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Article

Dunkl generalization of Phillips operators and approximation in weighted spaces

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Abstract: Purpose of this article is to introduce a modification of Phillips operators on the interval $\left[\frac{1}{2},\infty\right)$ via Dunkl generalization. This type of modification enables to provide an error estimation on the interval $\left[\frac{1}{2},\infty\right)$. We discuss the convergence results and obtain the degrees of approximations. Furthermore, we calculate the rate of convergence by means of modulus of continuity, Lipschitz type maximal functions, Peetre's K-functional and second order modulus of continuity.

Keywords: Szász operator; Dunkl analogue; generalization of exponential function; Korovkin type theorem; modulus of continuity; order of convergence.

1. Preliminaries and Introduction

Szász operators [20] provide an extension to Bernstein operators [4] on the interval $[0,\infty)$. In the present years, several authors have studied the Dunkl type generalization of Szász operators (see [1,2,9,10,12,13,15–17,21]).

In [11] the Phillips operator is defined by

$$\mathcal{S}_n(f;x) = \int_0^\infty e^{-n(x+t)}\left(\sum_{u=1}^{\infty}\frac{(n^2x)^u t^{u-1}}{u!(u-1)!}\right) f(t)\mathrm{d}t + e^{-nx}f(0), \tag{1}$$

degree of approximation in concerned, the behavior of this operator is close similar to that of Bernstein polynomials and Szász operators. For details see also in [18].

Sucu [19] introduced Dunkl analogue of Szász operators. That is, for $f \in C[0,\infty)$, $x \geq 0$, $v \geq 0$ and $n \in \mathbb{N}$,

$$\mathcal{S}_n^*(f;x) := \frac{1}{E_v(nx)}\sum_{\ell=0}^{\infty}\frac{(nx)^\ell}{\gamma_v(\ell)} f\left(\frac{\ell+2v\theta_\ell}{n}\right), \tag{2}$$

where $\mathbb{N}$ is the set of all natural numbers and

$$E_v(x) = \sum_{\ell=0}^{\infty}\frac{x^\ell}{\gamma_v(\ell)}. \tag{3}$$

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The coefficients γ_v are given as

$$\gamma_v(2\ell) = \frac{2^{2\ell}\ell!\Gamma\left(\ell+v+\frac{1}{2}\right)}{\Gamma\left(v+\frac{1}{2}\right)}, \quad \gamma_v(2\ell+1) = \frac{2^{2\ell+1}\ell!\Gamma\left(\ell+v+\frac{3}{2}\right)}{\Gamma\left(v+\frac{1}{2}\right)} \tag{4}$$

with recursion

$$\frac{\gamma_v(\ell+1)}{(\ell+1+2v\theta_{\ell+1})} = \gamma_v(\ell), \tag{5}$$

$$\Gamma(\ell+1) = \int_0^\infty e^{-t}t^\ell \mathrm{d}t = \ell!, \quad \ell > 0 \tag{6}$$

where

$$\theta_\ell = \begin{cases} 0 & \text{if } \ell = 0,2,4,\cdots, \\ 1 & \text{if } \ell = 1,3,5,\cdots,. \end{cases} \tag{7}$$

Studies on Dunkl type generalizations [14] and previous studies of Szász type operators [6,7] demonstrate an error estimation to the operators which allow us much faster approximation to the function which is being approximated. In this paper, we modify the Phillips operators given by [14] via Dunkl generalization. Our main idea is to approximate these operators by well known Korovkin's and weighted Korovkin's theorems. We estimate the degrees of approximations and calculate the rate of convergence by means of modulus of continuity, Lipschitz type maximal functions, Peetre's K-functional and second order modulus of continuity.

2. New operators and their moments

Let $\{\chi_n(x)\}$ be a sequence of nonnegative continuous functions on $[0,\infty)$ as

$$\chi_n(x) = \left(x - \frac{1}{2n}\right)_+, \quad n = 1,2,3,\cdots, \tag{8}$$

where

$$\kappa_+ = \begin{cases} \kappa & \text{if } \kappa \geq 0, \\ 0 & \text{if } \kappa < 0. \end{cases} \tag{9}$$

Moreover, suppose

$$\mathcal{J}_{n,v}(x) = \frac{E_v(-n\chi_n(x))}{E_v(n\chi_n(x))} \tag{10}$$

For $f \in C_\zeta(\mathbb{R}^+) = \{f \in C[0,\infty) :| f(t) |\leq K(1+t)^\zeta, \quad \zeta > 0, \ \ K > 0\}$, we define

$$\mathcal{P}_{n,v}(f;x) = \frac{n^2}{E_v(n\chi_n(x))} \sum_{\ell=0}^{\infty} \frac{(n\chi_n(x))^\ell}{\gamma_v(\ell)} \int_0^\infty \frac{e^{-nt}n^{\ell+2v\theta_\ell-1}t^{\ell+2v\theta_\ell}}{\gamma_v(\ell)} f(t)\mathrm{d}t, \ v \geq 0 \tag{11}$$

where $E_v(x)$, γ_v and θ_ℓ are defined as in [19] by (3), (4) and (7), respectively.

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Lemma 1. *Let $e_\ell = t^{\ell-1}$, $\ell = 1,2,3,4,5$ and $\mathcal{J}_{n,v}(x)$ defined by (10). Then for $x \geq 0$, $\mathcal{P}_{n,v}(e_1;x) = 1$ and for any $x \geq \frac{1}{2n}$ we have*

$$\begin{aligned}
(1)\quad \mathcal{P}_{n,v}(e_2;x) &= x + \frac{1}{2n},\\
(2)\quad \mathcal{P}_{n,v}(e_3;x) &= x^2 + \frac{1}{n}\left(3 + 2v\mathcal{J}_{n,v}(x)\right)x - \frac{1}{4n^2}\left(1 - 4v\mathcal{J}_{n,v}(x)\right),\\
(3)\quad \mathcal{P}_{n,v}(e_4;x) &= x^3 + \frac{1}{2n}\left(15 - 4v\mathcal{J}_{n,v}(x)\right)x^2\\
&+ \frac{1}{4n^2}\left(39 + 16v^2 + 72v\mathcal{J}_{n,v}(x)\right)x\\
&- \frac{1}{8n^3}\left(7 + 16v^2 + 68v\mathcal{J}_{n,v}(x)\right),\\
(4)\quad \mathcal{P}_{n,v}(e_5;x) &= x^4 + \frac{1}{n}\left(14 + 4v\mathcal{J}_{n,v}(x)\right)x^3\\
&+ \frac{1}{2n^2}\left(99 - 68v\mathcal{J}_{n,v}(x) + 8v^2\right)x^2\\
&+ \frac{1}{2n^3}\left(47 + 294v\mathcal{J}_{n,v}(x) + 96v^2 + 16v^3\mathcal{J}_{n,v}(x)\right)x\\
&- \frac{1}{16n^4}\left(959 + 120v\mathcal{J}_{n,v}(x) + 432v^2 + 64v^3\mathcal{J}_{n,v}(x)\right).
\end{aligned}$$

