

GLOBAL IN TIME EXISTENCE OF STRONG SOLUTION TO 3D NAVIER-STOKES EQUATIONS

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ABSTRACT. The purpose of this paper is to bring to light a method through which the global in time existence for arbitrary large in H^1 initial data of a strong solution to 3D periodic Navier-Stokes equations follows. The method consists of subdividing the time interval of existence into smaller sub-intervals carefully chosen. These sub-intervals are chosen based on the hypothesis that for any wavenumber m , one can find an interval of time on which the energy quantized in low-frequency components (up to m) of the solution u is lesser than the energy quantized in high-frequency components (down to m) or otherwise the opposite. We associate then a suitable number m to each one of the intervals and we prove that the norm $\|u(t)\|_{\dot{H}^1}$ is bounded in both mentioned cases. The process can be continued until reaching the maximal time of existence T_{max} which yields the global in time existence of strong solution.

1. INTRODUCTION

Let us consider the following incompressible Navier-Stokes equations:

$$(NSE) \quad \begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla)u + \nabla p = 0, & (x, t) \in \mathbb{T}^3 \times \mathbb{R}_+ \\ \nabla \cdot u = 0, & (x, t) \in \mathbb{T}^3 \times \mathbb{R}_+ \\ u|_{t=0} = u_0(x), & x \in \mathbb{T}^3, \end{cases}$$

where the constant $\nu > 0$ is the viscosity of the fluid, and $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ is the three-dimensional torus with periodic boundary conditions. Here u is a three-dimensional vector field $u = (u_1, u_2, u_3)$ representing the velocity of the fluid, and p is a scalar denoting the pressure, both are unknown functions of the space variable x and time variable t . We recall that the pressure can be eliminated by projecting (NSE) onto the space of free divergence vector fields, using the Leray projector

$$\mathbb{P} = Id - \nabla \Delta^{-1} \nabla \cdot .$$

Thus, it will be convenient using the following equivalent system

$$(NS) \quad \begin{cases} \partial_t u - \nu \Delta u + \mathbb{P}(u \cdot \nabla)u = 0, & (x, t) \in \mathbb{T}^3 \times \mathbb{R}_+ \\ \nabla \cdot u = 0, & (x, t) \in \mathbb{T}^3 \times \mathbb{R}_+ \\ u|_{t=0} = u_0(x), & x \in \mathbb{T}^3. \end{cases}$$

We define the Sobolev spaces $H^s(\mathbb{T}^3)$ for $s \geq 0$ by the Fourier expansion

$$H^s(\mathbb{T}^3) = \left\{ u \in L^2(\mathbb{T}^3) : u(x) := \sum_{k \in \mathbb{Z}^3} \hat{u}(k, t) e^{ikx}, \hat{u}(k, t) = \overline{\hat{u}(k, t)}, \|u\|_{H^s} < \infty \right\},$$

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where

$$\|u\|_{\dot{H}^s}^2 := \sum_{k \in \mathbb{Z}^3} (1 + |k|^{2s}) |\hat{u}(k, t)|^2$$

and

$$\hat{u}(k, t) = \int_{\mathbb{T}^3} u(x) e^{-ikx} dx.$$

We also give the definition of homogeneous Sobolev space:

$$\dot{H}^s(\mathbb{T}^3) = \left\{ u \in L^2(\mathbb{T}^3) : u(x) := \sum_{k \in \mathbb{Z}^3} \hat{u}(k, t) e^{ikx}, \hat{u}(k, t) = \overline{\hat{u}(k, t)}, \|u\|_{\dot{H}^s} < \infty \right\},$$

and endowed by the norm

$$\|u\|_{\dot{H}^s} := \|\Lambda^s u\|_{L^2} = \left(\sum_{k \in \mathbb{Z}^3} |k|^{2s} |\hat{u}(k, t)|^2 \right)^{1/2},$$

where by Λ we refer to the operator $\sqrt{-\Delta}$.

