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On Symmetrical Deformation of Toroidal Shell with Circular Cross-Section

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By introducing a variable transformation $\xi = \frac{1}{2}(\sin\theta + 1)$, the symmetrical deformation equation of elastic toroidal shells is successfully transferred into a well-known equation, namely Heun's equation of ordinary differential equation, whose exact solution is obtained in terms of Heun's functions. The computation of the problem can be carried out by symbolic software that is able to with the Heun's function, such as Maple. The Gauss curvature of the elastic toroidal shells shows that the internal portion of the toroidal shells has better bending capacity than the outer portion, which might be useful for the design of metamaterials with toroidal shells cells. Through numerical comparison study, the mechanics of elastic toroidal shells is sensitive to the radius ratio. By slightly adjustment of the ratio might get a desired high performance shell structure.

Keywords: toroidal shells, deformation, Gauss curvature, Heun's function, hypergeometric function, Mathieu function, Maple

INTRODUCTION

Among of most regular shell, such as circular cylindrical shells, conical shells, spherical shells and toroidal shells, the deformation of toroidal shells is one of upmost difficulty problem due to its complicated topology. Up to date, its exact solution at general radius ration has not been obtained yet.

Toroidal shells, in full or partial geometric form as shown in Figures 1, is widely used in structural engineering and have been extensively investigated [1–20].

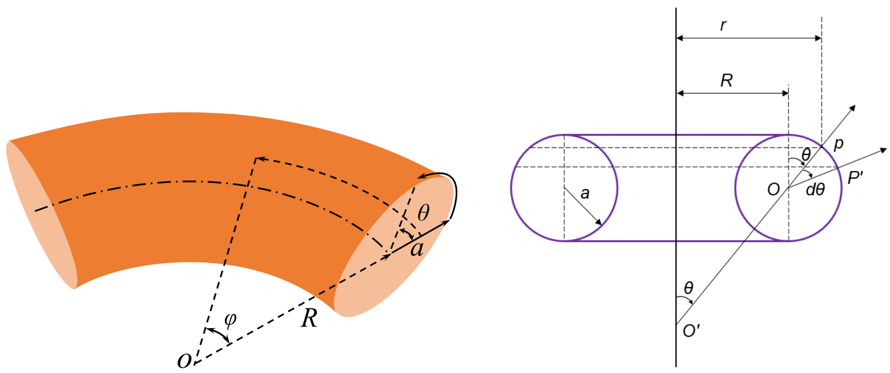

FIG. 1: Toroidal shells and cross-sectional view. Toroidal shells and geometry. The principal radius of curvature $R_1 = a$ and $R_2 = a + \frac{R}{\sin\theta}$; the principal curvature $K_1 = \frac{1}{a}$, $K_2 = \frac{\sin\theta}{R + a\sin\theta}$; the Gauss curvature $K = K_1 K_2 = \frac{\sin\theta}{a(R + a\sin\theta)}$.

The difficulty source of finding solution comes from the geometric feature of the toroidal shells. The toroidal shells's Gauss curvature K changes its sign as principal radius of curvature R_θ when the angle θ goes from 0 to 2π, it means that Gauss curvature has a turning point, namely $K = 0$ at $\theta = \pi$ as shown in Fig. 1. Recall that the partial differential equations governing the elasticity of elliptic shells $K > 0$ are themselves elliptic while those for hyperbolic shells $K < 0$ are hyperbolic. This means that the equations for a toroidal shell are of a mixed type, namely elliptic in the outer half of the toroidal shells, and hyperbolic in the inner one. The exist of the turning point in a complete toroidal shells is one source of the difficulty to find a solution. Owing to the difficulty of solving partial differential