Proof of Lemma 1.

$$\begin{aligned}
\mathcal{P}_{n,v}(e_1;x) &= \frac{n^2}{E_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^\ell}{\gamma_v(\ell)}\int_0^\infty \frac{e^{-nt}n^{\ell+2v\theta_\ell-1}t^{\ell+2v\theta_\ell}}{\gamma_v(\ell)}\mathrm{d}t\\
&= \frac{1}{E_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^\ell}{\gamma_v(\ell)}\int_0^\infty \frac{e^{-t}t^{\ell+2v\theta_\ell}}{\gamma_v(\ell)}\mathrm{d}t\\
&= \frac{1}{E_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^\ell}{\gamma_v(\ell)}\frac{\Gamma(\ell+2v\theta_\ell+1}{\gamma_v(\ell)}\\
&= \frac{1}{E_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^\ell}{\gamma_v(\ell)}\\
&= 1,
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{n,v}(e_2;x) &= \frac{n^2}{E_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^\ell}{\gamma_v(\ell)}\int_0^\infty \frac{e^{-t}t^{\ell+2v\theta_\ell+1}}{\gamma_v(\ell)}\mathrm{d}t\\
&= \frac{1}{nE_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^\ell}{\gamma_v(\ell)}\frac{\Gamma(\ell+2v\theta_\ell+2)}{\gamma_v(\ell)}\\
&= \frac{1}{nE_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^\ell}{\gamma_v(\ell)}(\ell+2v\theta_\ell+1)\\
&= \frac{1}{nE_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^\ell}{\gamma_v(\ell)}\\
&+ \frac{1}{nE_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^\ell}{\gamma_v(\ell)}(\ell+2v\theta_\ell)\\
&= \chi_n(x) + \frac{1}{n},
\end{aligned}$$

$$\begin{aligned}\mathcal{P}_{n,v}(e_3;x) &= \frac{1}{n^2 E_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^\ell}{\gamma_v(\ell)}(\ell+2v\theta_\ell)^2 \\ &+ \frac{3}{n^2 E_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^\ell}{\gamma_v(\ell)}(\ell+2v\theta_\ell) \\ &+ \frac{2}{n^2 E_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^\ell}{\gamma_v(\ell)} \\ &= \chi_n^2(x)+\frac{2}{n}\left(2+v\frac{E_v(-n\chi_n(x))}{E_v(n\chi_n(x))}\right)\chi_n(x)+\frac{2}{n^2}.\end{aligned}$$

Similar we have

$$\mathcal{P}_{n,v}(e_4;x) = \chi_n^3(x)+\frac{1}{n}\left(9-2v\frac{E_v(-n\chi_n(x))}{E_v(n\chi_n(x))}\right)\chi_n^2(x)+\frac{2}{n^2}\left(9+2v+8v\frac{E_v(-n\chi_n(x))}{E_v(n\chi_n(x))}\right)\chi_n(x)+\frac{6}{n^3}$$

$$\begin{aligned}\mathcal{P}_{n,v}(e_5;x) &= \chi_n^4(x)+\frac{4}{n}\left(4+v\frac{E_v(-n\chi_n(x))}{E_v(n\chi_n(x))}\right)\chi_n^3(x)+\frac{4}{n^2}\left(18+v^2-7v\frac{E_v(-n\chi_n(x))}{E_v(n\chi_n(x))}\right)\chi_n^2(x) \\ &+ \frac{2}{n^3}\left(63+26v^2+2v(29+2v^2)\frac{E_v(-n\chi_n(x))}{E_v(n\chi_n(x))}\right)\chi_n(x)+\frac{24}{n^2}.\end{aligned}$$

□

Remark 1. *For all* $0 \le x < \frac{1}{2n}$, *from* (8), (9) *and Lemma 1 we have* $\mathcal{P}_{n,v}(e_2;x)=\frac{1}{n}$; $\mathcal{P}_{n,v}(e_3;x)=\frac{2}{n^2}$; $\mathcal{P}_{n,v}(e_4;x)=\frac{6}{n^3}$; $\mathcal{P}_{n,v}(e_5;x)=\frac{24}{n^4}$.

Here we also introduce the Stancu type generalization to the operators defined by (11). Thus, for each $f \in C_\zeta(\mathbb{R}^+)$ the modified version of the operators (11) is defined as

$$\mathcal{S}^*_{n,v}(f;x)=\frac{n^2}{E_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^\ell}{\gamma_v(\ell)}\int_0^\infty\frac{e^{-nt}n^{\ell+2v\theta_\ell-1}t^{\ell+2v\theta_\ell}}{\gamma_v(\ell)}f\left(\frac{nt+\alpha}{n+\beta}\right)dt \tag{12}$$

where $0 \le \alpha \le \beta$. Note that if we take $\alpha=\beta=0$ in (12), then the operators $\mathcal{S}^*_{n,v}$ reduce to operators defined by (11) and if take $\chi_n(x)=x$ in $\mathcal{P}_{n,v}$, then we get the operators defined studied in [14].

Lemma 2. *Let* $f(t)=t^k$, *for* $k=0,1,2,3,4$ *in* (12), *then for all* $x \ge 0$, $\mathcal{S}^*_{n,v}(e_1;x)=1$ *and each* $x \ge \frac{1}{2n}$, *we have*

$$\begin{aligned}1^\circ\quad \mathcal{S}^*_{n,v}(t;x) &= \frac{n}{n+\beta}\mathcal{P}_{n,v}(e_2;x)+\frac{\alpha}{n+\beta}, \\ 2^\circ\quad \mathcal{S}^*_{n,v}(t^2;x) &= \frac{n^2}{(n+\beta)^2}\mathcal{P}_{n,v}(e_3;x)+\frac{2\alpha n}{(n+\beta)^2}\mathcal{P}_{n,v}(e_2;x)+\frac{\alpha^2}{(n+\beta)^2}, \\ 3^\circ\quad \mathcal{S}^*_{n,v}(t^3;x) &= \frac{n^3}{(n+\beta)^3}\mathcal{P}_{n,v}(e_4;x)+\frac{3\alpha n^2}{(n+\beta)^3}\mathcal{P}_{n,v}(e_3;x)+\frac{3\alpha^2 n}{(n+\beta)^3}\mathcal{P}_{n,v}(e_2;x)+\frac{\alpha^3}{(n+\beta)^3}, \\ 4^\circ\quad \mathcal{S}^*_{n,v}(t^4;x) &= \frac{n^4}{(n+\beta)^4}\mathcal{P}_{n,v}(e_5;x)+\frac{4\alpha n^3}{(n+\beta)^4}\mathcal{P}_{n,v}(e_4;x)+\frac{6\alpha^2 n^2}{(n+\beta)^4}\mathcal{P}_{n,v}(e_3;x) \\ &+ \frac{4\alpha^3 n}{(n+\beta)^4}\mathcal{P}_{n,v}(e_2;x)+\frac{\alpha^4}{(n+\beta)^4},\end{aligned}$$