We will also use the following function spaces:

$$\begin{aligned} \mathcal{D}_\sigma &:= \{\varphi \in [C_c^\infty(\mathbb{T}^3)]^3 : \nabla \cdot \varphi = 0\} \\ L_\sigma^2(\mathbb{T}^3) &:= \text{closure of } \mathcal{D}_\sigma \text{ in } L^2 \\ H_\sigma^1(\mathbb{T}^3) &:= \text{closure of } \mathcal{D}_\sigma \text{ in } H^1 \end{aligned}$$

For an initial data $u_0 \in L_\sigma^2(\mathbb{T}^3)$, it was proven by Leray and Hopf that there exists a global weak solution $u \in L_t^\infty(L_\sigma^2) \cap L_t^2(H_\sigma^1)$.

Theorem 1.1. *For every $u_0 \in L_\sigma^2(\mathbb{T}^3)$ there exists at least one global in time weak solution $u \in L^\infty(0, \infty; L_\sigma^2(\mathbb{T}^3)) \cap L^2(0, \infty; H_\sigma^1(\mathbb{T}^3))$ of the Navier-Stokes equations satisfying the initial condition u_0 . In particular, u satisfies the energy inequality*

$$(1.1) \quad \frac{1}{2} \|u(t)\|_{L_\sigma^2(\mathbb{T}^3)}^2 + \nu \int_0^t \|\nabla u(\tau)\|_{L_\sigma^2(\mathbb{T}^3)}^2 d\tau \leq \frac{1}{2} \|u_0\|_{L_\sigma^2(\mathbb{T}^3)}^2.$$

This result was proved by [6] as a generalisation of a previous existence theorem due to [8] for the whole space \mathbb{R}^3 .

It is also known that local in time strong solutions exist on the whole space due to [8], while the case of a bounded domain is due to [7].

Theorem 1.2. *There is a constant $C > 0$ such that any initial condition $u_0 \in H_\sigma^1(\mathbb{T}^3)$ gives rise to a strong solution of the Navier-Stokes equations*

$$u \in L^\infty(0, T_{max}; H_\sigma^1) \cap L^2(0, T_{max}; H^2), \text{ where } T_{max} = \frac{C}{\|\nabla u_0\|_{L^2}^4}.$$

The existence of global in time strong solution is known to occur for small initial data due to [5] and [2]. However, it remains the major open problem as to whether these solutions can be extended to be global in time for arbitrary large in H^1 initial data. Originally, the problem is the question of global existence of smooth solutions to the Navier-Stokes equations satisfying bounded energy condition (i.e.: $u \in C^\infty(\mathbb{T}^3 \times \mathbb{R}_+)$ and $\int_{\mathbb{T}^3} |u(x)|^2 dx < \infty$) or otherwise a breakdown. The official description has been given by [4]. The official Clay Millennium problem is to give a proof of one of the four following statements:

- (A) Existence and smoothness of Navier–Stokes solutions on \mathbb{R}^3
- (B) Existence and smoothness of Navier–Stokes solutions on $\mathbb{R}^3/\mathbb{Z}^3$

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- (C) Breakdown of Navier–Stokes solutions on \mathbb{R}^3
- (C) Breakdown of Navier–Stokes solutions on $\mathbb{R}^3/\mathbb{Z}^3$

In this paper, we prove the statement (B) which can be alternatively formulated as follows:

Theorem 1.3. *For every $u_0 \in H^1_\sigma(\mathbb{T}^3)$ there exists a unique global in time strong solution $u \in L^\infty(0, \infty; H^1_\sigma(\mathbb{T}^3)) \cap L^2(0, \infty; H^2(\mathbb{T}^3))$ of the Navier-Stokes equations.*

