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ON BOUNDS OF THE SINE AND COSINE ALONG A CIRCLE ON THE COMPLEX PLANE

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ABSTRACT. In the paper, the author finds bounds of the sine and cosine along a circle on the complex plane in terms of two double inequalities for the norms of the sine and cosine along a circle on the complex plane.

CONTENTS

1. MOTIVATIONS

In the theory of complex functions, the sine and cosine functions $\sin z$ and $\cos z$ on the complex plane $\mathbb{C}$ are defined by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

respectively, where $z = x + iy$, $x, y \in \mathbb{R}$, and $i = \sqrt{-1}$ is the imaginary unit. They have the least positive periodicity 2π, that is,

$$\sin(z + 2k\pi) = \sin z \quad \text{and} \quad \cos(z + 2k\pi) = \cos z$$

for $k \in \mathbb{Z}$.

When restricting $z = x \in \mathbb{R}$, the sine and cosine functions $\sin z$ and $\cos z$ become $\sin x$ and $\cos x$ and satisfy

$$0 \le |\sin x| \le 1 \quad \text{and} \quad 0 \le |\cos x| \le 1.$$

When restricting $z = iy$ for $y \in \mathbb{R}$, the sine and cosine functions $\sin z$ and $\cos z$ reduce to

$$\sin(iy) = \frac{e^{-y} - e^{y}}{2i} = i \sinh y \to \pm i\infty$$

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and

$$\cos(iy)=\frac{e^{-y}+e^{y}}{2}=\cosh y\to+\infty$$

as $y\to\pm\infty$. These imply that the sine and cosine are bounded on the real x-axis, but unbounded on the imaginary y-axis.

In the textbook [8, p. 93], Excise 6 states that, if $z\in\mathbb{C}$ and $|z|\le R$, then

$$|\sin z|\le\cosh R\quad\text{and}\quad|\cos z|\le\cosh R.$$

In [6], a criterion to justify a holomorphic function was discussed.

In [5], the author discussed and computed bounds of the sine and cosine functions $\sin z$ and $\cos z$ along straight lines on the complex plane $\mathbb{C}$. The main results in the paper [5] can be recited as follows.

(1) The complex functions $\sin z$ and $\cos z$ are bounded along straight lines parallel to the real x-axis on the complex plane $\mathbb{C}$:
   (a) along the horizontal straight line $y=\alpha$ on the complex plane $\mathbb{C}$,
$$|\sinh\alpha|\le|\sin(x+i\alpha)|\le\cosh\alpha \tag{1}$$
   and
$$|\sinh\alpha|\le|\cos(x+i\alpha)|\le\cosh\alpha, \tag{2}$$
   where $\alpha\in\mathbb{R}$ is a constant and $x\in\mathbb{R}$;
   (b) the equalities in the left hand side of (1) and in the right hand side of (2) hold if and only if $x=k\pi$ for $k\in\mathbb{Z}$;
   (c) the equalities in the right hand side of (1) and in the left hand side of (2) hold if and only if $x=k\pi+\frac{\pi}{2}$ for $k\in\mathbb{Z}$.

(2) The complex functions $\sin z$ and $\cos z$ are unbounded along straight lines whose slopes are not horizontal:
   (a) along the sloped straight line $y=\alpha+\beta x$ on the complex plane $\mathbb{C}$,
$$|\sin z|\ge|\sinh(\alpha+\beta x)|\quad\text{and}\quad|\cos z|\ge|\sinh(\alpha+\beta x)|,$$
   where $\alpha\in\mathbb{R}$ and $\beta\ne0$ are constants;
   (b) along the vertical straight line $x=\gamma$ on the complex plane $\mathbb{C}$,
$$|\sin z|\ge|\sinh y|\quad\text{and}\quad|\cos z|\ge|\sinh y|;$$
   where $\gamma\in\mathbb{R}$ is a constant;

In this paper, we find bounds of the sine and cosine functions $\sin z$ and $\cos z$ along a circle centered at the origin $z=0$ of radius r on the complex plane $\mathbb{C}$ in terms of double inequalities for their norms.

2. A double inequality for the norm of sine along a circle

In this section, we find a double inequality for the sine function.