Proof of Lemma 2.

$$\begin{aligned}\mathcal{S}^*_{n,v}\left(\frac{nt+\alpha}{n+\beta};x\right) &= \frac{n}{n+\beta}\frac{n^2}{E_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^{\ell}}{\gamma_v(\ell)}\int_0^{\infty}\frac{e^{-nt}n^{\ell+2v\theta_\ell-1}t^{\ell+2v\theta_\ell}}{\gamma_v(\ell)}t\mathrm{d}t\\ &+ \frac{\alpha}{n+\beta}\frac{n^2}{E_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^{\ell}}{\gamma_v(\ell)}\int_0^{\infty}\frac{e^{-nt}n^{\ell+2v\theta_\ell-1}t^{\ell+2v\theta_\ell}}{\gamma_v(\ell)}\mathrm{d}t\\ &= \frac{n}{n+\beta}\mathcal{P}_{n,v}(e_2;x)+\frac{\alpha}{n+\beta}\mathcal{P}_{n,v}(e_1;x),\end{aligned}$$

$$\begin{aligned}\mathcal{S}^*_{n,v}\left(\left(\frac{nt+\alpha}{n+\beta}\right)^2;x\right) &= \left(\frac{n}{n+\beta}\right)^2\frac{n^2}{E_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^{\ell}}{\gamma_v(\ell)}\int_0^{\infty}\frac{e^{-nt}n^{\ell+2v\theta_\ell-1}t^{\ell+2v\theta_\ell}}{\gamma_v(\ell)}t^2\mathrm{d}t\\ &+ \frac{2n\alpha}{(n+\beta)^2}\frac{n^2}{E_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^{\ell}}{\gamma_v(\ell)}\int_0^{\infty}\frac{e^{-nt}n^{\ell+2v\theta_\ell-1}t^{\ell+2v\theta_\ell}}{\gamma_v(\ell)}t\mathrm{d}t\\ &+ \left(\frac{\alpha}{n+\beta}\right)^2\frac{n^2}{E_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^{\ell}}{\gamma_v(\ell)}\int_0^{\infty}\frac{e^{-nt}n^{\ell+2v\theta_\ell-1}t^{\ell+2v\theta_\ell}}{\gamma_v(\ell)}\mathrm{d}t\\ &= \left(\frac{n}{n+\beta}\right)^2\mathcal{P}_{n,v}(e_3;x)+\frac{2n\alpha}{(n+\beta)^2}\mathcal{P}_{n,v}(e_2;x)+\left(\frac{\alpha}{n+\beta}\right)^2\mathcal{P}_{n,v}(e_1;x).\end{aligned}$$

Similar other estimates we can be obtained. □

Lemma 3. *For $0 \leq x < \frac{1}{2n}$, we have*

$$\begin{aligned}(1)^{\circ\circ}\quad \mathcal{S}^*_{n,v}(t;x) &= \frac{\alpha+1}{n+\beta},\\ (2)^{\circ\circ}\quad \mathcal{S}^*_{n,v}(t^2;x) &= \frac{2+\alpha+\alpha^2}{(n+\beta)^2},\\ (3)^{\circ\circ}\quad \mathcal{S}^*_{n,v}(t^3;x) &= \frac{6+6\alpha+3\alpha^2+\alpha^3}{(n+\beta)^3},\\ (4)^{\circ\circ}\quad \mathcal{S}^*_{n,v}(t^4;x) &= \frac{24+4\alpha+12\alpha^2+4\alpha^3+\alpha^4}{(n+\beta)^4}.\end{aligned}$$

Proof of Lemma 3. For the proof this Lemma clearly, we use Remark 1 and Lemma 2. □

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Lemma 4. *Suppose $\eta_j = (e_2 - x)^j$ for $j = 1, 2, 3, 4$, where e_2 is defined in Lemma 2. Then, for $x \geq \frac{1}{2n}$ we have*

$$
\begin{aligned}
1^* \quad \mathcal{S}^*_{n,v}(\eta_1; x) &= \left(\frac{n}{n+\beta} - 1\right) x + \frac{1+2\alpha}{2(n+\beta)}, \\
2^* \quad \mathcal{S}^*_{n,v}(\eta_2; x) &= \left[\frac{n^2}{(n+\beta)^2} - \frac{2n}{n+\beta} + 1\right] x^2 \\
&+ \left[\frac{n}{(n+\beta)^2}\left(3 + 2v\mathcal{J}_{n,v}(x)\right) + \frac{2\alpha n}{(n+\beta)^2} - \frac{2\alpha+1}{n+\beta}\right] x \\
&+ \frac{\alpha+\alpha^2}{(n+\beta)^2} - \frac{1}{4(n+\beta)^2}\left(1 - 4v\mathcal{J}_{n,v}(x)\right), \\
3^* \quad \mathcal{S}^*_{n,v}(\eta_4; x) &= \left[\frac{n^4}{(n+\beta)^4} - \frac{4n^3}{(n+\beta)^3} + \frac{6n^2}{(n+\beta)^2} - \frac{4n}{n+\beta} + 1\right] x^4 \\
&+ \left[\frac{n^3}{(n+\beta)^4}\left(14 + 4\alpha + 4v\mathcal{J}_{n,v}(x)\right) - \frac{2n^2}{(n+\beta)^3}\left(15 + 6\alpha - 4v\mathcal{J}_{n,v}(x)\right)\right. \\
&+ \left.\frac{6n}{(n+\beta)^2}\left(3 + 2\alpha + 2v\mathcal{J}_{n,v}(x)\right) - \frac{2+4\alpha}{n+\beta}\right] x^3 \\
&+ \left[\frac{n^2}{2(n+\beta)^4}\Big(99 + 60\alpha + 12\alpha^2 + 8v^2 - 4(17+4\alpha)4v\mathcal{J}_{n,v}(x)\Big)\right. \\
&- \frac{n}{(n+\beta)^3}\Big(39 + 16v^2 + 36\alpha + 12\alpha^2 + 24(3+\alpha)v\mathcal{J}_{n,v}(x)\Big) \\
&+ \left.\frac{1}{2(n+\beta)^2}\Big(-3 + 12\alpha + 12\alpha^2 + 12v\mathcal{J}_{n,v}(x)\Big)\right] x^2 \\
&+ \left[\frac{n}{2(n+\beta)^4}\Big(47 + 78\alpha + 36\alpha^2 + 8\alpha^3 + 16v^3 + 96v^2 + 16\alpha v^2 + 24\alpha(\alpha+6)v\mathcal{J}_{n,v}(x)\Big)\right. \\
&+ \left.\frac{1}{2(n+\beta)^3}\Big(7 + 16v^2 + 6\alpha - 12\alpha^2 - 8\alpha^3 + 4(17-6\alpha)v\mathcal{J}_{n,v}(x)\Big)\right] x \\
&+ \frac{1}{16(n+\beta)^4}\Big(-959 - 432v^2 - 56\alpha + 24\alpha^2 + 32\alpha^3 + 16\alpha^4 \\
&- 128\alpha v^2 - 8(15 + 68\alpha + 12\alpha^2 + 8v^2)v\mathcal{J}_{n,v}(x)\Big).
\end{aligned}
$$