The method used to extend the solution into a global one is to prove that on an interval of strictly positive length $[t_0, t_1] \subset (0, T_{max})$ under a first condition on $\sum_{k \in \mathbb{Z}^3} |\hat{u}(k, t)|$ among two possible ones:

$$\underbrace{\sum_{|k|>m} |\hat{u}(k, t)| \leq \sum_{|k|\leq m} |\hat{u}(k, t)|}_{\text{condition 1}} \quad \text{or} \quad \underbrace{\sum_{|k|\leq m} |\hat{u}(k, t)| \leq \sum_{|k|>m} |\hat{u}(k, t)|}_{\text{condition 2}}$$

the solution will be controlled in H^1 by a suitable function defined in terms of time t , $\|\nabla u(t_0)\|_{L^2}$, $\|u_0\|_{L^2}$ and a finite number m (depending on the viscosity ν and $\|\nabla u(t_0)\|_{L^2}$), until reaching t_1 . Otherwise, that is if condition 2 holds true $\forall t \in [t_0, t_1]$, then the norm $\|\nabla u(t)\|_{L^2}^2$ is non-increasing on $[t_0, t_1]$. We continue then in this vein until reaching T_{max} . To be more precise, we subdivide the interval $(0, T_{max})$ into a series of successive sub-intervals each of them is akin to $[t_0, t_1]$, i.e.: on each of them either condition 1 or 2 holds. It should be emphasized that the number m may change from an interval to another.

We quote the following two results, the proof of which is given in [9] and based on that of (Theorem 10.6 [3]).

Theorem 1.4. *Let u be a strong solution of the Navier-Stokes equations (NS) on the time interval $[0, T]$, with initial condition $u_0 \in H^1$. Then for all $0 < \varepsilon < T$ we have $u \in C([\varepsilon, T]; H^p)$ for all $p \in \mathbb{N}$. In particular, for all $t \in [0, T]$ the function $u(t)$ is smooth with respect to the space variables.*

The Theorem 1.4 together with the following lemma constitute a cornerstone in establishing the proof of Theorem 1.3.

Lemma 1.5. *Let u be a strong solution of the Navier-Stokes equations on the time interval $[0, T]$. Then for every $\varepsilon > 0$, and all $p, l \in \mathbb{N}$ we have*

$$\partial_t^l u \in L^\infty(\varepsilon; T; H^p).$$

The rest of the paper is dedicated to give the proof of Theorem 1.3.

2. THE PROOF

The analysis can be started by sketching the procedure through which the existence of local in time strong solution to (NS) follows. To this end, let P_n be the projection onto the Fourier modes of order up to $n \in \mathbb{N}$, that is

$$P_n \left(\sum_{k \in \mathbb{Z}^3} \hat{v}_k e^{ixk} \right) = \sum_{|k| \leq n} \hat{v}_k e^{ixk}.$$

Let $u_n = P_n u$ be the solution to

$$(NS_n) \quad \begin{cases} \partial_t u_n - \nu \Delta u_n + P_n[(u_n \cdot \nabla)u_n] = 0, & (x, t) \in \mathbb{T}^3 \times \mathbb{R}_+ \\ \nabla \cdot u_n = 0, & (x, t) \in \mathbb{T}^3 \times \mathbb{R}_+ \\ u_n|_{t=0}(x) = P_n(u^{in})(x), & x \in \mathbb{T}^3. \end{cases}$$

For some T_n , there exists a solution $u_n \in C^\infty([0, T_n] \times \mathbb{T}^3)$ to this finite-dimensional locally-Lipschitz system of ODEs. We take the L^2 -inner product of the first equation in (NS_n) against $-\Delta u_n$ to obtain

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \|\nabla u_n(t)\|_{L^2}^2 + \nu \|\Delta u_n(t)\|_{L^2}^2 &\leq | \langle (u_n \cdot \nabla u_n), \Delta u_n \rangle_{L^2(\mathbb{T}^3)} | \\
 &\leq \|u_n(t)\|_{L^\infty(\mathbb{T}^3)} \|\nabla u_n(t)\|_{L^2(\mathbb{T}^3)} \|\Delta u_n(t)\|_{L^2(\mathbb{T}^3)} \\
 &\leq c \|u_n(t)\|_{H^1(\mathbb{T}^3)}^{1/2} \|u_n(t)\|_{H^2(\mathbb{T}^3)}^{1/2} \|\nabla u_n(t)\|_{L^2(\mathbb{T}^3)} \|\Delta u_n(t)\|_{L^2(\mathbb{T}^3)} \\
 (2.1) \qquad \qquad \qquad &\leq c \|\nabla u_n(t)\|_{L^2(\mathbb{T}^3)}^{3/2} \|\Delta u_n(t)\|_{L^2(\mathbb{T}^3)}^{3/2},
 \end{aligned}$$

