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— Theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Frontier Operators — *Definitions, Essential Properties and, Consistent, Independent Axioms*

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Abstract. In a generalized topological space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, generalized interior and generalized closure operators $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, are merely two of a number of generalized primitive operators which may be employed to topologize the underlying set Ω in the generalized sense. Generalized exterior and generalized frontier operators $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, are other generalized primitive operators by means of which characterizations of generalized operations under $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ can be given without even realizing generalized interior and generalized closure operations first in order to topologize Ω in the generalized sense. In a recent work, the present authors have defined novel types of generalized interior and generalized closure operators $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in $\mathfrak{T}_{\mathfrak{g}}$ and studied their essential properties and commutativity. In this work, they propose to present novel definitions of generalized exterior and generalized frontier operators $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, a set of consistent, independent axioms after studying their essential properties, and established further characterizations of generalized operations under $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in $\mathfrak{T}_{\mathfrak{g}}$.

Key words and phrases. *Generalized topological space, generalized sets, generalized exterior operator, generalized frontier operator, consistent, independent axioms*

Contents

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1. Introduction

Any $\mathfrak{T}$-set[1] in a $\mathscr{T}$-space or $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space generates a natural partition of points in its $\mathscr{T}$-space or $\mathscr{T}_{\mathfrak{g}}$-space into three pairwise disjoint classes whose union is the underlying set of the $\mathscr{T}$-space or $\mathscr{T}_{\mathfrak{g}}$-space. In the $\mathscr{T}$-space, an ordinary partition is realized by the $\mathfrak{T}$-operators int, ext, fr : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (ordinary interior, ordinary exterior and ordinary frontier operators in ordinary topological spaces) [Dix84, Gab64, Kur22, Lev61, Rad80, Wil70] and a generalized partition by the $\mathfrak{g}$-$\mathfrak{T}$-operators $\mathfrak{g}$-Int, $\mathfrak{g}$-Ext, $\mathfrak{g}$-Fr : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (generalized interior, generalized exterior and generalized frontier operators in ordinary topological spaces) [CJK04, Cs8, Cs7, JN19, LZ19]. In the $\mathscr{T}_{\mathfrak{g}}$-space, an ordinary partition is realized by the $\mathfrak{T}_{\mathfrak{g}}$-operators $\mathrm{int}_{\mathfrak{g}}$, $\mathrm{ext}_{\mathfrak{g}}$, $\mathrm{fr}_{\mathfrak{g}}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (ordinary interior, ordinary exterior and ordinary frontier operators in generalized topological spaces) [Cs5, TC16] and a generalized partition by the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Ext}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Fr}_{\mathfrak{g}}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (generalized interior, generalized exterior and generalized frontier operators in generalized topological spaces) [Cs6, Mod17, SKK15]. Thus, $\mathrm{int}(\mathscr{S}) \cup \mathrm{ext}(\mathscr{S}) \cup \mathrm{fr}(\mathscr{S}) = \mathfrak{g}\text{-Int}(\mathscr{S}) \cup \mathfrak{g}\text{-Ext}(\mathscr{S}) \cup \mathfrak{g}\text{-Fr}(\mathscr{S})$ for any $\mathscr{S}$ in the $\mathscr{T}$-space and likewise, $\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathrm{ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathrm{fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ for any $\mathscr{S}_{\mathfrak{g}}$ in the $\mathscr{T}_{\mathfrak{g}}$-space. From the first set-theoretic $\cup$-relation, it follows that $\mathfrak{g}$-Ext, $\mathfrak{g}$-Fr : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are also essential primitive $\mathfrak{g}$-$\mathfrak{T}$-operators in the study of $\mathfrak{T}$-sets in $\mathscr{T}$-spaces, and from the second, it follows that $\mathfrak{g}$-$\mathrm{Ext}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Fr}_{\mathfrak{g}}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are also essential primitive $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators in the study of $\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathscr{T}_{\mathfrak{g}}$-spaces.

Intuitively, the images of any $\mathfrak{T}$-set in a $\mathscr{T}$-space under the $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-exterior operators ext, $\mathfrak{g}$-Ext : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, are the complements corresponding to the images of the $\mathfrak{T}$-set under the $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-closure operators cl, $\mathfrak{g}$-Cl : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ [Gab64, LZ19]. The images of the $\mathfrak{T}$-set in the $\mathscr{T}$-space under the $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-frontier operators fr, $\mathfrak{g}$-Fr : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, are the intersections of its image under cl : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ with the image of its complement under cl : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and, its image under $\mathfrak{g}$-Cl : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ with the image of its complement under $\mathfrak{g}$-Cl : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ [Gab64, LZ19]. When $(\mathscr{T}, \mathfrak{T}, \mathfrak{g}\text{-}\mathfrak{T}) \longmapsto (\mathscr{T}_{\mathfrak{g}}, \mathfrak{T}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}})$, the intuitive descriptions as to the images of any $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space under the $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operators ext, $\mathfrak{g}$-Ext : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and the $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators fr, $\mathfrak{g}$-Fr : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, follow.

Hence, in a $\mathscr{T}$-space, the ordinary topologization of a set can be characterized by specifying a $\mathfrak{T}$-exterior operator ext : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or a $\mathfrak{T}$-frontier operator fr : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and the generalized topologization of a set can be characterized by specifying a $\mathfrak{g}$-$\mathfrak{T}$-exterior operator $\mathfrak{g}$-Ext : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or a $\mathfrak{g}$-$\mathfrak{T}$-frontier operator $\mathfrak{g}$-Fr : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ [Gab64]. Similarly, in a $\mathscr{T}_{\mathfrak{g}}$-space, the ordinary

[1]Notes to the reader: The structures $\mathfrak{T} = (\Omega, \mathscr{T})$ and $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, respectively, are called ordinary and generalized topological spaces (briefly, $\mathscr{T}$-space and $\mathscr{T}_{\mathfrak{g}}$-space). The symbols $\mathscr{T}$ and $\mathscr{T}_{\mathfrak{g}}$, respectively, are called ordinary topology and generalized topology (briefly, topology and $\mathfrak{g}$-topology). Subsets of $\mathfrak{T}$ and $\mathfrak{T}_{\mathfrak{g}}$, respectively, are called $\mathfrak{T}$-sets and $\mathfrak{T}_{\mathfrak{g}}$-sets; subsets of $\mathscr{T}$ and $\mathscr{T}_{\mathfrak{g}}$, respectively, are called $\mathscr{T}$-open and $\mathscr{T}_{\mathfrak{g}}$-open sets, and their complements are called $\mathscr{T}$-closed and $\mathscr{T}_{\mathfrak{g}}$-closed sets. Generalizations of $\mathfrak{T}$-sets, $\mathscr{T}$-open and $\mathscr{T}$-closed sets in $\mathscr{T}$, respectively, are called $\mathfrak{g}$-$\mathfrak{T}$-sets, $\mathfrak{g}$-$\mathscr{T}$-open and $\mathfrak{g}$-$\mathscr{T}$-closed sets; generalizations of $\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets in $\mathscr{T}_{\mathfrak{g}}$, respectively, are called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-closed sets. By a Λ-operator is meant an operator using Λ-sets to characterize its argument, where $\Lambda \in \{\mathscr{T}, \mathfrak{T}, \mathfrak{g}\text{-}\mathscr{T}, \mathfrak{g}\text{-}\mathfrak{T}\} \cup \{\mathscr{T}_{\mathfrak{g}}, \mathfrak{T}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}\}$.

topologization of a set can be characterized by specifying a $\mathfrak{T}_{\mathfrak{g}}$-exterior operator $\mathrm{ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or a $\mathfrak{T}_{\mathfrak{g}}$-frontier operator $\mathrm{fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and the generalized topologization of a set can be characterized by specifying a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ [LZ19]. Of all such primitive operators ext, fr, $\mathfrak{g}$-Ext, $\mathfrak{g}$-Fr $: \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in $\mathscr{T}$-spaces and $\mathrm{ext}_{\mathfrak{g}}$, $\mathrm{fr}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in $\mathscr{T}_{\mathfrak{g}}$-spaces, ext, fr $: \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are the oldest, and $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are the newest. Thus, the studies of primitive operators of these kinds have evolved from the studies of ordinary exterior and ordinary frontier operators in ordinary topological spaces to the studies of generalized exterior and generalized frontier operators in generalized topological spaces.

In the literature of $\mathscr{T}_{\mathfrak{g}}$-spaces on $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators, few Mathematicians have introduced some new types of one-valued maps $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, studied and related to one another some $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators properties in $\mathscr{T}_{\mathfrak{g}}$-spaces similar in descriptions to $\mathfrak{g}$-$\mathfrak{T}$-exterior and $\mathfrak{g}$-$\mathfrak{T}$-frontier operators properties in $\mathscr{T}$-spaces [Kle77, Mod17].

In studying the properties of $\tilde{\mu}$-open sets in $\mathscr{T}_{\mathfrak{g}}$-spaces, [SKK15] have also used these $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets to define new $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators called $\tilde{\mu}$-exterior and $\tilde{\mu}$-frontier operators and characterized by $\mathrm{ext}_{\tilde{\mu}}$, $\mathrm{bd}_{\tilde{\mu}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and studied some of their properties. In studying the properties of λ_{κ}-closed sets in $\mathscr{T}_{\mathfrak{g}}$-spaces, [JJV14] have also used these $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets to define new $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators called λ_{κ}-exterior and λ_{κ}-frontier operators and characterized by $\mathrm{E}_{\lambda_{\kappa}}$, $\mathrm{F}_{\lambda_{\kappa}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and studied some of their properties. In a paper on base for $\mathscr{T}_{\mathfrak{g}}$-spaces, [KM11] have defined a new $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator which may be called ∂-operator and characterized by $\partial : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and studied some of its properties. In the paper of [Boo18], the author gave the definition of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator called $(\zeta, \delta(\mu))$-frontier operator and characterized by $\mathrm{Fr}_{(\zeta,\delta(\mu))} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and utilized it to study the properties of $\zeta_{\delta(\mu)}$ and $(\zeta, \delta(\mu))$-closed sets in strong $\mathscr{T}_{\mathfrak{g}}$-spaces.

In view of the above few references, it follows that the subject-matter was given little attention. In this paper titled *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Exterior and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Frontier Operators* and subtitled *Definitions, Essential Properties and, Consistent, Independent Axioms*, the authors attempt to add, in as unique and unified a way as possible so as to offer a unified approach to many $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators properties, a further contribution to the field with these two research objectives in mind:

- I. To present the definitions and the essential properties of a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.
- II. To discuss the consistency, independency of some sets of axioms for the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

These two research objectives form properly two separate sections and the rest of this paper is structured in this manner: In SECT. 2, preliminary notions are described in SUBSECT. 2.1 (APPX. A contains pre-preliminary notions extracted from the pre-preliminary and preliminary sections of our sixth work titled *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Interior and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Closure Operators*) and the main results of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators in $\mathscr{T}_{\mathfrak{g}}$-spaces are reported in SECT. 3: results associated with essential properties are given in SUBSECT. 3.1 and those associated with the notions of consistent, independent axioms are given in SUBSECT.

3.2. In Sect. 4, the establishment of the various relationships between these $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators are discussed in Sects 4.1. To support the work, a nice application of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators in a $\mathscr{T}_{\mathfrak{g}}$-space is presented in Subsect. 4.2. Finally, Subsect. 4.3 provides concluding remarks and future directions of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

2. Theory

2.1. Preliminaries. Foreign terms employed below are extracted from the preliminary section of our sixth work titled *Theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Closure Operators* and are presented in Appx. A.

The discussion starts by introducing the definitions of the notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators of category ν in $\mathscr{T}_{\mathfrak{g}}$-spaces. Taking $\mathfrak{g}\text{-Int}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ as the primitive $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-operator, they may be defined as thus.

Definition 2.1 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Exterior, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Frontier Operators). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then:

- I. The one-valued map $\mathfrak{g}\text{-Ext}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator of category ν" if and only if

$$(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)) \left[\mathfrak{g}\text{-Ext}_{\mathfrak{g},\nu} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g},\nu} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right]. \tag{2.1}$$

- II. The one-valued map $\mathfrak{g}\text{-Fr}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator of category ν" if and only if

$$(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)) \left[\mathfrak{g}\text{-Fr}_{\mathfrak{g},\nu} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\bigcup_{\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}(\mathscr{R}_{\mathfrak{g}})\right)\right]. \tag{2.2}$$

The classes $\mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-Ext}_{\mathfrak{g},\nu} : \ \nu \in I_3^0\}$ and $\mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-Fr}_{\mathfrak{g},\nu} : \ \nu \in I_3^0\}$ are called, respectively, the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operators and the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators in $\mathfrak{T}_{\mathfrak{g}}$.

The $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Ext}_{\mathfrak{g},\nu}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ may be chosen, evidently, as primitive $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators of category ν in $\mathfrak{T}_{\mathfrak{g}}$.

Remark 2.2. Observing that, for every $\nu \in I_3^*$, the $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators $\mathfrak{g}\text{-Ext}_{\mathfrak{g},\nu}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are based on the $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, it follows that:

- I. $(\mathfrak{g}\text{-Ext}_{\mathfrak{g},\nu}, \mathfrak{g}\text{-Fr}_{\mathfrak{g},\nu}) \stackrel{\text{def}}{=} (\text{ext}_{\mathfrak{g}}, \text{fr}_{\mathfrak{g}})$ if based on the $\mathfrak{T}_{\mathfrak{g}}$-operators $\text{int}_{\mathfrak{g}}$, $\text{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$;
- II. $(\mathfrak{g}\text{-Ext}_{\mathfrak{g},\nu}, \mathfrak{g}\text{-Fr}_{\mathfrak{g},\nu}) \stackrel{\text{def}}{=} (\mathfrak{g}\text{-Ext}_{\nu}, \mathfrak{g}\text{-Fr}_{\nu})$ if based on the $\mathfrak{g}$-$\mathfrak{T}$-operators $\mathfrak{g}\text{-Int}_{\nu}$, $\mathfrak{g}\text{-Cl}_{\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$;
- III. $(\mathfrak{g}\text{-Ext}_{\mathfrak{g},\nu}, \mathfrak{g}\text{-Fr}_{\mathfrak{g},\nu}) \stackrel{\text{def}}{=} (\text{ext}, \text{fr})$ if based on the $\mathfrak{T}$-operators int, cl : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

In this way, $(\text{ext}_{\mathfrak{g}}, \text{fr}_{\mathfrak{g}})$ is called a pair of $\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{T}_{\mathfrak{g}}$-frontier operators in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$; $(\mathfrak{g}\text{-Ext}_{\nu}, \mathfrak{g}\text{-Fr}_{\nu})$, a pair of $\mathfrak{g}$-$\mathfrak{T}$-exterior and $\mathfrak{g}$-$\mathfrak{T}$-frontier operators of category ν in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$ and (ext, fr), a pair of $\mathfrak{T}$-exterior and $\mathfrak{T}$-frontier operators in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$. Accordingly, $\mathfrak{g}\text{-E}[\mathfrak{T}] \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-Ext}_{\nu} :$

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$\nu \in I_3^0\}$ and $\mathfrak{g}\text{-F}[\mathfrak{T}] \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-Fr}_\nu :\ \nu \in I_3^0\}$. Then, $\mathfrak{g}\text{-E}[\mathfrak{T}]$ and $\mathfrak{g}\text{-F}[\mathfrak{T}]$, respectively, denote the classes of all $\mathfrak{g}$-$\mathfrak{T}$-exterior operators and $\mathfrak{g}$-$\mathfrak{T}$-frontier operators in the $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$.

3. Main Results

Using the foregoing definitions, some essential properties as well as the consistency, independency of some sets of axioms for the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators in $\mathscr{T}_{\mathfrak{g}}$-spaces are presented below.

3.1. Essential Properties. The discussion begins by giving some of the basic consequences resulting from the foregoing definition.

In a $\mathscr{T}_{\mathfrak{g}}$-space, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operator may also be used to define the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators as proved in the following proposition.

Proposition 3.1. *If* $(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}, \mathfrak{g}\text{-Fr}_{\mathfrak{g}}) \in \mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*exterior and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*frontier operators* $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *respectively, in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then*

$$\begin{aligned}&(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega))\big[\big(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\\ &\wedge \big(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \textstyle\bigcap_{\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\big].\end{aligned} \tag{3.1}$$

Proof. Let $(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}, \mathfrak{g}\text{-Fr}_{\mathfrak{g}}) \in \mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary. Then, by virtue of the definition of $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ together with the relation $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}$, it results that

$$\begin{aligned}\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} &\longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Hence, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Likewise, by virtue of the definition of $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ together with $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}$, it results that

$$\begin{aligned}\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} &\longmapsto \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Big(\bigcup_{\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\Big)\\ &\longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Big(\bigcup_{\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\Big)\\ &\longleftrightarrow \bigcap_{\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\\ &\longleftrightarrow \bigcap_{\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).\end{aligned}$$

Hence, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \bigcap_{\mathscr{R}_{\mathfrak{g}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. The proof of the proposition is complete. Q.E.D.

That the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator may be considered to establish the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators follows from the following theorem.

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THEOREM 3.2. *If* $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \in \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ *be a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-frontier operator* $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$,
- II. $\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

PROOF. Let $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \in \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-frontier operator $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary. Then, since $\mathscr{S}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}$ and $\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \emptyset$, it follows that

$$\begin{aligned}
\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cap \Bigg(\bigcup_{\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \Bigg) \\
&\longleftrightarrow \bigcup_{\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \\
&\longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).
\end{aligned}$$

Thus, $\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. On the other hand, since $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}$, it follows that

$$\begin{aligned}
\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Op}_{\mathfrak{g}} \Bigg(\bigcup_{\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \Bigg) \\
&\longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cup \Bigg(\bigcap_{\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \Bigg) \\
&\longleftrightarrow \bigcap_{\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \big(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \\
&\longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).
\end{aligned}$$

Thus, $\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the theorem is complete. Q.E.D.

Observing that, for any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, it holds that

$$\begin{aligned}
\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \cap \big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \\
&\longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cap \big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big),
\end{aligned}$$

and, for any $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, it also holds that

$$\begin{aligned}
&\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\
\longleftrightarrow\ &\big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\
\longleftrightarrow\ &\mathscr{R}_{\mathfrak{g}} \cup \big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) = \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}),
\end{aligned}$$

an immediate consequence of the above theorem is the following corollary.

COROLLARY 3.3. *If* $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \in \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ *be a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-frontier operator* $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$,
- II. $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

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For any $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space, its image under $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is a *super-image* of its image under $\text{ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ whereas, its image under $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is a *sub-image* of its image under $\text{fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. These are embodied in the theorem that follows.

THEOREM 3.4. *If* $\left(\text{ext}_{\mathfrak{g}}, \mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right) \in \mathrm{E}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-exterior operators* $\text{ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *respectively, and* $\left(\text{fr}_{\mathfrak{g}}, \mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) \in \mathrm{F}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-frontier operators* fr, $\mathfrak{g}\text{-Fr} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *respectively, in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

- I. $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \text{ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$,
- II. $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

PROOF. Let $\left(\text{ext}_{\mathfrak{g}}, \mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right) \in \mathrm{E}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}]$ and $\left(\text{fr}_{\mathfrak{g}}, \mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) \in \mathrm{F}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ be pairs of $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operators and $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary. Then, since $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it results that

$$\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \text{int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \text{ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).$$

Thus, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \text{ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. On the other hand, since $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ implies $\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it follows that

$$\begin{aligned} \mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \quad \longmapsto \quad & \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\bigcup_{\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \\ \subseteq \quad & \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\bigcup_{\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \longleftrightarrow \text{fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}). \end{aligned}$$

Hence, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the theorem is complete. Q.E.D.

REMARK 3.5. If the relation "$\mathfrak{g}\text{-Ext}_{\mathfrak{g}} \gtrsim \text{ext}_{\mathfrak{g}}$" stands for "$\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \text{ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$" and "$\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \precsim \text{fr}_{\mathfrak{g}}$," for "$\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$," then the outstanding facts are: $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\text{ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\text{ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\text{fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\text{fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

The images of $\emptyset \in \mathscr{P}(\Omega)$ under $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space are, respectively, the underlying set and itself as proved in the following proposition.

PROPOSITION 3.6. *If* $\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}, \mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-exterior and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-frontier operators* $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *respectively, in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

- I. $\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \emptyset \longmapsto \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset)\right) \wedge \left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \emptyset \longmapsto \emptyset\right)$,
- II. $\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} : \emptyset \longmapsto \emptyset\right) \wedge \left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} : \emptyset \longmapsto \emptyset\right)$.

PROOF. Let $\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}, \mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

I. Since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, implying $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\Omega) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset)$, it follows that

$$\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \emptyset \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset) \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\Omega) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset).$$

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Therefore, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}:\emptyset\longmapsto\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset)$. On the other hand, since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, implying $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\emptyset)=\emptyset$, by the relations $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\longleftrightarrow\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}$ and $\bigcup_{\mathscr{R}_{\mathfrak{g}}=\emptyset,\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset)}\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})=\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset)$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Fr}_{\mathfrak{g}}:\emptyset &\longmapsto \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{R}_{\mathfrak{g}}=\emptyset,\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset)}\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\Bigg)\\ &\longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset)\longleftrightarrow\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\emptyset)=\emptyset.\end{aligned}$$

Hence, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}:\emptyset\longmapsto\emptyset$.

II. Since $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\emptyset)=\emptyset$ in $\mathfrak{T}_{\mathfrak{g}}$,

$$\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}:\emptyset\longmapsto\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset)\longleftrightarrow\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\emptyset)=\emptyset.$$

Thus, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}:\emptyset\longmapsto\emptyset$. On the other hand,

$$\begin{aligned}\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}:\emptyset &\longmapsto \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{R}_{\mathfrak{g}}=\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset),\emptyset}\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\Bigg)\\ &\longleftrightarrow \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\emptyset)=\emptyset.\end{aligned}$$

Hence, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}:\emptyset\longmapsto\emptyset$. The proof of the proposition is complete. Q.E.D.

For any $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space, its image under $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is the image of the complement of this image under $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ whereas, its image under $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is the image of its complement under $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$. In the following theorem are proved these facts.

THEOREM 3.7. *If* $(\mathfrak{g}\text{-Ext}_{\mathfrak{g}},\mathfrak{g}\text{-Fr}_{\mathfrak{g}})\in\mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-exterior and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-frontier operators* $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, *respectively, in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, *then:*

- I. $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\quad\forall\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$,
- II. $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\quad\forall\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$.

PROOF. Let $(\mathfrak{g}\text{-Ext}_{\mathfrak{g}},\mathfrak{g}\text{-Fr}_{\mathfrak{g}})\in\mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ and, let $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$ be arbitrary. Then, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operation on $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)$ under $\mathfrak{g}\text{-Ext}\in\mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}]$, taking into account the definition of $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, gives

$$\begin{aligned}\mathfrak{g}\text{-Ext}_{\mathfrak{g}}:\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Thus, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. In a similar fashion, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operation on $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)$ under $\mathfrak{g}\text{-Fr}\in\mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ gives

$$\begin{aligned}\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{R}_{\mathfrak{g}}=\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})}\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\Bigg)\\ &\longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{R}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})}\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\Bigg)\longleftrightarrow\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

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Hence, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the theorem is complete. Q.E.D.

For an arbitrary $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ containing another arbitrary $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$, the statements given in the following theorem holds.

THEOREM 3.8. *If* $(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}, \mathfrak{g}\text{-Fr}_{\mathfrak{g}}) \in \mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-exterior and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-frontier operators* $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *respectively, in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *then:*

- I. $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longrightarrow \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$,
- II. $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longrightarrow \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$.

PROOF. Let $(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}, \mathfrak{g}\text{-Fr}_{\mathfrak{g}}) \in \mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ such that $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$ be arbitrary. Then, since condition $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$ implies $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it follows that

$$\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).$$

Therefore, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. On the other hand, since $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$ implies $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ and $\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it results that

$$\begin{aligned} \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) &\longleftrightarrow \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\ &\supseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}). \end{aligned}$$

Thus, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the theorem is complete. Q.E.D.

For any $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space, the union of its images under $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$ $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is the underlying set itself as proved in the following proposition.

PROPOSITION 3.9. *If* $(\mathfrak{g}\text{-Ext}, \mathfrak{g}\text{-Fr}, \mathfrak{g}\text{-Int}) \in \mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g}}]$ *be a triple of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-exterior,* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-frontier and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-interior operators* $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *respectively, in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

$$(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega))\left[\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \Omega\right]. \tag{3.2}$$

PROOF. Let $(\mathfrak{g}\text{-Ext}, \mathfrak{g}\text{-Fr}, \mathfrak{g}\text{-Int}) \in \mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g}}]$ be a triple of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary. Then, since the relation $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}$ implies $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}$, it follows that

$$\bigcap_{\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\Bigg)$$

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and consequently,

$$\begin{aligned}
\Omega &= \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \Bigg(\Bigg(\bigcap_{\mathscr{R}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\Bigg) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\Bigg) \\
&\longleftrightarrow \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{R}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\Bigg) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).
\end{aligned}$$

Hence, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \Omega$. The proof of the proposition is complete. Q.E.D.

For a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space to be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set, it is necessary and sufficient that its image under $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be contained either in itself or in its complement. The proof of this statement is contained in the following lemma, which will be helpful in the proof of the next theorem.

LEMMA 3.10. *Let* $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-frontier operator and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. *Then:*

- I. $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \longleftrightarrow \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$,
- II. $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \longleftrightarrow \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}}$.

PROOF. Let $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator and suppose $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

I. The supposition $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and hence, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Conversely, suppose the relation $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ hold. Then, $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$. By virtue of the relations $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \Omega$ and $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it follows, obviously, that $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Hence, $\mathscr{S}_{\mathfrak{g}} = \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathscr{S}_{\mathfrak{g}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$.

II. The supposition $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Therefore, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and hence, $\mathscr{S}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Conversely, suppose $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}}$. Then, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}}$ is equivalent to $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}}$. By virtue of the relations $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \Omega$ and $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it follows, consequently, that $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Thus, $\mathscr{S}_{\mathfrak{g}} = \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathscr{S}_{\mathfrak{g}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. The proof of the lemma is complete. Q.E.D.

In connection with the lemma just proved, the theorem follows.

THEOREM 3.11. *Let* $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-frontier operator and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. *Then:*

- I. $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$,
- II. $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$.

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PROOF. Let $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator and suppose $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

I. The supposition $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ is equivalent to $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But since $\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \big(\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\ &= \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Thus, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$.

II. The supposition $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ is equivalent to $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}}$. But since $\mathscr{S}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathscr{S}_{\mathfrak{g}}\\ &\longleftrightarrow \big(\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \cap \mathscr{S}_{\mathfrak{g}}\\ &= \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Thus, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the theorem is complete. Q.E.D.

In connection with the theorem just proved, notice that the statement given below holds:

$$\begin{aligned}\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \cap \big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) = \emptyset.\end{aligned}$$

Accordingly, an immediate consequence of the above theorem is the following corollary.

COROLLARY 3.12. *If $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:*

$$(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega))\big[\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \emptyset\big]. \tag{3.3}$$

For any $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space, the images of its $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure points under $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are both contained in its image under $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. Such statement is proved in the theorem that follows.

THEOREM 3.13. *If $\mathfrak{g}\text{-Fr} \in \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:*

- I. $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$,
- II. $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

PROOF. Let $\mathfrak{g}\text{-Fr} \in \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary. Then, the following proofs follow.

I. Since $\mathscr{S}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ implies $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{R}_{\mathfrak{g}}=\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\Bigg)\\ &\subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{R}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\Bigg) = \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

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Thus, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ for every $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

II. Since $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ implies $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it results that

$$\begin{aligned}
\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{R}_{\mathfrak{g}}=\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\Bigg) \\
&\subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{R}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\Bigg) = \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).
\end{aligned}$$

Hence, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ for every $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. The proof of the theorem is complete. Q.E.D.

THEOREM 3.14. *If* $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-frontier operator in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \subseteq \bigcup_{(\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}})=(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}),(\mathscr{S}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}})} \big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})\big)$,
- II. $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \subseteq \bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$,

for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$.

PROOF. Let $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator and suppose $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}} &\longmapsto \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\Bigg) \\
&\longleftrightarrow \bigcap_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \\
&\subseteq \Bigg(\bigcap_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\Bigg) \cap \Bigg(\bigcup_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})\Bigg) \\
&\longleftrightarrow \bigcup_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \Bigg(\Bigg(\bigcap_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\Bigg) \cap \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})\Bigg) \\
&\longleftrightarrow \bigcup_{(\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}})=(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}),(\mathscr{S}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}})} \big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})\big).
\end{aligned}$$

Thus,

$$\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \subseteq \bigcup_{(\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}})=(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}),(\mathscr{S}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}})} \big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})\big).$$

On the other hand,

$$\begin{aligned}
\mathfrak{g}\text{-Fr}_{\mathfrak{g}}:\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}} &\longmapsto \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})}\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\Bigg)\\
&\longleftrightarrow \bigcap_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})}\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\\
&\longleftrightarrow \Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\Bigg)\cap\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\Bigg(\bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})\Bigg)\\
&\subseteq \Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\Bigg)\cap\Bigg(\bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})\Bigg)\\
&\subseteq \bigcup_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\Bigg(\bigcap_{\mathscr{U}_{\mathfrak{g}}=\mathscr{V}_{\mathfrak{g}},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})}\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\Bigg)\\
&\longleftrightarrow \bigcup_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{V}_{\mathfrak{g}},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})}\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\Bigg)\\
&\longleftrightarrow \bigcup_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}).
\end{aligned}$$

Hence, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})\subseteq\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. The proof of the theorem is complete. Q.E.D.