Proof of Lemma 4. In the light of Lemma 1 we prove it and we know

$$
\mathcal{S}^*_{n,v}(\eta_1; x) = \frac{n}{n+\beta}\mathcal{P}_{n,v}(e_2; x) + \frac{\alpha}{n+\beta} - x,
$$

$$
\mathcal{S}^*_{n,v}(\eta_2; x) = \frac{n^2}{(n+\beta)^2}\mathcal{P}_{n,v}(e_3; x) + \frac{2\alpha n}{(n+\beta)^2}\mathcal{P}_{n,v}(e_2; x) + \frac{\alpha^2}{(n+\beta)^2} - \frac{2nx}{n+\beta}\mathcal{P}_{n,v}(e_2; x) - \frac{2nx\alpha}{n+\beta} + x^2.
$$

Similarly in the light of Lemma 2 we obtain for $\mathcal{S}^*_{n,v}(\eta_3; x)$ and $\mathcal{S}^*_{n,v}(\eta_4; x)$. □

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Lemma 5. *Suppose $\eta_j = (e_2 - x)^j$ for $j = 1,2,3,4$. Then, for all $0 \le x < \frac{1}{2n}$ in the view of Lemma 3 we have*

$$\begin{aligned}
(1)^{**} \quad \mathcal{S}^*_{n,v}(\eta_1;x) &= \frac{\alpha+1}{n+\beta} - x,\\
(2)^{**} \quad \mathcal{S}^*_{n,v}(\eta_2;x) &= x^2 - \frac{2(\alpha+1)}{(n+\beta)}x + \frac{2+\alpha+\alpha^2}{(n+\beta)^2},\\
(3)^{**} \quad \mathcal{S}^*_{n,v}(\eta_4;x) &= x^4 - \frac{4(\alpha+1)}{(n+\beta)}x^3 + \frac{6(2+\alpha+\alpha^2)}{(n+\beta)^2}x^2\\
&- \frac{4(6+6\alpha+3\alpha^2+\alpha^3)}{(n+\beta)^3}x + \frac{24+24\alpha+12\alpha^2+4\alpha^3+\alpha^4}{(n+\beta)^4}.
\end{aligned}$$

Remark 2. *In the light of Lemma 4 and 5 we use the notation*

$$\sqrt{\mathcal{S}^*_{n,v}(\eta_2;x)} = \tilde{\delta}_{n,v}(x), \tag{13}$$

where

$$\left(\tilde{\delta}_{n,v}(x)\right)^2 = \begin{cases} x^2 - \frac{2(\alpha+1)}{(n+\beta)}x + \frac{2+\alpha+\alpha^2}{(n+\beta)^2}; & \text{if} \quad 0 \le x < \frac{1}{2n},\\ \left[\frac{n^2}{(n+\beta)^2} - \frac{2n}{n+\beta} + 1\right]x^2 & \\ + \left[\frac{n}{(n+\beta)^2}\left(3 + 2v\mathcal{J}_{n,v}(x)\right) + \frac{2\alpha n}{(n+\beta)^2} - \frac{2\alpha+1}{n+\beta}\right]x & \\ + \frac{\alpha+\alpha^2}{(n+\beta)^2} - \frac{1}{4(n+\beta)^2}\left(1 - 4v\mathcal{J}_{n,v}(x)\right); & \text{if} \quad x \ge \frac{1}{2n} \end{cases} \tag{14}$$

and

$$\mathcal{J}_{n,v}(x) = \mathcal{J}^*_{n,v}(n\chi_n(x)) = \begin{cases} 1; & \text{if} \quad 0 \le x < \frac{1}{2n},\\ \mathcal{J}^*_{n,v}\left(nx - \frac{1}{2}\right); & \text{if} \quad x \ge \frac{1}{2n}. \end{cases} \tag{15}$$

3. Korovkin type approximation

In the present section the results related to uniform convergence of the operators defined by (12) are given via well-known Korovkin's and weighted Korovkin's type theorems.

Let $\mathbb{R}^+ = [0,\infty)$ and $C_B(\mathbb{R}^+)$ denote the linear normed space with the norm

$$\| f \|_{C_{B(\mathbb{R}^+)}} = \sup_{x \ge 0} | f(x) | .$$

Let

$$\mathcal{H} := \{f : \lim_{x\to\infty} \frac{f(x)}{1+x^2} \text{ exists, } x \in [0,\infty)\}.$$

Theorem 1. *Let the function $f \in C[0,\infty) \cap \mathcal{H}$ and the operators $\mathcal{S}^*_{n,v}$ defined by (12). Then*

$$\lim_{n\to\infty} \mathcal{S}^*_{n,v}(f;x) = f(x)$$

uniformly on U, where U is any compact subset of $[0,\infty)$.

Proof of Theorem 1. We apply the well-known Korovkin's theorem to prove the uniform convergence of operators $\mathcal{S}^*_{n,v}$. Therefore, for $\ell = 1,2,3$, we see $\lim_{n\to\infty} \mathcal{S}^*_{n,v}(e_\ell;x) = x^{\ell-1}$ uniformly. Therefore, we have

$$\lim_{n\to\infty} \mathcal{S}^*_{n,v}(e_1;x) = 1; \quad \lim_{n\to\infty} \mathcal{S}^*_{n,v}(e_2;x) = x; \quad \lim_{n\to\infty} \mathcal{S}^*_{n,v}(e_3;x) = x^2.$$