where we used Hölder's inequality together with Agmon's inequality [1] and the Poincaré inequality.

Using Young's inequality with exponents 4 and 4/3 yields

$$| \langle (u_n \cdot \nabla u_n), \Delta u_n \rangle_{L^2(\mathbb{T}^3)} | \leq c \|\nabla u_n\|^6 + \frac{\nu}{2} \|\Delta u_n(t)\|_{L^2}^2,$$

where c is a positive constant that does not depend on n . It turns out that

$$\frac{d}{dt} \|\nabla u_n(t)\|_{L^2}^2 + \nu \|\Delta u_n(t)\|_{L^2}^2 \leq c \|\nabla u_n\|^6.$$

By comparing the function $\|\nabla u_n(t)\|_{L^2}^2$ with the solution of the ODE:

$$\frac{dx}{dt} = cx^3, \quad x(0) = \|\nabla u_0\|_{L^2}^2,$$

we infer that as long as $0 \leq t < \frac{1}{2c\|\nabla u_0\|_{L^2}^4}$, the following holds

$$(2.2) \qquad \qquad \qquad \|\nabla u_n(t)\|_{L^2}^2 \leq \underbrace{\frac{\|\nabla u_0\|_{L^2}^2}{\sqrt{1 - 2ct\|\nabla u_0\|_{L^2}^4}}}_{\mathcal{W}(t)}.$$

From (2.2) and (2.1) we now have uniform bounds on $u_n \in L^\infty([0, T_{max}); H^1)$ and on $u_n \in L^2([0, T_{max}); H^2)$ where $T_{max} \sim \frac{1}{\|\nabla u_0\|_{L^2}^4}$. Those uniform bounds together with (NS_n) and a standard procedure allows to take the limit as $n \rightarrow \infty$ (see [9] and references therein). The standard method shows that the limit u is a strong solution on $[0, T_{max})$. However, what happens after time T_{max} is unknown.

We turn now to the question of whether a local in time strong solution can be extended into a global solution. To this end, let us make estimates directly for u instead of using the Galerkin approximation. We know that u_0 gives rise to a strong solution that exists at least on a certain time interval $[0, T_{max})$. On this time interval for each time $t \in (0, T_{max})$ we take the L^2 -inner product of (NS) against $-\Delta u$, we obtain

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \nu \|\Delta u(t)\|_{L^2}^2 &\leq | \langle (u \cdot \nabla u), \Delta u \rangle_{L^2(\mathbb{T}^3)} | \\
 &\leq \|u(t)\|_{L^\infty(\mathbb{T}^3)} \|\nabla u(t)\|_{L^2(\mathbb{T}^3)} \|\Delta u(t)\|_{L^2(\mathbb{T}^3)}.
 \end{aligned}$$

The Fourier expansion of $u(x, t)$ is given by

$$u(x, t) = \sum_{k \in \mathbb{Z}^3} \hat{u}(k, t) e^{ikx}.$$

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For a certain number m (to be discussed later on), we have

$$\begin{aligned} \|u(t)\|_{L^\infty(\mathbb{T}^3)} &\leq \sum_{k \in \mathbb{Z}^3} |\hat{u}(k, t)| \\ &= \sum_{|k| \leq m} |\hat{u}(k, t)| + \sum_{|k| > m} |\hat{u}(k, t)|. \end{aligned}$$