If $\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})=\emptyset=\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ be taken as additional condition, then $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ instead of $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Accordingly, the following corollary is an immediate consequence of the above theorem.

COROLLARY 3.15. *If $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ and $\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})=\emptyset=\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ holds for any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$, then:*

- I. $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})=\displaystyle\bigcup_{(\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}})=(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}),(\mathscr{S}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}})}\big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})\big)$,
- II. $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})=\displaystyle\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$,

for any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$.

THEOREM 3.16. *If $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, then:*

- I. $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})\supseteq\displaystyle\bigcap_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\ \forall(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$,
- II. $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})\subseteq\displaystyle\bigcup_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})\ \forall(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$.

PROOF. Let $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator and suppose $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then:

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I. Since $\mathscr{U}_{\mathfrak{g}} \subseteq \mathscr{V}_{\mathfrak{g}}$ implies $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$ for any $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, it follows, consequently, that $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and the relation $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ hold. Hence, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \supseteq \bigcap_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$.

II. Since $\mathscr{U}_{\mathfrak{g}} \subseteq \mathscr{V}_{\mathfrak{g}}$ implies $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$ for any $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, it follows that $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}$ and $\mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}$ imply $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}})$, respectively. Thus, it follows that $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \subseteq \bigcup_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$. The proof of the theorem is complete. Q.E.D.

Proposition 3.17. *If* $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*exterior operator in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

$$\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \bigcap_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}). \tag{3.4}$$

for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$.

Proof. Let $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator and suppose $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

$$\begin{aligned} \mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}} \quad &\longmapsto \quad \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathscr{U}_{\mathfrak{g}}\Bigg) \\ &\longleftrightarrow \quad \mathfrak{g}\text{-Int}_{\mathfrak{g}}\Bigg(\bigcap_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\Bigg) \\ &\longleftrightarrow \quad \bigcap_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) = \bigcap_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}). \end{aligned}$$

Hence, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$. The proof of the proposition is complete. Q.E.D.

Proposition 3.18. *If* $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*exterior operator in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

$$\big(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\big)\big[\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \emptyset\big]. \tag{3.5}$$

Proof. Let $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator and suppose $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Thus $\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \emptyset$. The proof of the proposition is complete. Q.E.D.

In a $\mathscr{T}_{\mathfrak{g}}$-space, $\mathfrak{g}$-topologies can be constructed by means of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively as demonstrated in the following theorem.

Theorem 3.19. *If* $\big(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}, \mathfrak{g}\text{-Fr}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*exterior and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*frontier operators* $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *respectively, in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then the one-valued maps* $\mathscr{T}_{\mathfrak{g},\mathrm{E}}$, $\mathscr{T}_{\mathfrak{g},\mathrm{F}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *on* $\mathscr{P}(\Omega)$ *ranging in* $\mathscr{P}(\Omega)$ *defined as*

- I. $\mathscr{T}_{\mathfrak{g},\mathrm{E}}(\Omega) \stackrel{\text{def}}{=} \big\{\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \ \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g}}\big\}$,

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- II. $\mathscr{T}_{\mathfrak{g},\mathrm{F}}(\Omega) \stackrel{\text{def}}{=} \{\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\}$,

respectively, form $\mathfrak{g}$*-topologies on* Ω *in the* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}$.

Proof. Let $(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mathrm{F}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, and suppose $\mathscr{T}_{\mathfrak{g},\mathrm{E}}$, $\mathscr{T}_{\mathfrak{g},\mathrm{F}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ known. Then, for $\mathscr{T}_{\mathfrak{g},\mathrm{E}}$, $\mathscr{T}_{\mathfrak{g},\mathrm{F}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ to be $\mathfrak{g}$-topologies on Ω, both must satisfy the axioms for a $\mathfrak{g}$-topology on Ω: $\mathscr{T}_{\mathfrak{g},\mathrm{A}}(\emptyset) = \emptyset$, $\mathscr{T}_{\mathfrak{g},\mathrm{A}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathscr{O}_{\mathfrak{g}}$ and $\mathscr{T}_{\mathfrak{g},\mathrm{A}}\big(\bigcup_{\nu \in I^*_{\infty}} \mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu \in I^*_{\infty}} \mathscr{T}_{\mathfrak{g},\mathrm{A}}(\mathscr{O}_{\mathfrak{g},\nu})$ for every $\mathrm{A} \in \{\mathrm{E},\mathrm{F}\}$; evidently, $\{\mathscr{O}_{\mathfrak{g},\nu} : \ \nu \in I^*_{\infty}\} \subseteq \mathscr{P}(\Omega)$. The corresponding proofs follow.

I. Since $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}$ and, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathscr{O}_{\mathfrak{g}}$ for any $\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, it follows that

$$\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathscr{O}_{\mathfrak{g}},$$

implying $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g}}$ holds when $\mathscr{O}_{\mathfrak{g}} = \emptyset$. Thus, $\mathscr{T}_{\mathfrak{g},\mathrm{E}}(\emptyset) = \emptyset$. Since $\mathscr{T}_{\mathfrak{g},\mathrm{E}}(\Omega) \subseteq \{\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g}}\}$, it results that, for every $(\mathscr{O}_{\mathfrak{g}}, \mathscr{T}_{\mathfrak{g},\mathrm{E}}(\mathscr{O}_{\mathfrak{g}})) \in \mathscr{P}(\Omega) \times \mathscr{T}_{\mathfrak{g},\mathrm{E}}(\Omega)$, the relation $\mathscr{T}_{\mathfrak{g},\mathrm{E}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathscr{O}_{\mathfrak{g}}$ holds. Suppose $\{\mathscr{O}_{\mathfrak{g},\nu} : \ \nu \in I^*_{\infty}\} \subseteq \mathscr{P}(\Omega)$ such that, for every $\nu \in I^*_{\infty}$, the relation $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) = \mathscr{O}_{\mathfrak{g},\nu}$ holds. Then, since $\bigcup_{\nu \in I^*_{\infty}} \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) = \bigcup_{\nu \in I^*_{\infty}} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) = \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\bigcup_{\nu \in I^*_{\infty}} \mathscr{O}_{\mathfrak{g},\nu}\big)$, it results that

$$\begin{aligned} \textstyle\bigcup_{\nu \in I^*_{\infty}} \mathscr{O}_{\mathfrak{g},\nu} &= \bigcup_{\nu \in I^*_{\infty}} \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) \\ &= \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\Big(\bigcup_{\nu \in I^*_{\infty}} \mathscr{O}_{\mathfrak{g},\nu}\Big) = \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Big(\bigcup_{\nu \in I^*_{\infty}} \mathscr{O}_{\mathfrak{g},\nu}\Big) \\ &\qquad = \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Big(\bigcup_{\nu \in I^*_{\infty}} \mathscr{O}_{\mathfrak{g},\nu}\Big). \end{aligned}$$

Hence, $\mathscr{T}_{\mathfrak{g},\mathrm{E}}\big(\bigcup_{\nu \in I^*_{\infty}} \mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu \in I^*_{\infty}} \mathscr{T}_{\mathfrak{g},\mathrm{E}}(\mathscr{O}_{\mathfrak{g},\nu})$ and $\mathscr{T}_{\mathfrak{g},\mathrm{E}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is a $\mathfrak{g}$-topology on Ω in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

II. Since $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}$ and, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathscr{O}_{\mathfrak{g}}$ for any $\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, it follows that

$$\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{R}_{\mathfrak{g}} = \mathscr{O}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\Bigg) \subseteq \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}),$$

implying $\mathscr{O}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \supseteq \bigcup_{\mathscr{R}_{\mathfrak{g}} = \mathscr{O}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{O}_{\mathfrak{g}}$. Consequently, $\mathscr{O}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \supseteq \mathscr{O}_{\mathfrak{g}}$ and this relation holds when $\mathscr{O}_{\mathfrak{g}} = \emptyset$. Hence, $\mathscr{T}_{\mathfrak{g},\mathrm{F}}(\emptyset) = \emptyset$. On the other hand, because $\mathscr{T}_{\mathfrak{g},\mathrm{F}}(\Omega) \subseteq \{\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\}$, it follows that, for every $(\mathscr{O}_{\mathfrak{g}}, \mathscr{T}_{\mathfrak{g},\mathrm{F}}(\mathscr{O}_{\mathfrak{g}})) \in \mathscr{P}(\Omega) \times \mathscr{T}_{\mathfrak{g},\mathrm{F}}(\Omega)$, the relation $\mathscr{T}_{\mathfrak{g},\mathrm{F}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathscr{O}_{\mathfrak{g}}$ holds. Suppose $\{\mathscr{O}_{\mathfrak{g},\nu} : \ \nu \in I^*_{\infty}\} \subseteq \mathscr{P}(\Omega)$ such that, for every $\nu \in I^*_{\infty}$, the relation $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) \subseteq \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ holds. Then, since $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}$ and $\bigcap_{\nu \in I^*_{\infty}} \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\big(\bigcup_{\nu \in I^*_{\infty}} \mathscr{O}_{\mathfrak{g},\nu}\big)$, it results that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\textstyle\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{O}_{\mathfrak{g},\nu}\right) &\supseteq \bigcap_{\nu\in I_{\infty}^{*}} \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\nu}\right) \\
&\longleftrightarrow \bigcap_{\nu\in I_{\infty}^{*}} \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\nu}\right) \\
&\longleftrightarrow \bigcap_{\nu\in I_{\infty}^{*}} \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{R}_{\mathfrak{g},\nu}=\mathscr{O}_{\mathfrak{g},\nu},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g},\nu}\right)\Bigg) \\
&\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Bigg(\bigcup_{(\nu,\mathscr{R}_{\mathfrak{g},\nu})\in I_{\infty}^{*}\times\{\mathscr{O}_{\mathfrak{g},\nu},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})\}} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g},\nu}\right)\Bigg) \\
&\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{R}_{\mathfrak{g}}=\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{O}_{\mathfrak{g},\nu},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{O}_{\mathfrak{g},\nu}\right)} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\Bigg) \\
&\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\Bigg(\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{O}_{\mathfrak{g},\nu}\Bigg) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Bigg(\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{O}_{\mathfrak{g},\nu}\Bigg)
\end{aligned}$$

Hence, $\mathscr{T}_{\mathfrak{g},\mathrm{F}}\left(\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{O}_{\mathfrak{g},\nu}\right) = \bigcup_{\nu\in I_{\infty}^{*}}\mathscr{T}_{\mathfrak{g},\mathrm{F}}\left(\mathscr{O}_{\mathfrak{g},\nu}\right)$ and $\mathscr{T}_{\mathfrak{g},\mathrm{F}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is a $\mathfrak{g}$-topology on Ω in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. The proof of the theorem is complete. Q.E.D.

PROPOSITION 3.20. *If* $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-exterior operator in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then the following logical implications hold for any* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$*:*

$$\begin{gathered}
\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right) \supseteq \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right) \\
\wedge\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right) \\
\downarrow \\
\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right) \supseteq \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
\downarrow \\
\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right) \supseteq \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right).
\end{gathered} \tag{3.6}$$

PROOF. Let $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, suppose the following logical conjunction holds:

$$\begin{gathered}
\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right) \supseteq \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right) \\
\wedge\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right).
\end{gathered}$$

Set $\mathscr{R}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)$. Then, by virtue of the logical statement following $\wedge$, it results that

$$\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right) \supseteq \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right).$$

But, $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)$ by virtue of that preceding $\wedge$. Thus, $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right) \supseteq \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)$. On the other hand, since $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)$ for any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, it

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follows that

$$\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\supseteq \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\supseteq \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right).$$

Hence, $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\supseteq \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)$. The proof of the proposition is complete. Q.E.D.

Proposition 3.21. *If* $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-exterior operator in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$*, then the following logical implication holds for any* $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$*:*

$$\begin{aligned}&\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\subseteq \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right)\\ &\qquad\wedge\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}\right)=\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\\ &\downarrow\\ &\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right)=\emptyset\\ &\downarrow\\ &\left(\mathscr{R}_{\mathfrak{g}}\cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right)\cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right)=\emptyset.\end{aligned}\tag{3.7}$$

Proof. Let $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ and, for any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$, suppose the following logical conjunction holds:

$$\begin{aligned}&\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\subseteq \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right)\\ &\qquad\wedge\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}\right)=\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right).\end{aligned}$$

Set $\mathscr{S}_{\mathfrak{g}}=\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)$. Then, by virtue of the logical statements following and preceding the logical connective $\wedge$, $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right)=\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)=\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)$. Therefore,

$$\begin{aligned}&\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right)\\ &\qquad=\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right).\end{aligned}$$

But, $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)=\emptyset$. Thus,

$$\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right)=\emptyset.$$

Since $\mathscr{R}_{\mathfrak{g}}\cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right)\longleftrightarrow\mathscr{R}_{\mathfrak{g}}\cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)=\emptyset$ obviously, $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right)=\emptyset$ implies

$$\left(\mathscr{R}_{\mathfrak{g}}\cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right)\cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right)=\emptyset.$$

The proof of the proposition is complete. Q.E.D.

Proposition 3.22. *If* $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-exterior operator in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$*, then the following logical implication holds for any* $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$*:*

$$\begin{aligned}&\left(\mathscr{R}_{\mathfrak{g}}\cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)=\emptyset\right)\wedge\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\subseteq \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right)\\ &\quad\wedge\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}\right)=\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\\ &\downarrow\\ &\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right)=\emptyset.\end{aligned}\tag{3.8}$$

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Proof. Let $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ and, for any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$, suppose the following logical conjunction holds:

$$\begin{aligned}&\big(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})=\emptyset\big)\wedge\big(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\\&\quad\wedge\big(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big).\end{aligned}$$

Then, by virtue of the logical statement preceding the first logical connective $\wedge$, $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})=\emptyset$. Taking into account the setting $\mathscr{S}_{\mathfrak{g}}=\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ in the logical statement following the second logical connective $\wedge$ yields

$$\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)=\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).$$

Consequently,

$$\begin{aligned}&\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\\&\qquad=\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\\&\qquad\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})=\emptyset.\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)=\emptyset$. The proof of the proposition is complete. Q.E.D.

Proposition 3.23. *If* $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*exterior operator in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, *then the following logical implication holds for any* $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$:

$$\begin{aligned}&\big(\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)=\emptyset\big)\\&\qquad\wedge\big(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\\&\qquad\wedge\big(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\\&\qquad\qquad\Downarrow\\&\qquad\qquad\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})=\emptyset.\end{aligned}\tag{3.9}$$

Proof. Let $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ and, for any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$, suppose the following logical conjunction holds:

$$\begin{aligned}&\big(\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)=\emptyset\big)\\&\qquad\wedge\big(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\\&\qquad\wedge\big(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big).\end{aligned}$$

Then, by virtue of the logical statements preceding the first and following the last logical connective $\wedge$, it follows that

$$\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\big(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)=\emptyset.$$

By virtue of the logical statement following the first logical connective $\wedge$, it results that $\mathscr{R}_{\mathfrak{g}}\supseteq\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. Consequently,

$$\begin{aligned}\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\ &\subseteq\ \big(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\\&\cap\ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})=\emptyset.\end{aligned}$$

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Hence, $\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \emptyset$. The proof of the proposition is complete. Q.E.D.

THEOREM 3.24. *If* $(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}, \mathfrak{g}\text{-Fr}_{\mathfrak{g}}) \in \mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-exterior and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-frontier operators* $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$*, respectively, in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}\, \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \quad \forall \mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$,
- II. $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

PROOF. Let $(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}, \mathfrak{g}\text{-Fr}_{\mathfrak{g}}) \in \mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and suppose $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\mathscr{U}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ for any $\mathscr{U}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, by virtue of the definition of $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ it results that

$$\begin{aligned}\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ &\subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Thus, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. On the other hand, by virtue of the definition of $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ it follows that

$$\begin{aligned}\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\, \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\Bigg) \\ &\longleftrightarrow \bigcap_{\mathscr{U}_{\mathfrak{g}} = \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\, \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \\ &\subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Hence, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the theorem is complete. Q.E.D.

PROPOSITION 3.25. *If* $(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}, \mathfrak{g}\text{-Fr}_{\mathfrak{g}}) \in \mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-exterior and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-frontier operators* $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$*, respectively, in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$,
- II. $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

PROOF. Let $(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}, \mathfrak{g}\text{-Fr}_{\mathfrak{g}}) \in \mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and suppose $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then,

$$\mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).$$

Therefore, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. But, for any $\mathscr{U}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, it holds that $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. Thus, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. On the other hand, the following relation holds:

$$\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).$$

Hence, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. The proof of the proposition is complete. Q.E.D.

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PROPOSITION 3.26. *If* $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-frontier operator in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then the following logical implications hold for any* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$*:*

$$\begin{gathered}
\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \wedge \left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right) \\
\Big\downarrow \\
\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right) \subseteq \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
\Big\downarrow \\
\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).
\end{gathered} \tag{3.10}$$

PROOF. Let $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-frontier operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, suppose the following logical conjunction holds:

$$\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \wedge \left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right).$$

Then, by virtue of the logical statement following $\wedge$, it follows that

$$\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \subseteq \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).$$

By virtue of that preceding $\wedge$, if follows that $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. Thus, $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right) \subseteq \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Since $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ for any $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) &\subseteq \left(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \\
&\subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).
\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. The proof of the proposition is complete. Q.E.D.

PROPOSITION 3.27. *If* $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-frontier operator in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then the following logical implication holds for any* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$*:*

$$\begin{gathered}
\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \wedge \left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \subseteq \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \\
\wedge\left(\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \longrightarrow \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right) \\
\Big\downarrow \\
\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).
\end{gathered} \tag{3.11}$$

PROOF. Let $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-frontier operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, suppose the following logical conjunction holds:

$$\begin{gathered}
\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \wedge \left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \subseteq \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \\
\wedge\left(\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \longrightarrow \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right).
\end{gathered}$$

Then, by virtue of the logical statements preceding the first and following the last logical connective $\wedge$, it results that $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ for any $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. Because $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \subseteq \mathscr{V}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$ holds for any $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times$

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$\mathscr{P}(\Omega)$ such that $\mathscr{U}_{\mathfrak{g}} \subseteq \mathscr{V}_{\mathfrak{g}}$, setting $\mathscr{U}_{\mathfrak{g}} = \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ and $\mathscr{V}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ in $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \subseteq \mathscr{V}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$ yields

$$\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \left(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right).$$

But $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \subseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. Therefore, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. Since $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ is arbitrary, it holds, in particular, for its complement. Thus, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ which, by virtue of $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ reduces to $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. Consequently,

$$\begin{aligned}
\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\
&\subseteq \left(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \cap \left(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \\
&\longleftrightarrow \left(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).
\end{aligned}$$

Hence, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is complete. Q.E.D.

Our first research objective concerning the definitions and the essential properties of a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators in $\mathscr{T}_{\mathfrak{g}}$-spaces is now complete. We conclude the present section with two corollaries which will be useful in the section following it.

In view of Cor. 3.15, Props 3.6, 3.17, 3.18, and Thms 3.7, 3.24, it follows at once that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}$: $\mathrm{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, also possess similar properties analogous to the *Kuratowski closure Axioms* which can be grouped and stated in the form of a corollary.

Corollary 3.28. *Let* $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-exterior and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-frontier operators in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. *Then:*

- *For every* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$,
  - i. $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
  - ii. $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$,
  - iii. $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset)$,
  - iv. $\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \emptyset$.
- *For every* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$,
  - v. $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
  - vi. $(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \left(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)$,
  - vii. $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\emptyset) = \emptyset$,
  - viii. $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$.

For each of Items i.–iv. of Cor. 3.28, it may be well to give a concrete interpretation whenever it is satisfied by $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; likewise for a concrete interpretation whenever each of Items v.–viii. of Cor. 3.28 is satisfied by $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. Accordingly, the following nice Mathematical vocabulary presents itself: The $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is said to be *contracting*, $\cup$*-destroyed*, Ω*-grounded relative to* $\emptyset$ and *disjointed relative to its argument* if and only if it satisfies Items i., ii., iii. and iv., respectively, of Cor. 3.28; the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is said to be *contracting*, $\cup$*-preserved*, $\emptyset$*-grounded* and *complement-invariant* if and only if it satisfies Items v, vi., vii. and viii., respectively, of Cor. 3.28.

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Because Items i.–viii. of Cor. 3.28 also hold when the role of $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$ is given to $\left(\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}),\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$, another corollary equivalent to Cor. 3.28 is the following.

Corollary 3.29. *Let* $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *be a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-exterior and a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-frontier operators in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. *Then:*

- *For every* $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$,
  - i. $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
  - ii. $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})=\bigcap_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$,
  - iii. $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\emptyset)=\emptyset$,
  - iv. $\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})=\emptyset$.
- *For every* $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$,
  - v. $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\supseteq\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
  - vi. $\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})=\bigcap_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\left(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)$,
  - vii. $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\emptyset)=\emptyset$,
  - viii. $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,

  where $\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}$.

Hence, in a $\mathscr{T}_{\mathfrak{g}}$-space, a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-exterior operator is a one-valued map $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ satisfying a list of *derived set* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-exterior operator conditions* (Items i.–iv. of Cor. 3.28 or Cor. 3.29), and a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-frontier operator is a one-valued map $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ satisfying a list of *derived set* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-frontier operator conditions* (Items v.–viii. of Cor. 3.28 or 3.29). For the derived set $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-exterior operator conditions to stand as *fundamental* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-exterior operator axioms* and the derived set $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-frontier operator conditions to stand as *fundamental* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-frontier operator axioms*, they must not, of course, be inconsistent nor be free from redundancies.

What is presented below concerns the study related to the consistency, independency of the two lists of conditions.

3.2. Consistent, Independent Axioms. In this section, the focus is on the construction of some sets of axioms for the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-frontier operators in $\mathscr{T}_{\mathfrak{g}}$-spaces. Proofs related to the consistency, independency of those sets of axioms are given.

The discussion will be facilitated by the following definition in which is introduced a list of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms.

Definition 3.30. Let $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-frontier operator and let $(\mathscr{R}_{\mathfrak{g}}\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$ be an arbitrary pair in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$.

Then, $\mathrm{Ax}_{\mathrm{E},\nu} : \mathfrak{g}\text{-}\mathrm{E}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longrightarrow \mathbb{B} \stackrel{\text{def}}{=} \{0,1\}$, $\nu \in I_6^*$, defined as

$$\begin{aligned}
\mathrm{Ax}_{\mathrm{E},1}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) &\stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \subseteq \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right), \\
\mathrm{Ax}_{\mathrm{E},2}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) &\stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right) \cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right), \\
\mathrm{Ax}_{\mathrm{E},3}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) &\stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\emptyset) = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\emptyset), \\
\mathrm{Ax}_{\mathrm{E},4}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) &\stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Ext} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right) \subseteq \mathfrak{g}\text{-}\mathrm{Ext} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right), \\
\mathrm{Ax}_{\mathrm{E},5}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) &\stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Ext} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right) \cap \mathfrak{g}\text{-}\mathrm{Ext} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
&= \mathfrak{g}\text{-}\mathrm{Ext} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right), \\
\mathrm{Ax}_{\mathrm{E},6}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) &\stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\emptyset) = \emptyset,
\end{aligned}$$

and belonging to $\mathrm{AX}\big[\mathfrak{g}\text{-}\mathrm{E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\big] \stackrel{\text{def}}{=} \left\{\mathrm{Ax}_{\mathrm{E},\nu}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) :\ \nu \in I_6^*\right\}$, are called "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms."

Thus, in a $\mathscr{T}_{\mathfrak{g}}$-space, $\mathrm{AX}\big[\mathfrak{g}\text{-}\mathrm{E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\big]$ forms a class of derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms.

Lemma 3.31. *Let $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ such that, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, the following statements hold:*

$$\left(\forall \nu \in I_3^*\right)\left[\mathrm{Ax}_{\mathrm{E},\nu}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1\right]. \tag{3.12}$$

Then, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operator and the associated $\mathfrak{g}$-topology is $\mathscr{T}_{\mathfrak{g},\mathrm{Cl}}^{\mathrm{E}}(\Omega) \stackrel{\text{def}}{=} \left\{\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega) :\ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g}}\right)\right\}$.

Proof. Let $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ such that, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, the following $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms hold: $\mathrm{Ax}_{\mathrm{E},\nu}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1$ for every $\nu \in I_3^*$. Then, by virtue of $\mathrm{Ax}_{\mathrm{E},3}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right)$, it follows that $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\emptyset) = \emptyset$. Thus, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}}(\emptyset) = \emptyset$ and $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is $\emptyset$-grounded. From the definition of $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, it follows that $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathscr{S}_{\mathfrak{g}}$. Hence, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}}\left(\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathscr{S}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is expansive. By virtue of $\mathrm{Ax}_{\mathrm{E},1}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right)$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}}\left(\mathscr{S}_{\mathfrak{g}}\right) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
&\supseteq \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}}\left(\mathscr{S}_{\mathfrak{g}}\right).
\end{aligned}$$

Thus, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}}\left(\mathscr{S}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}}\left(\mathscr{S}_{\mathfrak{g}}\right)$ and $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is idempotent. By virtue of $\mathrm{Ax}_{\mathrm{E},2}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right)$, it follows that $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right) \cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)$. Hence, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}}\left(\mathscr{R}_{\mathfrak{g}}\right) \cup \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}}\left(\mathscr{S}_{\mathfrak{g}}\right)$ and $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is $\cup$-additive.

For $\mathscr{T}_{\mathfrak{g},\mathrm{Cl}}^{\mathrm{E}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ to be a $\mathfrak{g}$-topology on Ω, it must satisfy the axioms for a $\mathfrak{g}$-topology on Ω: $\mathscr{T}_{\mathfrak{g},\mathrm{Cl}}^{\mathrm{E}}(\emptyset) = \emptyset$, $\mathscr{T}_{\mathfrak{g},\mathrm{Cl}}^{\mathrm{E}}\left(\mathscr{O}_{\mathfrak{g}}\right) \subseteq \mathscr{O}_{\mathfrak{g}}$ and $\mathscr{T}_{\mathfrak{g},\mathrm{Cl}}^{\mathrm{E}}\left(\bigcup_{\nu \in I_\infty^*} \mathscr{O}_{\mathfrak{g},\nu}\right) = \bigcup_{\nu \in I_\infty^*} \mathscr{T}_{\mathfrak{g},\mathrm{Cl}}^{\mathrm{E}}\left(\mathscr{O}_{\mathfrak{g},\nu}\right)$; evidently, $\left\{\mathscr{O}_{\mathfrak{g},\nu} :\ \nu \in I_\infty^*\right\} \subseteq \mathscr{P}(\Omega)$. Clearly, the relation $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g}}\right)$ holds when $\mathscr{O}_{\mathfrak{g}} = \emptyset$. Thus, $\mathscr{T}_{\mathfrak{g},\mathrm{Cl}}^{\mathrm{E}}(\emptyset) = \emptyset$. Since

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$\mathscr{T}^{\mathrm{E}}_{\mathfrak{g},\mathrm{Cl}}(\Omega) \subseteq \{\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \mathfrak{g}\text{-}\mathrm{Cl}^{\mathrm{E}}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\}$, it results that, for every $(\mathscr{O}_{\mathfrak{g}}, \mathscr{T}^{\mathrm{E}}_{\mathfrak{g},\mathrm{Cl}}(\mathscr{O}_{\mathfrak{g}})) \in \mathscr{P}(\Omega) \times \mathscr{T}^{\mathrm{E}}_{\mathfrak{g},\mathrm{Cl}}(\Omega)$, the relation $\mathscr{T}^{\mathrm{E}}_{\mathfrak{g},\mathrm{Cl}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathscr{O}_{\mathfrak{g}}$ holds. Suppose $\{\mathscr{O}_{\mathfrak{g},\nu} : \nu \in I^{*}_{\infty}\} \subseteq \mathscr{P}(\Omega)$ such that, for every $\nu \in I^{*}_{\infty}$, the relation $\mathfrak{g}\text{-}\mathrm{Cl}^{\mathrm{E}}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ holds. Then, since $\bigcap_{\nu\in I^{*}_{\infty}} \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\bigcup_{\nu\in I^{*}_{\infty}} \mathscr{O}_{\mathfrak{g},\nu})$, it results that

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\big(\textstyle\bigcup_{\nu\in I^{*}_{\infty}} \mathscr{O}_{\mathfrak{g},\nu}\big) &= \bigcap_{\nu\in I^{*}_{\infty}} \mathfrak{g}\text{-}\mathrm{Cl}^{\mathrm{E}}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) \\ &= \mathfrak{g}\text{-}\mathrm{Cl}^{\mathrm{E}}_{\mathfrak{g}}\Big(\bigcap_{\nu\in I^{*}_{\infty}} \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})\Big) = \mathfrak{g}\text{-}\mathrm{Cl}^{\mathrm{E}}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Big(\bigcup_{\nu\in I^{*}_{\infty}} \mathscr{O}_{\mathfrak{g},\nu}\Big). \end{aligned}$$

Therefore, $\mathscr{T}^{\mathrm{E}}_{\mathfrak{g},\mathrm{Cl}}(\bigcup_{\nu\in I^{*}_{\infty}} \mathscr{O}_{\mathfrak{g},\nu}) = \bigcup_{\nu\in I^{*}_{\infty}} \mathscr{T}^{\mathrm{E}}_{\mathfrak{g},\mathrm{Cl}}(\mathscr{O}_{\mathfrak{g},\nu})$ holds. Hence, $\mathfrak{g}\text{-}\mathrm{Cl}^{\mathrm{E}}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operator and the associated $\mathfrak{g}$-topology is $\mathscr{T}^{\mathrm{E}}_{\mathfrak{g},\mathrm{Cl}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. The proof of the lemma is complete.