Which completes the proof of Theorem 1. □

We recall the weighted spaces defined by Gadžiev [8]. We write $B_\sigma(\mathbb{R}^+)$ for the set of all functions such that

$$| f(x) | \leq m_f \sigma(x),$$

where m_f is a constant depends on f and $||f||_\sigma = \sup_{x\geq 0} \frac{|f(x)|}{\sigma(x)}$. Let $x \to \phi(x)$ be a continuous and strictly increasing function such that $\sigma(x) = 1 + \phi^2(x)$ and $\lim_{x\to\infty} \sigma(x) = \infty$. Consequently, we suppose $C_\sigma(\mathbb{R}^+) = B_\sigma(\mathbb{R}^+) \cap C(\mathbb{R}^+)$. From [8] we note that the sequence of positive linear operators $\{L_n\}_{n\geq 1}$ maps $C_\sigma(\mathbb{R}^+)$ into $B_\sigma(\mathbb{R}^+)$ if and only if

$$|L_n(\sigma; x)| \leq K\sigma(x),$$

where $\sigma(x) = 1 + \phi^2(x)$, $x \in \mathbb{R}^+$ and K is a positive constant. Let $C^0_\sigma(\mathbb{R}^+)$ be a subset of $C_\sigma(\mathbb{R}^+)$ such that

$$\lim_{x\to\infty} \frac{f(x)}{\sigma(x)} = k_f < \infty.$$

Theorem 2. *Let $\mathcal{S}^*_{n,v}$ be the sequence of positive linear operators acting from $C_\sigma(\mathbb{R}^+)$ into $B_\sigma(\mathbb{R}^+)$ such that*

$$\lim_{n\to\infty} \| \mathcal{S}^*_{n,v}(\varphi^k) - \varphi^k \|_\sigma = 0, \quad k = 0, 1, 2.$$

Then for all $f \in C^0_\sigma(\mathbb{R}^+)$, we have

$$\lim_{n\to\infty} \| \mathcal{S}^*_{n,v}(f) - f \|_\sigma = 0.$$

Proof of Theorem 2. Consider $\varphi(x) = x$, $\sigma(x) = 1 + x^2$ and

$$\left|\left|\mathcal{S}^*_{n,v}(e_\ell) - x^\ell\right|\right|_\sigma = \sup_{x\geq 0} \frac{\left|\mathcal{S}^*_{n,v}(e_\ell) - x^\ell\right|}{1 + x^2}.$$

Then by Korovkin's theorem, it is easily proved that $\lim_{n\to\infty} \left|\left|\mathcal{S}^*_{n,v}(e_\ell) - x^\ell\right|\right|_\sigma = 0$, for $\ell = 0, 1, 2$. Hence for any $f \in C^0_\sigma(\mathbb{R}^+)$, we get

$$\lim_{n\to\infty} \left|\left|\mathcal{S}^*_{n,v}(f) - f\right|\right|_\sigma = 0.$$

□

Theorem 3. *Let $\mathcal{S}^*_{n,v}$ be the operators defined by (12). Then for every $f \in C^0_\sigma(\mathbb{R}^+)$, we have*

$$\lim_{n\to\infty} \| \mathcal{S}^*_{n,v}(f; x) - f \|_\sigma = 0.$$

Proof of Theorem 3. We prove this theorem in the light of well-known Korovkin's theorem. Take $f(t) = t^k$, for $k = 0, 1, 2$, defined by Lemma 2. Then, for any $f(t) \in C^0_\sigma(\mathbb{R}^+)$, $\mathcal{S}^*_{n,v}(t^k; x) \to x^k$ $(k = 0, 1, 2,)$ uniformly as $n \to \infty$. For $k = 0$, by applying Lemma 2, we get $\mathcal{S}^*_{n,v}(1; x) = 1$, so that

$$\lim_{n\to\infty} \left|\left|\mathcal{S}^*_{n,v}(1; x) - 1\right|\right|_\sigma = 0. \tag{16}$$

Take $k = 1$ and $x \geq \frac{1}{2n}$, we get

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$$
\begin{aligned}
&\left|\left|\mathcal{S}^{*}_{n,v}\left(t;x\right)-x\right|\right|_{\sigma}\\
&=\sup_{x\geq 0}\frac{\left|\mathcal{S}^{*}_{n,v}(t;x)-x\right|}{1+x^{2}}\\
&=\sup_{x\geq 0}\frac{\left|\frac{n}{n+\beta}\mathcal{P}_{n,v}(e_{2};x)-x+\frac{\alpha}{n+\beta}\right|}{1+x^{2}}\\
&\leq\left(\frac{n}{n+\beta}-1\right)\sup_{x\geq 0}\frac{x}{1+x^{2}}+\frac{1+2\alpha}{2(n+\beta)}\sup_{x\geq 0}\frac{1}{1+x^{2}}.
\end{aligned}
$$

In case of $0\leq x\leq\frac{1}{2n}$, we get

$$
\begin{aligned}
&\left|\left|\mathcal{S}^{*}_{n,v}\left(t;x\right)-x\right|\right|_{\sigma}\\
&=\max_{0\leq x\leq\frac{1}{2n}}\frac{\left|\mathcal{S}^{*}_{n,v}(t;x)-x\right|}{1+x^{2}}\\
&\leq\max_{0\leq x\leq\frac{1}{2n}}\left|\mathcal{S}^{*}_{n,v}(e_{2};x)-x\right|\\
&\leq\max_{0\leq x\leq\frac{1}{2n}}\left|\frac{\alpha+1}{n+\beta}-x\right|\\
&=\frac{1}{n+\beta}\max\left\{\alpha+1,\left|\alpha-\frac{\beta}{2n}\right|\right\}.
\end{aligned}
$$

Then

$$
\lim_{n\to\infty}\left|\left|\mathcal{S}^{*}_{n,v}\left(t;x\right)-x\right|\right|_{\sigma}=0. \tag{17}
$$

In similar way if take $k=2$ and $x\geq\frac{1}{2n}$, we get

$$
\begin{aligned}
&\left|\left|\mathcal{S}^{*}_{n,v}\left(t^{2};x\right)-x^{2}\right|\right|_{\sigma}\\
&=\sup_{x\geq 0}\frac{\left|\mathcal{S}^{*}_{n,v}(t^{2};x)-x^{2}\right|}{1+x^{2}}\\
&=\sup_{x\geq 0}\frac{\left|\frac{n^{2}}{(n+\beta)^{2}}\mathcal{P}_{n,v}(e_{3};x)+\frac{2\alpha n}{(n+\beta)^{2}}\mathcal{P}_{n,v}(e_{2};x)+\frac{\alpha^{2}}{(n+\beta)^{2}}-x^{2}\right|}{1+x^{2}}\\
&\leq\left(\frac{n^{2}}{(n+\beta)^{2}}-1\right)\sup_{x\geq 0}\frac{x^{2}}{1+x^{2}}+\frac{n}{(n+\beta)^{2}}\left(2\alpha+3+2v\mathcal{J}_{n,v}(x)\right)\sup_{x\geq 0}\frac{x}{1+x^{2}}\\
&+\frac{1}{4(n+\beta)^{2}}\left(4\alpha+4\alpha^{2}-\mathcal{J}_{n,v}(x)\right)\sup_{x\geq 0}\frac{1}{1+x^{2}},
\end{aligned}
$$