Theorem 2.1. *Let $r>0$ be a constant and let $C(0,r):z=re^{i\theta}$ for $\theta\in[0,2\pi)$ denote a circle centered at the origin $z=0$ of radius r. Then*

$$|\sin r|\le\bigl|\sin\bigl(re^{i\theta}\bigr)\bigr|\le\sinh r,\quad\theta\in[0,2\pi). \tag{3}$$

The left equality is valid if and only if $\theta=0,\pi$ while the right equality is valid if and only if $\theta=\frac{\pi}{2},\frac{3\pi}{2}$.

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Proof. The circle $C(0,r)$ can be represented by

$$z = re^{i\theta}, \quad \theta \in [0, 2\pi).$$

It is not difficult to see that, for fixed $r > 0$, $\left|\sin\left(re^{i\theta}\right)\right| = |\sin r|$ for $\theta = 0, \pi$, $\left|\sin\left(re^{i\theta}\right)\right| = \sinh r$ for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, and $\left|\sin\left(re^{i\theta}\right)\right|$ has a least positive periodicity π with respect to the argument θ.

Straightforward computation yields

$$\begin{aligned}
\sin z = \sin\left(re^{i\theta}\right) &= \sin\left(r\cos\theta + ir\sin\theta\right) \\
&= \frac{e^{i(r\cos\theta + ir\sin\theta)} - e^{-i(r\cos\theta + ir\sin\theta)}}{2i} \\
&= \frac{e^{-(r\sin\theta - ir\cos\theta)} - e^{r\sin\theta - ir\cos\theta}}{2i} \\
&= \frac{e^{-r\sin\theta}[\cos(r\cos\theta) + i\sin(r\cos\theta)] - e^{r\sin\theta}[\cos(r\cos\theta) - i\sin(r\cos\theta)]}{2i} \\
&= \frac{\left(e^{-r\sin\theta} - e^{r\sin\theta}\right)\cos(r\cos\theta) + i\left(e^{-r\sin\theta} + e^{r\sin\theta}\right)\sin(r\cos\theta)]}{2i} \\
&= \cosh(r\sin\theta)\sin(r\cos\theta) + i\sinh(r\sin\theta)\cos(r\cos\theta)
\end{aligned}$$

and

$$\left|\sin\left(re^{i\theta}\right)\right| = \sqrt{[\cosh(r\sin\theta)\sin(r\cos\theta)]^2 + [\sinh(r\sin\theta)\cos(r\cos\theta)]^2}\,.$$

In Figure 1, we plot the 3D graph of $\left|\sin\left(re^{i\theta}\right)\right|$ for $r \in [0, 5]$ and $\theta \in [0, 2\pi)$. In Figure 2, we plot the graph of $\left|\sin\left(\pi e^{i\theta}\right)\right|$ for $\theta \in [0, 2\pi)$. These two figures are

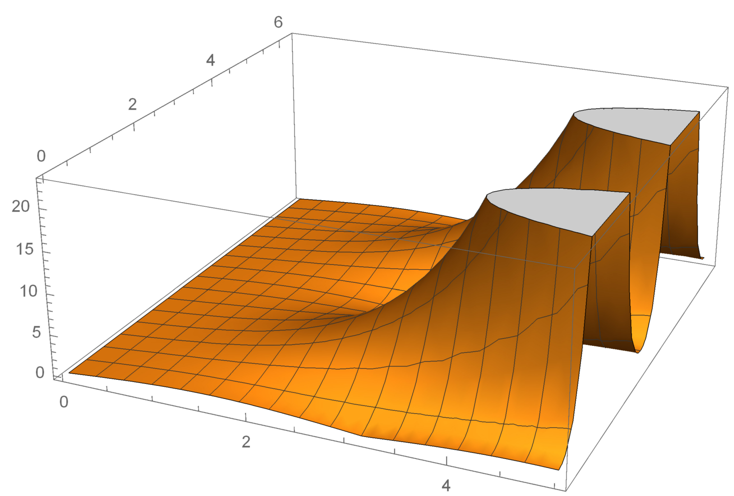

FIGURE 1. The 3D graph of $\left|\sin\left(re^{i\theta}\right)\right|$ for $r \in [0, 5]$ and $\theta \in [0, 2\pi)$

helpful for analyzing and understanding the behaviour of the sine function $\sin z$ along the circle $C(0,r)$ centered at the origin $z = 0$ of radius r.

From Figure 2, we can see that the norm $\left|\sin\left(\pi e^{i\theta}\right)\right|$ has only two maximums at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, while it has only two minimums at $\theta = 0, \pi$ on the interval $[0, 2\pi)$.