Q.E.D.

Lemma 3.32. *Let $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ such that, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, the following statements hold:*

$$(\forall \nu \in I^{*}_{3})\big[\mathrm{Ax}_{\mathrm{E},3+\nu}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}) = 1\big]. \tag{3.13}$$

Then, $\mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operator and the associated $\mathfrak{g}$-topology is $\mathscr{T}^{\mathrm{E}}_{\mathfrak{g},\mathrm{Int}}(\Omega) \stackrel{\mathrm{def}}{=} \{\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathscr{O}_{\mathfrak{g}}\}$.

Proof. Let $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ such that, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, the following $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms hold: $\mathrm{Ax}_{\mathrm{E},3+\nu}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}) = 1$ for every $\nu \in I^{*}_{3}$. Then, by virtue of $\mathrm{Ax}_{\mathrm{E},6}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})$, $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\emptyset) = \emptyset$ is equivalent to $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\Omega) = \Omega$ and, $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\Omega) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}}(\Omega)$. Thus, it results that $\mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}}(\Omega) = \Omega$ and $\mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is Ω-grounded. From the definition of $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, it follows that $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}}$. Hence, $\mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is non-expansive. By virtue of $\mathrm{Ax}_{\mathrm{E},4}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})$,

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Ext} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\subseteq \mathfrak{g}\text{-}\mathrm{Ext} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}). \end{aligned}$$

But, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}}$. Hence, it follows that $\mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is idempotent. By virtue of $\mathrm{Ax}_{\mathrm{E},5}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})$,

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Ext} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Ext} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ &= \mathfrak{g}\text{-}\mathrm{Ext} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}). \end{aligned}$$

Thus, $\mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-}\mathrm{Int}^{\mathrm{E}}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is $\cap$-additive.

For $\mathscr{T}^{\mathrm{E}}_{\mathfrak{g},\mathrm{Int}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ to be a $\mathfrak{g}$-topology on Ω, it must satisfy the axioms for a $\mathfrak{g}$-topology on Ω: $\mathscr{T}^{\mathrm{E}}_{\mathfrak{g},\mathrm{Int}}(\emptyset) = \emptyset$, $\mathscr{T}^{\mathrm{E}}_{\mathfrak{g},\mathrm{Int}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathscr{O}_{\mathfrak{g}}$ and $\mathscr{T}^{\mathrm{E}}_{\mathfrak{g},\mathrm{Int}}(\bigcup_{\nu\in I^{*}_{\infty}} \mathscr{O}_{\mathfrak{g},\nu}) =$

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$\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{E}}(\mathscr{O}_{\mathfrak{g},\nu})$; evidently, $\{\mathscr{O}_{\mathfrak{g},\nu}:\ \nu\in I_{\infty}^{*}\}\subseteq\mathscr{P}(\Omega)$. Clearly, the relation $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{E}}(\mathscr{O}_{\mathfrak{g}})\subseteq\mathscr{O}_{\mathfrak{g}}$ holds when $\mathscr{O}_{\mathfrak{g}}=\emptyset$. Thus, $\mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{E}}(\emptyset)=\emptyset$. Since $\mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{E}}(\Omega)\subseteq\{\mathscr{O}_{\mathfrak{g}}\in\mathscr{P}(\Omega):\ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{E}}(\mathscr{O}_{\mathfrak{g}})\subseteq\mathscr{O}_{\mathfrak{g}}\}$, it results that, for every $(\mathscr{O}_{\mathfrak{g}},\mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{E}}(\mathscr{O}_{\mathfrak{g}}))\in\mathscr{P}(\Omega)\times\mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{E}}(\Omega)$, the relation $\mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{E}}(\mathscr{O}_{\mathfrak{g}})\subseteq\mathscr{O}_{\mathfrak{g}}$ holds. Suppose $\{\mathscr{O}_{\mathfrak{g},\nu}:\ \nu\in I_{\infty}^{*}\}\subseteq\mathscr{P}(\Omega)$ such that, for every $\nu\in I_{\infty}^{*}$, the relation $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{E}}(\mathscr{O}_{\mathfrak{g},\nu})\subseteq\mathscr{O}_{\mathfrak{g},\nu}$ holds. Then,

$$\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{E}}\big(\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{O}_{\mathfrak{g},\nu}\big)\longleftrightarrow\bigcup_{\nu\in I_{\infty}^{*}}\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{E}}(\mathscr{O}_{\mathfrak{g},\nu})\subseteq\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{O}_{\mathfrak{g},\nu}.$$

Consequently, $\mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{E}}\big(\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{O}_{\mathfrak{g},\nu}\big)=\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{E}}(\mathscr{O}_{\mathfrak{g},\nu})$. Hence, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{E}}:\mathscr{S}_{\mathfrak{g}}\longmapsto\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operator and the associated $\mathfrak{g}$-topology is $\mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{E}}:\ \mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. The proof of the lemma is complete. Q.E.D.

THEOREM 3.33. *Let* $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-exterior operator in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ *and, let* $\mathrm{Ax}_{\mathrm{E},7}:\mathfrak{g}\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{g}}]\longrightarrow\mathbb{B}$ *such that, for any* $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$, $\mathrm{Ax}_{\mathrm{E},7}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})\overset{\mathrm{def}}{\longleftrightarrow}\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})=\emptyset$. *If* $\mathrm{Ax}_{\mathrm{E},7}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})=1$, *then*

$$(\forall\nu\in I_{3}^{*})\big[\mathrm{Ax}_{\mathrm{E},\nu}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})=1\ \longleftrightarrow\ \mathrm{Ax}_{\mathrm{E},3+\nu}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})=1\big].\tag{3.14}$$

PROOF. Let $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ and, let $\mathrm{Ax}_{\mathrm{E},7}:\mathfrak{g}\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{g}}]\longrightarrow\mathbb{B}$ such that, for any $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$, $\mathrm{Ax}_{\mathrm{E},7}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})\overset{\mathrm{def}}{\longleftrightarrow}\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})=\emptyset$. Suppose the relation $\mathrm{Ax}_{\mathrm{E},7}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})=1$ holds. Then, the statement that, for any $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$, $\mathrm{Ax}_{\mathrm{E},7}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})\overset{\mathrm{def}}{\longleftrightarrow}\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})=\emptyset$ holds. For $\nu=1$, set $\mathscr{R}_{\mathfrak{g}}=\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Then,

$$\begin{aligned}\mathrm{Ax}_{\mathrm{E},1}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})&\overset{\mathrm{def}}{\longleftrightarrow}\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\\&\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\&\subseteq\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\&\overset{\mathrm{def}}{\longleftrightarrow}\mathrm{Ax}_{\mathrm{E},4}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}).\end{aligned}$$

Thus, $\mathrm{Ax}_{\mathrm{E},1}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})=1\longleftrightarrow\mathrm{Ax}_{\mathrm{E},4}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})=1$. For $\nu=2$, set $\mathscr{R}_{\mathfrak{g}}=\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ and $\mathscr{S}_{\mathfrak{g}}=\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$. Then,

$$\begin{aligned}\mathrm{Ax}_{\mathrm{E},2}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})&\overset{\mathrm{def}}{\longleftrightarrow}\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\&\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Ext}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}\cap\mathscr{V}_{\mathfrak{g}})\\&=\mathfrak{g}\text{-}\mathrm{Ext}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\cap\mathfrak{g}\text{-}\mathrm{Ext}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})\\&\overset{\mathrm{def}}{\longleftrightarrow}\mathrm{Ax}_{\mathrm{E},5}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}).\end{aligned}$$

Thus, $\mathrm{Ax}_{\mathrm{E},2}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})=1\longleftrightarrow\mathrm{Ax}_{\mathrm{E},5}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})=1$. For $\nu=3$, set $\emptyset=\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\Omega)$. Then,

$$\mathrm{Ax}_{\mathrm{E},3}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})\overset{\mathrm{def}}{\longleftrightarrow}\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\emptyset)=\emptyset\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Ext}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\Omega)=\Omega.$$

But, the relation $\mathfrak{g}\text{-}\mathrm{Ext}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\Omega)=\Omega$ is equivalent $\mathfrak{g}\text{-}\mathrm{Ext}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\emptyset)=\emptyset$ and hence, $\mathrm{Ax}_{\mathrm{E},3}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})=1\longleftrightarrow\mathrm{Ax}_{\mathrm{E},6}(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}})=1$. The proof of the theorem is complete. Q.E.D.

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For every $\mu \in I_8^*$, set $\mathrm{Ax}_{\mathrm{E},\mu} = \mathrm{Ax}_{\mathrm{E},\mu}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right)$ and, consider $I_{\varepsilon(1)}^* \stackrel{\mathrm{def}}{=} \{1,2,3,7\}$, $I_{\varepsilon(2)}^* \stackrel{\mathrm{def}}{=} \{1,2,6,7\}$, $I_{\varepsilon(3)}^* \stackrel{\mathrm{def}}{=} \{1,3,5,7\}$, $I_{\varepsilon(4)}^* \stackrel{\mathrm{def}}{=} \{1,5,6,7\}$, $I_{\varepsilon(5)}^* \stackrel{\mathrm{def}}{=} \{2,3,4,7\}$, $I_{\varepsilon(6)}^* \stackrel{\mathrm{def}}{=} \{2,4,6,7\}$, $I_{\varepsilon(7)}^* \stackrel{\mathrm{def}}{=} \{3,4,5,7\}$, and $I_{\varepsilon(8)}^* \stackrel{\mathrm{def}}{=} \{4,5,6,7\}$. Furthermore, for every $\mu \in I_8^*$, let $I_{\varepsilon(\mu)}^* \stackrel{\mathrm{def}}{=} \{\alpha(\mu), \beta(\mu), \delta(\mu), \varepsilon(\mu)\}$. The proposition follows.

Proposition 3.34. *Let $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-exterior operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and let $\left\{\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right] : \ \mu \in I_8^*\right\}$ be a collection of classes of derived set $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms:*

$$(3.15) \qquad \left(\forall \mu \in I_8^*\right)\left[\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right] \stackrel{\mathrm{def}}{=} \left\{\mathrm{Ax}_{\mathrm{E},\nu} : \ \nu \in I_{\varepsilon(\mu)}^*\right\}\right].$$

Suppose there exist a $\mu \in I_8^*$ such that $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ satisfies the derived set $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms in $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$, then:

- I. *$\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{E}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closure operators and the associated $\mathfrak{g}$-topologies are $\mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{E}}$, $\mathscr{T}_{\mathfrak{g},\mathrm{Cl}}^{\mathrm{E}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively;*
- II. *$\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-exterior operator and the associated $\mathfrak{g}$-topology is $\mathscr{T}_{\mathfrak{g},\mathrm{Ext}} = \mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{E}} = \mathscr{T}_{\mathfrak{g},\mathrm{Cl}}^{\mathrm{E}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.*

Proof. Let $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-exterior operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and let $\left\{\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right] : \ \mu \in I_8^*\right\}$ be a collection of classes $\mathrm{AX}_1\left[\mathfrak{g}\text{-}\mathrm{E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right] \stackrel{\mathrm{def}}{=} \left\{\mathrm{Ax}_{\mathrm{E},\nu} : \ \nu \in I_{\varepsilon(1)}^*\right\}$, $\mathrm{AX}_2\left[\mathfrak{g}\text{-}\mathrm{E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right] \stackrel{\mathrm{def}}{=} \left\{\mathrm{Ax}_{\mathrm{E},\nu} : \ \nu \in I_{\varepsilon(2)}^*\right\}$, $\ldots$, $\mathrm{AX}_8\left[\mathfrak{g}\text{-}\mathrm{E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right] \stackrel{\mathrm{def}}{=} \left\{\mathrm{Ax}_{\mathrm{E},\nu} : \ \nu \in I_{\varepsilon(8)}^*\right\}$ of derived set $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms. Suppose

$$\left(\exists \mu \in I_8^*\right)\left(\forall \nu \in I_{\varepsilon(\mu)}^*\right)\left[\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right] \ni \mathrm{Ax}_{\mathrm{E},\nu}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1\right].$$

Then:

I. Since $\bigcap_{\mu \in I_8^*} \mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right] = \{\mathrm{Ax}_{\mathrm{E},7}\}$ and $\mathrm{Ax}_{\mathrm{E},7}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1$ implies

$$\left(\forall \nu \in I_3^*\right)\left[\mathrm{Ax}_{\mathrm{E},\nu}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1 \ \longleftrightarrow \ \mathrm{Ax}_{\mathrm{E},3+\nu}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1\right],$$

it follows that, for each $\nu \in I_3^*$, the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-exterior operator $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ satisfies both $\mathrm{Ax}_{\mathrm{E},\nu}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1$ and $\mathrm{Ax}_{\mathrm{E},3+\nu}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1$. Hence, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{E}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ are the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closure operators and, $\mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{E}}(\Omega) \stackrel{\mathrm{def}}{=} \left\{\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{E}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathscr{O}_{\mathfrak{g}}\right\}$ and $\mathscr{T}_{\mathfrak{g},\mathrm{Cl}}^{\mathrm{E}}(\Omega) \stackrel{\mathrm{def}}{=} \left\{\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\right\}$ are the associated $\mathfrak{g}$-topologies, respectively.

II. Because $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{E}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}}(\mathscr{S}_{\mathfrak{g}})$ holds for any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, it follows that $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-exterior operator. Whenever $\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, it follows that

$$\begin{aligned}
\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\mathrm{Cl}}^{\mathrm{E}} \quad &\longleftrightarrow \quad \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{E}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \\
&\longleftrightarrow \quad \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \\
&\longleftrightarrow \quad \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g}} \\
&\longleftrightarrow \quad \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{E}}(\mathscr{O}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g}} \ \longleftrightarrow \ \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{E}}.
\end{aligned}$$

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Hence, the associated $\mathfrak{g}$-topology is $\mathscr{T}_{\mathfrak{g},\mathrm{Ext}} = \mathscr{T}^{\mathrm{E}}_{\mathfrak{g},\mathrm{Int}} = \mathscr{T}^{\mathrm{E}}_{\mathfrak{g},\mathrm{Cl}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. The proof of the proposition is complete. Q.E.D.

For every $\mu \in I^*_8$, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms in $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$ are not inconsistent among themselves, as proved in the following proposition.

Proposition 3.35. *The classes* $\mathrm{AX}_1\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right], \ldots, \mathrm{AX}_8\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *such that, for every* $\mu \in I^*_8$, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *satisfies the axioms in* $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right] \stackrel{\mathrm{def}}{=} \left\{\mathrm{Ax}_{\mathrm{E},\nu} : \ \nu \in I^*_{\varepsilon(\mu)}\right\}$ *are composed of consistent derived set* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-exterior operator axioms.*

Proof. Let $\mathrm{AX}_1\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right], \ldots, \mathrm{AX}_8\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$ be given classes in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ such that, for every $\mu \in I^*_8$, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ satisfies the axioms in $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right] \stackrel{\mathrm{def}}{=} \left\{\mathrm{Ax}_{\mathrm{E},\nu} : \ \nu \in I^*_{\varepsilon(\mu)}\right\}$. Since for each $(\mu, \alpha(\mu)) \in I^*_8 \times I^*_{\varepsilon(\mu)}$ there exists exactly one $(\nu, \alpha(\nu)) \in I^*_8 \times I^*_{\varepsilon(\nu)}$ such that

$$\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right] \ni \mathrm{Ax}_{\mathrm{E},\alpha(\mu)} \longleftrightarrow \mathrm{Ax}_{\mathrm{E},\alpha(\nu)} \in \mathrm{AX}_{\nu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right],$$

to prove the proposition, it suffices to select a $\mu \in I^*_8$ and then show that the elements of $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right] = \left\{\mathrm{Ax}_{\mathrm{E},\alpha(\mu)}, \mathrm{Ax}_{\mathrm{E},\beta(\mu)}, \mathrm{Ax}_{\mathrm{E},\delta(\mu)}, \mathrm{Ax}_{\mathrm{E},\varepsilon(\mu)}\right\}$ are consistent. Select $\mu \in I^*_8 \setminus I^*_7$ so that $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$ becomes $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right] = \left\{\mathrm{Ax}_{\mathrm{E},\alpha(1)}, \mathrm{Ax}_{\mathrm{E},\beta(1)}, \mathrm{Ax}_{\mathrm{E},\delta(1)}, \mathrm{Ax}_{\mathrm{E},\varepsilon(1)}\right\}$. Suppose $\left(\Omega, \mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)$ be a mathematical system such that, for any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is defined as thus: $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\mathrm{def}}{=} \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Then:

$$\begin{aligned}
\mathrm{Ax}_{\mathrm{E},1}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right) &\stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}});\\
\mathrm{Ax}_{\mathrm{E},2}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right) &\stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}});\\
\mathrm{Ax}_{\mathrm{E},3}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right) &\stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset) \\
&\longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset);\\
\mathrm{Ax}_{\mathrm{E},7}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right) &\stackrel{\mathrm{def}}{\longleftrightarrow} \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \emptyset \\
&\longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \emptyset.
\end{aligned}$$

Thus, $\mathrm{Ax}_{\mathrm{E},\nu}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right) = 1$ for every $\nu \in I^*_{\varepsilon(1)}$. Therefore, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms $\mathrm{Ax}_{\mathrm{E},\alpha(1)}$, $\mathrm{Ax}_{\mathrm{E},\beta(1)}$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}$ themselves, and all their consequences, are simply set-theoretic expressions of the properties of the structure $\left(\Omega, \mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)$, and thus cannot involve contradiction, since no structure which really exists can have contradictory properties. Hence, for every $\mu \in I^*_8$, the class $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right] \stackrel{\mathrm{def}}{=} \left\{\mathrm{Ax}_{\mathrm{E},\nu} : \ \nu \in I^*_{\varepsilon(\mu)}\right\}$ are composed of consistent derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms. The proof of the proposition is complete. Q.E.D.

For the sake of elegance, for each $\mu \in I^*_8$, every class $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$ of derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms should be free from redundancies; that is, for each $\mu \in I^*_8$, the derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms of every class

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$\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$ should be independent, no one of them deductible from the rest. The theorem follows.

THEOREM 3.36. *The classes* $\mathrm{AX}_{1}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$, …, $\mathrm{AX}_{8}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ *such that, for every* $\mu\in I_{8}^{*}$, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *satisfies the axioms in* $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]\stackrel{\text{def}}{=}\left\{\mathrm{Ax}_{\mathrm{E},\nu}:\ \nu\in I_{\varepsilon(\mu)}^{*}\right\}$ *are composed of independent derived set* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-exterior operator axioms.*

PROOF. Let $\mathrm{AX}_{1}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$, …, $\mathrm{AX}_{8}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$ be given classes in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ such that, for every $\mu\in I_{8}^{*}$, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ satisfies the axioms in $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]\stackrel{\text{def}}{=}\left\{\mathrm{Ax}_{\mathrm{E},\nu}:\ \nu\in I_{\varepsilon(\mu)}^{*}\right\}$. Since for each $(\mu,\alpha(\mu))\in I_{8}^{*}\times I_{\varepsilon(\mu)}^{*}$ there exists exactly one $(\nu,\alpha(\nu))\in I_{8}^{*}\times I_{\varepsilon(\nu)}^{*}$ such that

$$\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]\ni\mathrm{Ax}_{\mathrm{E},\alpha(\mu)}\longleftrightarrow\mathrm{Ax}_{\mathrm{E},\alpha(\nu)}\in\mathrm{AX}_{\nu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right],$$

to prove the theorem, it suffices to select a $\mu\in I_{8}^{*}$ and then show that the elements of $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]=\left\{\mathrm{Ax}_{\mathrm{E},\alpha(\mu)},\mathrm{Ax}_{\mathrm{E},\beta(\mu)},\mathrm{Ax}_{\mathrm{E},\delta(\mu)},\mathrm{Ax}_{\mathrm{E},\varepsilon(\mu)}\right\}$ are independent by exhibiting, in the case of each $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-exterior operator axiom, a structure $\left(\Omega,\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)$ which satisfies all the other $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms, but not the one in question.

Set $\mu\in I_{8}^{*}\setminus I_{7}^{*}$ then, $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]=\left\{\mathrm{Ax}_{\mathrm{E},\alpha(1)},\mathrm{Ax}_{\mathrm{E},\beta(1)},\mathrm{Ax}_{\mathrm{E},\delta(1)},\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}\right\}$. Then:

CASE I. Consider the structure $\left(\Omega,\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)$ where, $\Omega\stackrel{\text{def}}{=}\bigcup_{\nu\in I_{2}^{*}}\Omega_{\nu}$ and, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is defined as thus: $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\stackrel{\text{def}}{=}\mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\cup\Omega_{2}\right)$ for every $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$. Then, $\mathrm{Ax}_{\mathrm{E},\alpha(1)}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)$, $\mathrm{Ax}_{\mathrm{E},\beta(1)}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)=1$ but $\mathrm{Ax}_{\mathrm{E},\delta(1)}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)=0$ because

$$\begin{aligned}\mathrm{Ax}_{\mathrm{E},\delta(1)}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right) &\stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset)=\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset)\\ &\longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\emptyset\cup\Omega_{2}\right)=\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset),\end{aligned}$$

and $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\emptyset\cup\Omega_{2}\right)=\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset)$ is untrue. Consequently, $\left(\Omega,\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)$ is a mathematical system such that $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ satisfies $\mathrm{Ax}_{\mathrm{E},\alpha(1)}$, $\mathrm{Ax}_{\mathrm{E},\beta(1)}$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}$ but not $\mathrm{Ax}_{\mathrm{E},\delta(1)}$. Therefore, $\mathrm{Ax}_{\mathrm{E},\delta(1)}$, then, cannot be a consequence of $\mathrm{Ax}_{\mathrm{E},\alpha(1)}$, $\mathrm{Ax}_{\mathrm{E},\beta(1)}$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}$. Hence, the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-exterior operator axiom $\mathrm{Ax}_{\mathrm{E},\delta(1)}$ is independent of the other $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms $\mathrm{Ax}_{\mathrm{E},\alpha(1)}$, $\mathrm{Ax}_{\mathrm{E},\beta(1)}$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}$.

CASE II. Consider the structure $\left(\Omega,\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)$ where, $\Omega\stackrel{\text{def}}{=}\bigcup_{\nu\in I_{3}^{*}}\Omega_{\nu}$ and, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is defined as thus: $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\stackrel{\text{def}}{=}\mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)$ for every $\mathscr{S}_{\mathfrak{g}}\in\{\emptyset,\Omega\}$, and every $\mathscr{S}_{\mathfrak{g}}\subseteq\Omega_{\nu}\cup\Omega_{\mu}$ such that, for each $(\nu,\mu)\in\{(1,2),(1,3),(2,3)\}$, the conditions $\mathscr{S}_{\mathfrak{g}}\cap\Omega_{\nu}$, $\mathscr{S}_{\mathfrak{g}}\cap\Omega_{\mu}\neq\emptyset$ hold; $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\stackrel{\text{def}}{=}\emptyset,\Omega_{3},\Omega_{2}$ if $\mathscr{S}_{\mathfrak{g}}\subseteq\Omega_{3},\Omega_{2}$, Ω_{1}, respectively. Then, $\mathrm{Ax}_{\mathrm{E},\alpha(1)}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)=1$ but $\mathrm{Ax}_{\mathrm{E},\beta(1)}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)=0$ because for all $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\subseteq(\Omega_{1},\Omega_{2})$, (Ω_{1},Ω_{3}), (Ω_{2},Ω_{3}) such that $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\notin\left\{(\emptyset,\Omega),(\Omega,\emptyset)\right\}$, and for all $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\subseteq\left(\Omega_{1},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\Omega_{3})\right)$, $\left(\Omega_{2},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\Omega_{1})\right)$, $\left(\Omega_{3},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\Omega_{2})\right)$, $\left(\Omega_{3},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\Omega_{1})\right)$ such that the non-membership condition $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\notin\left\{(\emptyset,\Omega),(\Omega,\emptyset)\right\}$ is satisfied, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})\neq\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Thus,

$$\mathrm{Ax}_{\mathrm{E},\beta(1)}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)\stackrel{\text{def}}{\longleftrightarrow}\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$$

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is untrue. Consequently, $(\Omega, \mathfrak{g}\text{-Ext}_{\mathfrak{g}})$ is a mathematical system such that $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ satisfies $\mathrm{Ax}_{\mathrm{E},\alpha(1)}$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}$ but not $\mathrm{Ax}_{\mathrm{E},\beta(1)}$. Therefore, $\mathrm{Ax}_{\mathrm{E},\beta(1)}$, then, cannot be a consequence of $\mathrm{Ax}_{\mathrm{E},\alpha(1)}$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}$. Hence, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axiom $\mathrm{Ax}_{\mathrm{E},\beta(1)}$ is independent of the other $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms $\mathrm{Ax}_{\mathrm{E},\alpha(1)}$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}$.

Case III. Consider the structure $(\Omega, \mathfrak{g}\text{-Ext}_{\mathfrak{g}})$ where, $\Omega \stackrel{\text{def}}{=} \bigcup_{\nu\in I_3^*} \Omega_\nu$ and, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is defined as thus: $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \Omega_2, \Omega_3, \Omega_1$ if $\mathscr{S}_{\mathfrak{g}} \subseteq \Omega_1, \Omega_2, \Omega_3$, respectively; $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \Omega$ if $\mathscr{S}_{\mathfrak{g}} = \emptyset$, and $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \emptyset$ if $\mathscr{S}_{\mathfrak{g}} = \Omega$, or $\mathscr{S}_{\mathfrak{g}} = \Omega \setminus \Omega_\nu$ for any $\nu \in I_3^*$. Then, $\mathrm{Ax}_{\mathrm{E},\beta(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}})$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}})$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}) = 1$ but $\mathrm{Ax}_{\mathrm{E},\alpha(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}) = 0$ because for all $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{\nu=1,2} \Omega_\nu$, $\bigcup_{\nu=1,3} \Omega_\nu$, $\bigcup_{\nu=2,3} \Omega_\nu$ such that $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \notin \{(\emptyset, \Omega), (\Omega, \emptyset)\}$, it results that the relation $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \nsubseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds. Thus,

$$\mathrm{Ax}_{\mathrm{E},\alpha(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$$

is untrue. Consequently, $(\Omega, \mathfrak{g}\text{-Ext}_{\mathfrak{g}})$ is a mathematical system such that $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ satisfies $\mathrm{Ax}_{\mathrm{E},\beta(1)}$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}$ but not $\mathrm{Ax}_{\mathrm{E},\alpha(1)}$. Therefore, $\mathrm{Ax}_{\mathrm{E},\alpha(1)}$, then, cannot be a consequence of $\mathrm{Ax}_{\mathrm{E},\beta(1)}$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}$. Hence, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axiom $\mathrm{Ax}_{\mathrm{E},\alpha(1)}$ is independent of the other $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms $\mathrm{Ax}_{\mathrm{E},\beta(1)}$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}$.