Two possible natural cases may occur. The first is when the major amount of energy at the instant t is quantized in low-frequency components. This case can be represented by the following inequality:

$$(2.3) \quad \sum_{|k| > m} |\hat{u}(k, t)| \leq \sum_{|k| \leq m} |\hat{u}(k, t)|.$$

The second case is when the major amount of energy at time t is quantized in high-frequency components. That is to say:

$$(2.4) \quad \sum_{|k| \leq m} |\hat{u}(k, t)| \leq \sum_{|k| > m} |\hat{u}(k, t)|.$$

We state here the Agmon's inequality [1] which reads:

$$\sum_{k \in \mathbb{Z}^3} |\hat{u}(k, t)| \leq c \|u(t)\|_{H^1(\mathbb{T}^3)}^{1/2} \|u(t)\|_{H^2(\mathbb{T}^3)}^{1/2}.$$

By Theorem 1.4 we have $u \in C([0, T_{max}); H^1)$ and $C((0, T_{max}); H^2)$, then by a continuity argument one can always find at least a small interval of strictly positive length $[t_0, t_1] \subset (0, T_{max})$ on which either (2.3) or (2.4) occurs for any positive number m . Since $[t_0, t_1]$ will serve as a test interval to examine case (2.3) and case (2.4), one can choose without loss of generality t_0 very close to the instant zero. To be more precise, let t_0 be the instant of time immediately after $t = 0$, there exists $t_1 > t_0$ such that we have either (2.3) for all $t \in [t_0, t_1]$ or (2.4) for all $t \in [t_0, t_1]$. We need also to make use of Lemma 1.5, which combined with the fact that $\partial_t |\hat{u}(k, t)| \leq |\partial_t \hat{u}(k, t)|$ yields

$$(2.5) \quad \partial_t \sum_{k \in \mathbb{Z}^3} |\hat{u}(k, t)| \in L^\infty(\varepsilon, T_{max}; H^p), \quad \forall p \in \mathbb{N}.$$

Property (2.5) is useful because it assures the smoothness of $\sum_{k \in \mathbb{Z}^3} |\hat{u}(k, t)|$ with respect to time and hence that of $\sum_{|k| \leq m} |\hat{u}(k, t)|$ and $\sum_{|k| > m} |\hat{u}(k, t)|$. This prevents the abrupt bends of the function $t \mapsto F_m(t) = \sum_{|k| \leq m} |\hat{u}(k, t)| - \sum_{|k| > m} |\hat{u}(k, t)|$. Let us now discuss both cases on $[t_0, t_1]$. To be more precise, if (2.3) holds true on $[t_0, t_1]$ what will happen and if (2.4) holds true on $[t_0, t_1]$ what will happen.

If condition (2.3) holds:

By using (2.3), the Cauchy-Schwarz inequality and Young's inequality, we get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \nu \|\Delta u(t)\|_{L^2}^2 \\
& \leq 2 \sum_{|k| \leq m} |\hat{u}(k, t)| \|\nabla u(t)\|_{L^2(\mathbb{T}^3)} \|\Delta u(t)\|_{L^2(\mathbb{T}^3)} \\
& \leq 2 \left(\sum_{|k| \leq m} 1 \right)^{1/2} \left(\sum_{|k| \leq m} |\hat{u}(k, t)|^2 \right)^{1/2} \\
& \times \|\nabla u(t)\|_{L^2(\mathbb{T}^3)} \|\Delta u(t)\|_{L^2(\mathbb{T}^3)} \\
& \leq 2 \left(\sum_{|k| \leq m} 1 \right)^{1/2} \|u(t)\|_{L^2} \|\nabla u(t)\|_{L^2(\mathbb{T}^3)} \|\Delta u(t)\|_{L^2(\mathbb{T}^3)} \\
& \leq C(m) \|u(t)\|_{L^2}^2 \|\nabla u(t)\|_{L^2(\mathbb{T}^3)}^2 + \frac{\nu}{2} \|\Delta u(t)\|_{L^2(\mathbb{T}^3)}^2,
\end{aligned}$$

where $C(m) = \frac{2 \sum_{|k| \leq m} 1}{\nu}$. By using the energy inequality for weak solutions (1.1) and dropping the viscous term from both sides above, we obtain

$$\frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 \leq 2C(m) \|u_0\|_{L^2}^2 \|\nabla u(t)\|_{L^2(\mathbb{T}^3)}^2.$$