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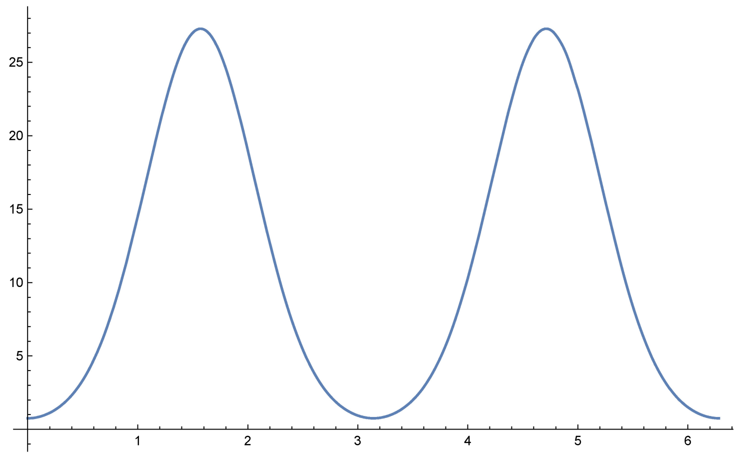

FIGURE 2. The graph of $\left|\sin\left(\pi e^{i\theta}\right)\right|$ for $\theta\in[0,2\pi)$

Differentiating the square of $\left|\sin\left(re^{i\theta}\right)\right|$ yields

$$
\begin{aligned}
\frac{\mathrm{d}\left|\sin\left(re^{i\theta}\right)\right|^2}{\mathrm{d}\theta}&=r[\cos\theta\sinh(2r\sin\theta)-\sin\theta\sin(2r\cos\theta)]\\
&=r[\sinh(2r\sin\theta)-\tan\theta\sin(2r\cos\theta)]\cos\theta\\
&=r[\cot\theta\sinh(2r\sin\theta)-\sin(2r\cos\theta)]\sin\theta\\
&=r^2\left[\frac{\sinh(2r\sin\theta)}{2r\sin\theta}-\frac{\sin(2r\cos\theta)}{2r\cos\theta}\right]\sin(2\theta).
\end{aligned}
$$

From the first three expressions above, we conclude that the derivative $\frac{\mathrm{d}|\sin(re^{i\theta})|^2}{\mathrm{d}\theta}$ is equal to 0 at $\theta=0,\frac{\pi}{2},\pi,\frac{3\pi}{2}$. Considering the fourth expression above on the intervals $\left(k\frac{\pi}{2},(k+1)\frac{\pi}{2}\right)$ for $k=0,1,2,3$, in order that $\frac{\mathrm{d}|\sin(re^{i\theta})|^2}{\mathrm{d}\theta}\ne0$ for $\theta\in\left(k\frac{\pi}{2},(k+1)\frac{\pi}{2}\right)$ and $r>0$, it is sufficient to find

$$
\frac{\sinh(2r\sin\theta)}{2r\sin\theta}>1 \tag{4}
$$

and

$$
\frac{\sin(2r\cos\theta)}{2r\cos\theta}<1 \tag{5}
$$

for $\theta\in\left(k\frac{\pi}{2},(k+1)\frac{\pi}{2}\right)$ and $r>0$. Then, for fixed $r>0$, the square $|\sin(re^{i\theta})|^2$ and the norm $\left|\sin\left(re^{i\theta}\right)\right|$ have only two maximums at $\theta=\frac{\pi}{2},\frac{3\pi}{2}$, while they have only two minimums at $\theta=0,\pi$ on the interval $[0,2\pi)$. At $\theta=\frac{\pi}{2},\frac{3\pi}{2}$, the values of $\left|\sin\left(re^{i\theta}\right)\right|$ are both $\sinh r$; at $\theta=0,\pi$, the values of $\left|\sin\left(re^{i\theta}\right)\right|$ are both $|\sin r|$.

Considering the odevity of $\sinh t$ and $\sin t$, we see that two inequalities in (4) and (5) are equivalent to

$$
\frac{\sinh t}{t}>1 \quad \text{and} \quad \frac{\sin t}{t}<1 \tag{6}
$$

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for $t \in (0, \infty)$. The first inequality in (6) follows from $\cosh x > 1$ for $x \neq 0$ and the Lazarević inequality

$$\cosh x < \left(\frac{\sinh x}{x}\right)^3 \tag{7}$$

in [2, p. 270, 3.6.9]. When $t \in (0, \frac{\pi}{2})$, the second inequality in (6) follows from the right hand side of the Jordan inequality

$$\frac{\pi}{2} \le \frac{\sin t}{t} < 1, \quad 0 < |t| \le \frac{\pi}{2} \tag{8}$$

in [2, Section 2.3] and the papers [1, 3, 4, 7]. When $t > \frac{\pi}{2}$, the second inequality in (6) follows from $\sin t \le 1$ on $(0, \infty)$ and standard argument. The double inequality (3) is thus proved. The proof of Theorem 2.1 is complete. □

3. A double inequality for the norm of cosine along a circle

In this section, we find a double inequality for the cosine function.