Case IV. Consider the structure $(\Omega, \mathfrak{g}\text{-Ext}_{\mathfrak{g}})$ where, $\Omega \stackrel{\text{def}}{=} \bigcup_{\nu\in I_2^*} \Omega_\nu$, satisfying $\Omega_1 \not\sim I_\sigma^*$ for all $\sigma \in I_\infty^*$ and $\mathrm{card}(\Omega_2) = 1$, and $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is defined as thus: $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ for every $\mathscr{S}_{\mathfrak{g}} \subset \bigcup_{\nu\in I_2^*} \Omega_\nu$; $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \Omega_2$ if $\mathscr{S}_{\mathfrak{g}} = \bigcup_{\nu\in I_2^*} \Omega_\nu$. Then, it follows, consequently, that $\mathrm{Ax}_{\mathrm{E},\alpha(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}})$, $\mathrm{Ax}_{\mathrm{E},\beta(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}})$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}) = 1$ hold but $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}) = 0$ because, $\mathscr{S}_{\mathfrak{g}} = \bigcup_{\nu=1,2} \Omega_\nu$ implies $\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \neq \emptyset$. Thus,

$$\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \emptyset$$

is untrue. Consequently, $(\Omega, \mathfrak{g}\text{-Ext}_{\mathfrak{g}})$ is a mathematical system such that $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ satisfies $\mathrm{Ax}_{\mathrm{E},\alpha(1)}$, $\mathrm{Ax}_{\mathrm{E},\beta(1)}$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}$ but not $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}$. Therefore, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}$, then, cannot be a consequence of $\mathrm{Ax}_{\mathrm{E},\alpha(1)}$, $\mathrm{Ax}_{\mathrm{E},\beta(1)}$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}$. Hence, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axiom $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}$ is independent of the other $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms $\mathrm{Ax}_{\mathrm{E},\alpha(1)}$, $\mathrm{Ax}_{\mathrm{E},\beta(1)}$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}$.

Thus, for every $\mu \in I_8^*$, the class $\mathrm{AX}_\mu[\mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}] \stackrel{\text{def}}{=} \{\mathrm{Ax}_{\mathrm{E},\nu} : \nu \in I_{\varepsilon(\mu)}^*\}$ are composed of independent derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms. The proof of the theorem is complete. Q.E.D.

The $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms in $\mathrm{AX}_\mu[\mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}]$ can be replaced by just one $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axiom such that each axioms in $\mathrm{AX}_\mu[\mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}]$ can be derived as a consequence of that one axiom. The proposition follows.

Proposition 3.37. *Let* $\mathrm{AX}_\mu[\mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}] \stackrel{\text{def}}{=} \{\mathrm{Ax}_{\mathrm{E},\nu} : \nu \in I_{\varepsilon(\mu)}^*\}$ *be a class of fundamental* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*exterior operator axioms of type* $\mu \in I_8^*$ *in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} =$

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$(\Omega, \mathscr{T}_{\mathfrak{g}})$ *and, let* $\mathrm{Ax}_{\mathrm{E}} : \mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longrightarrow \mathbb{B}$ *such that, for any* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$,

$$\begin{aligned}\mathrm{Ax}_{\mathrm{E}}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right) \stackrel{\text{def}}{\longleftrightarrow} & \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\ & \cup \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & = \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \setminus \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset).\end{aligned} \tag{3.16}$$

Then, $\mathrm{Ax}_{\mathrm{E}}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right) = 1 \longrightarrow \bigwedge_{\nu \in I^{*}_{\varepsilon(\mu)}} \mathrm{Ax}_{\mathrm{E},\nu}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right) = 1$.

Proof. Let $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right]; \mathbb{B}\right] \stackrel{\text{def}}{=} \left\{\mathrm{Ax}_{\mathrm{E},\nu} : \ \nu \in I^{*}_{\varepsilon(\mu)}\right\}$ be a class of fundamental $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms of type $\mu \in I^{*}_{3}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, let $\mathrm{Ax}_{\mathrm{E}} : \mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longrightarrow \mathbb{B}$ such that, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$,

$$\begin{aligned}\mathrm{Ax}_{\mathrm{E}}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right) \stackrel{\text{def}}{\longleftrightarrow} & \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\ & \cup \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & = \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \setminus \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset).\end{aligned}$$

Suppose $\mathrm{Ax}_{\mathrm{E}}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right) = 1$ holds. Then, since

$$\begin{aligned}\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \\ \supseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), & \\ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \setminus \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset) \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) &\end{aligned}$$

hold, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ such that $\mathscr{S}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}$ it results, consequently, that

$$\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).$$

Therefore, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{E},\alpha(1)}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)$ and hence, $\mathrm{Ax}_{\mathrm{E}} \longrightarrow \mathrm{Ax}_{\mathrm{E},\alpha(1)}$.

If $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) = (\emptyset, \emptyset)$, then

$$\begin{aligned}\mathrm{Ax}_{\mathrm{E}}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right) & \stackrel{\text{def}}{\longleftrightarrow} \emptyset \cup \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset) \cup \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset) \\ & = \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset \cup \emptyset) \setminus \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset).\end{aligned}$$

Consequently, $\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset)$, $\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset) = \emptyset$. Therefore, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset) \stackrel{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{E},\delta(1)}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)$ and thus, $\mathrm{Ax}_{\mathrm{E}} \longrightarrow \mathrm{Ax}_{\mathrm{E},\delta(1)}$.

If $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ be arbitrary, then

$$\begin{aligned}\mathrm{Ax}_{\mathrm{E}}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right) \stackrel{\text{def}}{\longleftrightarrow} & \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\ & \cup \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & = \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \setminus \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset).\end{aligned}$$

But, $\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Moreover, by virtue of $\mathrm{Ax}_{\mathrm{E},\delta(1)}$ the relation $\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset) = \emptyset$ holds. Consequently,

$$\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}).$$

Therefore, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{E},\beta(1)}\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\right)$ and hence, $\mathrm{Ax}_{\mathrm{E}} \longrightarrow \mathrm{Ax}_{\mathrm{E},\beta(1)}$.

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For an arbitrary $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ and $\mathscr{S}_{\mathfrak{g}} = \emptyset$, it results that

$$\begin{aligned} \mathrm{Ax}_{\mathrm{E}}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) \stackrel{\text{def}}{\longleftrightarrow} & \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\ & \cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\emptyset) \\ & = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \emptyset) \setminus \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\emptyset). \end{aligned}$$

Since $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\emptyset) = \emptyset$ by virtue of $\mathrm{Ax}_{\mathrm{E},\delta(1)}$, it follows that

$$\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\emptyset) = \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).$$

But, $\mathscr{R}_{\mathfrak{g}} \cap \left(\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\emptyset)\right) = \emptyset$. Therefore, $\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \emptyset \stackrel{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{E},\varepsilon(1)}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right)$ and thus, $\mathrm{Ax}_{\mathrm{E}} \longrightarrow \mathrm{Ax}_{\mathrm{E},\varepsilon(1)}$.

Hence, $\mathrm{Ax}_{\mathrm{E}}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1 \longrightarrow \bigwedge_{\nu \in I^{*}_{\varepsilon(\mu)}} \mathrm{Ax}_{\mathrm{E},\nu}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1$ and the proof of the proposition is complete. Q.E.D.

Having shown the consistency, independency of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms, the axiomatic definition of the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior structure in $\mathscr{T}_{\mathfrak{g}}$-spaces can now be given and is contained in the following statement.

Definition 3.38 (Axiomatic Definition: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Exterior Structure). A "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior structure of type $\mu \in I^{*}_{8}$" in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a pair $\mathfrak{E}_{\mathfrak{g}} = \left(\Omega, \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right)$ consisting of a nonempty set Ω and a unary operation $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ such that the following "fundamental $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms" are satisfied:

$$\text{(3.17)} \quad \begin{array}{llll} \text{I.} & \mathrm{Ax}_{\mathrm{E},\alpha(\mu)}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1, & \text{III.} & \mathrm{Ax}_{\mathrm{E},\delta(\mu)}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1, \\ \text{II.} & \mathrm{Ax}_{\mathrm{E},\beta(\mu)}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1, & \text{IV.} & \mathrm{Ax}_{\mathrm{E},\varepsilon(\mu)}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1. \end{array}$$

For an arbitrary $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, let it be granted the following definition:

$$\begin{aligned} \mathrm{Ax}_{\mathrm{E},\hat{\varepsilon}(\mu)}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) \quad \stackrel{\text{def}}{\longleftrightarrow} \quad & \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\ \cap \quad & \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right). \end{aligned}$$

Then, by the aid of Def. 3.30, the logical implications in Props 3.22, 3.23 reduce to

$$\begin{aligned} \left(\mathrm{Ax}_{\mathrm{E},\hat{\varepsilon}(\mu)} = 1\right) \quad \longleftarrow \quad & \left(\mathrm{Ax}_{\mathrm{E},\alpha(\mu)} = 1\right) \wedge \left(\mathrm{Ax}_{\mathrm{E},\beta(\mu)} = 1\right) \wedge \left(\mathrm{Ax}_{\mathrm{E},\varepsilon(\mu)} = 1\right), \\ \left(\mathrm{Ax}_{\mathrm{E},\varepsilon(\mu)} = 1\right) \quad \longleftarrow \quad & \left(\mathrm{Ax}_{\mathrm{E},\alpha(\mu)} = 1\right) \wedge \left(\mathrm{Ax}_{\mathrm{E},\beta(\mu)} = 1\right) \wedge \left(\mathrm{Ax}_{\mathrm{E},\hat{\varepsilon}(\mu)} = 1\right), \end{aligned}$$

respectively. Consequently, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axiom $\mathrm{Ax}_{\mathrm{E},\varepsilon(\mu)}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right)$ can be equivalently replaced by the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axiom $\mathrm{Ax}_{\mathrm{E},\hat{\varepsilon}(\mu)}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right)$. Hence, an equivalent axiomatic definition of the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior structure follows.

Definition 3.39 (Equivalent Axiomatic Definition: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Exterior Structure). A "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior structure of type $\mu \in I^{*}_{8}$" in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a pair $\mathfrak{E}_{\mathfrak{g}} = \left(\Omega, \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right)$ consisting of a nonempty set Ω and a unary operation $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ such that the following "fundamental $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms" are satisfied:

$$\text{(3.18)} \quad \begin{array}{llll} \text{I.} & \mathrm{Ax}_{\mathrm{E},\alpha(\mu)}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1, & \text{III.} & \mathrm{Ax}_{\mathrm{E},\delta(\mu)}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1, \\ \text{II.} & \mathrm{Ax}_{\mathrm{E},\beta(\mu)}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1, & \text{IV.} & \mathrm{Ax}_{\mathrm{E},\hat{\varepsilon}(\mu)}\left(\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\right) = 1. \end{array}$$

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As above, the discussion will be facilitated by the following definition in which is introduced a list of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms.

DEFINITION 3.40. Let $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator and let $(\mathscr{R}_{\mathfrak{g}}\mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ be an arbitrary pair in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, $\text{Ax}_{\text{F},\nu} : \mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B} \stackrel{\text{def}}{=} \{0,1\}$, $\nu \in I_6^*$, defined as

$$\begin{aligned}
\text{Ax}_{\text{F},1}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}) &\stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \\
\text{Ax}_{\text{F},2}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}) &\stackrel{\text{def}}{\longleftrightarrow} (\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})) \cup (\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \\
&= (\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}), \\
\text{Ax}_{\text{F},3}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}) &\stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\emptyset) = \emptyset, \\
\text{Ax}_{\text{F},4}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}) &\stackrel{\text{def}}{\longleftrightarrow} \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Op} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}), \\
\text{Ax}_{\text{F},5}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}) &\stackrel{\text{def}}{\longleftrightarrow} (\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})) \cap (\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \\
&= (\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}), \\
\text{Ax}_{\text{F},6}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}) &\stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset) = \emptyset,
\end{aligned}$$

and belonging to $\text{AX}[\mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}] \stackrel{\text{def}}{=} \{\text{Ax}_{\text{F},\nu} :\ \nu \in I_6^*\}$, are called "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms."

Hence, in a $\mathscr{T}_{\mathfrak{g}}$-space, $\text{AX}[\mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}]$ forms a class of derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms.

LEMMA 3.41. *Let $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ such that, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, the following statements hold:*

$$(\forall \nu \in I_3^*)\left[\text{Ax}_{\text{F},\nu}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}) = 1\right]. \tag{3.19}$$

Then, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operator and the associated $\mathfrak{g}$-topology is $\mathscr{T}_{\mathfrak{g},\text{Cl}}^{\text{F}}(\Omega) \stackrel{\text{def}}{=} \{\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega) :\ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g}}\}$.

PROOF. Let $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ such that, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, the following $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms hold: $\text{Ax}_{\text{F},\nu}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}) = 1$ for every $\nu \in I_3^*$. Then, by virtue of $\text{Ax}_{\text{F},3}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}})$, it follows that $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\emptyset) \longleftrightarrow \emptyset \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\emptyset) = \emptyset$. Thus, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\emptyset) = \emptyset$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is $\emptyset$-grounded. Clearly, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}}$. Hence, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is expansive. By virtue of $\text{Ax}_{\text{F},1}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}})$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{S}_{\mathfrak{g}}) \cup (\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \\
&\subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{S}_{\mathfrak{g}}) \cup (\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

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Therefore, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{S}_{\mathfrak{g}})$. But, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathscr{S}_{\mathfrak{g}}$. Hence, it follows that $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is idempotent. By virtue of $\text{Ax}_{\text{F},2}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}})$, it is clear that

$$\begin{aligned}\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{R}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow (\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}))\cup(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}))\\ &= (\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Thus, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{R}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is $\cup$-additive.

For $\mathscr{T}_{\mathfrak{g},\text{Cl}}^{\text{F}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ to be a $\mathfrak{g}$-topology on Ω, it must satisfy the axioms for a $\mathfrak{g}$-topology on Ω: $\mathscr{T}_{\mathfrak{g},\text{Cl}}^{\text{F}}(\emptyset)=\emptyset$, $\mathscr{T}_{\mathfrak{g},\text{Cl}}^{\text{F}}(\mathscr{O}_{\mathfrak{g}})\subseteq\mathscr{O}_{\mathfrak{g}}$ and $\mathscr{T}_{\mathfrak{g},\text{Cl}}^{\text{F}}(\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{O}_{\mathfrak{g},\nu})=\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{T}_{\mathfrak{g},\text{Cl}}^{\text{F}}(\mathscr{O}_{\mathfrak{g},\nu})$; evidently, $\{\mathscr{O}_{\mathfrak{g},\nu}:\ \nu\in I_{\infty}^{*}\}\subseteq\mathscr{P}(\Omega)$. Since $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}(\mathscr{O}_{\mathfrak{g}})\longleftrightarrow\mathscr{O}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})$, it follows, consequently, that $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})$. It is clear that the relation $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})$ holds when $\mathscr{O}_{\mathfrak{g}}=\emptyset$. Thus, $\mathscr{T}_{\mathfrak{g},\text{Cl}}^{\text{F}}(\emptyset)=\emptyset$. Since $\mathscr{T}_{\mathfrak{g},\text{Cl}}^{\text{F}}(\Omega)\subseteq\{\mathscr{O}_{\mathfrak{g}}\in\mathscr{P}(\Omega):\ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})=\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\}$, it follows that, for every $(\mathscr{O}_{\mathfrak{g}},\mathscr{T}_{\mathfrak{g},\text{Cl}}^{\text{F}}(\mathscr{O}_{\mathfrak{g}}))\in\mathscr{P}(\Omega)\times\mathscr{T}_{\mathfrak{g},\text{Cl}}^{\text{F}}(\Omega)$, $\mathscr{T}_{\mathfrak{g},\text{Cl}}^{\text{F}}(\mathscr{O}_{\mathfrak{g}})\subseteq\mathscr{O}_{\mathfrak{g}}$ holds. Suppose $\{\mathscr{O}_{\mathfrak{g},\nu}:\ \nu\in I_{\infty}^{*}\}\subseteq\mathscr{P}(\Omega)$ such that, for every $\nu\in I_{\infty}^{*}$, the relation $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})=\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ holds. Then, since $\bigcap_{\nu\in I_{\infty}^{*}}\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})\longleftrightarrow\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{O}_{\mathfrak{g},\nu})$, it results that

$$\begin{aligned}\mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\textstyle\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{O}_{\mathfrak{g},\nu}\big) &= \bigcap_{\nu\in I_{\infty}^{*}}\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})\\ &= \mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}\Big(\bigcap_{\nu\in I_{\infty}^{*}}\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})\Big)=\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}\Big(\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{O}_{\mathfrak{g},\nu}\Big).\end{aligned}$$

Therefore, $\mathscr{T}_{\mathfrak{g},\text{Cl}}^{\text{F}}(\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{O}_{\mathfrak{g},\nu})=\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{T}_{\mathfrak{g},\text{Cl}}^{\text{F}}(\mathscr{O}_{\mathfrak{g},\nu})$ holds. Hence, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{\text{F}}:\mathscr{S}_{\mathfrak{g}}\longmapsto\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operator and the associated $\mathfrak{g}$-topology is $\mathscr{T}_{\mathfrak{g},\text{Cl}}^{\text{F}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. The proof of the lemma is complete. Q.E.D.

LEMMA 3.42. *Let* $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*frontier operator in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ *such that, for any* $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$, *the following statements hold:*

$$(\forall\nu\in I_{3}^{*})\,\big[\text{Ax}_{\text{F},3+\nu}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}})=1\big]. \tag{3.20}$$

Then, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}^{\text{F}}:\mathscr{S}_{\mathfrak{g}}\longmapsto\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ *is a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*interior operator and the associated* $\mathfrak{g}$-*topology is* $\mathscr{T}_{\mathfrak{g},\text{Int}}^{\text{F}}(\Omega)\stackrel{\text{def}}{=}\{\mathscr{O}_{\mathfrak{g}}\in\mathscr{P}(\Omega):\ \mathfrak{g}\text{-Int}_{\mathfrak{g}}^{\text{F}}(\mathscr{O}_{\mathfrak{g}})\subseteq\mathscr{O}_{\mathfrak{g}}\}$.

PROOF. Let $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ such that, for any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$, the following $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms hold: $\text{Ax}_{\text{F},3+\nu}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}})=1$ for every $\nu\in I_{3}^{*}$. Then, by virtue of $\text{Ax}_{\text{F},6}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}})$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset)=\emptyset$ implies $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset)=\Omega$ and consequently,

$$\mathfrak{g}\text{-Int}_{\mathfrak{g}}^{\text{F}}(\Omega)\longleftrightarrow\Omega\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\Omega)\longleftrightarrow\Omega\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset)=\Omega.$$

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Hence, it results that $\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\Omega)=\Omega$ and $\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is Ω-grounded. From the definition of $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathscr{S}_{\mathfrak{g}}\cap\Bigg(\bigcup_{\mathscr{R}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})}\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\Bigg)\\ &= \mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}.\end{aligned}$$

Thus, $\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is non-expansive. By virtue of $\mathrm{Ax}_{\mathrm{F},4}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}})$,

$$\begin{aligned}\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}))\\ &\longleftrightarrow \mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow (\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}))\\ &\cap \quad \mathfrak{g}\text{-Op}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Thus, $\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is idempotent. By virtue of $\mathrm{Ax}_{\mathrm{F},5}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}})$,

$$\begin{aligned}\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow (\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}))\\ &\cap \quad (\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}))\\ &= (\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Hence, $\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is $\cap$-additive.

For $\mathscr{T}^{\mathrm{F}}_{\mathfrak{g},\mathrm{Int}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ to be a $\mathfrak{g}$-topology on Ω, it must satisfy the axioms for a $\mathfrak{g}$-topology on Ω: $\mathscr{T}^{\mathrm{F}}_{\mathfrak{g},\mathrm{Int}}(\emptyset)=\emptyset$, $\mathscr{T}^{\mathrm{F}}_{\mathfrak{g},\mathrm{Int}}(\mathscr{O}_{\mathfrak{g}})\subseteq\mathscr{O}_{\mathfrak{g}}$ and $\mathscr{T}^{\mathrm{F}}_{\mathfrak{g},\mathrm{Int}}\big(\bigcup_{\nu\in I^{*}_{\infty}}\mathscr{O}_{\mathfrak{g},\nu}\big)=\bigcup_{\nu\in I^{*}_{\infty}}\mathscr{T}^{\mathrm{F}}_{\mathfrak{g},\mathrm{Int}}(\mathscr{O}_{\mathfrak{g},\nu})$; evidently, $\{\mathscr{O}_{\mathfrak{g},\nu}:\ \nu\in I^{*}_{\infty}\}\subseteq\mathscr{P}(\Omega)$. Clearly, the relation $\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\subseteq\mathscr{O}_{\mathfrak{g}}$ holds when $\mathscr{O}_{\mathfrak{g}}=\emptyset$. Thus, $\mathscr{T}^{\mathrm{F}}_{\mathfrak{g},\mathrm{Int}}(\emptyset)=\emptyset$. Since $\mathscr{T}^{\mathrm{F}}_{\mathfrak{g},\mathrm{Int}}(\Omega)\subseteq\{\mathscr{O}_{\mathfrak{g}}\in\mathscr{P}(\Omega):\ \mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\subseteq\mathscr{O}_{\mathfrak{g}}\}$, it results that, for every $(\mathscr{O}_{\mathfrak{g}},\mathscr{T}^{\mathrm{F}}_{\mathfrak{g},\mathrm{Int}}(\mathscr{O}_{\mathfrak{g}}))\in\mathscr{P}(\Omega)\times\mathscr{T}^{\mathrm{F}}_{\mathfrak{g},\mathrm{Int}}(\Omega)$, the relation $\mathscr{T}^{\mathrm{F}}_{\mathfrak{g},\mathrm{Int}}(\mathscr{O}_{\mathfrak{g}})\subseteq\mathscr{O}_{\mathfrak{g}}$ holds. Suppose $\{\mathscr{O}_{\mathfrak{g},\nu}:\ \nu\in I^{*}_{\infty}\}\subseteq\mathscr{P}(\Omega)$ such that, for every $\nu\in I^{*}_{\infty}$, the relation $\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})\subseteq\mathscr{O}_{\mathfrak{g},\nu}$ holds. Then,

$$\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}\big(\textstyle\bigcup_{\nu\in I^{*}_{\infty}}\mathscr{O}_{\mathfrak{g},\nu}\big)\longleftrightarrow\displaystyle\bigcup_{\nu\in I^{*}_{\infty}}\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})\subseteq\bigcup_{\nu\in I^{*}_{\infty}}\mathscr{O}_{\mathfrak{g},\nu}.$$

Consequently, $\mathscr{T}^{\mathrm{F}}_{\mathfrak{g},\mathrm{Int}}\big(\bigcup_{\nu\in I^{*}_{\infty}}\mathscr{O}_{\mathfrak{g},\nu}\big)=\bigcup_{\nu\in I^{*}_{\infty}}\mathscr{T}^{\mathrm{F}}_{\mathfrak{g},\mathrm{Int}}(\mathscr{O}_{\mathfrak{g},\nu})$. Hence, $\mathfrak{g}\text{-Int}^{\mathrm{F}}_{\mathfrak{g}}:\mathscr{S}_{\mathfrak{g}}\longmapsto\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operator and the associated $\mathfrak{g}$-topology is $\mathscr{T}^{\mathrm{F}}_{\mathfrak{g},\mathrm{Int}}:\ \mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. The proof of the lemma is complete. Q.E.D.

THEOREM 3.43. *Let* $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-frontier operator in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ *and, let* $\mathrm{Ax}_{\mathrm{F},7}:\mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]\longrightarrow\mathbb{B}$ *such that* $\mathrm{Ax}_{\mathrm{F},7}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}})\overset{\mathrm{def}}{\longleftrightarrow}$

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$\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ *for any* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. *If* $\mathrm{Ax}_{\mathrm{F},7}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}) = 1$, *then*

$$(\forall \nu \in I_3^*)\left[\mathrm{Ax}_{\mathrm{F},\nu}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}) = 1 \longleftrightarrow \mathrm{Ax}_{\mathrm{F},3+\nu}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}) = 1\right]. \tag{3.21}$$

Proof. Let $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, let $\mathrm{Ax}_{\mathrm{F},7} : \mathfrak{g}\text{-}\mathrm{F}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B}$ such that, for some $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, $\mathrm{Ax}_{\mathrm{F},7}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Suppose $\mathrm{Ax}_{\mathrm{F},7}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}) = 1$. Then, the statement $\mathrm{Ax}_{\mathrm{F},7}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds for any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. For $\nu = 1$, set $\mathscr{R}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Then,

$$\begin{aligned}
\mathrm{Ax}_{\mathrm{F},1}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}) &\overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \\
&\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\supseteq \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Op} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\
&\overset{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{F},4}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}).
\end{aligned}$$

Thus, it follows that $\mathrm{Ax}_{\mathrm{F},1}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}) = 1 \longleftrightarrow \mathrm{Ax}_{\mathrm{F},4}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}) = 1$. For $\nu = 2$, set $\mathscr{R}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ and $\mathscr{S}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$. Then,

$$\begin{aligned}
\mathrm{Ax}_{\mathrm{F},2}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}) &\overset{\text{def}}{\longleftrightarrow} \bigcup_{\mathscr{Z}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} (\mathscr{Z}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{Z}_{\mathfrak{g}})) \\
&= (\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \bigcup_{\mathscr{Z}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} (\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Z}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Z}_{\mathfrak{g}})) \\
&= (\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})) \\
&\longleftrightarrow \bigcup_{\mathscr{Z}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Z}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Z}_{\mathfrak{g}})) \\
&= \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\bigcap_{\mathscr{Z}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} (\mathscr{Z}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Z}_{\mathfrak{g}}))\right) \\
&= \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}((\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}})) \\
&\longleftrightarrow \bigcap_{\mathscr{Z}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} (\mathscr{Z}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Z}_{\mathfrak{g}})) \\
&= (\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) \\
&\longleftrightarrow (\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})) \cap (\mathscr{V}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})) \\
&= (\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{F},5}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}).
\end{aligned}$$

Hence, it follows that $\mathrm{Ax}_{\mathrm{F},2}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}) = 1 \longleftrightarrow \mathrm{Ax}_{\mathrm{F},5}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}) = 1$. Clearly, the relation $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds for every $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. Therefore, for $\nu = 3$, it results that

$$\mathrm{Ax}_{\mathrm{F},3}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\emptyset) = \emptyset \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\emptyset) = \emptyset \overset{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{F},6}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}).$$

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Hence, $\mathrm{Ax}_{\mathrm{F},3}\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)=1\longleftrightarrow\mathrm{Ax}_{\mathrm{F},6}\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)=1$. The proof of the theorem is complete. Q.E.D.

For every $\mu\in I_7^*$, set $\mathrm{Ax}_{\mathrm{F},\mu}=\mathrm{Ax}_{\mathrm{F},\mu}\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)$ and, consider $I_{\varphi(1)}^*\overset{\text{def}}{=}\{1,2,3,7\}$, $I_{\varphi(2)}^*\overset{\text{def}}{=}\{1,2,6,7\}$, $I_{\varphi(3)}^*\overset{\text{def}}{=}\{1,3,5,7\}$, $I_{\varphi(4)}^*\overset{\text{def}}{=}\{1,5,6,7\}$, $I_{\varphi(5)}^*\overset{\text{def}}{=}\{2,3,4,7\}$, $I_{\varphi(6)}^*\overset{\text{def}}{=}\{2,4,6,7\}$, $I_{\varphi(7)}^*\overset{\text{def}}{=}\{3,4,5,7\}$, and $I_{\varphi(8)}^*\overset{\text{def}}{=}\{4,5,6,7\}$. Moreover, for each $\mu\in I_8^*$, let $I_{\varphi(\mu)}^*\overset{\text{def}}{=}\{\alpha(\mu),\beta(\mu),\delta(\mu),\varphi(\mu)\}$. The proposition follows.

Proposition 3.44. *Let* $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-frontier operator in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ *and let* $\left\{\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]:\ \mu\in I_8^*\right\}$ *be a collection of classes of derived set* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-frontier operator axioms:*

$$(3.22)\qquad \left(\forall\mu\in I_8^*\right)\left[\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]\overset{\text{def}}{=}\left\{\mathrm{Ax}_{\mathrm{F},\nu}:\ \nu\in I_{\varphi(\mu)}^*\right\}\right].$$

Suppose there exist a $\mu\in I_8^*$ *such that* $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *satisfies the derived set* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-exterior operator axioms in* $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$*, then:*

- I. $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{F}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{F}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *are the* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-interior and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-closure operators and the associated* $\mathfrak{g}$*-topologies are* $\mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{F}}$, $\mathscr{T}_{\mathfrak{g},\mathrm{Cl}}^{\mathrm{F}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$*, respectively;*
- II. $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *is the* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-exterior operator and the associated* $\mathfrak{g}$*-topology is* $\mathscr{T}_{\mathfrak{g},\mathrm{Fr}}=\mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{F}}=\mathscr{T}_{\mathfrak{g},\mathrm{Cl}}^{\mathrm{F}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$.