The Gronwall's inequality yields

$$\|\nabla u(t)\|_{L^2}^2 \leq \|\nabla u(t_0)\|_{L^2}^2 \exp\{2C(m) \|u_0\|_{L^2}^2 (t - t_0)\}, \text{ for all } t \in [t_0, t_1].$$

If condition (2.4) holds:

By using (2.4) and the Cauchy-Schwarz inequality we infer that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \nu \|\Delta u(t)\|_{L^2}^2 \\
& \leq 2 \sum_{|k| > m} |\hat{u}(k, t)| \|\nabla u(t)\|_{L^2(\mathbb{T}^3)} \|\Delta u(t)\|_{L^2(\mathbb{T}^3)} \\
& = 2 \sum_{|k| > m} |k|^{-2} |k|^2 |\hat{u}(k, t)| \|\nabla u(t)\|_{L^2(\mathbb{T}^3)} \|\Delta u(t)\|_{L^2(\mathbb{T}^3)} \\
& \leq 2 \left(\sum_{|k| > m} |k|^{-4} \right)^{1/2} \left(\sum_{|k| > m} |k|^4 |\hat{u}(k, t)|^2 \right)^{1/2} \\
& \times \|\nabla u(t)\|_{L^2(\mathbb{T}^3)} \|\Delta u(t)\|_{L^2(\mathbb{T}^3)}.
\end{aligned}$$

Since $\sum_{|k| > m} |k|^4 |\hat{u}(k, t)|^2 \leq \sum_{k \in \mathbb{Z}^3} |k|^4 |\hat{u}(k, t)|^2 = \|\Delta u(t)\|_{L^2(\mathbb{T}^3)}^2$, it turns out

$$\frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \{\nu - 2c(m)\} \|\nabla u(t)\|_{L^2} \|\Delta u(t)\|_{L^2} \leq 0,$$

where $c(m) = (\sum_{|k| > m} |k|^{-4})^{1/2}$. Notice that $\lim_{m \rightarrow \infty} c(m) = 0$, then one can choose the number m such that

$$(2.6) \quad c(m) < \frac{\nu}{2} \|\nabla u(t_0)\|_{L^2(\mathbb{T}^3)}^{-1}.$$

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In such a way, the factor $\{\nu - 2c(m)\|\nabla u(t)\|_{L^2(\mathbb{T}^3)}\}$ would still be positive at least over a short interval of time $[t_0, \tau_1] \subset [t_0, t_1]$. Consequently, it turns out that

$$\frac{d}{dt}\|\nabla u(t)\|_{L^2(\mathbb{T}^3)}^2 \leq 0, \text{ and } \|\nabla u(t)\|_{L^2(\mathbb{T}^3)} \leq \|\nabla u(t_0)\|_{L^2(\mathbb{T}^3)} \text{ for all } t \in [t_0, \tau_1].$$

But as $\|\nabla u(t)\|_{L^2(\mathbb{T}^3)}$ is continuous on $[t_0, t_1]$, we obtain

$$\|\nabla u(t)\|_{L^2(\mathbb{T}^3)} \leq \|\nabla u(t_0)\|_{L^2(\mathbb{T}^3)} \text{ for all } t \in [t_0, t_1].$$