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In case of $0 \leq x \leq \frac{1}{2n}$, we get

$$\begin{aligned}
&\left|\left|\mathcal{S}_{n,v}^{*}\left(t^2;x\right)-x\right|\right|_{\sigma} \\
&= \max_{0\leq x\leq \frac{1}{2n}} \frac{\left|\mathcal{S}_{n,v}^{*}(e_2;x)-x\right|}{1+x^2} \\
&\leq \max_{0\leq x\leq \frac{1}{2n}} \left|\mathcal{S}_{n,v}^{*}(e_2;x)-x\right| \\
&\leq \max_{0\leq x\leq \frac{1}{2n}} \left|\frac{2+\alpha+\alpha^2}{(n+\beta)^2}-x^2\right| \\
&= \frac{2+\alpha+\alpha^2}{(n+\beta)^2}.
\end{aligned}$$

$$\lim_{n\to\infty}\left|\left|\mathcal{S}_{n,v}^{*}\left(t^2;x\right)-x^2\right|\right|_{\sigma}=0. \tag{18}$$

This proves the theorem. □

4. Order of approximation of $\mathcal{S}_{n,v}^{*}(\cdot\,;\cdot)$

We denote the set of all uniformly continuous functions by $\tilde{C}[0,\infty)$. Let $\tilde{\omega}(f;\tilde{\delta})$ denote the modulus of continuity of $f \in \tilde{C}[0,\infty)$, i.e.

$$\tilde{\omega}(f;\tilde{\delta}) = \sup_{|x_1-x_2|\leq\delta} \mid f(x_1)-f(x_2)\mid; \quad x_1,x_2 \in [0,\infty),\ \tilde{\delta}>0. \tag{19}$$

which satisfies that $\lim_{\tilde{\delta}\to 0+}\tilde{\omega}(f;\tilde{\delta})=0$, and

$$\mid f(x_1)-f(x_2)\mid \leq \left(\frac{\mid x_1-x_2\mid}{\tilde{\delta}}+1\right)\tilde{\omega}(f;\tilde{\delta}). \tag{20}$$

Theorem 4. *For any $f \in \tilde{C}[0,\infty)$,*

$$\mid \mathcal{S}_{n,v}^{*}(f;x)-f(x)\mid \leq 2\tilde{\omega}\left(f;\tilde{\delta}_{n,v}(x)\right),$$

where $\tilde{\delta}_{n,v}(x)$ is defined by (14).

Proof of Theorem 4. Using (19) and (20), we get

$$\begin{aligned}
\mathcal{S}_{n,v}^{*}(f;x)-f(x) &= \mathcal{S}_{n,v}^{*}(f;x)-f(x)\mathcal{S}_{n,v}^{*}(1;x) \\
&= \mathcal{S}_{n,v}^{*}\left(f(t)-f(x);x\right) \\
&\leq \mathcal{S}_{n,v}^{*}\left(|f(t)-f(x)|\,;x\right)
\end{aligned}$$

Since $\mathcal{S}_{n,v}^{*}(1;x)=1$, by (20) we get

$$\begin{aligned}
\left|\mathcal{S}_{n,v}^{*}(f;x)-f(x)\right| &\leq \mathcal{S}_{n,v}^{*}\left(1+\frac{|(t-x)|}{\tilde{\delta}};x\right)\tilde{\omega}(f;\tilde{\delta}) \\
&= \left(1+\frac{1}{\tilde{\delta}}\mathcal{S}_{n,v}^{*}(|(t-x)|\,;x)\right)\tilde{\omega}(f;\tilde{\delta}).
\end{aligned}$$

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From the Cauchy-Schwarz inequality we conclude that

$$\begin{aligned} \mathcal{S}_{n,v}^{*}\left(|(t-x)|\,;x\right) &\leq \mathcal{S}_{n,v}^{*}\left((t-x);x\right)^{\frac{1}{2}}\mathcal{S}_{n,v}^{*}\left((t-x)^{2};x\right)^{\frac{1}{2}} \\ &= \mathcal{S}_{n,v}^{*}\left((t-x)^{2};x\right)^{\frac{1}{2}}. \end{aligned}$$

Therefore,

$$\left|\mathcal{S}_{n,v}^{*}(f;x)-f(x)\right| \leq \left(1+\frac{1}{\tilde{\delta}}\mathcal{S}_{n,v}^{*}\left(\eta_{2};x\right)^{\frac{1}{2}}\right)\tilde{\omega}(f;\tilde{\delta}).$$

Choose $\tilde{\delta}=\tilde{\delta}_{n,v}(x)=\sqrt{\mathcal{S}_{n,v}^{*}\left(\eta_{2};x\right)}$, then we get the result. □

Here we use the usual class of Lipschitz functions and obtain the rate of convergence of the sequence of positive linear operators $\mathcal{S}_{n,v}^{*}$ (12). For $\mathcal{L}>0,\quad 0<\varrho\leq 1$ and for the continuous functions f on $[0,\infty)$, the class of Lipschitz functions $Lip_{\mathcal{L},\varrho}(f)$ is

$$Lip_{\mathcal{L},\varrho}(f)=\{f:|\,f(\varsigma_{1})-f(\varsigma_{2})\,|\leq\mathcal{L}\,|\,\varsigma_{1}-\varsigma_{2}\,|^{\varrho};\ \mathcal{L}>0,0<\varrho\leq 1\quad(\varsigma_{1},\varsigma_{2}\in[0,\infty))\} \tag{21}$$

Theorem 5. *For any $f\in Lip_{\mathcal{L},\varrho}$, we have*

$$|\,\mathcal{S}_{n,v}^{*}(f;x)-f(x)\,|\leq\mathcal{L}\left(\tilde{\delta}_{n,v}(x)\right)^{\varrho}$$

where $\tilde{\delta}_{n,v}(x)$ is defined by (14).