Proof. Let $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ and let $\left\{\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]:\ \mu\in I_8^*\right\}$ be a collection of classes $\mathrm{AX}_1\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]\overset{\text{def}}{=}\left\{\mathrm{Ax}_{\mathrm{F},\nu}:\ \nu\in I_{\varphi(1)}^*\right\}$, $\mathrm{AX}_2\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]\overset{\text{def}}{=}\left\{\mathrm{Ax}_{\mathrm{F},\nu}:\ \nu\in I_{\varphi(2)}^*\right\}$, ..., $\mathrm{AX}_8\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]\overset{\text{def}}{=}\left\{\mathrm{Ax}_{\mathrm{F},\nu}:\ \nu\in I_{\varphi(8)}^*\right\}$ of derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms. Suppose

$$\left(\exists\mu\in I_8^*\right)\left(\forall\nu\in I_{\varphi(\mu)}^*\right)\left[\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]\ni\mathrm{Ax}_{\mathrm{F},\nu}\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)=1\right].$$

Then:

I. Since $\bigcap_{\mu\in I_8^*}\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]=\{\mathrm{Ax}_{\mathrm{F},7}\}$ holds and on the other hand, the relation $\mathrm{Ax}_{\mathrm{E},7}\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)=1$ implies

$$\left(\forall\nu\in I_3^*\right)\left[\mathrm{Ax}_{\mathrm{F},\nu}\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)=1\ \longleftrightarrow\ \mathrm{Ax}_{\mathrm{F},3+\nu}\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)=1\right],$$

it follows that, for each $\nu\in I_3^*$, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ satisfies both $\mathrm{Ax}_{\mathrm{F},\nu}\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)=1$ and $\mathrm{Ax}_{\mathrm{F},3+\nu}\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)=1$. Hence, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{F}}:\mathscr{S}_{\mathfrak{g}}\longmapsto\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{F}}:\mathscr{S}_{\mathfrak{g}}\longmapsto\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ are the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators and, $\mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{F}}(\Omega)\overset{\text{def}}{=}\left\{\mathscr{O}_{\mathfrak{g}}\in\mathscr{P}(\Omega):\ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{F}}(\mathscr{O}_{\mathfrak{g}})\subseteq\mathscr{O}_{\mathfrak{g}}\right\}$ and $\mathscr{T}_{\mathfrak{g},\mathrm{Cl}}^{\mathrm{F}}(\Omega)\overset{\text{def}}{=}\left\{\mathscr{O}_{\mathfrak{g}}\in\mathscr{P}(\Omega):\ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{F}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\right\}$ are the associated $\mathfrak{g}$-topologies, respectively.

II. Whenever $\mathscr{O}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$,

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\ &\longleftrightarrow\ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{R}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})}\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\Bigg)\\ &\longleftrightarrow\ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{F}}(\mathscr{S}_{\mathfrak{g}})\cap\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{F}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

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Because $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{F}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{F}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators, respectively, it follows that $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator. Whenever $\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, it follows that

$$\begin{aligned}
\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\mathrm{Cl}}^{\mathrm{F}} \quad &\longleftrightarrow \quad \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{\mathrm{F}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \\
&\longleftrightarrow \quad \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{F}}(\mathscr{O}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \\
&\longleftrightarrow \quad \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{\mathrm{F}}(\mathscr{O}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g}} \ \longleftrightarrow \ \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{F}}.
\end{aligned}$$

Hence, the associated $\mathfrak{g}$-topology is $\mathscr{T}_{\mathfrak{g},\mathrm{Fr}} = \mathscr{T}_{\mathfrak{g},\mathrm{Int}}^{\mathrm{F}} = \mathscr{T}_{\mathfrak{g},\mathrm{Cl}}^{\mathrm{F}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. The proof of the proposition is complete. Q.E.D.

As above, for every $\mu \in I_8^*$, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms in $\mathrm{AX}_{\mu}\big[\mathfrak{g}\text{-}\mathrm{F}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big]$ are not inconsistent among themselves, as proved in the following proposition.

Proposition 3.45. *The classes* $\mathrm{AX}_1\big[\mathfrak{g}\text{-}\mathrm{F}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big], \ldots, \mathrm{AX}_8\big[\mathfrak{g}\text{-}\mathrm{F}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big]$ *in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *such that, for every* $\mu \in I_8^*$, $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *satisfies the axioms in* $\mathrm{AX}_{\mu}\big[\mathfrak{g}\text{-}\mathrm{F}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big] \stackrel{\mathrm{def}}{=} \big\{\mathrm{Ax}_{\mathrm{F},\nu} : \ \nu \in I_{\varphi(\mu)}^*\big\}$ *are composed of consistent derived set* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*frontier operator axioms.*

Proof. Let $\mathrm{AX}_1\big[\mathfrak{g}\text{-}\mathrm{F}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big], \ldots, \mathrm{AX}_8\big[\mathfrak{g}\text{-}\mathrm{F}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big]$ be given classes in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ such that, for every $\mu \in I_8^*$, $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ satisfies the axioms in $\mathrm{AX}_{\mu}\big[\mathfrak{g}\text{-}\mathrm{F}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big] \stackrel{\mathrm{def}}{=} \big\{\mathrm{Ax}_{\mathrm{F},\nu} : \ \nu \in I_{\varphi(\mu)}^*\big\}$. Since for each $(\mu, \alpha(\mu)) \in I_8^* \times I_{\varphi(\mu)}^*$ there exists exactly one $(\nu, \alpha(\nu)) \in I_8^* \times I_{\varphi(\nu)}^*$ such that

$$\mathrm{AX}_{\mu}\big[\mathfrak{g}\text{-}\mathrm{F}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big] \ni \mathrm{Ax}_{\mathrm{F},\alpha(\mu)} \longleftrightarrow \mathrm{Ax}_{\mathrm{F},\alpha(\nu)} \in \mathrm{AX}_{\nu}\big[\mathfrak{g}\text{-}\mathrm{F}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big],$$

to prove the proposition, it suffices to select a $\mu \in I_8^*$ and then show that the elements of $\mathrm{AX}_{\mu}\big[\mathfrak{g}\text{-}\mathrm{F}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big] = \big\{\mathrm{Ax}_{\mathrm{F},\alpha(\mu)}, \mathrm{Ax}_{\mathrm{F},\beta(\mu)}, \mathrm{Ax}_{\mathrm{F},\delta(\mu)}, \mathrm{Ax}_{\mathrm{F},\varphi(\mu)}\big\}$ are not inconsistent. Select $\mu \in I_8^* \setminus I_7^*$. Consequently, it follows that $\mathrm{AX}_{\mu}\big[\mathfrak{g}\text{-}\mathrm{F}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big] = \big\{\mathrm{Ax}_{\mathrm{F},\alpha(1)}, \mathrm{Ax}_{\mathrm{F},\beta(1)}, \mathrm{Ax}_{\mathrm{F},\delta(1)}, \mathrm{Ax}_{\mathrm{F},\varphi(1)}\big\}$. Suppose $\big(\Omega, \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\big)$ be a mathematical system such that, for any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is defined as thus: $\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\mathrm{def}}{=} \mathscr{S}_{\mathfrak{g}}$. Then:

$$\begin{aligned}
\mathrm{Ax}_{\mathrm{F},1}\big(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\big) \quad &\stackrel{\mathrm{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \quad \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}; \\
\mathrm{Ax}_{\mathrm{F},2}\big(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\big) \quad &\stackrel{\mathrm{def}}{\longleftrightarrow} \quad \big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \cup \big(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \\
&= \quad \big(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}\big) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}\big) \\
&\longleftrightarrow \quad \mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}; \\
\mathrm{Ax}_{\mathrm{F},3}\big(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\big) \quad &\stackrel{\mathrm{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\emptyset) = \emptyset \\
&\longleftrightarrow \quad \emptyset \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\emptyset) = \emptyset; \\
\mathrm{Ax}_{\mathrm{F},7}\big(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\big) \quad &\stackrel{\mathrm{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-}\mathrm{Fr} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \quad \mathscr{S}_{\mathfrak{g}} \cup \big(\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \\
&= \quad \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \big(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \\
&\longleftrightarrow \quad \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathscr{S}_{\mathfrak{g}}.
\end{aligned}$$

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Thus, $\mathrm{Ax}_{\mathrm{F},\nu}\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)=1$ for every $\nu\in I_{\varphi(1)}^{*}$. Therefore, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms $\mathrm{Ax}_{\mathrm{F},\alpha(1)}$, $\mathrm{Ax}_{\mathrm{F},\beta(1)}$, $\mathrm{Ax}_{\mathrm{F},\delta(1)}$, $\mathrm{Ax}_{\mathrm{F},(1)}$ themselves, and all their consequences, are simply set-theoretic expressions of the properties of the structure $\left(\Omega,\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)$, and thus cannot involve contradiction, since no structure which really exists can have contradictory properties. Hence, for every $\mu\in I_{8}^{*}$, the class $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]\stackrel{\mathrm{def}}{=}\left\{\mathrm{Ax}_{\mathrm{F},\nu}:\ \nu\in I_{\varphi(\mu)}^{*}\right\}$ are composed of consistent derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms. The proof of the proposition is complete. Q.E.D.

Again for the sake of elegance, for each $\mu\in I_{8}^{*}$, every class $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$ of derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms should be free from redundancies; that is, for each $\mu\in I_{8}^{*}$, the derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms of every class $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$ should be independent, no one of them deductible from the rest. The theorem follows.

Theorem 3.46. *The classes* $\mathrm{AX}_{1}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$, …, $\mathrm{AX}_{8}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ *such that, for every* $\mu\in I_{8}^{*}$, $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *satisfies the axioms in* $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$ *are composed of independent derived set* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-frontier operator axioms.*

Proof. Let $\mathrm{AX}_{1}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$, …, $\mathrm{AX}_{8}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]$ be given classes in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ such that, for every $\mu\in I_{8}^{*}$, $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ satisfies the axioms in $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]\stackrel{\mathrm{def}}{=}\left\{\mathrm{Ax}_{\mathrm{F},\nu}:\ \nu\in I_{\varphi(\mu)}^{*}\right\}$. Since for each $(\mu,\alpha(\mu))\in I_{8}^{*}\times I_{\varphi(\mu)}^{*}$ there exists exactly one $(\nu,\alpha(\nu))\in I_{8}^{*}\times I_{\varphi(\nu)}^{*}$ such that

$$\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]\ni\mathrm{Ax}_{\mathrm{F},\alpha(\mu)}\longleftrightarrow\mathrm{Ax}_{\mathrm{F},\alpha(\nu)}\in\mathrm{AX}_{\nu}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right],$$

to prove the theorem, it suffices to select a $\mu\in I_{8}^{*}$ and then show that the elements of $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]=\left\{\mathrm{Ax}_{\mathrm{F},\alpha(\mu)},\mathrm{Ax}_{\mathrm{E},\beta(\mu)},\mathrm{Ax}_{\mathrm{F},\delta(\mu)},\mathrm{Ax}_{\mathrm{F},\varphi(\mu)}\right\}$ are independent by exhibiting, in the case of each $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axiom, a structure $\left(\Omega,\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)$ which satisfies all the other $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms, but not the one in question.

Set $\mu\in I_{8}^{*}\setminus I_{7}^{*}$ then, $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-}\mathrm{F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right]=\left\{\mathrm{Ax}_{\mathrm{F},\alpha(1)},\mathrm{Ax}_{\mathrm{F},\beta(1)},\mathrm{Ax}_{\mathrm{F},\delta(1)},\mathrm{Ax}_{\mathrm{F},\varphi(1)}\right\}$. Then:

Case i. Consider the structure $\left(\Omega,\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)$ where, $\Omega\stackrel{\mathrm{def}}{=}\bigcup_{\nu\in I_{2}^{*}}\Omega_{\nu}$ and, $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is defined as thus: $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\stackrel{\mathrm{def}}{=}\Omega_{1}$ for every $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$. Then, $\mathrm{Ax}_{\mathrm{F},\alpha(1)}\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)$, $\mathrm{Ax}_{\mathrm{F},\beta(1)}\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)$, $\mathrm{Ax}_{\mathrm{F},\varphi(1)}\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)=1$ but $\mathrm{Ax}_{\mathrm{F},\delta(1)}\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)=0$ because

$$\begin{aligned}\mathrm{Ax}_{\mathrm{F},\delta(1)}\left(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)&\stackrel{\mathrm{def}}{\longleftrightarrow}\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\emptyset)=\emptyset\\&\longleftrightarrow\Omega_{1}=\emptyset,\end{aligned}$$

and $\Omega_{1}=\emptyset$ is untrue. Consequently, $\left(\Omega,\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)$ is a mathematical system such that $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ satisfies $\mathrm{Ax}_{\mathrm{F},\alpha(1)}$, $\mathrm{Ax}_{\mathrm{F},\beta(1)}$, $\mathrm{Ax}_{\mathrm{F},\varphi(1)}$ but not $\mathrm{Ax}_{\mathrm{F},\delta(1)}$. Therefore, $\mathrm{Ax}_{\mathrm{F},\delta(1)}$, then, cannot be a consequence of $\mathrm{Ax}_{\mathrm{F},\alpha(1)}$, $\mathrm{Ax}_{\mathrm{F},\beta(1)}$, $\mathrm{Ax}_{\mathrm{F},\varphi(1)}$. Hence, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axiom $\mathrm{Ax}_{\mathrm{F},\delta(1)}$ is independent of the other $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms $\mathrm{Ax}_{\mathrm{F},\alpha(1)}$, $\mathrm{Ax}_{\mathrm{F},\beta(1)}$, $\mathrm{Ax}_{\mathrm{F},\varphi(1)}$.

Case ii. Consider the structure $\left(\Omega,\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\right)$ where, $\Omega\stackrel{\mathrm{def}}{=}\bigcup_{\nu\in I_{3}^{*}}\Omega_{\nu}$ and, $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is defined as thus: $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\stackrel{\mathrm{def}}{=}\emptyset$ for every $\mathscr{S}_{\mathfrak{g}}\in\{\emptyset,\Omega\}$; $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\stackrel{\mathrm{def}}{=}\bigcup_{\nu=1,2}\Omega_{\nu}$ if $\mathscr{S}_{\mathfrak{g}}\subseteq\Omega_{1}$; $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\stackrel{\mathrm{def}}{=}\Omega$ if $\mathscr{S}_{\mathfrak{g}}\subseteq\Omega_{2}$, Ω_{3}, $\bigcup_{\nu=1,2}\Omega_{\nu}$

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$,\bigcup_{\nu=1,3}\Omega_{\nu}$; $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcup_{\nu=1,2}\Omega_{\nu}$ if $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{\nu=2,3}\Omega_{\nu}$. Then, $\text{Ax}_{\text{F},\beta(1)}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}})$, $\text{Ax}_{\text{F},\delta(1)}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}})$, $\text{Ax}_{\text{F},\varphi(1)}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}) = 1$ but $\text{Ax}_{\text{F},\alpha(1)}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}) = 0$ because for all $\mathscr{S}_{\mathfrak{g}} \subseteq \Omega_1$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \not\subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Thus,

$$\text{Ax}_{\text{F},\alpha(1)}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$$

is untrue. Consequently, $(\Omega, \mathfrak{g}\text{-Fr}_{\mathfrak{g}})$ is a mathematical system such that $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ satisfies $\text{Ax}_{\text{F},\beta(1)}$, $\text{Ax}_{\text{F},\delta(1)}$, $\text{Ax}_{\text{F},\varphi(1)}$ but not $\text{Ax}_{\text{F},\alpha(1)}$. Therefore, $\text{Ax}_{\text{F},\alpha(1)}$, then, cannot be a consequence of $\text{Ax}_{\text{F},\beta(1)}$, $\text{Ax}_{\text{F},\delta(1)}$, $\text{Ax}_{\text{F},\varphi(1)}$. Hence, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axiom $\text{Ax}_{\text{E},\alpha(1)}$ is independent of the other $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms $\text{Ax}_{\text{F},\beta(1)}$, $\text{Ax}_{\text{F},\delta(1)}$, $\text{Ax}_{\text{F},\varphi(1)}$.

Case III. Consider the structure $(\Omega, \mathfrak{g}\text{-Ext}_{\mathfrak{g}})$ where, $\Omega \stackrel{\text{def}}{=} \bigcup_{\nu\in I_3^*}\Omega_{\nu}$ and, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is defined as thus: $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \emptyset$ for every $\mathscr{S}_{\mathfrak{g}} \in \{\emptyset, \Omega\}$; $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \emptyset$ if $\mathscr{S}_{\mathfrak{g}} \subseteq \Omega_1$; for each $\mu \in I_3^* \setminus \{1\}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \Omega_{\mu}$ if $\mathscr{S}_{\mathfrak{g}} \subseteq \Omega_{\mu}$ and $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \Omega_{\mu}$ if $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{\nu\in I_3^*\setminus\{\mu\}}\Omega_{\nu}$; $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \emptyset$ if $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{\nu\in I_3^*\setminus\{1\}}\Omega_{\nu}$. Then, it follows that $\text{Ax}_{\text{F},\alpha(1)}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}})$, $\text{Ax}_{\text{F},\delta(1)}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}})$, $\text{Ax}_{\text{F},\varphi(1)}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}) = 1$ but $\text{Ax}_{\text{F},\beta(1)}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}) = 0$ because for all $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subseteq (\Omega_1, \Omega_2)$, (Ω_1, Ω_3), it results that $(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})) \cup (\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \neq (\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}})$. Thus,

$$\begin{aligned}\text{Ax}_{\text{F},\beta(1)}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}) &\stackrel{\text{def}}{\longleftrightarrow} (\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})) \cup (\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \\ &= (\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}})\end{aligned}$$

is untrue. Consequently, $(\Omega, \mathfrak{g}\text{-Fr}_{\mathfrak{g}})$ is a mathematical system such that $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ satisfies $\text{Ax}_{\text{F},\alpha(1)}$, $\text{Ax}_{\text{F},\delta(1)}$, $\text{Ax}_{\text{F},\varphi(1)}$ but not $\text{Ax}_{\text{F},\beta(1)}$. Therefore, $\text{Ax}_{\text{F},\beta(1)}$, then, cannot be a consequence of $\text{Ax}_{\text{F},\alpha(1)}$, $\text{Ax}_{\text{F},\delta(1)}$, $\text{Ax}_{\text{F},\varphi(1)}$. Hence, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axiom $\text{Ax}_{\text{F},\beta(1)}$ is independent of the other $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms $\text{Ax}_{\text{F},\alpha(1)}$, $\text{Ax}_{\text{F},\delta(1)}$, $\text{Ax}_{\text{F},\varepsilon(1)}$.

Case IV. Consider the structure $(\Omega, \mathfrak{g}\text{-Fr}_{\mathfrak{g}})$ where, $\Omega \stackrel{\text{def}}{=} \bigcup_{\nu\in I_2^*}\Omega_{\nu}$ and, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is defined as thus: $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \emptyset$ for every $\mathscr{S}_{\mathfrak{g}} \in \{\emptyset, \Omega\}$; for each $\mu \in I_2^*$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \Omega_{\mu}$ for every $\mathscr{S}_{\mathfrak{g}} \subseteq \Omega_{\mu}$. Then, it follows that $\text{Ax}_{\text{F},\alpha(1)}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}})$, $\text{Ax}_{\text{F},\beta(1)}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}})$, $\text{Ax}_{\text{F},\delta(1)}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}) = 1$ but $\text{Ax}_{\text{F},\varphi(1)}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}) = 0$ because for each $\mu \in I_2^*$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \neq \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ for all $\mathscr{S}_{\mathfrak{g}} \subseteq \Omega_{\mu}$. Thus,

$$\text{Ax}_{\text{F},\varphi(1)}(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$$

is untrue. Consequently, $(\Omega, \mathfrak{g}\text{-Fr}_{\mathfrak{g}})$ is a mathematical system such that $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ satisfies $\text{Ax}_{\text{F},\alpha(1)}$, $\text{Ax}_{\text{F},\beta(1)}$, $\text{Ax}_{\text{F},\delta(1)}$ but not $\text{Ax}_{\text{F},\varphi(1)}$. Therefore, $\text{Ax}_{\text{F},\varphi(1)}$, then, cannot be a consequence of $\text{Ax}_{\text{F},\alpha(1)}$, $\text{Ax}_{\text{F},\beta(1)}$, $\text{Ax}_{\text{F},\delta(1)}$. Hence, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axiom $\text{Ax}_{\text{E},\varphi(1)}$ is independent of the other $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms $\text{Ax}_{\text{F},\alpha(1)}$, $\text{Ax}_{\text{F},\beta(1)}$, $\text{Ax}_{\text{F},\delta(1)}$.

Thus, for every $\mu \in I_8^*$, the class $\text{AX}_{\mu}[\mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]; \mathbb{B}] \stackrel{\text{def}}{=} \{\text{Ax}_{\text{F},\nu} : \nu \in I_{\varphi(\mu)}^*\}$ are composed of independent derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms. The proof of the theorem is complete. Q.E.D.

For every $\mu \in I_8^*$, the four $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms in $\text{AX}_{\mu}[\mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]; \mathbb{B}] = \{\text{Ax}_{\text{F},\alpha(\mu)}, \text{Ax}_{\text{F},\beta(\mu)}, \text{Ax}_{\text{F},\delta(\mu)}, \text{Ax}_{\text{F},\varphi(\mu)}\}$ may be replaced by a single $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axiom as shown in the following proposition.

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Proposition 3.47. *Let* $\mathrm{AX}_{\mu}\big[\mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big] \overset{\text{def}}{=} \big\{\mathrm{Ax}_{\mathrm{F},\nu}:\ \nu\in I^{*}_{\varphi(\mu)}\big\}$ *be a class of fundamental* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-frontier operator axioms of type* $\mu\in I^{*}_{8}$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ *and, let* $\mathrm{Ax}_{\mathrm{F}}:\mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]\longrightarrow\mathbb{B}$ *such that, for any* $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$,

$$\begin{aligned}\mathrm{Ax}_{\mathrm{F}}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) &\overset{\text{def}}{\longleftrightarrow} \left(\bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\left(\mathscr{W}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\right)\right)\right)\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\\ &= \left(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}\right)\right)\setminus\left(\emptyset\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\emptyset\right)\right). \end{aligned} \tag{3.23}$$

Then, $\mathrm{Ax}_{\mathrm{F}}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right)=1\ \longrightarrow\ \bigwedge_{\nu\in I^{*}_{\varphi(\mu)}}\mathrm{Ax}_{\mathrm{F},\nu}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right)=1.$

Proof. Let $\mathrm{AX}_{\mu}\big[\mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big] \overset{\text{def}}{=} \big\{\mathrm{Ax}_{\mathrm{F},\nu}:\ \nu\in I^{*}_{\varphi(\mu)}\big\}$ be a class of fundamental $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms of type $\mu\in I^{*}_{8}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ and, let $\mathrm{Ax}_{\mathrm{F}}:\mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}]\longrightarrow\mathbb{B}$ such that, for any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$,

$$\begin{aligned}\mathrm{Ax}_{\mathrm{F}}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) &\overset{\text{def}}{\longleftrightarrow} \left(\bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\left(\mathscr{W}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\right)\right)\right)\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\\ &= \left(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}\right)\right)\setminus\left(\emptyset\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\emptyset\right)\right). \end{aligned}$$

Suppose $\mathrm{Ax}_{\mathrm{F}}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right)=1$ holds. Then, since

$$\begin{aligned}&\left(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}\right)\right)\setminus\left(\emptyset\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\emptyset\right)\right)\subseteq\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right),\\ &\left(\bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\left(\mathscr{W}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\right)\right)\right)\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\\ &\qquad\supseteq\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\\ &\qquad\supseteq\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\end{aligned}$$

hold, for any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$ such that $\mathscr{S}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}}$ it results, consequently, that $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\subseteq\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\overset{\text{def}}{\longleftrightarrow}\mathrm{Ax}_{\mathrm{F},\alpha(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right)$ and hence, $\mathrm{Ax}_{\mathrm{F}}\longrightarrow\mathrm{Ax}_{\mathrm{F},\alpha(1)}$.

If $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})=(\emptyset,\emptyset)$, then

$$\begin{aligned}\mathrm{Ax}_{\mathrm{F}}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) &\overset{\text{def}}{\longleftrightarrow} \left(\bigcup_{\mathscr{W}_{\mathfrak{g}}=\emptyset,\emptyset}\left(\mathscr{W}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\right)\right)\right)\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\emptyset\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\emptyset\right)\right)\\ &= \left(\emptyset\cup\emptyset\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\emptyset\cup\emptyset\right)\right)\setminus\left(\emptyset\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\emptyset\right)\right). \end{aligned}$$

Consequently, it follows that $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\emptyset\right)\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\emptyset\right)=\emptyset$, implying $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\emptyset\right)=\emptyset\overset{\text{def}}{\longleftrightarrow}\mathrm{Ax}_{\mathrm{F},\delta(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right)$ and thus, $\mathrm{Ax}_{\mathrm{F}}\longrightarrow\mathrm{Ax}_{\mathrm{F},\delta(1)}$.

If $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$ be arbitrary, then

$$\begin{aligned}\mathrm{Ax}_{\mathrm{F}}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) &\overset{\text{def}}{\longleftrightarrow} \left(\bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\left(\mathscr{W}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\right)\right)\right)\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\\ &= \left(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}\right)\right)\setminus\left(\emptyset\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\left(\emptyset\right)\right). \end{aligned}$$

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But,

$$\begin{aligned}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}})) \setminus (\emptyset \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\emptyset)) &\subseteq (\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \\ &\quad \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}), \\ \left(\bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} (\mathscr{W}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})) \right) &\cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \\ &\subseteq \bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} (\mathscr{W}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}))\end{aligned}$$

and $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}})$. Therefore, $(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})) \cup (\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) = (\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{F},\beta(1)}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}})$ and hence, $\mathrm{Ax}_{\mathrm{F}} \longrightarrow \mathrm{Ax}_{\mathrm{F},\beta(1)}$.

For an arbitrary $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ such that $\mathscr{S}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, it results that

$$\begin{aligned}\mathrm{Ax}_{\mathrm{F}}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} & \left(\bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})} (\mathscr{W}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})) \right) \\ \cup \quad & \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})) \\ = \quad & (\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}))) \\ \setminus \quad & (\emptyset \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\emptyset)).\end{aligned}$$

Consequently,

$$\begin{aligned}\left(\bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})} \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) \right) & \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})) \\ &= \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})).\end{aligned}$$

Since $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cup \mathscr{V}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$ for any $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, it follows that

$$\bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})} \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}),$$

implying $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ which, by $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ for any $\mathscr{U}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, implies

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) &= \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\ &= \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\ &\cap \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).\end{aligned}$$

Therefore, $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{F},\varphi(1)}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}})$ and thus, $\mathrm{Ax}_{\mathrm{F}} \longrightarrow \mathrm{Ax}_{\mathrm{F},\varphi(1)}$.

Hence, $\mathrm{Ax}_{\mathrm{F}}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}) = 1 \longrightarrow \bigwedge_{\nu \in I^{*}_{\varphi(\mu)}} \mathrm{Ax}_{\mathrm{F},\nu}(\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}) = 1$ and the proof of the proposition is complete.

Q.E.D.

As above, having shown the consistency, independency of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms, the axiomatic definition of the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier structure in $\mathscr{T}_{\mathfrak{g}}$-spaces can now be given and is contained in the following statement.

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DEFINITION 3.48 (Axiomatic Definition: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Frontier Structure). A "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier structure of type $\mu \in I_8^*$" in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a pair $\mathfrak{F}_{\mathfrak{g}} = (\Omega, \mathfrak{g}\text{-Fr}_{\mathfrak{g}})$ consisting of a nonempty set Ω and a unary operation $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ such that the following "fundamental $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms" are satisfied:

$$(3.24)\quad \begin{array}{llll} \text{I.} & \mathrm{Ax}_{\mathrm{F},\alpha(\mu)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) = 1, & \text{III.} & \mathrm{Ax}_{\mathrm{F},\delta(\mu)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) = 1, \\ \text{II.} & \mathrm{Ax}_{\mathrm{F},\beta(\mu)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) = 1, & \text{IV.} & \mathrm{Ax}_{\mathrm{F},\varphi(\mu)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) = 1. \end{array}$$

For an arbitrary $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, let it be granted the following definition:

$$\mathrm{Ax}_{\mathrm{F},\hat{\alpha}(\mu)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).$$

Then, by the aid of DEF. 3.40, the logical implications in PROPS 3.26, 3.27 reduce to

$$\begin{aligned} \left(\mathrm{Ax}_{\mathrm{F},\hat{\alpha}(\mu)} = 1\right) &\longleftarrow \left(\mathrm{Ax}_{\mathrm{F},\alpha(\mu)} = 1\right) \wedge \left(\mathrm{Ax}_{\mathrm{F},\beta(\mu)} = 1\right), \\ \left(\mathrm{Ax}_{\mathrm{F},\alpha(\mu)} = 1\right) &\longleftarrow \left(\mathrm{Ax}_{\mathrm{F},\hat{\alpha}(\mu)} = 1\right) \wedge \left(\mathrm{Ax}_{\mathrm{F},\beta(\mu)} = 1\right) \wedge \left(\mathrm{Ax}_{\mathrm{F},\varphi(\mu)} = 1\right), \end{aligned}$$

respectively. Therefore, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axiom $\mathrm{Ax}_{\mathrm{F},\alpha(\mu)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right)$ can be equivalently replaced by the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axiom $\mathrm{Ax}_{\mathrm{F},\hat{\alpha}(\mu)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right)$. Hence, an equivalent axiomatic definition of the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier structure follows.