Theorem 3.1. *Let $r > 0$ be a constant and let $C(0, r) : z = re^{i\theta}$ for $\theta \in [0, 2\pi)$ denote a circle centered at the origin $z = 0$ of radius r. Then*

$$|\cos r| \le \left|\cos\left(re^{i\theta}\right)\right| \le \cosh r, \quad \theta \in [0, 2\pi). \tag{9}$$

The left equality is valid if and only if $\theta = 0, \pi$ while the right equality is valid if and only if $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Proof. It is easy to see that, for fixed $r > 0$, $\left|\cos\left(re^{i\theta}\right)\right| = |\cos r|$ for $\theta = 0, \pi$, $\left|\cos\left(re^{i\theta}\right)\right| = \cosh r$ for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, and $\left|\cos\left(re^{i\theta}\right)\right|$ has a least positive periodicity π with respect to the argument θ.

Direct calculation yields

$$\begin{aligned}
\cos z = \cos\left(re^{i\theta}\right) &= \cos\left(r\cos\theta + ir\sin\theta\right)\\
&= \frac{e^{i(r\cos\theta + ir\sin\theta)} + e^{-i(r\cos\theta + ir\sin\theta)}}{2}\\
&= \frac{e^{-(r\sin\theta - ir\cos\theta)} + e^{r\sin\theta - ir\cos\theta}}{2}\\
&= \frac{e^{-r\sin\theta}[\cos(r\cos\theta) + i\sin(r\cos\theta)] + e^{r\sin\theta}[\cos(r\cos\theta) - i\sin(r\cos\theta)]}{2}\\
&= \frac{\left(e^{-r\sin\theta} + e^{r\sin\theta}\right)\cos(r\cos\theta) + i\left(e^{-r\sin\theta} - e^{r\sin\theta}\right)\sin(r\cos\theta)]}{2}\\
&= \cosh(r\sin\theta)\cos(r\cos\theta) - i\sinh(r\sin\theta)\sin(r\cos\theta)
\end{aligned}$$

and

$$\left|\cos\left(re^{i\theta}\right)\right| = \sqrt{[\cosh(r\sin\theta)\cos(r\cos\theta)]^2 + [\sinh(r\sin\theta)\sin(r\cos\theta)]^2}\,.$$

In Figure 3, we plot the 3D graph of $\left|\cos\left(re^{i\theta}\right)\right|$ for $r \in [0, 5]$ and $\theta \in [0, 2\pi)$. In Figure 4, we plot the graph of $\left|\cos\left(re^{i\theta}\right)\right|$ for $r = \pi$ and $\theta \in [0, 2\pi)$. These two figures are helpful for analyzing and understanding the behaviour of the cosine function $\cos z$ along the circle $C(0, r)$ centered at the origin $z = 0$ of radius r.

From Figure 4, we can see that the norm $\left|\cos\left(\pi e^{i\theta}\right)\right|$ has only two maximums at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, while it has only two minimums at $\theta = 0, \pi$ on the interval $[0, 2\pi)$.

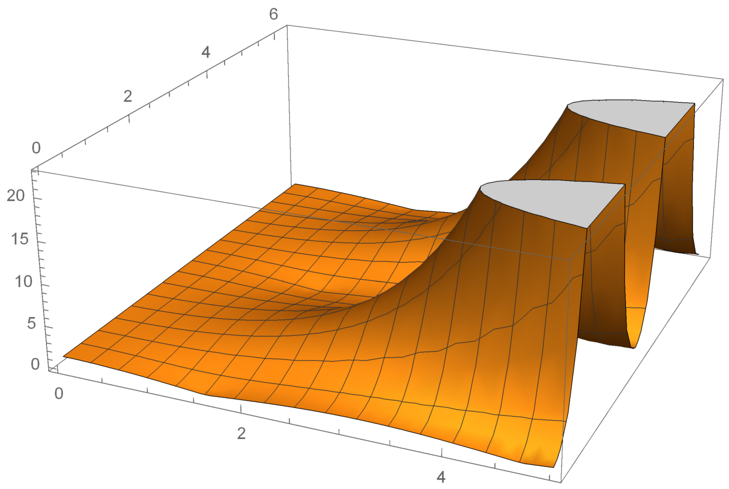

FIGURE 3. The 3D graph of $\left|\cos\left(re^{i\theta}\right)\right|$ for $r \in [0,5]$ and $\theta \in [0,2\pi)$

