epsilon) - \sum_{m=0}^{N-1} H_m(\boldsymbol{t}, z) \frac{\epsilon^m}{m!} \right| \leq C M^N \Gamma\left(1 + \frac{N}{\alpha}\right) |\epsilon|^N, \tag{68}$$

for every integer $N \geq 0$, $0 \leq p \leq \iota - 1$ and all $\epsilon \in \mathcal{E}_p$. In case the opening of $\mathcal{E}_{p_0}$ is larger than π/α for some $0 \leq p_0 \leq \iota - 1$, then $u(\boldsymbol{t}, z, \epsilon)$ turns out to be the $\alpha-$sum of $\hat{u}(\boldsymbol{t}, z, \epsilon)$ in $\mathcal{E}_{p_0}$.

Proof For every $0 \leq p \leq \iota - 1$, let G_p be the function $\epsilon \mapsto u_p(\boldsymbol{t}, z, \epsilon)$. It holds that $G_p : \mathcal{E}_p \to \mathbb{E}$ is a holomorphic function in $\mathcal{E}_p$ and moreover, in view of (56), it holds that

$$\|G_{p+1}(\epsilon) - G_p(\epsilon)\|_{\mathbb{E}} \leq K \exp\left(-\frac{M}{|\epsilon|^\alpha}\right),$$

for some $K, M > 0$, and all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$. Regarding Ramis-Sibuya Theorem (RS), this entails the existence of a formal power series in the form (67), such that $\epsilon \mapsto u_p(\boldsymbol{t}, z, \epsilon)$ admits $\epsilon \mapsto \hat{u}(\boldsymbol{t}, z, \epsilon)$ as its Gevrey asymptotic expansion of order $1/\alpha$. The function $u_{p_0}(\boldsymbol{t}, z, \epsilon)$ is the $\alpha-$sum of $\hat{u}(\boldsymbol{t}, z, \epsilon)$ if the opening of $\mathcal{E}_{p_0}$ is larger than π/α, for some $0 \leq p_0 \leq \iota - 1$.

It is straight to check that the formal power series (67) is a formal solution of (8) by plugging it into (8) and taking into account that, in accordance to the existence of the asymptotic expansion in (68), it holds that

$$\lim_{\substack{\epsilon \to 0, \epsilon \in \mathcal{E}_p \\ (\boldsymbol{t}, z) \in (\mathcal{T}_1 \cap D(0,h')) \times (\mathcal{T}_2 \cap D(0,h')) \times H_{\beta'}}} |\partial_\epsilon^m u_p(\boldsymbol{t}, z, \epsilon) - H_m(\boldsymbol{t}, z)| = 0, \quad m \geq 0.$$

We refer to Theorem 2 [6] for further details on this last part of the proof, which follows usual reasonings. □

Remark: An example of equation which can be considered in this study is the following:

$$\begin{aligned} (\partial_z^8 + M) u(\boldsymbol{t}, z, \epsilon) = {} & \epsilon^{12} (t_1^4 \partial_{t_1})^2 (t_2^3 \partial_{t_2})^3 (\partial_z^4 + 1) u(\boldsymbol{t}, z, \epsilon) \\ & + \epsilon^7 (t_1^4 \partial_{t_1}) t_2^7 \partial_{t_2} c_{11}(z, \epsilon) R_{11}(\partial_z) u(\boldsymbol{t}, z, \epsilon) + f(\boldsymbol{t}, z, \epsilon), \end{aligned} \tag{69}$$

for some large $M > 0$, $R_{11}(X) \in \mathbb{C}[X]$ with $\deg(R_{11}) \leq 4$, and for some $c_{11}(z, \epsilon)$ and $f(\boldsymbol{t}, z, \epsilon)$ constructed as in Section 2.2.

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