 doi:10.20944/preprints202003.0156.v2

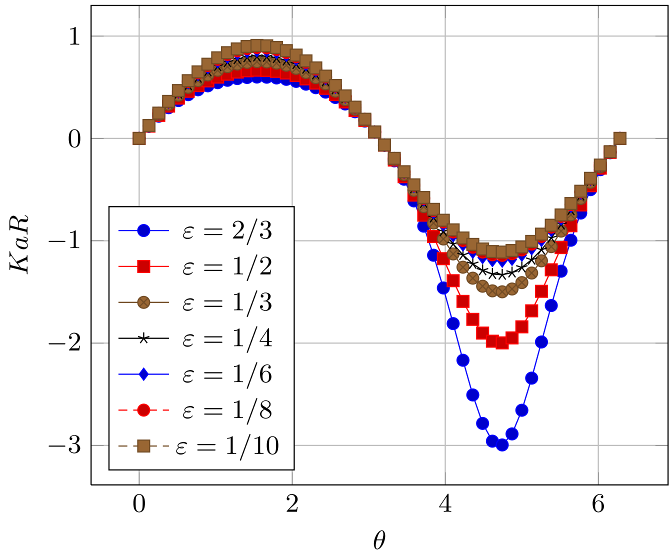

FIG. 2: The curves of $RaK = \frac{\sin\theta}{1+\varepsilon\sin\theta}$, where $\varepsilon = \frac{a}{R}$. The geometry of toroidal shells surface is elliptic in $\theta \in [0, \pi]$, parabolic at $\theta = 0$ and hyperbolic $\theta \in [0, -\pi]$.

equations with both hyperbolic and elliptic regions, few general results are known for shells with mixed type [15–19]. To find the solution of elastic toroidal shells still remains a great challenge.

The Gauss curvature change sign also provides some special feature of mechanics. Since the bending capacity of the shell is proportional to the Gauss curvature, the Fig.2 reveals that the inner part of toroidal shells ($\theta \in [\pi, 2\pi]$) is much stronger than outer part of the shell ($\theta \in [0, \pi]$). Therefore, for given amount of materials, to construct a high bending performance toroidal shells, the internal toroidal shells topology is the best choice. This secret mechanical performance might be useful to the design of metamaterials with toroidal shells cell.

To attack the problem, the high order and complicated governing equations of toroidal shells under symmetric loads is reduced to a single equation of lower order complex equation. The complex form governing equations of toroidal shells of revolution were formulated firstly by Reissner (1912)[1], Wissler (1916) [3] and later finalized by Novozhilov (1959) [7] a follows:

$$(1+\varepsilon\sin\theta)\frac{d^2V}{d\theta^2} - \varepsilon\cos\theta\frac{dV}{d\theta} + 2id^2\sin\theta V = P(\theta). \tag{1}$$

Eq.1 is called Reissner-Novozhilov equation of toroidal shells, in which, $P(\theta) = -2d^2(2d^2A + \frac{1}{2}i\varepsilon qa)\cos\theta$, $2d^2 = \frac{a^2}{Rh}\sqrt{12(1-\mu^2)}$, μ is the Poisson's ratio, h is thickness, and q is distributed loads, A is an integration constant.

Beside the turning point issue, it is clear that the challenge is also comes from the variable coefficients of the differential equation in Eq.1.

Nevertheless, some kind solutions have been proposed successfully. The first exact series solution of toroidal equations was obtained by Wissler (1916) [3], however, Wissler's series solution has little practical value due to its slow convergency and not established a linkage with any special known functions as well.

To provide practical solution, thus a various of asymptotic solutions have been proposed [14–16], however, the turning point of the Gauss curvature makes all proposed asymptotic solutions invalid near the point. Up to date, no exact solution in terms of special functions has been obtained for symmetrical deformation of toroidal shells, except in the case of slender toroidal shells, whose displacement type closed-form solution is obtained by Sun (2011) [17, 19].

To find an exact solution that can be expressed in a special function is still a open question even after more than centenary development of the theory of toroidal shells. In this paper, we will shoulder this historical responsibility and propose an exact solution in terms of special functions. Once we obtain the solution in terms of well-known special functions, the convergency issue will be solved.

The paper is organized into various sections, namely: Section 2 finds exact solution; Section 3 presents two numerical cases; Section IV, finally, Section 4 concludes with future perspectives.