Thus, the condition on m has been determined successfully. To conclude, we have proved that there exists $t_1 > t_0$ such that we have either

$$\|\nabla u(t)\|_{L^2}^2 \leq \|\nabla u(t_0)\|_{L^2}^2 \exp\{2C(m_0)\|u_0\|_{L^2}^2(t - t_0)\}, \text{ for all } t \in [t_0, t_1]$$

or

$$\|\nabla u(t)\|_{L^2}^2 \leq \|\nabla u(t_0)\|_{L^2}^2, \text{ for all } t \in [t_0, t_1],$$

where

$$C(m_0) = \frac{2 \sum_{|k| \leq m_0} 1}{\nu}, \text{ and } m_0 = \inf \left\{ m; \left(\sum_{|k| > m} |k|^{-4} \right)^{1/2} \leq \frac{\nu}{4} \|\nabla u(t_0)\|_{L^2}^{-1} \right\}.$$

The statement above can be summarized by making use of the fact $\|\nabla u(t_0)\|_{L^2}^2 \leq \|\nabla u(t_0)\|_{L^2}^2 \exp\{2C(m_0)\|u_0\|_{L^2}^2(t - t_0)\}$ as follows: there exists $t_1 > t_0$ such that we have

$$\|\nabla u(t)\|_{L^2}^2 \leq \|\nabla u(t_0)\|_{L^2}^2 \exp\{2C(m_0)\|u_0\|_{L^2}^2(t - t_0)\}, \text{ for all } t \in [t_0, t_1].$$

Continuing in this vein, in the next interval we know already that m must be as large as $c(m) < \frac{\nu}{2} \|\nabla u(t_1)\|_{L^2}^{-1}$ which guarantees by continuity that in case (2.4) the function $\|\nabla u(t)\|_{L^2}$ is non-increasing on this interval. There exists then $t_2 > t_1$ such that

$$\|\nabla u(t)\|_{L^2}^2 \leq \|\nabla u(t_1)\|_{L^2}^2 \exp\{2C(m_1)\|u_0\|_{L^2}^2(t - t_1)\}, \text{ for all } t \in [t_1, t_2]$$

or

$$\|\nabla u(t)\|_{L^2}^2 \leq \|\nabla u(t_1)\|_{L^2}^2, \text{ for all } t \in [t_1, t_2],$$

where

$$C(m_1) = \frac{2 \sum_{|k| \leq m_1} 1}{\nu}, \text{ and } m_1 = \inf \left\{ m; \left(\sum_{|k| > m} |k|^{-4} \right)^{1/2} \leq \frac{\nu}{4} \|\nabla u(t_1)\|_{L^2}^{-1} \right\}.$$

Repeating this process as many times as needed to obtain $[t_0, T_{max} - \epsilon] = \cup_{j=0}^N [t_j, t_{j+1}]$ (where ϵ is an arbitrary small constant and $[t_j, t_{j+1}]$ are successive intervals), such that for all $t \in [t_j, t_{j+1}]$ we have either

$$(2.7) \quad \|\nabla u(t)\|_{L^2}^2 \leq \|\nabla u(t_j)\|_{L^2}^2 \exp\{2C(m_j)\|u_0\|_{L^2}^2(t - t_j)\}$$

or

$$(2.8) \quad \|\nabla u(t)\|_{L^2}^2 \leq \|\nabla u(t_j)\|_{L^2}^2,$$

where

$$C(m_j) = \frac{2 \sum_{|k| \leq m_j} 1}{\nu}, \text{ and } m_j = \inf \left\{ m; \left(\sum_{|k| > m} |k|^{-4} \right)^{1/2} \leq \frac{\nu}{4} \|\nabla u(t_j)\|_{L^2}^{-1} \right\}.$$

We remark that the number m_j may change from an interval $[t_j, t_{j+1}]$ to the next one $[t_{j+1}, t_{j+2}]$, but it remains finite as long as $\|\nabla u(t_{j+1})\|_{L^2}$ is finite which is guaranteed by the estimates (2.7) or (2.8). This process would certainly control the norm $\|\nabla u(t)\|_{L^2}^2$ and rules out the blowup of u in $H^1(\mathbb{T}^3)$ as t approaches T_{max} . Therefore, the solution u can be extended into a global in time strong solution.