DEFINITION 3.49 (Equivalent Axiomatic Definition: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Frontier Structure). A "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier structure of type $\mu \in I_8^*$" in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a pair $\mathfrak{F}_{\mathfrak{g}} = (\Omega, \mathfrak{g}\text{-Fr}_{\mathfrak{g}})$ consisting of a nonempty set Ω and a unary operation $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ such that the following "fundamental $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms" are satisfied:

$$(3.25)\quad \begin{array}{llll} \text{I.} & \mathrm{Ax}_{\mathrm{F},\hat{\alpha}(\mu)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) = 1, & \text{III.} & \mathrm{Ax}_{\mathrm{F},\delta(\mu)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) = 1, \\ \text{II.} & \mathrm{Ax}_{\mathrm{F},\beta(\mu)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) = 1, & \text{IV.} & \mathrm{Ax}_{\mathrm{F},\varphi(\mu)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) = 1. \end{array}$$

Our last research objective concerning the consistency, independency of some sets of axioms for the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators in $\mathscr{T}_{\mathfrak{g}}$-spaces is now complete and the discussion of the present section terminates with the remark below.

REMARK 3.50. It is plain that, $\mathfrak{E}_{\mathfrak{g}} = (\Omega, \mathfrak{g}\text{-Ext}_{\mathfrak{g}})$ together with its axiom-set $\{\mathrm{Ax}_{\mathrm{E},\alpha(\mu)}, \mathrm{Ax}_{\mathrm{E},\beta(\mu)}, \mathrm{Ax}_{\mathrm{E},\delta(\mu)}, \mathrm{Ax}_{\mathrm{E},\varepsilon(\mu)}\}$ and $\mathfrak{F}_{\mathfrak{g}} = (\Omega, \mathfrak{g}\text{-Fr}_{\mathfrak{g}})$ together with its axiom-set $\{\mathrm{Ax}_{\mathrm{F},\alpha(\mu)}, \mathrm{Ax}_{\mathrm{F},\beta(\mu)}, \mathrm{Ax}_{\mathrm{F},\delta(\mu)}, \mathrm{Ax}_{\mathrm{F},\varphi(\mu)}\}$ will give rise to corresponding algebras: *Algebra of the* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Exterior Structure* $\mathfrak{E}_{\mathfrak{g}} = (\Omega, \mathfrak{g}\text{-Ext}_{\mathfrak{g}})$ and *Algebra of the* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Frontier Structure* $\mathfrak{F}_{\mathfrak{g}} = (\Omega, \mathfrak{g}\text{-Fr}_{\mathfrak{g}})$ – namely, the totality of propositions which follow from their sets of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-axioms by logical deduction.

The categorical classifications of $\mathfrak{g}$-$\mathfrak{T}$-exterior and $\mathfrak{g}$-$\mathfrak{T}$-frontier operators in the $\mathscr{T}$-space $\mathfrak{T} \subset \mathfrak{T}_{\mathfrak{g}}$ and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ are discussed and diagrammed on this ground in the next section.

4. DISCUSSION

4.1. CATEGORICAL CLASSIFICATIONS. Having adopted a categorical approach in the classifications of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators, the twofold purposes here are to establish the various relationships between the classes of $\mathfrak{g}$-$\mathfrak{T}$-exterior operators in the $\mathscr{T}$-space $\mathfrak{T}$ and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operators in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, and the classes of $\mathfrak{g}$-$\mathfrak{T}$-frontier operators in the $\mathscr{T}$-space $\mathfrak{T}$ and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, and to illustrate them through diagrams.

In a $\mathscr{T}$-space $\mathfrak{T}$, $\mathfrak{g}\text{-Int}_0(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_1(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_3(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Int}_2(\mathscr{S}_{\mathfrak{g}})$ holds for any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ because, for every $\mathscr{O}_{\mathfrak{g}} \in \mathrm{O}[\mathfrak{T}]$, $\mathrm{op}_0(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathrm{op}_1(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathrm{op}_3(\mathscr{O}_{\mathfrak{g}}) \supseteq \mathrm{op}_2(\mathscr{O}_{\mathfrak{g}})$ holds. Consequently, it results that the relation $\mathfrak{g}\text{-Int}_0 \circ \mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_1 \circ \mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_3 \circ \mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Int}_2 \circ \mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}})$ and the relation

$$\begin{aligned}
&\mathfrak{g}\text{-Op}\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_0(\mathscr{U}_{\mathfrak{g}})\Bigg)\\
&\quad\supseteq \mathfrak{g}\text{-Op}\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_1(\mathscr{U}_{\mathfrak{g}})\Bigg)\\
&\qquad\supseteq \mathfrak{g}\text{-Op}\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_3(\mathscr{U}_{\mathfrak{g}})\Bigg)\\
&\qquad\quad\subseteq \mathfrak{g}\text{-Op}\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_2(\mathscr{U}_{\mathfrak{g}})\Bigg)
\end{aligned}$$

holds. Hence, both $\mathfrak{g}\text{-Ext}_0(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Ext}_1(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Ext}_3(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Ext}_2(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Fr}_0(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Fr}_1(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Fr}_3(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Fr}_2(\mathscr{S}_{\mathfrak{g}})$ hold for any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

Similarly, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Int}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Int}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}})$ holds for any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ because, for every $\mathscr{O}_{\mathfrak{g}} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$, $\mathrm{op}_{\mathfrak{g},0}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},1}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},3}(\mathscr{O}_{\mathfrak{g}}) \supseteq \mathrm{op}_{\mathfrak{g},2}(\mathscr{O}_{\mathfrak{g}})$ holds. Consequently, $\mathfrak{g}\text{-Int}_{\mathfrak{g},0} \circ \mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g},1} \circ \mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g},3} \circ \mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Int}_{\mathfrak{g},2} \circ \mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}})$ and the relation

$$\begin{aligned}
&\mathfrak{g}\text{-Op}\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g},0}(\mathscr{U}_{\mathfrak{g}})\Bigg)\\
&\quad\supseteq \mathfrak{g}\text{-Op}\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g},1}(\mathscr{U}_{\mathfrak{g}})\Bigg)\\
&\qquad\supseteq \mathfrak{g}\text{-Op}\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g},3}(\mathscr{U}_{\mathfrak{g}})\Bigg)\\
&\qquad\quad\subseteq \mathfrak{g}\text{-Op}\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Int}_{\mathfrak{g},2}(\mathscr{U}_{\mathfrak{g}})\Bigg)
\end{aligned}$$

holds. Thus, both $\mathfrak{g}\text{-Ext}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Fr}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Fr}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}})$ hold for any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

But, $\mathfrak{g}\text{-Int}_{\nu}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$ holds for any $(\nu, \mathscr{S}_{\mathfrak{g}}) \in I_3^0 \times \mathscr{P}(\Omega)$ because, for every $\nu \in I_3^0$, $\mathscr{O}_{\mathfrak{g}} \subseteq \mathrm{op}_{\nu}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}})$ holds. Consequently, $\mathfrak{g}\text{-Ext}_{\nu}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow$

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$\mathfrak{g}\text{-Int}_{\nu}\circ\mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}\circ\mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Ext}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$ and

$$\begin{aligned}\mathfrak{g}\text{-Fr}_{\nu}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Op}\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}})}\mathfrak{g}\text{-Int}_{\nu}(\mathscr{U}_{\mathfrak{g}})\Bigg)\\ &\supseteq \mathfrak{g}\text{-Op}\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Op}(\mathscr{S}_{\mathfrak{g}})}\mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}(\mathscr{U}_{\mathfrak{g}})\Bigg)\longleftrightarrow\mathfrak{g}\text{-Fr}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Hence, both $\mathfrak{g}\text{-Ext}_{\nu}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Ext}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Fr}_{\nu}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Fr}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$ hold for any $(\nu,\mathscr{S}_{\mathfrak{g}})\in I_3^0\times\mathscr{P}(\Omega)$. Taking all these lines of reasoning into account, the following diagrams, which are both to be read horizontally, from left to right and vertically, from top to bottom, present themselves:

$$\begin{array}{ccccccc}\mathfrak{g}\text{-Ext}_0(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-Ext}_1(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-Ext}_3(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-Ext}_2(\mathscr{S}_{\mathfrak{g}})\\ \cap\!\!\mid & & \cap\!\!\mid & & \cap\!\!\mid & & \cap\!\!\mid\\ \mathfrak{g}\text{-Ext}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-Ext}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-Ext}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-Ext}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}});\end{array}\tag{4.1}$$

$$\begin{array}{ccccccc}\mathfrak{g}\text{-Fr}_0(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-Fr}_1(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-Fr}_3(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-Fr}_2(\mathscr{S}_{\mathfrak{g}})\\ \cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid\\ \mathfrak{g}\text{-Fr}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-Fr}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-Fr}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-Fr}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}}).\end{array}\tag{4.2}$$

In Fig. 1 are presented the relationships between the elements of the collections $\{\mathfrak{g}\text{-Ext}_{\nu}(\mathscr{S}_{\mathfrak{g}}):\ \nu\in I_3^0\}$ in the $\mathscr{T}$-space $\mathfrak{T}$ and $\{\mathfrak{g}\text{-Ext}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}):\ \nu\in I_3^0\}$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$; Fig. 1 may well be called a $(\mathfrak{g}\text{-Ext},\mathfrak{g}\text{-Ext}_{\mathfrak{g}})$*-valued diagram.*

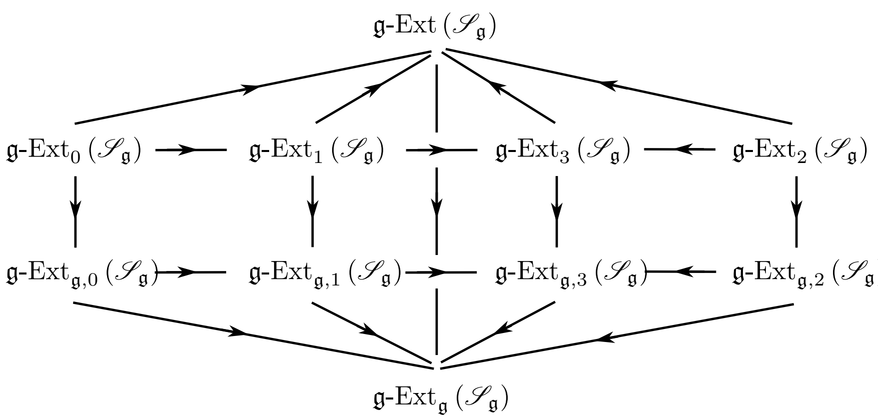

Figure 1. Relationships: $\mathfrak{g}$-$\mathfrak{T}$-exterior operators in $\mathscr{T}$-spaces and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

In Fig. 2 are presented the relationships between the elements of the collections $\{\mathfrak{g}\text{-Fr}_{\nu}(\mathscr{S}_{\mathfrak{g}}):\ \nu\in I_3^0\}$ in the $\mathscr{T}$-space $\mathfrak{T}$ and $\{\mathfrak{g}\text{-Fr}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}):\ \nu\in I_3^0\}$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$; Fig. 2 may well be called a $(\mathfrak{g}\text{-Fr},\mathfrak{g}\text{-Fr}_{\mathfrak{g}})$*-valued diagram.*

As in the works of other authors [CJS05, Don97, JJLL08, TC16], the manner we have positioned the arrows in the $(\mathfrak{g}\text{-Ext},\mathfrak{g}\text{-Ext}_{\mathfrak{g}})$, $(\mathfrak{g}\text{-Fr},\mathfrak{g}\text{-Fr}_{\mathfrak{g}})$*-valued diagrams* (Figs 1, 2) is solely to stress that, in general, the implications in Figs 1, 2 are irreversible.

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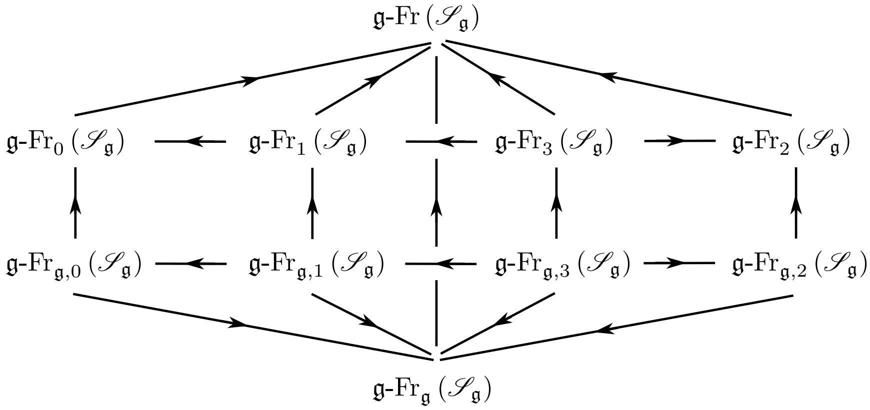

FIGURE 2. Relationships: $\mathfrak{g}$-$\mathfrak{T}$-frontier operators in $\mathscr{T}$-spaces and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

At this stage, a nice application is worth considering, and is presented in the following section.

4.2. A NICE APPLICATION. Focusing on fundamental concepts from the standpoint of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators in $\mathscr{T}_{\mathfrak{g}}$-spaces in an attempt to shed lights on some of the essential properties and such notions as consistency, independency of the axioms associated with the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators, we shall now present a nice application comprising of some interesting cases.

CASE I. Let $\Omega = \{\xi_\nu : \ \nu \in I_5^*\}$ denotes the underlying set and consider the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, where Ω is topologized by the choice:

$$\begin{aligned} \mathscr{T}_{\mathfrak{g}}(\Omega) &= \big\{\emptyset,\ \{\xi_2\},\ \{\xi_2, \xi_4, \xi_5\},\ \Omega\big\} \\ &= \big\{\mathscr{O}_{\mathfrak{g},1},\ \mathscr{O}_{\mathfrak{g},2},\ \mathscr{O}_{\mathfrak{g},3},\ \mathscr{O}_{\mathfrak{g},4}\big\}; \end{aligned} \tag{4.3}$$

$$\begin{aligned} \neg\mathscr{T}_{\mathfrak{g}}(\Omega) &= \big\{\Omega,\ \{\xi_1, \xi_3, \xi_4, \xi_5\},\ \{\xi_1, \xi_3\}, \emptyset\big\} \\ &= \big\{\mathscr{K}_{\mathfrak{g},1},\ \mathscr{K}_{\mathfrak{g},2},\ \mathscr{K}_{\mathfrak{g},3},\ \mathscr{K}_{\mathfrak{g},4}\big\}. \end{aligned} \tag{4.4}$$

Evidently, the set-valued set maps $\mathscr{T}_{\mathfrak{g}}, \neg\mathscr{T}_{\mathfrak{g}} : \ \mathscr{P}(\Omega) \longrightarrow \mathscr{P}\big(\{\xi_\nu : \ \nu \in I_5^*\}\big)$ establish the classes of $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets, respectively. Since conditions $\mathscr{T}_{\mathfrak{g}}(\emptyset) = \emptyset$, $\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) \subseteq \mathscr{O}_{\mathfrak{g},\nu}$ for every $\nu \in I_4^*$, $\mathscr{T}_{\mathfrak{g}}(\Omega) = \Omega$, and $\mathscr{T}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_4^*} \mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu\in I_4^*} \mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ are satisfied, it is clear that the one-valued map $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}\big(\{\xi_\nu : \ \nu \in I_5^*\}\big)$ is a strong $\mathfrak{g}$-topology and hence, $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space. On the other hand, because the additional condition $\mathscr{T}_{\mathfrak{g}}\big(\bigcap_{\nu\in I_4^*} \mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcap_{\nu\in I_4^*} \mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ is satisfied, $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}\big(\{\xi_\nu : \ \nu \in I_5^*\}\big)$ is also a topology and thus, $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$. Moreover, it is easily checked that $\mathscr{O}_{\mathfrak{g},\mu} \in \mathfrak{g}$-$\nu$-$\mathrm{O}[\mathfrak{T}]$ for every $(\nu, \mu) \in I_3^0 \times I_4^*$. Thus, the $\mathscr{T}_{\mathfrak{g}}$-open sets forming the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}\big(\{\xi_\nu : \ \nu \in I_5^*\}\big)$ of the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ are $\mathfrak{g}$-$\mathscr{T}$-open sets relative to the $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$.

Clearly, the cardinality $\mathrm{card}\big(\mathscr{P}(\Omega)\big) = 2^{\mathrm{card}(\Omega)}$ is very large. For convenience of notation, express $\mathscr{P}(\Omega)$ in set-builder notation as a collection indexed by the

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Cartesian product $I^{*}_{\mathrm{card}(\mathscr{P}(\Omega))}\times I^{0}_{\mathrm{card}(\Omega)}$:

$$\mathscr{P}(\Omega) = \left\{\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\in\mathscr{P}(\Omega):\ (\nu,\mu)\in I^{*}_{\mathrm{card}(\mathscr{P}(\Omega))}\times I^{0}_{\mathrm{card}(\Omega)}\right\}, \tag{4.5}$$

where $\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\in\mathscr{P}(\Omega)$ denotes a $\mathfrak{T}_{\mathfrak{g}}$-set labeled $\nu\in I^{*}_{\mathrm{card}(\mathscr{P}(\Omega))}$ and containing $\mu\in I^{0}_{\mathrm{card}(\Omega)}$ elements. Below is established the indexing by the Cartesian product $I^{*}_{\mathrm{card}(\mathscr{P}(\Omega))}\times I^{0}_{\mathrm{card}(\Omega)}$ by the choice: $\mathscr{S}_{\mathfrak{g},(1,0)}=\emptyset$, …, $\mathscr{S}_{\mathfrak{g},(\nu,\mu)}=\{\xi_1,\xi_2,\ldots,\xi_\mu\}$, …, $\mathscr{S}_{\mathfrak{g},(32,5)}=\Omega$.

For $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$ such that $\mathrm{card}(\mathscr{S}_{\mathfrak{g}})\in\{0,5\}$, let $\mathscr{S}_{\mathfrak{g},(1,0)}=\emptyset$ and $\mathscr{S}_{\mathfrak{g},(32,5)}=\Omega$. For $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$ such that $\mathrm{card}(\mathscr{S}_{\mathfrak{g}})\in\{1,4\}$, let $\mathscr{S}_{\mathfrak{g},(2,1)}=\{\xi_1\}$, $\mathscr{S}_{\mathfrak{g},(3,1)}=\{\xi_2\}$, $\mathscr{S}_{\mathfrak{g},(4,1)}=\{\xi_3\}$, $\mathscr{S}_{\mathfrak{g},(5,1)}=\{\xi_4\}$, and $\mathscr{S}_{\mathfrak{g},(6,1)}=\{\xi_5\}$; $\mathscr{S}_{\mathfrak{g},(27,4)}=\{\xi_1,\xi_2,\xi_3,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(28,4)}=\{\xi_2,\xi_3,\xi_4,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(29,4)}=\{\xi_1,\xi_3,\xi_4,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(30,4)}=\{\xi_1,\xi_2,\xi_3,\xi_5\}$, and $\mathscr{S}_{\mathfrak{g},(31,4)}=\{\xi_1,\xi_2,\xi_4,\xi_5\}$. For $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$ such that $\mathrm{card}(\mathscr{S}_{\mathfrak{g}})\in\{2,3\}$, let $\mathscr{S}_{\mathfrak{g},(7,2)}=\{\xi_1,\xi_2\}$, $\mathscr{S}_{\mathfrak{g},(8,2)}=\{\xi_1,\xi_3\}$, $\mathscr{S}_{\mathfrak{g},(9,2)}=\{\xi_1,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(10,2)}=\{\xi_1,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(11,2)}=\{\xi_2,\xi_3\}$, $\mathscr{S}_{\mathfrak{g},(12,2)}=\{\xi_2,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(13,2)}=\{\xi_2,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(14,2)}=\{\xi_3,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(15,2)}=\{\xi_3,\xi_5\}$, and $\mathscr{S}_{\mathfrak{g},(16,2)}=\{\xi_4,\xi_5\}$; $\mathscr{S}_{\mathfrak{g},(17,3)}=\{\xi_1,\xi_2,\xi_3\}$, $\mathscr{S}_{\mathfrak{g},(18,3)}=\{\xi_1,\xi_3,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(19,3)}=\{\xi_1,\xi_4,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(20,3)}=\{\xi_1,\xi_2,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(21,3)}=\{\xi_1,\xi_2,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(22,3)}=\{\xi_1,\xi_3,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(23,3)}=\{\xi_2,\xi_3,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(24,3)}=\{\xi_2,\xi_3,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(25,3)}=\{\xi_3,\xi_4,\xi_5\}$, and $\mathscr{S}_{\mathfrak{g},(26,3)}=\{\xi_2,\xi_4,\xi_5\}$.

A first series of calculations shows that, for every $(\nu,\mu)\in I^{*}_{\mathrm{card}(\mathscr{P}(\Omega))}\times I^{0}_{\mathrm{card}(\Omega)}$,

$$\begin{array}{ccc}\mathrm{ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\right) & \longleftrightarrow & \mathrm{int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\right)\\ \cap\!\!| & & \cap\!\!|\\ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\right) & \longleftrightarrow & \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\right)\end{array} \tag{4.6}$$

and

$$\begin{array}{ccc}\mathrm{fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\right) & \longleftrightarrow & \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\bigcup_{\mathscr{R}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g},(\nu,\mu)},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\right)}\mathrm{int}_{\mathfrak{g},\nu}\left(\mathscr{R}_{\mathfrak{g}}\right)\right)\\ \cup\!\!| & & \cup\!\!|\\ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\right) & \longleftrightarrow & \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\bigcup_{\mathscr{R}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g},(\nu,\mu)},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\right)}\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu}\left(\mathscr{R}_{\mathfrak{g}}\right)\right),\end{array} \tag{4.7}$$

meaning that, $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathrm{ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ or, $\mathrm{ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$; $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathrm{fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ or, $\mathrm{fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$. Thus, EQS (4.6), (4.7) validate ITEMS I., II., respectively, of THM. 3.4. Moreover, for each $(\sigma,\eta)\in\{(0,1),(1,3),(2,3)\}$,

$$\begin{array}{ccc}\mathfrak{g}\text{-}\mathrm{Ext}_{\sigma}\left(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\right) & \subseteq & \mathfrak{g}\text{-}\mathrm{Ext}_{\eta}\left(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\right)\\ \cap\!\!| & & \cap\!\!|\\ \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g},\sigma}\left(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\right) & \subseteq & \mathfrak{g}\text{-}\mathrm{Ext}_{\mathfrak{g},\eta}\left(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\right)\end{array} \tag{4.8}$$

and

$$\begin{array}{ccc}\mathfrak{g}\text{-}\mathrm{Fr}_{\sigma}\left(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\right) & \supseteq & \mathfrak{g}\text{-}\mathrm{Fr}_{\eta}\left(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\right)\\ \cup\!\!| & & \cup\!\!|\\ \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g},\sigma}\left(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\right) & \supseteq & \mathfrak{g}\text{-}\mathrm{Fr}_{\mathfrak{g},\eta}\left(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\right)\end{array} \tag{4.9}$$

for all $(\nu,\mu)\in I^{*}_{\mathrm{card}(\mathscr{P}(\Omega))}\times I^{0}_{\mathrm{card}(\Omega)}$. Equivalently stated, it means that, for each $(\sigma,\eta)\in\{(0,1),(1,3),(2,3)\}$, $\mathfrak{g}\text{-Ext}_{\sigma}\gtrsim\mathfrak{g}\text{-Ext}_{\mathfrak{g},\sigma}$, $\mathfrak{g}\text{-Ext}_{\eta}\gtrsim\mathfrak{g}\text{-Ext}_{\mathfrak{g},\eta}$ and $\mathfrak{g}\text{-Fr}_{\sigma}\lesssim\mathfrak{g}\text{-Fr}_{\mathfrak{g},\sigma}$, $\mathfrak{g}\text{-Fr}_{\eta}\lesssim\mathfrak{g}\text{-Fr}_{\mathfrak{g},\eta}$ for all $(\nu,\mu)\in I^{*}_{\mathrm{card}(\mathscr{P}(\Omega))}\times I^{0}_{\mathrm{card}(\Omega)}$. Thus, EQ. (4.8) validates the $(\mathfrak{g}\text{-Ext},\mathfrak{g}\text{-Ext}_{\mathfrak{g}})$-valued diagram present in FIG. 1, and EQ. (4.9) validates the $(\mathfrak{g}\text{-Fr},\mathfrak{g}\text{-Fr}_{\mathfrak{g}})$-valued diagram present in FIG. 2.

A second series of calculations shows that, for every $(\sigma,\nu,\mu,\eta)\in I^{*}_{\mathrm{card}(\mathscr{P}(\Omega))}\times I^{*}_{\mathrm{card}(\Omega)}\times I^{0}_{\mathrm{card}(\mathscr{P}(\Omega))}\times I^{0}_{\mathrm{card}(\Omega)}$,

$$\begin{aligned}
\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\sigma,\eta)}\big) &\subseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\sigma,\eta)}\big),\\
\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\sigma,\eta)}\cup\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big) &= \mathfrak{g}\text{-Ext}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\sigma,\eta)}\big)\cap\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big),\\
\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(1,0)}\big) &= \mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(1,0)}\big),\\
\mathscr{S}_{\mathfrak{g},(\sigma,\eta)}\cap\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\sigma,\eta)}\big) &= \emptyset
\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}
\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\sigma,\eta)}\big) &\subseteq \mathscr{S}_{\mathfrak{g},(\sigma,\eta)}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\sigma,\eta)}\big),\\
\big(\mathscr{S}_{\mathfrak{g},(\sigma,\eta)}\cup\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big)\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\sigma,\eta)}\cup\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big) &= \\
&\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g},(\sigma,\eta)},\mathscr{S}_{\mathfrak{g},(\nu,\mu)}}\big(\mathscr{U}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big),\\
\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(1,0)}\big) &= \emptyset,\\
\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\sigma,\eta)}\big) &= \mathfrak{g}\text{-Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\sigma,\eta)}\big).
\end{aligned}\tag{4.11}$$

Hence, EQS (4.10) validate ITEMS I.–IV., and EQS (4.11) validate ITEMS V.–VIII. of COR. 3.28.

CASE II. Let $(\Omega,\mathfrak{g}\text{-Ext}_{\mathfrak{g}})$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior structure and let $(\Omega,\mathfrak{g}\text{-Fr}_{\mathfrak{g}})$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier structure in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, where $\Omega=\{\xi_{\nu}:\ \nu\in I^{*}_{3}\}$ denote the underlying set. Then:

(1) If $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ be defined as, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_{\nu}\})\overset{\text{def}}{=}\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi_{\nu}\})$ for any $\nu\in I^{*}_{3}$, and $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset)\overset{\text{def}}{=}\Omega$. Then, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset)=\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset)$ is a direct consequence of the definition of $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ and, for every $(\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ also satisfies the relations $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Ext}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}\cup\mathscr{V}_{\mathfrak{g}})=\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$, and $\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})=\emptyset$. Hence, for every $\mu\in I^{*}_{8}$, the class $\mathrm{AX}_{\mu}[\mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}]\overset{\text{def}}{=}\{\mathrm{Ax}_{\mathrm{E},\nu}:\ \nu\in I^{*}_{\varepsilon(\mu)}\}$ are composed of consistent derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms.

(2) If $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ be defined as, $\{\xi_{\nu}\}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\{\xi_{\nu}\})\overset{\text{def}}{=}\{\xi_{\nu}\}$ for any $\nu\in I^{*}_{3}$, and $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\emptyset)\overset{\text{def}}{=}\emptyset$. Then, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\emptyset)=\emptyset$ is a direct consequence of the definition of $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ and, for every $(\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ also satisfies the relations $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\subseteq\mathscr{U}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, $(\mathscr{U}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}))\cup(\mathscr{V}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}))=(\mathscr{U}_{\mathfrak{g}}\cup\mathscr{V}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}\cup\mathscr{V}_{\mathfrak{g}})$, and $\mathfrak{g}\text{-Fr}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})=\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. Hence, for every $\mu\in I^{*}_{8}$, the class $\mathrm{AX}_{\mu}[\mathfrak{g}\text{-F}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}]\overset{\text{def}}{=}\{\mathrm{Ax}_{\mathrm{F},\nu}:$

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$\nu \in I^{*}_{\varepsilon(\mu)}\}$ are composed of consistent derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms.

Hence, ITEM (1) of CASE II. validates PROP. 3.35 and ITEM (2) of CASE II. validates PROP. 3.45.

CASE III. Let $(\Omega_{\sigma}, \mathfrak{g}\text{-Ext}_{\mathfrak{g}})$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior structure in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\sigma} = (\Omega_{\sigma}, \mathscr{T}_{\mathfrak{g},\sigma})$, where $\Omega_{\sigma} = \{\xi_{\nu} :\ \nu \in I^{*}_{\sigma}\}$ denote the underlying set, conditioned by the parameter $\sigma > 1$. Then:

(1) If $\sigma = 2$ and $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega_2) \longrightarrow \mathscr{P}(\Omega_2)$ is defined as thus: $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}} \cup \{\xi_2\})$ for every $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega_2)$. Then, it follows that the statements $\mathrm{Ax}_{\mathrm{E},\alpha(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}})$, $\mathrm{Ax}_{\mathrm{E},\beta(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}})$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}) = 1$ hold but $\mathrm{Ax}_{\mathrm{E},\delta(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}) = 0$ because, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset) = \{\xi_1\} \neq \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset)$. Hence, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axiom $\mathrm{Ax}_{\mathrm{E},\delta(1)}$ is independent of the other $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms $\mathrm{Ax}_{\mathrm{E},\alpha(1)}$, $\mathrm{Ax}_{\mathrm{E},\beta(1)}$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}$.