Proof of Theorem 5. By Hölder inequality and (21), we get

$$\begin{aligned} |\,\mathcal{S}_{n,v}^{*}(f;x)-f(x)\,| &\leq |\,\mathcal{S}_{n,v}^{*}(f(t)-f(x);x)\,| \\ &\leq \mathcal{S}_{n,v}^{*}\left(|\,f(t)-f(x)\,|;x\right) \\ &\leq \mathcal{L}\mathcal{S}_{n,v}^{*}\left(|\,(t-x)\,|^{\varrho};x\right) \\ &\leq \mathcal{L}\left(\mathcal{S}_{n,v}^{*}(1;x)\right)^{\frac{2-\varrho}{2}}\left(\mathcal{S}_{n,v}^{*}(|\,(t-x)\,|^{2};x)\right)^{\frac{\varrho}{2}} \\ &= \mathcal{L}\left(\mathcal{S}_{n,v}^{*}((t-x)^{2};x)\right)^{\frac{\varrho}{2}}. \end{aligned}$$

□

The space of all that continuous and bounded functions on $\mathbb{R}^{+}$ is denoted by $C_{B}(\mathbb{R}^{+})$ and

$$C_{B}^{2}(\mathbb{R}^{+})=\{\psi\in C_{B}(\mathbb{R}^{+}):\psi',\psi''\in C_{B}(\mathbb{R}^{+})\}. \tag{22}$$

The norm on $C_{B}^{2}(\mathbb{R}^{+})$ is given by

$$\|\,\psi\,\|_{C_{B}^{2}(\mathbb{R}^{+})}=\|\,\psi''\,\|_{C_{B}(\mathbb{R}^{+})}+\|\,\psi'\,\|_{C_{B}(\mathbb{R}^{+})}+\|\,\psi\,\|_{C_{B}(\mathbb{R}^{+})}, \tag{23}$$

where the norm for $C_{B}(\mathbb{R}^{+})$ is

$$\|\,\psi\,\|_{C_{B}(\mathbb{R}^{+})}=\sup_{x\geq 0}|\,\psi(x)\,|\,. \tag{24}$$

Theorem 6. *Let $\psi\in C_{B}^{2}(\mathbb{R}^{+})$. Then*

$$|\,\mathcal{S}_{n,v}^{*}(\psi;x)-\psi(x)\,|\leq\mu_{n,v}(x)||\psi||_{C_{B}^{2}(\mathbb{R}^{+})},$$

where $\mu_{n,v}(x)=\tilde{\delta}_{n,v}(x)+\frac{(\tilde{\delta}_{n,v}(x))^{2}}{2}$.

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Proof of Theorem 6. By Taylor series expansion for $\psi \in C_B^2(\mathbb{R}^+)$ we obtain

$$\psi(t) = \psi(x) + \psi'(x)(t-x) + \psi''(\varphi)\,\frac{(t-x)^2}{2}$$

$$|\psi(t) - \psi(x)| \leq \mathcal{U}_1 \mid (t-x) \mid + \frac{1}{2}\mathcal{U}_2 (t-x)^2,$$

where

$$\mathcal{U}_1 = \sup_{x \geq 0} |\psi'(x)| = ||\psi'||_{C_B(\mathbb{R}^+)} \leq ||\psi||_{C_B^2(\mathbb{R}^+)},$$

$$\mathcal{U}_2 = \sup_{x \geq 0} |\psi''(x)| = ||\psi''||_{C_B(\mathbb{R}^+)} \leq ||\psi||_{C_B^2(\mathbb{R}^+)}.$$

Therefore,

$$|\psi(t) - \psi(x)| \leq \left(\mid (t-x) \mid + \frac{1}{2}(t-x)^2 \right) ||\psi||_{C_B^2(\mathbb{R}^+)}.$$

By using linearity of $\mathcal{S}_{n,v}^*$ we get

$$\left|\mathcal{S}_{n,v}^*(\psi, x) - \psi(x)\right| \leq \left(\mathcal{S}_{n,v}^*(\mid (t-x) \mid; x) + \frac{1}{2}\mathcal{S}_{n,v}^*((t-x)^2; x) \right) ||\psi||_{C_B^2(\mathbb{R}^+)}.$$

From Cauchy-Schwarz inequality,

$$\mathcal{S}_{n,v}^*(\mid (t-x) \mid; x) \leq \left(\mathcal{S}_{n,v}^* \left((t-x)^2; x \right) \right)^{\frac{1}{2}}.$$

Thus, we have

$$\left|\mathcal{S}_{n,v}^*(\psi, x) - \psi(x)\right| \leq \left(\tilde{\delta}_{n,v}(x) + \frac{(\tilde{\delta}_{n,v}(x))^2}{2} \right) ||\psi||_{C_B^2(\mathbb{R}^+)}.$$

□

5. Peetre's K-functional and Direct theorem of $\mathcal{S}_{n,v}^*$

In 1968 a potential research work given by J. Peetre and known as Peetre's K-functional. Peetre investigate the interpolation between two Banach spaces and obtained an interactions to the real interpolation on K-functional. For any $f \in C_B(\mathbb{R}^+)$, the Peetre's, well-known K-functional is defined as:

$$\mathcal{K}_2(f, \breve{\delta}) = \inf \left\{ \left(\| f - \psi \|_{C_B(\mathbb{R}^+)} + \breve{\delta} \parallel \psi'' \parallel_{C_B^2(\mathbb{R}^+)} \right) : \psi \in \mathcal{W}^2 \right\}, \tag{25}$$

where

$$\mathcal{W}^2 = \{ \psi | \quad \psi, \psi' \quad \text{and} \quad \psi'' \in C_B(\mathbb{R}^+) \}. \tag{26}$$

For any $\breve{\delta} > 0$ and a positive constant $\mathcal{C}$ one has $\mathcal{K}_2(f; \breve{\delta}) \leq \mathcal{C}\omega_2(f; \breve{\delta}^{\frac{1}{2}})$, where

$$\omega_2(f; \breve{\delta}^{\frac{1}{2}}) = \sup_{0<h<\breve{\delta}^{\frac{1}{2}}} \sup_{t \geq 0} \mid f(t+2h) - 2f(t+h) + f(t) \mid. \tag{27}$$

Theorem 7. *Let $f \in C_B(\mathbb{R}^+)$. Then there exists a positive constant $\mathcal{D}$ such as*

$$\begin{aligned} &\left|\mathcal{S}_{n,v}^*(f; x) - f(x)\right| \\ &\leq 2\mathcal{D}\left\{ \omega_2 \left(f; \sqrt{\frac{\mu_{n,v}(x)}{2}} \right) + \min\left(1; \frac{\mu_{n,v}(x)}{2}\right) ||f||_{C_B(\mathbb{R}^+)} \right\}, \end{aligned}$$

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where $\mu_{n,v}(x)$ *is given by* 6 *and* $\omega_2\left(f;\frac{\mu_{n,v}(x)}{2}\right)$ *is given by* (14).