EXACT SOLUTION OF SYMMETRICAL DEOFRMATION OF TOROIDAL SHELLS OF REVOLUTION

To solve the Eq.1, Wissler [3] introduced a variable transformation, $x = \sin\theta$, thus leads $dx = \cos\theta d\theta$, $\frac{dV}{d\theta} = \frac{dV}{dx}\frac{dx}{d\theta} = \cos\theta\frac{dV}{dx}$ and $\frac{d^2V}{d\theta^2} = \cos^2\theta\frac{d2V}{dx^2} - \sin\theta\frac{dV}{dx} = (1-x^2)\frac{d^2V}{dx2} - x\frac{dV}{dx}$, hence the Eq.1 is transferred into following form

$$(1-x^2)(1+\varepsilon x)\frac{d^2V}{dx^2} - (x+\varepsilon)\frac{dV}{dx} + 2id^2xV = P(x), \tag{2}$$

where $P(x) = -2d^2(2d^2A + \frac{1}{2}i\varepsilon qa)\sqrt{1-x^2}$. The Eq.2 is a Fuchian type differential equation whose series solution was given by Wissler [3].

In order to establish Eq.2 with well-known equations, let's carry on and introducing another variable transformation,

$$\xi = \frac{1}{2}x + \frac{1}{2} = \frac{1}{2}(\sin\theta + 1), \tag{3}$$

thus $\frac{dV}{dx} = \frac{1}{2}\frac{dV}{d\xi}$, $\frac{d^2V}{dx^2} = \frac{1}{4}\frac{d^2V}{d\xi^2}$, and $1-x^2 = -4\xi(\xi-1)$, hence Eq.2 can be transferred into

$$\xi(\xi-1)[\xi - (\frac{1}{2} - \frac{1}{2\varepsilon})]\frac{d^2V}{d\xi^2} + \frac{1}{2\varepsilon}(\xi - \frac{1}{2} + \frac{\varepsilon}{2})\frac{dV}{d\xi} + (-\frac{4id^2}{\varepsilon}\xi + \frac{id^2}{\varepsilon})V = P(\xi), \tag{4}$$

where $P(\xi) = \frac{2d^2}{\varepsilon}(2d^2A + \frac{1}{2}i\varepsilon qa)\sqrt{\xi(1-\xi)}$; and or in another popular format as follows

$$\frac{d^2V}{d\xi^2} + (\frac{\frac{1}{2}}{\xi} + \frac{\frac{1}{2}}{\xi-1} + \frac{-1}{\xi - \frac{\varepsilon-1}{2\varepsilon}})\frac{dV}{d\xi} + \frac{-\frac{2id^2}{\varepsilon}\xi + \frac{id^2}{\varepsilon}}{\xi(\xi-1)(\xi-\frac{\varepsilon-1}{2\epsilon})}V = \frac{2d^2}{\varepsilon}\frac{2d^2A + \frac{1}{2}i\varepsilon aq}{\xi(\xi-1)(\xi-\frac{\varepsilon-1}{2\varepsilon})}. \tag{5}$$

The Eq.4 and/or 5 is a Fuchian type differential equation that has been studied by Heun (1889) [21]. Eq.4 is called general Heun's equation, whose solutions can be represented by the Heun's functions. It is clear that the numerical advantage of solution being expressed by Heun's functions is that software package able to work with the Heun functions, such as MAPLE [25], can be used for calculations, which by the way will sort out the convergency issue of the solution.

The Heun functions (named after Karl Heun: 1859-1929) are unique local Frobenius solutions of a second-order linear ordinary differential equation of the Fuchsian type which in the general case have 4 regular singular points. Heun's equation is an extension of the ${}_2F_1$ hypergeometric equation in that it is a second-order Fuchsian equation with four regular singular points. The ${}_2F_1$ equation has three regular singularities. The HeunG function, thus, contains as particular cases all the functions of the hypergeometric ${}_2F_1$ class [21–23].

The Heun functions generalize the hypergeometric function, the Láme function, Mathieu function [24] and the spheroidal wave functions. Because of the wide range of their applications, they can be considered as the 21st century successors of the hypergeometric functions. [25].