(2) If $\sigma = 3$ and $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega_3) \longrightarrow \mathscr{P}(\Omega_3)$ is defined as thus: $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ for all $\mathscr{S}_{\mathfrak{g}} \in \{\emptyset, \Omega_3\}$, and $\mathscr{S}_{\mathfrak{g}} \in \{\{\xi_{\nu}, \xi_{\mu}\} :\ (\nu, \mu) \in I^{*2}_{3} \setminus I^{*2}_{3,\Delta}\}$, where $I^{*2}_{3,\Delta} \stackrel{\text{def}}{=} \{(\nu, \nu) :\ \nu \in I^{*}_{3}\}$; $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \emptyset$, $\{\xi_3\}$, $\{\xi_2\}$ if $\mathscr{S}_{\mathfrak{g}} = \{\xi_3\}$, $\{\xi_2\}$, $\{\xi_1\}$, respectively. Then, it results that $\mathrm{Ax}_{\mathrm{E},\alpha(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}})$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}})$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}) = 1$ but $\mathrm{Ax}_{\mathrm{E},\beta(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}) = 0$ because, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_1\} \cup \{\xi_2\}) = \{\xi_3\}$ but $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_1\}) \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_2\}) = \emptyset$; $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_1\} \cup \{\xi_3\}) = \{\xi_2\}$ but $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_1\}) \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_3\}) = \emptyset$; also, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_2\} \cup \{\xi_3\}) = \{\xi_1\}$ but $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_2\}) \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_3\}) = \emptyset$; also, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_1\} \cup \{\xi_1, \xi_2\}) = \{\xi_3\}$ but $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_1\}) \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_1, \xi_2\}) = \emptyset$; $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_2\} \cup \{\xi_2, \xi_3\}) = \{\xi_1\}$ but $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_2\}) \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_2, \xi_3\}) = \emptyset$; $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_3\} \cup \{\xi_1, \xi_3\}) = \{\xi_2\}$ but $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_3\}) \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_1, \xi_3\}) = \emptyset$. Hence, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axiom $\mathrm{Ax}_{\mathrm{E},\beta(1)}$ is independent of the other $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms $\mathrm{Ax}_{\mathrm{E},\alpha(1)}$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}$.

(3) If $\sigma = 3$ and $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega_3) \longrightarrow \mathscr{P}(\Omega_3)$ is defined as thus: $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \{\xi_2\}$, $\{\xi_3\}$, $\{\xi_1\}$ if $\mathscr{S}_{\mathfrak{g}} = \{\xi_1\}$, $\{\xi_2\}$, $\{\xi_3\}$, respectively; $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \Omega_3$ if $\mathscr{S}_{\mathfrak{g}} = \emptyset$, and $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \emptyset$ if $\mathscr{S}_{\mathfrak{g}} = \Omega_3$, or $\mathscr{S}_{\mathfrak{g}} \in \{\Omega_3 \setminus \{\xi_{\nu}\} :\ \nu \in I^{*}_{3}\}$. Then, $\mathrm{Ax}_{\mathrm{E},\beta(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}})$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}})$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}) = 1$ but $\mathrm{Ax}_{\mathrm{E},\alpha(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}) = 0$ because, it follows that $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_1\}) = \{\xi_2\} \not\subseteq \emptyset = \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_1\})$; also, because $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_2\}) = \{\xi_3\} \not\subseteq \emptyset = \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_2\})$; finally, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_3\}) = \{\xi_1\} \not\subseteq \emptyset = \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_3\})$. Hence, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axiom $\mathrm{Ax}_{\mathrm{E},\alpha(1)}$ is independent of the other $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms $\mathrm{Ax}_{\mathrm{E},\beta(1)}$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}$, $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}$.

(4) If $\sigma = \infty$ and $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega_{\infty}) \longrightarrow \mathscr{P}(\Omega_{\infty})$ is defined as thus: For every $\mathscr{S}_{\mathfrak{g}} \subset \Omega_{\infty}$, $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$; $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \{\xi_{\infty}\}$ if $\mathscr{S}_{\mathfrak{g}} = \Omega_{\infty}$. Then, it follow that $\mathrm{Ax}_{\mathrm{E},\alpha(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}})$, $\mathrm{Ax}_{\mathrm{E},\beta(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}})$, $\mathrm{Ax}_{\mathrm{E},\delta(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}) = 1$ but $\mathrm{Ax}_{\mathrm{E},\varepsilon(1)}(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}) = 0$ because, it results that

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$\{\xi_\nu : \nu \in I_\infty^*\} \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\{\xi_\nu : \nu \in I_\infty^*\}) = \{\xi_\infty\} \neq \emptyset$. Hence, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axiom $\text{Ax}_{\text{E},\varepsilon(1)}$ is independent of the other $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms $\text{Ax}_{\text{E},\alpha(1)}$, $\text{Ax}_{\text{E},\beta(1)}$, $\text{Ax}_{\text{E},\delta(1)}$.

In view of ITEMS (1)–(4) of CASE III., for every $\mu \in I_8^*$, the class $\text{AX}_\mu\left[\mathfrak{g}\text{-E}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right] \stackrel{\text{def}}{=} \left\{\text{Ax}_{\text{E},\nu} : \nu \in I_{\varepsilon(\mu)}^*\right\}$ are composed of independent derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms. Hence, ITEMS (1)–(4) of CASE III. altogether validate THM. 3.36.

CASE IV. Let $\left(\Omega_\sigma, \mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier structure in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\sigma} = (\Omega_\sigma, \mathscr{T}_{\mathfrak{g},\sigma})$, where $\Omega_\sigma = \left\{\xi_\nu : \nu \in I_\sigma^*\right\}$ denote the underlying set, conditioned by the parameter $\sigma > 1$. Then:

(1) If $\sigma = 2$ and $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega_2) \longrightarrow \mathscr{P}(\Omega_2)$ be defined as thus: $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \{\xi_2\}$ for every $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. Then, $\text{Ax}_{\text{F},\alpha(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right)$, $\text{Ax}_{\text{F},\beta(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right)$, $\text{Ax}_{\text{F},\varphi(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) = 1$ but $\text{Ax}_{\text{F},\delta(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) = 0$ because, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\emptyset) = \{\xi_2\} \neq \emptyset$. Hence, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axiom $\text{Ax}_{\text{F},\delta(1)}$ is independent of the other $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms $\text{Ax}_{\text{F},\alpha(1)}$, $\text{Ax}_{\text{F},\beta(1)}$, $\text{Ax}_{\text{F},\varphi(1)}$.

(2) If $\sigma = 3$ and $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega_3) \longrightarrow \mathscr{P}(\Omega_3)$ be defined as thus: $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \emptyset$ for every $\mathscr{S}_{\mathfrak{g}} \in \{\emptyset, \Omega_3\}$; $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \{\xi_1, \xi_2\}$ if $\mathscr{S}_{\mathfrak{g}} = \{\xi_1\}$; $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \Omega$ if $\mathscr{S}_{\mathfrak{g}} \in \left\{\{\xi_2\}, \{\xi_3\}, \{\xi_1, \xi_2\}\right\}$,$\{\xi_1, \xi_3\}$; $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \{\xi_1, \xi_2\}$ if $\mathscr{S}_{\mathfrak{g}} = \{\xi_2, \xi_3\}$. Then, $\text{Ax}_{\text{F},\beta(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right)$, $\text{Ax}_{\text{F},\delta(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right)$, $\text{Ax}_{\text{F},\varphi(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) = 1$ but $\text{Ax}_{\text{F},\alpha(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) = 0$ because, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\{\xi_1\}) \not\subseteq \{\xi_1\} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\{\xi_1\})$. Hence, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axiom $\text{Ax}_{\text{E},\alpha(1)}$ is independent of the other $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms $\text{Ax}_{\text{F},\beta(1)}$, $\text{Ax}_{\text{F},\delta(1)}$, $\text{Ax}_{\text{F},\varphi(1)}$.

(3) If $\sigma = 3$ and $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega_3) \longrightarrow \mathscr{P}(\Omega_3)$ is defined as thus: $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \emptyset$ for every $\mathscr{S}_{\mathfrak{g}} \in \{\emptyset, \Omega_3\}$; $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \emptyset$ if $\mathscr{S}_{\mathfrak{g}} = \{\xi_1\}$; for each $\mu \in I_3^* \setminus \{1\}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \{\xi_\mu\}$ if $\mathscr{S}_{\mathfrak{g}} = \{\xi_\mu\}$ and $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \{\xi_\mu\}$ if $\mathscr{S}_{\mathfrak{g}} = \{\xi_\nu : \nu \in I_3^* \setminus \{\mu\}\}$; $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \emptyset$ if $\mathscr{S}_{\mathfrak{g}} = \{\xi_\nu : \nu \in I_3^* \setminus \{1\}\}$. Then, it follows that $\text{Ax}_{\text{F},\alpha(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right)$, $\text{Ax}_{\text{F},\delta(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right)$, $\text{Ax}_{\text{F},\varphi(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) = 1$ but $\text{Ax}_{\text{F},\beta(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) = 0$ because, for each $\mu \in I_3^* \setminus \{1\}$, $\left(\{\xi_1\} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\{\xi_1\})\right) \cup \left(\{\xi_\mu\} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\{\xi_\mu\})\right) = \{\xi_1, \xi_\mu\} \neq \Omega_3 = \left(\{\xi_1\} \cup \{\xi_\mu\}\right) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\{\xi_1\} \cup \{\xi_\mu\})$. Hence, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axiom $\text{Ax}_{\text{F},\beta(1)}$ is independent of the other $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms $\text{Ax}_{\text{F},\alpha(1)}$, $\text{Ax}_{\text{F},\delta(1)}$, $\text{Ax}_{\text{F},\varepsilon(1)}$.

(4) If $\sigma = 2$ and $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is defined as thus: $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \emptyset$ for every $\mathscr{S}_{\mathfrak{g}} \in \{\emptyset, \Omega_2\}$; for each $\mu \in I_2^*$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \Omega_\mu$ if $\mathscr{S}_{\mathfrak{g}} = \{\xi_\mu\}$. Then, it follows that $\text{Ax}_{\text{F},\alpha(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right)$, $\text{Ax}_{\text{F},\beta(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right)$, $\text{Ax}_{\text{F},\delta(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) = 1$ but $\text{Ax}_{\text{F},\varphi(1)}\left(\mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) = 0$ because, for each $\mu \in I_2^*$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi_\mu\}) = \Omega_2 \setminus \{\xi_\mu\} \neq \{\xi_\mu\} = \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\{\xi_\mu\})$. Hence, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axiom $\text{Ax}_{\text{E},\varphi(1)}$ is independent of the other $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms $\text{Ax}_{\text{F},\alpha(1)}$, $\text{Ax}_{\text{F},\beta(1)}$, $\text{Ax}_{\text{F},\delta(1)}$.

In view of ITEMS (1)–(4) of CASE IV., for every $\mu \in I_8^*$, the class $\mathrm{AX}_{\mu}\left[\mathfrak{g}\text{-F}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\right] \stackrel{\mathrm{def}}{=} \left\{\mathrm{Ax}_{\mathrm{F},\nu} : \nu \in I_{\varphi(\mu)}^*\right\}$ are composed of independent derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms. Hence, ITEMS (1)–(4) of CASE IV. altogether validate THM. 3.46.

If the discussions of this nice application be explore a step further, other interesting conclusions can be drawn. The next section provides concluding remarks and future directions of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators discussed in the preceding sections.

4.3. CONCLUDING REMARKS. In this Pure Mathematical manuscript titled *Theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Frontier Operators* and subtitled *Definitions, Essential Properties and, Consistent, Independent Axioms*, a new theory has been developed with the twofold objectives of, firstly, presenting the definitions and the essential properties of a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators in $\mathscr{T}_{\mathfrak{g}}$-spaces and, secondly, discussing the consistency, independency of some sets of axioms for the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

Concisely, the definitions of the notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces were presented in as general and unified a manner as possible [SUBSECT. 2.1: DEF. 2.1 & REM. 2.2]; the essential properties of such novel types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators were discussed in such a way as to show that much of the fundamental structure of $\mathscr{T}_{\mathfrak{g}}$-spaces is better considered for $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ than for the $\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{T}_{\mathfrak{g}}$-frontier operators $\mathrm{ext}_{\mathfrak{g}}$, $\mathrm{fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively [SUBSECT. 3.1: PROPS 3.1–3.27, THMS 3.2–3.24, CORS 3.3–3.29 & LEM. 3.10]; the notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators axioms in $\mathscr{T}_{\mathfrak{g}}$-spaces were presented from a purely mathematical or abstract point of view and, the consistency, independency of some sets of such axioms for the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators in $\mathscr{T}_{\mathfrak{g}}$-spaces were discussed in such a way as to show that they form fundamental sets of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-axioms [SUBSECT. 3.2: DEFS 3.30–3.40, LEMS 3.31–3.42, THMS 3.33–3.46 & PROPS 3.34–3.47].

Precisely, the outstanding facts are:

- I. If the definitions of $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are based on $\mathrm{int}_{\mathfrak{g}}$, $\mathrm{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ instead of $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, then $\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}, \mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) \stackrel{\mathrm{def}}{=} \left(\mathrm{ext}_{\mathfrak{g}}, \mathrm{fr}_{\mathfrak{g}}\right)$, and $\mathrm{ext}_{\mathfrak{g}}$, $\mathrm{fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are called, respectively, a $\mathfrak{T}_{\mathfrak{g}}$-exterior and a $\mathfrak{T}_{\mathfrak{g}}$-frontier operators in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$; if the definitions of $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are based on $\mathfrak{g}\text{-Int}$, $\mathfrak{g}\text{-Cl} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ instead of $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, then $\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}, \mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) \stackrel{\mathrm{def}}{=} \left(\mathfrak{g}\text{-Ext}, \mathfrak{g}\text{-Fr}\right)$, and $\mathfrak{g}\text{-Ext}$, $\mathfrak{g}\text{-Fr} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are called, respectively, a $\mathfrak{g}$-$\mathfrak{T}$-exterior and a $\mathfrak{g}$-$\mathfrak{T}$-frontier operators in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$; if the definitions of $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are based on int, cl $: \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ instead of $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, then $\left(\mathfrak{g}\text{-Ext}_{\mathfrak{g}}, \mathfrak{g}\text{-Fr}_{\mathfrak{g}}\right) \stackrel{\mathrm{def}}{=} \left(\mathrm{ext}, \mathrm{fr}\right)$, and ext, fr $: \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are called, respectively, a $\mathfrak{T}$-exterior and a $\mathfrak{T}$-frontier operators in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$.
- II. If the relation "$\mathfrak{g}\text{-Ext}_{\mathfrak{g}} \gtrsim \mathrm{ext}_{\mathfrak{g}}$" stands for "$\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{ext}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$" and "$\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \precsim \mathrm{fr}_{\mathfrak{g}}$," for "$\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{fr}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$," then the outstanding facts are: $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathrm{ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathrm{ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or,

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smaller, *weaker*) than $\mathfrak{g}\text{-Ext}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\text{fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\text{fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

- III. For every $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, the propositions
  - (*i.*) $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Ext}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
  - (*ii.*) $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$,
  - (*iii.*) $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\emptyset) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\emptyset)$,
  - (*iv.*) $\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Ext}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \emptyset$

  hold and form a set of consistent, independent $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior operator axioms.
- IV. For every $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, the propositions
  - (*i.*) $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
  - (*ii.*) $(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} (\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}))$,
  - (*iii.*) $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\emptyset) = \emptyset$,
  - (*iv.*) $\mathfrak{g}\text{-Fr}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Fr}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$

  hold and form a set of consistent, independent $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operator axioms.

Hence, the proposed theory, in its own rights, has several advantages. Indeed, in relation to ITEM I., the theory offers very nice features for the passage from $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators to $\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{T}_{\mathfrak{g}}$-frontier operators, $\mathfrak{g}$-$\mathfrak{T}$-exterior and $\mathfrak{g}$-$\mathfrak{T}$-frontier operators and $\mathfrak{T}$-exterior and $\mathfrak{T}$-frontier operators, respectively. Thus, the theory holds equally well when $(\Omega, \mathscr{T}_{\mathfrak{g}}) = (\Omega, \mathscr{T})$ and other features adapted on this ground, in which case it might be called *Theory of* $\mathfrak{g}$-$\mathfrak{T}$*-Exterior and* $\mathfrak{g}$-$\mathfrak{T}$*-Frontier Operators*. Therefore, in a $\mathscr{T}_{\mathfrak{g}}$-space the theoretical framework categorises the pair $(\mathfrak{g}\text{-Ext}_{\mathfrak{g},0}, \mathfrak{g}\text{-Fr}_{\mathfrak{g},0})$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-(exterior, frontier) operators*, the pair $(\mathfrak{g}\text{-Ext}_{\mathfrak{g},1}, \mathfrak{g}\text{-Fr}_{\mathfrak{g},1})$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-semi-(exterior, frontier) operators*, the pair $(\mathfrak{g}\text{-Ext}_{\mathfrak{g},2}, \mathfrak{g}\text{-Fr}_{\mathfrak{g},2})$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-pre-(exterior, frontier) operators* and the pair $(\mathfrak{g}\text{-Ext}_{\mathfrak{g},3}, \mathfrak{g}\text{-Fr}_{\mathfrak{g},3})$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-semi-pre-(exterior, frontier) operators* as pairs of $\mathfrak{g}$-$\mathfrak{T}$*-(exterior, frontier) operators of categories* 0, 1, 2, *and* 3, respectively, and theorises the concepts in a unified way.

In relation to ITEM II., the theory offers $\mathfrak{T}_{\mathfrak{g}}$-exterior structures as $(\Omega, \mathfrak{g}\text{-Ext}_{\mathfrak{g}})$ which are *finer* (or, *larger*, *stronger*) than $\mathfrak{T}_{\mathfrak{g}}$-exterior structures as $(\Omega, \text{ext}_{\mathfrak{g}})$ or, $(\Omega, \text{ext}_{\mathfrak{g}})$ is *coarser* (or, *smaller*, *weaker*) than $(\Omega, \mathfrak{g}\text{-Ext}_{\mathfrak{g}})$; $\mathfrak{T}_{\mathfrak{g}}$-frontier structures as $(\Omega, \mathfrak{g}\text{-Fr}_{\mathfrak{g}})$ which are *coarser* (or, *smaller*, *weaker*) than $\mathfrak{T}_{\mathfrak{g}}$-frontier structures as $(\Omega, \text{fr}_{\mathfrak{g}})$ or, $(\Omega, \text{fr}_{\mathfrak{g}})$ is *finer* (or, *larger*, *stronger*) than $(\Omega, \mathfrak{g}\text{-Fr}_{\mathfrak{g}})$. Hence, such $\mathfrak{T}_{\mathfrak{g}}$-structures can be considered as a means of handling certain problems in Functional Analysis. Finally, in relation to ITEMS III. (*i.*)–(*iv.*) and ITEMS IV. (*i.*)–(*iv.*), the theory offers fundamental $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-structures from which many other novel propositions can be deduced by means of their $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-axioms by purely logical processes. Thus, the construction of a purely *deductive theory* of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators a step further is made possible.

In view of the foregoing facts, making the theorization of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators of mixed categories in $\mathscr{T}_{\mathfrak{g}}$-spaces by purely formal processes a prime subject of inquiry for future research may therefore be not without interest. More precisely, for some pair $(\nu,\mu)\in I_3^0\times I_3^0$ such that $\nu\neq\mu$, to develop a purely deductive theory of $\mathfrak{g}$-(ν,μ)-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-(ν,μ)-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators $\mathfrak{g}\text{-Ext}_{\mathfrak{g},\nu\mu}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g},\nu\mu}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, respectively, in $\mathscr{T}_{\mathfrak{g}}$-spaces, where $\mathfrak{g}\text{-Ext}_{\mathfrak{g},\nu\mu}:\mathscr{S}_{\mathfrak{g}}\longmapsto\mathfrak{g}\text{-Ext}_{\mathfrak{g},\nu\mu}(\mathscr{S}_{\mathfrak{g}})$ describes a type of collection of points exterior to $\mathscr{S}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-Fr}_{\mathfrak{g},\nu\mu}:\mathscr{S}_{\mathfrak{g}}\longmapsto\mathfrak{g}\text{-Fr}_{\mathfrak{g},\nu\mu}(\mathscr{S}_{\mathfrak{g}})$ describes another type of collection of points at the frontier of $\mathscr{S}_{\mathfrak{g}}$, and both *exteriorness* and *frontierness* are characterized either by $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets belonging to the class $\{\mathscr{O}_{\mathfrak{g}}=\mathscr{O}_{\mathfrak{g},\nu}\cup\mathscr{O}_{\mathfrak{g},\mu}:\ (\mathscr{O}_{\mathfrak{g},\nu},\mathscr{O}_{\mathfrak{g},\mu})\in\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-}\mu\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\}$ or, by $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets belonging to the class $\{\mathscr{K}_{\mathfrak{g}}=\mathscr{K}_{\mathfrak{g},\nu}\cap\mathscr{K}_{\mathfrak{g},\mu}:\ (\mathscr{K}_{\mathfrak{g},\nu},\mathscr{K}_{\mathfrak{g},\mu})\in\mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-}\mu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\}$. Such an interestingly promising theory is what the present authors thought would certainly be worth considering, and the discussion of this paper ends here.

Appendix A. Pre-preliminaries

In this pre-preliminaries section, the elements accompanying the foregoing preliminary section are presented below. In actual fact, they are the elements extracted from the pre-preliminaries and preliminaries sections of the previous Pure Mathematical manuscript of the authors entitled *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Interior and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Closure Operators*. As in all the previous Pure Mathematical manuscripts of the authors (See, *Theories of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Sets*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Maps*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Connectedness*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Separation Axioms*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Compactness* and, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Interior and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Closure Operators*), $\mathfrak{U}$ is the *universe* of discourse, fixed within the framework of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators and containing as elements all sets (briefly, Ω, Γ-sets; $\mathscr{T}$, $\mathfrak{g}$-$\mathscr{T}$, $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-sets; $\mathscr{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$, $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets) considered in this theory, and $I_n^0\overset{\text{def}}{=}\{\nu\in\mathbb{N}^0:\ \nu\leq n\}$; index sets I_∞^0, I_n^*, I_∞^* are defined similarly. A set $\Gamma\subset\mathfrak{U}$ is a subset of the set $\Omega\subset\mathfrak{U}$ and, for some $\mathscr{T}_{\mathfrak{g}}$-open set $\mathscr{O}_{\mathfrak{g}}\in\mathscr{T}\cup\mathfrak{g}\text{-}\mathscr{T}\cup\mathscr{T}_{\mathfrak{g}}\cup\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}$, these implications hold:

$$\mathscr{O}_{\mathfrak{g}}\in\mathscr{T}\Rightarrow\mathscr{O}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathscr{T}\Rightarrow\mathscr{O}_{\mathfrak{g}}\in\mathscr{T}_{\mathfrak{g}}\Rightarrow\mathscr{O}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}\Rightarrow\mathscr{O}_{\mathfrak{g}}\subset\Omega\subset\mathfrak{U}. \tag{A.1}$$

In a natural way, a monotonic map $\mathscr{T}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ from the power set $\mathscr{P}(\Omega)$ of Ω into itself can be associated to a given mapping $\pi_{\mathfrak{g}}:\Omega\longrightarrow\Omega$, thereby inducing a $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}}\subset\mathscr{P}(\Omega)$ on the underlying set $\Omega\subset\mathfrak{U}$ [PC12]. When some further axioms [LR15] is specified for $\mathscr{T}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ with no separation axioms assumed unless otherwise stated, the notion of a $\mathscr{T}_{\mathfrak{g}}$-space follows.

Definition A.1 ($\mathscr{T}_{\mathfrak{g}}$-Space). Let $\Omega\subset\mathfrak{U}$ be a given set and let $\mathscr{P}(\Omega)\overset{\text{def}}{=}\{\mathscr{O}_{\mathfrak{g},\nu}\subseteq\Omega:\ \nu\in I_\infty^*\}$ be the family of all subsets $\mathscr{O}_{\mathfrak{g},1}$, $\mathscr{O}_{\mathfrak{g},2}$, ..., of Ω. Then every one-valued map of the type $\mathscr{T}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ satisfying the following axioms:

- Ax. i. $\mathscr{T}_{\mathfrak{g}}(\emptyset)=\emptyset$,
- Ax. ii. $\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\subseteq\mathscr{O}_{\mathfrak{g}}$,
- Ax. iii. $\mathscr{T}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_\infty^*}\mathscr{O}_{\mathfrak{g},\nu}\big)=\bigcup_{\nu\in I_\infty^*}\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$,

is called a "$\mathfrak{g}$-topology on Ω," and the structure $\mathfrak{T}_{\mathfrak{g}}\overset{\text{def}}{=}(\Omega,\mathscr{T}_{\mathfrak{g}})$ is called a "$\mathscr{T}_{\mathfrak{g}}$-space."

In Def. A.1, by Ax. i., Ax. ii. and Ax. iii., respectively, are meant that the unary operation $\mathscr{T}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ preserves nullary union, is contracting and preserves binary union. Any element $\mathscr{O}_{\mathfrak{g}}\in\mathscr{T}_{\mathfrak{g}}\overset{\text{def}}{=}\{\mathscr{O}_{\mathfrak{g}}:\ \mathscr{O}_{\mathfrak{g}}\in\mathscr{T}_{\mathfrak{g}}\}$ of the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ is called a $\mathscr{T}_{\mathfrak{g}}$-open set and its complement element $\complement_{\Omega}(\mathscr{O}_{\mathfrak{g}})=\mathscr{K}_{\mathfrak{g}}\in\neg\mathscr{T}_{\mathfrak{g}}\overset{\text{def}}{=}$

$\{\mathscr{K}_{\mathfrak{g}} : \complement(\mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g}}\}$, a $\mathscr{T}_{\mathfrak{g}}$-closed set; by convention, $\mathscr{T}_{\mathfrak{g}}$ and $\neg\mathscr{T}_{\mathfrak{g}}$, respectively, stand for the classes of all $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets relative to the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}}$. If there exists a $\nu \in I^{*}_{\infty}$ such that $\mathscr{O}_{\mathfrak{g},\nu} = \Omega$, then $\mathfrak{T}_{\mathfrak{g}}$ is called a strong $\mathscr{T}_{\mathfrak{g}}$-space [Cs5, PC12]. Moreover, if $\mathscr{T}_{\mathfrak{g}}\big(\bigcap_{\nu\in I^{*}_{n}} \mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcap_{\nu\in I^{*}_{n}} \mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ holds for any index set $I^{*}_{n} \subset I^{*}_{\infty}$ such that $n < \infty$, then $\mathfrak{T}_{\mathfrak{g}}$ is called a quasi $\mathscr{T}_{\mathfrak{g}}$-space [Cs8].