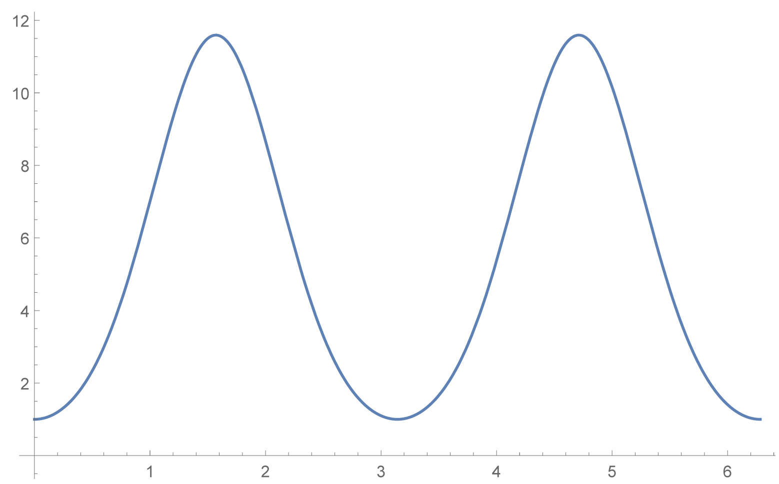

FIGURE 4. The graph of $\left|\cos\left(\pi e^{i\theta}\right)\right|$ for $\theta \in [0,2\pi)$

Differentiating the square of $\left|\cos\left(re^{i\theta}\right)\right|$ with respect to θ gives

$$\begin{aligned}
\frac{\mathrm{d}\left|\cos\left(re^{i\theta}\right)\right|^2}{\mathrm{d}\theta} &= r[\sin\theta\sin(2r\cos\theta) + \cos\theta\sinh(2r\sin\theta)] \\
&= r[\tan\theta\sin(2r\cos\theta) + \sinh(2r\sin\theta)]\cos\theta \\
&= r[\sin(2r\cos\theta) + \cot\theta\sinh(2r\sin\theta)]\sin\theta \\
&= r^2\left[\frac{\sin(2r\cos\theta)}{2r\cos\theta} + \frac{\sinh(2r\sin\theta)}{2r\sin\theta}\right]\sin(2\theta).
\end{aligned}$$

From the first three expressions above, we conclude that the derivative $\frac{\mathrm{d}|\cos(re^{i\theta})|^2}{\mathrm{d}\theta}$ is equal to 0 at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Considering the fourth expression above on the intervals $\left(k\frac{\pi}{2},(k+1)\frac{\pi}{2}\right)$ for $k=0,1,2,3$, in order that $\frac{\mathrm{d}|\cos(re^{i\theta})|^2}{\mathrm{d}\theta}\ne0$, it is sufficient to show

$$\frac{\sinh(2r\sin\theta)}{2r\sin\theta}>1 \tag{10}$$

and

$$\frac{\sin(2r\cos\theta)}{2r\cos\theta}>-1 \tag{11}$$

for $\theta\in\left(k\frac{\pi}{2},(k+1)\frac{\pi}{2}\right)$ and $r>0$. Then, for fixed $r>0$, the square $|\cos(re^{i\theta})|^2$ and the norm $\left|\cos\left(re^{i\theta}\right)\right|$ have only two maximums at $\theta=\frac{\pi}{2},\frac{3\pi}{2}$, while they have only two minimums at $\theta=0,\pi$ on the interval $[0,2\pi)$. At $\theta=\frac{\pi}{2},\frac{3\pi}{2}$, the values of $\left|\cos\left(re^{i\theta}\right)\right|$ are both $\cosh r$; at $\theta=0,\pi$, the values of $\left|\cos\left(re^{i\theta}\right)\right|$ are both $|\cos r|$.