The exact solution of Eq.4 is summation of homogenous solution $V^h(x)$ and particular solution $V^p(x)$, namely, $V = V^h + V^p$, both of them can be expressed by Heun functions. The homogenous solution can be given as follows

$$V^h(x) = C_1y_1(x) + C_2(x+1)^{\frac{1}{2}}y_2(x), \tag{6}$$

where

$$y_1(x) = HeunG(\frac{\varepsilon-1}{2\varepsilon}, -\frac{id^2}{\varepsilon}, -\frac{1}{2}\frac{\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}+\varepsilon}{\varepsilon}, \frac{1}{2}\frac{8id^2+\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}-\varepsilon}{\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}+2\varepsilon}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}(x+1)), \tag{7}$$

$$y_2(x) = HeunG(\frac{\varepsilon-1}{2\varepsilon}, -\frac{1}{8}\frac{8id^2+3\varepsilon+1}{\varepsilon}, -\frac{1}{2}\frac{\sqrt{8id^2+\varepsilon}}{\sqrt{\varepsilon}}, \frac{1}{2}\frac{8id^2+2\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}+\varepsilon}{\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}+2\varepsilon}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}(x+1)). \tag{8}$$

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where the general Heun's function HeunG can be computed for given loading condition.

Substitutes the Wissler's transformation $x = \sin\theta$ into the above solutions, we have

$$V^h(\theta) = C_1 y_1(\theta) + C_2(\sin\theta + 1)^{\frac{1}{2}} y_2(\theta), \tag{9}$$

where

$$y_1(\theta) = HeunG\left(\frac{\varepsilon-1}{2\varepsilon}, -\frac{id^2}{\varepsilon}, -\frac{1}{2}\frac{\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}+\varepsilon}{\varepsilon}, \frac{1}{2}\frac{8id^2+\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}-\varepsilon}{\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}+2\varepsilon}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}(\sin\theta+1)\right), \tag{10}$$

$$y_2(\theta) = HeunG\left(\frac{\varepsilon-1}{2\varepsilon}, -\frac{1}{8}\frac{8id^2+3\varepsilon+1}{\varepsilon}, -\frac{1}{2}\frac{\sqrt{8id^2+\varepsilon}}{\sqrt{\varepsilon}}, \frac{1}{2}\frac{8id^2+2\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}+\varepsilon}{\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}+2\varepsilon}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}(\sin\theta+1)\right). \tag{11}$$

After determination of the auxiliary function $V(\theta)$, all other quantities can be expressed as by the function $V(\theta)$, such as middle surface resultant force T_1, T_2

$$T_1 = -\frac{\varepsilon\cos\theta}{2d^2(1+\varepsilon\sin\theta)} Im(V) + \frac{qa}{2}\frac{2+\varepsilon\sin\theta}{1+\varepsilon\sin\theta} - \varepsilon A\frac{\varepsilon+\sin\theta}{(1+\varepsilon\sin\theta)^2}, \tag{12}$$

$$T_2 = -\frac{1}{2d^2} Im\left[\frac{d}{d\theta}\left(\frac{V}{1+\varepsilon\sin\theta}\right)\right] + \frac{qa}{2} + \varepsilon A\frac{\varepsilon+\sin\theta}{(1+\varepsilon\sin\theta)^2}, \tag{13}$$

resultant moments M_1

$$M_1 = -\frac{h}{2d^2\sqrt{12(1-\mu)}}\frac{\mu\varepsilon\cos\theta}{(1+\varepsilon\sin\theta)^2} Re(V) - \frac{h}{2d^2\sqrt{12(1-\mu)}} Re\left[\frac{d}{d\theta}\left(\frac{V}{1+\varepsilon\sin\theta}\right)\right]. \tag{14}$$

and resultant shear force N_1

$$N_1 = -\frac{h}{a\sqrt{12(1-\mu^2)}}\frac{\sin\theta Im(V) + 2d^2 A\cos\theta}{(1+\varepsilon\sin\theta)^2}, \tag{15}$$

NUMERICAL CASE STUDIES OF SYMMETRICAL DEFORMATION OF TOROIDAL SHELLS

A complete toroidal shells with a penetrate cut along the parallel $\theta = \frac{\pi}{2}$ (or $\theta = -\frac{\pi}{2}$) and loaded with distributed bending moment M_0, as shown in Fig.3

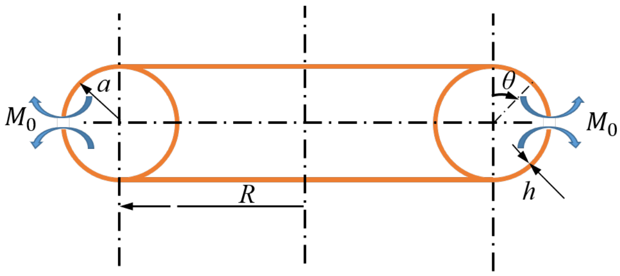

FIG. 3: Toroidal shells with a cut along its parallel at $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$ under a pure moment M_0