3. DISCUSSION

An interesting observation is the following: it suffice that (2.4) occurs only once on an interval of strictly positive length $[t_0, t_1] \subset (0, T_{max})$ for m such that $\left(\sum_{|k|>m} |k|^{-4}\right)^{1/2} < \frac{\nu}{2} \|\nabla u(t_0)\|_{L^2}^{-1}$ to extend the solution onto a larger interval of time $[0, T_{max} + t_1 - t_0)$. In fact, estimate (2.2) tells us that an initial data as large as $\|\nabla u_0\|_{L^2}$ gives rise to a solution u that would remain bounded on an interval $[0, T_{max})$ of length $T_{max} - 0$. Let $t_0 \in (0, T_{max})$, inequality (2.2) also tells us that an initial data as large as $\frac{\|\nabla u_0\|_{L^2}^2}{\sqrt{1-2ct_0\|\nabla u_0\|_{L^2}^4}}$ gives rise to a solution u that remains bounded on an interval of length $T_{max} - t_0$. According to the proof given in the previous section, if condition (2.4) holds true on $[t_0, t_1]$, it turns out

$$\frac{d}{dt} \|\nabla u(t)\|_{L^2(\mathbb{T}^3)}^2 \leq 0, \text{ and } \|\nabla u(t)\|_{L^2(\mathbb{T}^3)} \leq \|\nabla u(t_0)\|_{L^2(\mathbb{T}^3)} \text{ for all } t \in [t_0, t_1].$$

Now since

$$\|\nabla u(t_1)\|_{L^2}^2 \leq \|\nabla u(t_0)\|_{L^2}^2 \leq \frac{\|\nabla u_0\|_{L^2}^2}{\sqrt{1-2ct_0\|\nabla u_0\|_{L^2}^4}},$$

therefore by starting from t_1 the solution u would now still bounded on an interval of time of length $T_{max} - t_0$. In other words, the solution u is extended to the interval $[0, T_{max} + t_1 - t_0)$ which means that $u(T_{max}) \in H^1(\mathbb{T}^3)$.

At this point, one may ask the question under which condition the norm $\|\nabla u(t)\|_{L^2}$ keeps decreasing for all positive time $t \in \mathbb{R}_+$. In fact, this is possible when the distribution of energy in the initial data is extremely unbalanced (i.e. $\sum_{|k|\leq m} |\hat{u}(k, 0)| \ll \sum_{|k|>m} |\hat{u}(k, 0)|$). In that case, by smoothness of the function $\sum_{k \in \mathbb{Z}^p} |\hat{u}(k, t)|$ with respect to time, condition (2.4) keeps for a long interval of time until potentially $\|\nabla u(t)\|_{L^2}$ satisfies the smallness condition of [5].

Another aspect to discuss here is the motivation behind choosing the instant t_0 very close to zero. In fact, by doing so one can ensure via (2.2) the closeness of $\|\nabla u(t_0)\|_{L^2}$ to $\|\nabla u_0\|_{L^2}$ while holding the necessary regularity ($u(t_0) \in H^p(\mathbb{T}^3)$ for all $p \in \mathbb{N}$). This also guarantees the minimum worsening to $\|\nabla u(t_1)\|_{L^2}$ in case (2.3). However, it is needless to say that this was optional and that one can choose any instant $t_0 \in [0, T_{max})$ as initial time.

4. CONCLUSION

We have already proved that the local in time strong solution to (NS) can be extended to be global in time strong solution. This was done via making estimates to u in H^1 on a series of time intervals requiring that the function $t \mapsto F_m(t) = \sum_{|k|\leq m} |\hat{u}(k, t)| - \sum_{|k|>m} |\hat{u}(k, t)|$ keeps its sign constant (either positive or negative) on each of them. It is worth noting that when the major amount of energy is located in high-frequency components (i.e. $F_m(t) \leq 0$), the norm $\|\nabla u(t)\|_{L^2}$ decreases with time. This is in fact consistent with the phenomenology

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of the turbulent cascade which states that energy is dissipated at the small scales (i.e. higher frequencies).

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