Considering odevity of $\sinh t$ and $\sin t$, two inequalities in (10) and (11) are equivalent to

$$\frac{\sinh t}{t}>1 \quad\text{and}\quad \frac{\sin t}{t}>-1 \tag{12}$$

for $t\in(0,\infty)$. The first inequality in (12) follows from $\cosh x>1$ for $x\ne0$ and the Lazarević inequality (7). When $t\in(0,\frac{\pi}{2})$, the second inequality in (12) follows from the left hand side of the Jordan inequality (8). When $t>\frac{\pi}{2}$, the second inequality in (12) follows from $\sin t\ge-1$ on $(0,\infty)$ and simple argument. The double inequality (9) is thus proved. The proof of Theorem 3.1 is complete. □

4. Remarks

From Figures 1 and 3, it is not easy to see the difference between $\left|\sin\left(re^{i\theta}\right)\right|$ and $\left|\cos\left(re^{i\theta}\right)\right|$. In fact, the difference $\left|\sin\left(re^{i\theta}\right)\right|-\left|\cos\left(re^{i\theta}\right)\right|$ for $r\in[0,2\pi]$ and $\theta\in[0,2\pi)$ can be showed by Figure 5.

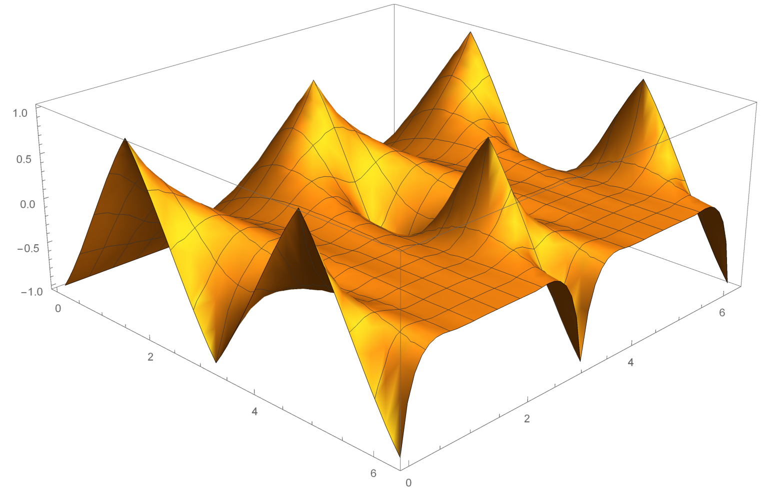

Figure 5. The difference $\left|\sin\left(re^{i\theta}\right)\right|-\left|\cos\left(re^{i\theta}\right)\right|$ for $r,\theta\in[0,2\pi)$

From Figures 2 and 4, it is not easy to see the difference between $\left|\sin\left(\pi e^{i\theta}\right)\right|$ and $\left|\cos\left(\pi e^{i\theta}\right)\right|$. In fact, the difference $\left|\sin\left(\pi e^{i\theta}\right)\right|-\left|\cos\left(\pi e^{i\theta}\right)\right|$ for $\theta\in[0,2\pi)$ can be demonstrated by Figure 6.

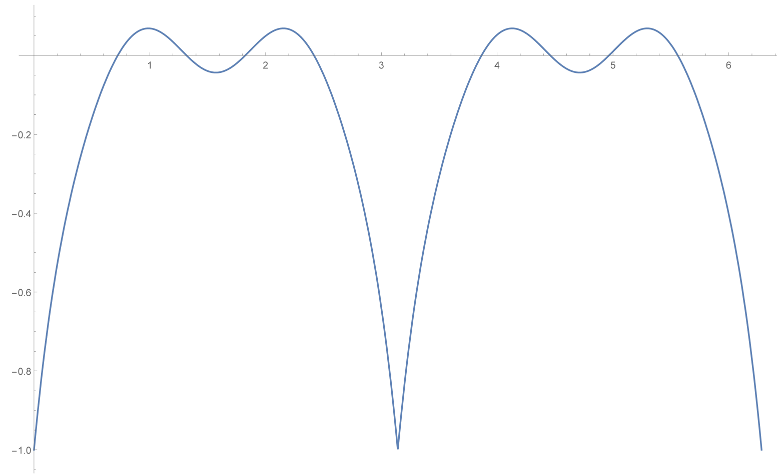

FIGURE 6. The difference $\left|\sin\left(\pi e^{i\theta}\right)\right| - \left|\cos\left(\pi e^{i\theta}\right)\right|$ for $\theta \in [0, 2\pi)$

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