Proof of Theorem 7. Take $\psi\in C_B^2(\mathbb{R}^+)$. Thus we get

$$\begin{aligned}\left|\mathcal{S}^*_{n,v}(f;x)-f(x)\right| &\leq \left|\mathcal{S}^*_{n,v}(f-\psi;x)\right|+\left|\mathcal{S}^*_{n,v}(\psi;x)-\psi(x)\right|+|f(x)-\psi(x)|\\ &\leq 2||f-\psi||_{C_B(\mathbb{R}^+)}+\mu_{n,v}(x)||\psi||_{C_B^2(\mathbb{R}^+)}\\ &=2\left(||f-\psi||_{C_B(\mathbb{R}^+)}+\frac{\mu_{n,v}(x)}{2}||\psi||_{C_B^2(\mathbb{R}^+)}\right).\end{aligned}$$

By taking the infimum over all $\psi\in C_B^2(\mathbb{R}^+)$ and by using (25), we get

$$\left|\mathcal{S}^*_{n,v}(f;x)-f(x)\right|\leq 2K_2\left(f;\frac{\mu_{n,v}(x)}{2}\right).$$

Now, for an absolute constant $\mathcal{D}>0$ in [5], we use the following relation:

$$K_2(f;\breve{\delta})\leq\mathcal{D}\{\omega_2(f;\sqrt{\breve{\delta}})+\min(1;\breve{\delta})||f||_{C_B(\mathbb{R}^+)}\}$$

where $\breve{\delta}=\frac{\mu_{n,v}(x)}{2}$. This completes the proof. □

For an arbitrary $f\in C_\sigma^k(\mathbb{R}^+)$ the following weighted modulus of continuity was defined in [3]

$$\bar{\Omega}(f;\hat{\delta})=\sup_{|h|\leq\hat{\delta},\quad x\geq 0}\frac{|\,f(x+h)-f(x)\,|}{(1+x^2)(1+h^2)},\tag{28}$$

satisfying

$$\lim_{\hat{\delta}\to 0}\bar{\Omega}(f;\hat{\delta})=0,\tag{29}$$

and

$$|\,f(t)-f(x)\,|\leq 2\left(\frac{|\,t-x\,|}{\hat{\delta}}+1\right)(1+\hat{\delta}^2)(1+x^2)\left((t-x)^2+1\right)\bar{\Omega}(f;\hat{\delta}).\tag{30}$$

Theorem 8. *For any* $f\in C_\sigma^k(\mathbb{R}^+)$*, we have*

$$\sup_{x\in[0,\mathcal{A}_{n,v})}\frac{\left|\mathcal{S}^*_{n,v}(f;x)-f(x)\right|}{1+x^2}\leq\mathcal{A}\Big(1+O(\mathcal{A}_{n,v})\Big)\Omega\Big(f;O(\sqrt{\mathcal{A}_{n,v}})\Big),$$

where $\mathcal{A}$ *is a positive constant and for* $x\geq\frac{1}{2n}$

$$\begin{aligned}\mathcal{A}_{n,v} &= \max\left\{\frac{n^2}{(n+\beta)^2}-\frac{2n}{n+\beta}+1,\frac{\alpha+\alpha^2}{(n+\beta)^2}-\frac{1}{4(n+\beta)^2}v\mathcal{J}_{n,v}(x),\right.\\ &\qquad\left.\frac{n}{(n+\beta)^2}(3+2v\mathcal{J}_{n,v}(x))+\frac{2\alpha n}{(n+\beta)^2}-\frac{2\alpha+1}{n+\beta}\right\},\end{aligned}$$

and for $0\leq x<\frac{1}{2n}$,

$$\mathcal{A}_{n,v} = \max\left\{1,\frac{2(\alpha+1)}{(n+\beta)},\frac{2+\alpha+\alpha^2}{(n+\beta)^2}\right\}.$$

Proof of Theorem 8. We prove it by using (28), (30) and Cauchy-Schwarz inequality. We have

$$\begin{aligned}&\left|\mathcal{S}^*_{n,v}(f;x)-f(x)\right|\\ &\quad\leq 2(1+\hat{\delta}^2)(1+x^2)\Omega(f;\hat{\delta})\left(1+\mathcal{S}^*_{n,v}((t-x)^2;x)+\mathcal{S}^*_{n,v}((1+(t-x)^2)\frac{|(t-x)|}{\hat{\delta}};x)\right).\end{aligned} \tag{31}$$

From the Lemma 4, 5 we easily conclude that,

$$\begin{aligned}\mathcal{S}^*_{n,v}((t-x)^2;x) &\leq \mathcal{A}_1\, O\Big(\mathcal{A}_{n,v}\Big)(1+x+x^2)\\ &\leq \mathcal{A}_2(1+x+x^2)\end{aligned}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are positive constants, and for $x \geq \frac{1}{2n}$

$$\begin{aligned}\mathcal{A}_{n,v} = \max\Bigg\{&\frac{n^2}{(n+\beta)^2}-\frac{2n}{n+\beta}+1, \frac{\alpha+\alpha^2}{(n+\beta)^2}-\frac{1}{4(n+\beta)^2}v\mathcal{J}_{n,v}(x),\\ &\frac{n}{(n+\beta)^2}(3+2v\mathcal{J}_{n,v}(x))+\frac{2\alpha n}{(n+\beta)^2}-\frac{2\alpha+1}{n+\beta}\Bigg\},\end{aligned} \tag{32}$$

and for $0 \leq x < \frac{1}{2n}$,

$$\mathcal{A}_{n,v} = \max\left\{1, \frac{2(\alpha+1)}{(n+\beta)}, \frac{2+\alpha+\alpha^2}{(n+\beta)^2}\right\}.$$

By apply the Cauchy-Schwarz inequality we easily see that

$$\begin{aligned}&\mathcal{S}^*_{n,v}\left((1+(t-x)^2)\frac{|(t-x)|}{\delta};x\right)\\ &\quad\leq 2\Big(\mathcal{S}^*_{n,v}(1+(t-x)^4;x)\Big)^{\frac{1}{2}}\left(\mathcal{S}^*_{n,v}\left(\frac{(t-x)^2}{\hat{\delta}^2};x\right)\right)^{\frac{1}{2}}.\end{aligned} \tag{33}$$

Similarly for the constants $\mathcal{A}_3 > 0$ and $\mathcal{A}_4 > 0$, we have

$$\Big(\mathcal{S}^*_{n,v}(1+(t-x)^4;x)\Big)^{\frac{1}{2}} \leq \mathcal{A}_3\Big(1+x^2+x^3+x^4\Big)^{\frac{1}{2}}$$

$$\left(\mathcal{S}^*_{n,v}\left(\frac{(t-x)^2}{\hat{\delta}^2};x\right)\right)^{\frac{1}{2}} \leq \frac{1}{\hat{\delta}}\,\mathcal{A}_4\, O\Big(\mathcal{A}_{n,v}\Big)^{\frac{1}{2}}\Big(1+x+x^2\Big)^{\frac{1}{2}}$$

Finally, in view of (31), (32) and (33), and choosing $\hat{\delta} = O\Big(\sqrt{\mathcal{A}_{n,v}}\Big)$, and $\mathcal{A} = 2(1+\mathcal{A}_2+2\mathcal{A}_3\mathcal{A}_4)$, we easily led to the desire result. □

6. Conflicts of Interest

The author declares no conflict of interest.

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