In the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, the operator $\mathrm{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ carrying each $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ into its interior $\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \Omega - \mathrm{cl}_{\mathfrak{g}}(\Omega \setminus \mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}}$ is called a "$\mathfrak{T}_{\mathfrak{g}}$-interior operator;" the operator $\mathrm{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ carrying each $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ into its closure $\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \Omega - \mathrm{int}_{\mathfrak{g}}\big(\Omega \setminus \mathscr{S}_{\mathfrak{g}}\big) \subset \mathfrak{T}_{\mathfrak{g}}$ is called a "$\mathfrak{T}_{\mathfrak{g}}$-closure operator." The classes $\mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}] \stackrel{\mathrm{def}}{=} \{\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}} : \mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}\}$ and $\mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}] \stackrel{\mathrm{def}}{=} \{\mathscr{K}_{\mathfrak{g}} \in \neg\mathscr{T}_{\mathfrak{g}} : \mathscr{K}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g}}\}$, respectively, denote the classes of $\mathscr{T}_{\mathfrak{g}}$-open subsets and $\mathscr{T}_{\mathfrak{g}}$-closed supersets of the $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ relative to the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}}$. That $\mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}] \subseteq \mathscr{T}_{\mathfrak{g}}(\Omega)$ and $\neg\mathscr{T}_{\mathfrak{g}}(\Omega) \supseteq \mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]$ are true for the $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in question are clear from the context. To this end, the $\mathfrak{T}_{\mathfrak{g}}$-closure and the $\mathfrak{T}_{\mathfrak{g}}$-interior of a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space define themselves as

$$\text{(A.2)} \qquad \mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\mathrm{def}}{=} \bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \quad \mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\mathrm{def}}{=} \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}},$$

respectively. We note in passing that, $\mathrm{cl}_{\mathfrak{g}}(\cdot) \neq \mathrm{cl}(\cdot)$ and $\mathrm{int}_{\mathfrak{g}}(\cdot) \neq \mathrm{int}(\cdot)$ in general, because the resulting sets obtained from the intersection of all $\mathscr{T}_{\mathfrak{g}}$-closed supersets and the union of all $\mathscr{T}_{\mathfrak{g}}$-open subsets, respectively, relative to the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}}$ are not necessarily equal to those which would be obtained from the intersection of all $\mathscr{T}$-closed supersets and the union of all $\mathscr{T}$-open subsets relative to the topology $\mathscr{T}$ [BKR13]. Throughout this work, by $\mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\cdot)$, $\mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\cdot)$, and $\mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\cdot)$, respectively, are meant $\mathrm{cl}_{\mathfrak{g}}(\mathrm{int}_{\mathfrak{g}}(\cdot))$, $\mathrm{int}_{\mathfrak{g}}(\mathrm{cl}_{\mathfrak{g}}(\cdot))$, and $\mathrm{cl}_{\mathfrak{g}}(\mathrm{int}_{\mathfrak{g}}(\mathrm{cl}_{\mathfrak{g}}(\cdot)))$; other composition operators are defined in a similar way. Also, the backslash $\Omega \setminus \mathscr{S}_{\mathfrak{g}}$ refers to the set-theoretic difference $\Omega - \mathscr{S}_{\mathfrak{g}}$. Finally, for convenience of notation, let $\mathscr{P}^{*}(\Omega) = \mathscr{P}(\Omega) \setminus \{\emptyset\}$, $\mathscr{T}^{*}_{\mathfrak{g}} = \mathscr{T}_{\mathfrak{g}} \setminus \{\emptyset\}$, and $\neg\mathscr{T}^{*}_{\mathfrak{g}} = \neg\mathscr{T}_{\mathfrak{g}} \setminus \{\emptyset\}$.

Definition A.2 ($\mathfrak{g}$-Operation). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then, a mapping $\mathrm{op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ is called a "$\mathfrak{g}$-operation" if and only if the following statements hold:

$$\text{(A.3)} \; \big(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}^{*}(\Omega)\big)\big(\exists(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}^{*}_{\mathfrak{g}} \times \neg\mathscr{T}^{*}_{\mathfrak{g}}\big)\big[\big(\mathrm{op}_{\mathfrak{g}}(\emptyset) = \emptyset\big) \vee \big(\neg\,\mathrm{op}_{\mathfrak{g}}(\emptyset) = \emptyset\big) \\ \vee\big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\big)\big],$$

where $\neg\,\mathrm{op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is called the "complementary $\mathfrak{g}$-operation" on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ and, for all $(\mathscr{S}_{\mathfrak{g}}, \mathscr{U}_{\mathfrak{g},\mu}, \mathscr{V}_{\mathfrak{g},\nu}) \in \times_{\alpha\in I^{*}_{3}} \mathscr{P}^{*}(\Omega)$ such that $\mathscr{W}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g},\mu} \cup \mathscr{V}_{\mathfrak{g},\nu}$, the following axioms are satisfied:

- Ax. I. $\big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\big)$,
- Ax. II. $\big(\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g}} \circ \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}} \circ \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\big)$,
- Ax. III. $\Big(\mathrm{op}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) \subseteq \bigcup_{\sigma=\mu,\nu} \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\sigma})\Big) \bigvee \Big(\neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) \supseteq \bigcup_{\sigma=\mu,\nu} \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\sigma})\Big)$,
- Ax. IV. $\big(\mathscr{U}_{\mathfrak{g},\mu} \subseteq \mathscr{V}_{\mathfrak{g},\nu} \longrightarrow \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\mu}) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})\big) \vee \big(\mathscr{U}_{\mathfrak{g},\mu} \supseteq \mathscr{V}_{\mathfrak{g},\nu} \longleftarrow \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\mu}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\nu})\big)$,

for some $(\mathscr{O}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g},\mu}, \mathscr{O}_{\mathfrak{g},\nu}) \in \times_{\alpha\in I^{*}_{3}} \mathscr{T}^{*}_{\mathfrak{g}}$ and $(\mathscr{K}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g},\mu}, \mathscr{K}_{\mathfrak{g},\nu}) \in \times_{\alpha\in I^{*}_{3}} \neg\mathscr{T}^{*}_{\mathfrak{g}}$.

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The formulation of Def. A.2 is based on the axioms of the Čech closure operator [Boo11] and the various axioms used by many mathematicians to define closure operators [MHD83].

Definition A.3 ($\mathbf{op}_{\mathfrak{g}}$-Elements). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then, the class $\mathscr{L}_{\mathfrak{g}}[\Omega] \stackrel{\text{def}}{=} \{\mathbf{op}_{\mathfrak{g},\nu} = (\mathrm{op}_{\mathfrak{g},\nu}, \neg\,\mathrm{op}_{\mathfrak{g},\nu}) : \nu \in I_3^0\} \subseteq \mathscr{L}_{\mathfrak{g}}^{\omega}[\Omega] \times \mathscr{L}_{\mathfrak{g}}^{\kappa}[\Omega]$, where

$$\mathrm{op}_{\mathfrak{g}} \in \mathscr{L}_{\mathfrak{g}}^{\omega}[\Omega] \stackrel{\text{def}}{=} \{\mathrm{op}_{\mathfrak{g},0},\ \mathrm{op}_{\mathfrak{g},1},\ \mathrm{op}_{\mathfrak{g},2},\ \mathrm{op}_{\mathfrak{g},3}\} = \{\mathrm{int}_{\mathfrak{g}},\ \mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}},\ \mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}},\ \mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\}, \tag{A.4}$$

$$\neg\,\mathrm{op}_{\mathfrak{g}} \in \mathscr{L}_{\mathfrak{g}}^{\kappa}[\Omega] \stackrel{\text{def}}{=} \{\neg\,\mathrm{op}_{\mathfrak{g},0},\ \neg\,\mathrm{op}_{\mathfrak{g},1},\ \neg\,\mathrm{op}_{\mathfrak{g},2},\ \neg\,\mathrm{op}_{\mathfrak{g},3}\} = \{\mathrm{cl}_{\mathfrak{g}},\ \mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}},\ \mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}},\ \mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\}, \tag{A.5}$$

stands for the class of all possible pairs of $\mathfrak{g}$-operators and its complementary $\mathfrak{g}$-operators in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

The use of $\mathbf{op}_{\mathfrak{g}} = (\mathrm{op}_{\mathfrak{g}}, \neg\,\mathrm{op}_{\mathfrak{g}}) \in \mathscr{L}_{\mathfrak{g}}[\Omega]$ on a class of $\mathfrak{T}_{\mathfrak{g}}$-sets will construct a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, just as the use of $\mathscr{L}[\Omega] \stackrel{\text{def}}{=} \{\mathbf{op}_{\nu} = (\mathrm{op}_{\nu}, \neg\,\mathrm{op}_{\nu}) : \nu \in I_3^0\}$ on the class of $\mathfrak{T}$-sets have constructed the new class of $\mathfrak{g}$-$\mathfrak{T}$-sets. But since $\mathrm{cl}_{\mathfrak{g}} \neq \mathrm{cl}$ and $\mathrm{int}_{\mathfrak{g}} \neq \mathrm{int}$, in general, it follows that $\mathbf{op}_{\mathfrak{g},\nu} \neq \mathbf{op}_{\nu}$ for some $\nu \in I_3^0$ and therefore, the new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets that will be obtained from the first construction will, in general, differ from the new class of $\mathfrak{g}$-$\mathfrak{T}$-sets that had been obtained from the second construction. Employing the set-builder notations, the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set of category ν may then be defined as thus:

Definition A.4. Let $(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathfrak{T}_{\mathfrak{g}} \times \mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}}$ and let $\mathbf{op}_{\mathfrak{g},\nu} \in \mathscr{L}_{\mathfrak{g}}[\Omega]$ be a $\mathfrak{g}$-operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose the predicates

$$\begin{aligned}
\mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \subseteq, \supseteq) &\stackrel{\text{def}}{=} \mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \subseteq) \vee \mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \supseteq),\\
\mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \subseteq) &\stackrel{\text{def}}{=} (\exists\,(\mathscr{O}_{\mathfrak{g}}, \mathrm{op}_{\mathfrak{g},\nu}) \in \mathscr{T}_{\mathfrak{g}} \times \mathscr{L}_{\mathfrak{g}}^{\omega}[\Omega])\\
&\qquad [\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}})],\\
\mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \supseteq) &\stackrel{\text{def}}{=} (\exists\,(\mathscr{K}_{\mathfrak{g}}, \neg\,\mathrm{op}_{\mathfrak{g},\nu}) \in \neg\mathscr{T}_{\mathfrak{g}} \times \mathscr{L}_{\mathfrak{g}}^{\kappa}[\Omega])\\
&\qquad [\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g}})]
\end{aligned} \tag{A.6}$$

be "Boolean-valued functions" on $\mathfrak{T}_{\mathfrak{g}} \times (\mathscr{T}_{\mathfrak{g}} \cup \neg\mathscr{T}_{\mathfrak{g}}) \times \mathscr{L}_{\mathfrak{g}}[\Omega] \times \{\subseteq, \supseteq\}$, then

$$\begin{aligned}
\mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \subseteq, \supseteq)\},\\
\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \subseteq)\},\\
\mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \supseteq)\},
\end{aligned} \tag{A.7}$$

respectively, are called the classes of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of category ν in $\mathfrak{T}_{\mathfrak{g}}$.

Thus, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set of category ν if and only if there exists a pair $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}}$ of $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets and a $\mathfrak{g}$-operator $\mathbf{op}_{\mathfrak{g},\nu} \in \mathscr{L}_{\mathfrak{g}}[\Omega]$ of category ν such that the following statement holds:

$$(\exists\xi)\,[(\xi \in \mathscr{S}_{\mathfrak{g}}) \wedge ((\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}})) \vee (\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g}})))]\,.$$

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Clearly,

$$\begin{aligned} \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g}}\right] \stackrel{\text{def}}{=} \bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-S}\left[\mathfrak{T}_{\mathfrak{g}}\right] &= \bigcup_{\nu\in I_3^0} \left(\mathfrak{g}\text{-}\nu\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \cup \mathfrak{g}\text{-}\nu\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right) \\ &= \left(\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right) \cup \left(\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right) \\ &\stackrel{\text{def}}{=} \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \cup \mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right], \end{aligned}$$

then, defines the class of all $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-sets as the union of the classes of all $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, defined by $\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ respectively.

It is interesting to view the concepts of open, semi-open, preopen, semi-preopen sets [And86, And84, CM64, Lev63, MEMED82, Nj5] as $\mathfrak{g}$-$\mathfrak{T}$-open sets of categories 0, 1, 2, and 3, respectively; likewise, to view the concepts of closed, semi-closed, preclosed, semi-preclosed sets [And96] as $\mathfrak{g}$-$\mathfrak{T}$-closed sets of categories 0, 1, 2, and 3, respectively. These can be realised by omitting the subscript "$\mathfrak{g}$" in all symbols of the above definitions. The remark follows.

Remark A.5. Observing that, for every $\nu \in I_3^*$, the first and second components of the $\mathfrak{g}$-vector operator $\mathbf{op}_{\mathfrak{g},\nu} = \left(\mathrm{op}_{\mathfrak{g},\nu}, \neg\,\mathrm{op}_{\mathfrak{g},\nu}\right) \in \mathscr{L}_{\mathfrak{g}}\left[\Omega\right]$ are based on $\mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}}$, respectively, it follows that $\mathbf{op}_{\mathfrak{g},\nu} = \mathbf{op}_{\nu} \stackrel{\text{def}}{=} \left(\mathrm{op}_{\nu}, \neg\,\mathrm{op}_{\nu}\right) \in \mathscr{L}\left[\Omega\right]$ if based on $\mathscr{T} \times \neg\mathscr{T}$, respectively. In this way, $\mathbf{op} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is called a $\mathfrak{g}$-vector operator in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$. Accordingly,

$$\text{(A.8)} \qquad \begin{aligned} \mathrm{op} \in \mathscr{L}^{\omega}\left[\Omega\right] &\stackrel{\text{def}}{=} \left\{\mathrm{op}_0,\ \mathrm{op}_1,\ \mathrm{op}_2,\ \mathrm{op}_3\right\} \\ &= \left\{\mathrm{int},\ \mathrm{cl}\circ\mathrm{int},\ \mathrm{int}\circ\mathrm{cl},\ \mathrm{cl}\circ\mathrm{int}\circ\mathrm{cl}\right\}, \end{aligned}$$

$$\text{(A.9)} \qquad \begin{aligned} \neg\,\mathrm{op} \in \mathscr{L}^{\kappa}\left[\Omega\right] &\stackrel{\text{def}}{=} \left\{\neg\,\mathrm{op}_0,\ \neg\,\mathrm{op}_1,\ \neg\,\mathrm{op}_2,\ \neg\,\mathrm{op}_3\right\} \\ &= \left\{\mathrm{cl},\ \mathrm{int}\circ\mathrm{cl},\ \mathrm{cl}\circ\mathrm{int},\ \mathrm{int}\circ\mathrm{cl}\circ\mathrm{int}\right\}, \end{aligned}$$

and, $\mathscr{L}_{\mathfrak{g}}\left[\Omega\right] \stackrel{\text{def}}{=} \left\{\mathbf{op}_{\mathfrak{g},\nu} = \left(\mathrm{op}_{\mathfrak{g},\nu}, \neg\,\mathrm{op}_{\mathfrak{g},\nu}\right) :\ \nu \in I_3^0\right\} \subseteq \mathscr{L}_{\mathfrak{g}}^{\omega}\left[\Omega\right] \times \mathscr{L}_{\mathfrak{g}}^{\kappa}\left[\Omega\right]$ stands for the class of all possible pairs of $\mathfrak{g}$-operators and its complementary $\mathfrak{g}$-operators in the $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$.

By virtue of the above remark, if $(\mathscr{S}, \mathscr{O}, \mathscr{K}) \in \mathfrak{T} \times \mathscr{T} \times \neg\mathscr{T}$ and $\mathbf{op}_{\nu} \in \mathscr{L}\left[\Omega\right]$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then the predicates

$$\begin{aligned} \mathrm{P}\left(\mathscr{S}, \mathscr{O}, \mathscr{K}; \mathbf{op}_{\nu}; \subseteq, \supseteq\right) &\stackrel{\text{def}}{=} \mathrm{P}\left(\mathscr{S}, \mathscr{O}; \mathbf{op}_{\nu}; \subseteq\right) \vee \mathrm{P}\left(\mathscr{S}, \mathscr{K}; \mathbf{op}_{\nu}; \supseteq\right), \\ \mathrm{P}\left(\mathscr{S}, \mathscr{O}; \mathbf{op}_{\nu}; \subseteq\right) &\stackrel{\text{def}}{=} \left(\exists\left(\mathscr{O}, \mathrm{op}_{\nu}\right) \in \mathscr{T} \times \mathscr{L}^{\omega}\left[\Omega\right]\right)\left[\mathscr{S} \subseteq \mathrm{op}_{\nu}\left(\mathscr{O}\right)\right], \\ \text{(A.10)}\ \mathrm{P}\left(\mathscr{S}, \mathscr{K}; \mathbf{op}_{\nu}; \supseteq\right) &\stackrel{\text{def}}{=} \left(\exists\left(\mathscr{K}, \neg\,\mathrm{op}_{\nu}\right) \in \neg\mathscr{T} \times \mathscr{L}^{\kappa}\left[\Omega\right]\right)\left[\mathscr{S} \supseteq \neg\,\mathrm{op}_{\nu}\left(\mathscr{K}\right)\right] \end{aligned}$$

are obviously "Boolean-valued functions" on $\mathfrak{T} \times (\mathscr{T} \cup \neg\mathscr{T}) \times \mathscr{L}\left[\Omega\right] \times \left\{\subseteq, \supseteq\right\}$ and,

$$\text{(A.11)} \qquad \begin{aligned} \mathfrak{g}\text{-}\nu\text{-S}\left[\mathfrak{T}\right] &\stackrel{\text{def}}{=} \left\{\mathscr{S} \subset \mathfrak{T} :\ \mathrm{P}\left(\mathscr{S}, \mathscr{O}, \mathscr{K}; \mathbf{op}_{\nu}; \subseteq, \supseteq\right)\right\}, \\ \mathfrak{g}\text{-}\nu\text{-O}\left[\mathfrak{T}\right] &\stackrel{\text{def}}{=} \left\{\mathscr{S} \subset \mathfrak{T} :\ \mathrm{P}\left(\mathscr{S}, \mathscr{O}; \mathbf{op}_{\nu}; \subseteq\right)\right\}, \\ \mathfrak{g}\text{-}\nu\text{-K}\left[\mathfrak{T}\right] &\stackrel{\text{def}}{=} \left\{\mathscr{S} \subset \mathfrak{T} :\ \mathrm{P}\left(\mathscr{S}, \mathscr{K}; \mathbf{op}_{\nu}; \supseteq\right)\right\}, \end{aligned}$$

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respectively, are called the classes of all $\mathfrak{g}$-$\mathfrak{T}$-sets, $\mathfrak{g}$-$\mathfrak{T}$-open sets and $\mathfrak{g}$-$\mathfrak{T}$-closed sets of category ν in $\mathfrak{T}$. Therefore, $\mathscr{S} \subset \mathfrak{T}$ is called a $\mathfrak{g}$-$\mathfrak{T}$-set of category ν if and only if there exist a pair $(\mathscr{O}, \mathscr{K}) \in \mathscr{T} \times \neg \mathscr{T}$ of $\mathscr{T}$-open and $\mathscr{T}$-closed sets and a $\mathfrak{g}$-operator $\mathbf{op}_{\nu} \in \mathscr{L}[\Omega]$ of category ν such that the following statement holds:

$$(\exists \xi) \left[(\xi \in \mathscr{S}) \wedge \big((\mathscr{S} \subseteq \mathrm{op}_{\nu}(\mathscr{O})) \vee (\mathscr{S} \supseteq \neg\, \mathrm{op}_{\nu}(\mathscr{K})) \big) \right].$$

Evidently,

$$\begin{aligned} \mathfrak{g}\text{-S}[\mathfrak{T}] \stackrel{\text{def}}{=} \textstyle\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}] &= \bigcup_{\nu \in I_3^0} \left(\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}] \cup \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}] \right) \\ &= \Big(\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}] \Big) \cup \Big(\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}] \Big) \\ &\stackrel{\text{def}}{=} \mathfrak{g}\text{-O}[\mathfrak{T}] \cup \mathfrak{g}\text{-K}[\mathfrak{T}], \end{aligned}$$

then, defines the class of all $\mathfrak{g}$-ν-$\mathfrak{T}$-sets as the union of the classes of all $\mathfrak{g}$-ν-$\mathfrak{T}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}$-closed sets, defined by $\mathfrak{g}$-O$[\mathfrak{T}]$ and $\mathfrak{g}$-K$[\mathfrak{T}]$ respectively.

Similar to the definitions of $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] = \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] = \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}$, those standing for $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}$ and $\mathrm{S}[\mathfrak{T}] = \mathrm{O}[\mathfrak{T}] \cup \mathrm{K}[\mathfrak{T}]$ in $\mathfrak{T}$ are defined as thus:

Definition A.6. If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space and $\mathfrak{T} = (\Omega, \mathscr{T})$ be a $\mathscr{T}$-space, then:

- I. $\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \ \mathrm{P}_{\mathfrak{g}}(\mathscr{S}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =)\}$ and $\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =)\}$ denote the classes of all $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets, respectively, in $\mathfrak{T}_{\mathfrak{g}}$, with $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$;
- II. $\mathrm{O}[\mathfrak{T}] \stackrel{\text{def}}{=} \{\mathscr{S} \subset \mathfrak{T} : \ \mathrm{P}(\mathscr{S}, \mathscr{S}; \mathbf{op}_0; =)\}$ and $\mathrm{K}[\mathfrak{T}] \stackrel{\text{def}}{=} \{\mathscr{S} \subset \mathfrak{T} : \ \mathrm{P}(\mathscr{S}, \mathscr{S}; \mathbf{op}_0; =)\}$ denote the classes of all $\mathfrak{T}$-open and $\mathfrak{T}$-closed sets, respectively, in $\mathfrak{T}$, with $\mathrm{S}[\mathfrak{T}] = \mathrm{O}[\mathfrak{T}] \cup \mathrm{K}[\mathfrak{T}]$.

Remark A.7. Since

$$\mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =, =) \stackrel{\text{def}}{=} \mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =) \vee \mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =),$$

it is plain that $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \ \mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =, =)\}$; likewise, since

$$\mathrm{P}(\mathscr{S}, \mathscr{S}, \mathscr{S}; \mathbf{op}_{\mathfrak{g},0}; =, =) \stackrel{\text{def}}{=} \mathrm{P}(\mathscr{S}, \mathscr{S}; \mathbf{op}_0; =) \vee \mathrm{P}(\mathscr{S}, \mathscr{S}; \mathbf{op}_0; =),$$

it follows that $\mathrm{S}[\mathfrak{T}] \stackrel{\text{def}}{=} \{\mathscr{S} \subset \mathfrak{T} : \ \mathrm{P}(\mathscr{S}, \mathscr{S}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =, =)\}$.

Given the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, $\mathscr{R}_{\mathfrak{g}}$ is said to be *equivalent* to $\mathscr{S}_{\mathfrak{g}}$, written $\mathscr{R}_{\mathfrak{g}} \sim \mathscr{S}_{\mathfrak{g}}$, if and only if, there exists a $\mathfrak{T}_{\mathfrak{g}}$-map $\pi_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} \longrightarrow \mathscr{S}_{\mathfrak{g}}$ which is bijective; the relation $\mathscr{R}_{\mathfrak{g}} \nsim \mathscr{S}_{\mathfrak{g}}$, then, holds whenever $\mathscr{R}_{\mathfrak{g}}$ is *not equivalent* to $\mathscr{S}_{\mathfrak{g}}$.

The definitions of the notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators of category ν in $\mathscr{T}_{\mathfrak{g}}$-spaces are now given.

Definition A.8 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Interior, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Closure Operators). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space, let $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \{\mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] : \ \mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}\}$ be the family of all $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open subsets of $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ relative to the class $\mathfrak{g}$-ν-O$[\mathfrak{T}_{\mathfrak{g}}]$ of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open sets, and let $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \{\mathscr{K}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}] : \ \mathscr{K}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g}}\}$ be the family

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of all $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed supersets of $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ relative to the class $\mathfrak{g}$-ν-$\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed sets. Then, the one-valued maps of the types

$$\text{(A.12)} \qquad \begin{aligned} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \overset{\text{def}}{=} \left\{\mathscr{S}_{\mathfrak{g},\mu} \subseteq \Omega : \ \mu \in I_{\infty}^{*}\right\} \\ \mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \end{aligned}$$

$$\text{(A.13)} \qquad \begin{aligned} \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \overset{\text{def}}{=} \left\{\mathscr{S}_{\mathfrak{g},\mu} \subseteq \Omega : \ \mu \in I_{\infty}^{*}\right\} \\ \mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \end{aligned}$$

on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ are called, respectively, a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operator of category ν" and a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operator of category ν." The classes $\mathfrak{g}\text{-}\mathrm{I}[\mathfrak{T}_{\mathfrak{g}}] \overset{\text{def}}{=} \left\{\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu} : \ \nu \in I_3^0\right\}$ and $\mathfrak{g}\text{-}\mathrm{C}[\mathfrak{T}_{\mathfrak{g}}] \overset{\text{def}}{=} \left\{\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu} : \ \nu \in I_3^0\right\}$, respectively, are called the classes of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators.

Remark A.9. According to their definitions, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is the *dual* of $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and conversely. For, the definition of the first rests on such concepts as $\cup$, $\subseteq$, $\mathscr{O}_{\mathfrak{g},1}$, $\mathscr{O}_{\mathfrak{g},2}$, ... whereas the second, on $\cap$, $\supseteq$, $\mathscr{K}_{\mathfrak{g},1}$, $\mathscr{K}_{\mathfrak{g},2}$, ..., which are dual concepts to $\cup$, $\subseteq$, $\mathscr{O}_{\mathfrak{g},1}$, $\mathscr{O}_{\mathfrak{g},2}$, ..., respectively.

It is interesting to view $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ as the components of some so-called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-vector operator.

Definition A.10 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Vector Operator). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then, an operator of the type

$$\text{(A.14)} \qquad \begin{aligned} \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) &\longmapsto \left(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu}(\mathscr{R}_{\mathfrak{g}}), \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})\right) \end{aligned}$$

on $\mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-vector operator of category ν." Then, $\mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}] \overset{\text{def}}{=} \left\{\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} = \left(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu}\right) : \ \nu \in I_3^0\right\}$ is called the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-vector operators.

The following remark is an immediate consequence of the above definition.

Remark A.11. Observing that, for every $\nu \in I_3^*$, the first and second components of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-vector operator $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} = \left(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu}\right)$ are based on $\mathfrak{g}$-ν-$\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}$-ν-$\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$, respectively, it follows that:

- I. $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} = \mathbf{ic}_{\mathfrak{g}} \overset{\text{def}}{=} \left(\mathrm{int}_{\mathfrak{g}}, \mathrm{cl}_{\mathfrak{g}}\right)$ if based on $\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$;
- II. $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} = \mathfrak{g}\text{-}\mathbf{Ic}_{\nu} \overset{\text{def}}{=} \left(\mathfrak{g}\text{-}\mathrm{Int}_{\nu}, \mathfrak{g}\text{-}\mathrm{Cl}_{\nu}\right)$ if based on $\mathfrak{g}$-ν-$\mathrm{O}[\mathfrak{T}]$ and $\mathfrak{g}$-ν-$\mathrm{K}[\mathfrak{T}]$;
- III. $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} = \mathbf{ic} \overset{\text{def}}{=} \left(\mathrm{int}, \mathrm{cl}\right)$ if based on $\mathrm{O}[\mathfrak{T}]$ and $\mathrm{K}[\mathfrak{T}]$.

In this way, $\mathbf{ic}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathbf{Ic}_{\nu}$, $\mathbf{ic} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ are called a $\mathfrak{T}_{\mathfrak{g}}$-vector operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, a $\mathfrak{g}$-$\mathfrak{T}$-vector operator of category ν in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$ and a $\mathfrak{T}$-vector operator in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$, respectively. Accordingly,

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}] &\overset{\text{def}}{=} \left\{\mathfrak{g}\text{-}\mathbf{Ic}_{\nu} = \left(\mathfrak{g}\text{-}\mathrm{Int}_{\nu}, \mathfrak{g}\text{-}\mathrm{Cl}_{\nu}\right) : \ \nu \in I_3^0\right\} \\ \text{(A.15)} \qquad &\subseteq \left\{\mathfrak{g}\text{-}\mathrm{Int}_{\nu} : \ \nu \in I_3^0\right\} \times \left\{\mathfrak{g}\text{-}\mathrm{Cl}_{\nu} : \ \nu \in I_3^0\right\} \overset{\text{def}}{=} \mathfrak{g}\text{-}\mathrm{I}[\mathfrak{T}] \times \mathfrak{g}\text{-}\mathrm{C}[\mathfrak{T}]. \end{aligned}$$

Then, $\mathfrak{g}$-IC $[\mathfrak{T}]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}$-vector operators in the $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$; $\mathfrak{g}$-I $[\mathfrak{T}]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}$-interior operators while $\mathfrak{g}$-C $[\mathfrak{T}]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}$-closure operators in the $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$.

Definition A.12 (Complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Operator). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then, the one-valued map

$$\text{(A.16)} \qquad \begin{aligned} \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{g}} &\longmapsto \complement_{\mathscr{R}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g}}), \end{aligned}$$

where $\complement_{\mathscr{R}_{\mathfrak{g}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ denotes the relative complement of its operand with respect to $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$, is called a "natural complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator" on $\mathscr{P}(\Omega)$.

For clarity, the notation $\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}$ is employed whenever $\mathscr{R}_{\mathfrak{g}} = \Omega$ or $\mathscr{R}_{\mathfrak{g}}$ is understood from the context. When $\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is with respect to $\mathscr{R}_{\mathfrak{g}} \in \text{S}[\mathfrak{T}_{\mathfrak{g}}]$, $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}]$ and $\mathscr{R}_{\mathfrak{g}} \in \text{S}[\mathfrak{T}]$, the terms natural complement $\mathfrak{T}_{\mathfrak{g}}$-operator, natural complement $\mathfrak{g}$-$\mathfrak{T}$-operator and natural complement $\mathfrak{T}$-operator are employed and these terms stand for $\text{Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}$, $\mathfrak{g}\text{-Op}_{\mathscr{R}_{\mathfrak{g}}}$, $\text{Op}_{\mathscr{R}_{\mathfrak{g}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively.

The pre-preliminaries section ends here.

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