The toroidal shells's geometric and material properties data are listed in table below:

TABLE I: Data of slender toroidal shells

R	a	h	E	μ	M_0
m	m	m	N/m^2		N
1	0.1	0.0004	2.07×10^{11}	0.3	M_0

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The bending stiffness $D = \frac{Eh^3}{12(1-\mu^2)} = 1.213186813 \times 10^6$ and middle surface stiffness $K = \frac{Eh}{1-\mu^2} = 9.098901099 \times 10^9$. The boundary is:

$$\varphi = \frac{\pi}{2} : M_1 = M_0,\ N_1 = 0,\ T_1 = 0,\ T_2 = 0. \tag{16}$$

Under the above loading and boundary condition, the constant of integration $A = 0$ and distributed load $q = 0$.

Case 1: Slender toroidal shells with the radius ratio ($\varepsilon = \frac{a}{R} = 0$)

To demonstrate the analytical results. let's apply to a slender toroidal shells, namely $\varepsilon = 0$, with bending moment loading M_0. The governing equation can be reduced further to

$$(1 - x^2)\frac{d^2V}{dx^2} - x\frac{dV}{dx} + i\frac{a\sqrt{12(1-\mu^2)}}{Rh}xV = 0, \tag{17}$$

whose solution is given by

$$V(\theta) = C_1 MathieuC(0, \frac{413067791}{5}, \frac{1}{2}\theta - \frac{\pi}{4}) + C_2 MathieuS(0, \frac{413067791}{5}, \frac{1}{2}\theta - \frac{\pi}{4}), \tag{18}$$

where the transformation $x = \sin\theta$ has been used. With the obtained $V(\theta)$, we can compute other quantifies, such as

$$M_1 = -\frac{Rh^2}{12a^2(1-\mu^2)}Re(\frac{dV}{d\theta}), \quad N_1 = -\frac{h\sin\theta}{a\sqrt{12(1-mu^2)}}Im(V),$$
$$T_1 = 0, \quad T_2 = \frac{Rh\sqrt{12(1-\mu^2)}}{a^2}Im(V), \tag{19}$$

where, Re and Im are real and imaginary part of a complex. Applying the boundary condition in Eq.16, we can find $C_1 = 0$ and $C_2 = -34.125M_0$

Using Maple [25] to compute the Mathieu's functions, we can get the resultant bending moment M_1 and resultant shear force N_1 as follows

$$M_1 = M_0 Re\left(MathieuSPrime(0, 8.2613558i, 0.785398 + \frac{1}{2}\theta),\right) \tag{20}$$

$$N_1 = 8.26M_0 \sin\theta Im\left(MathieuS(0, 8.2613558i, 0.785398 + \frac{1}{2}\theta)\right), \tag{21}$$

$$T_2 = 902.14M_0 Im\left(MathieuSPrime(0, 8.2613558i, 0.785398 + \frac{1}{2}\theta)\right). \tag{22}$$

The resultant bending moment M_1 and resultant shear force N_1 are plotted in Fig.4 as below.

The solution of the slender toroidal shells increase dramatically in the range of $[-\frac{\pi}{2}, \frac{\pi}{8}]$, the slender solution is not valid in the domain, which implies that slender model might be not a proper approximation of toroidal shells.

Case 2: Toroidal shells with the radius ratio ($\varepsilon = \frac{a}{R} = \frac{1}{20}$)

In order to see the influence of the radius ratio $\varepsilon = \frac{a}{R}$, let's investigate a case of $\varepsilon = \frac{1}{20}$. In this case, the governing equation is given as

$$(1 - x^2)(1 + \varepsilon x)\frac{d^2V}{dx^2} - (x + \varepsilon)\frac{dV}{dx} + i\frac{a^2}{Rh}\sqrt{12(1-\mu^2)}xV = 0, \tag{23}$$

whose solution is obtained as follows

$$V(\theta) = C_1 V_1 + 2\sqrt{2}C_2 \cos\theta V_2. \tag{24}$$

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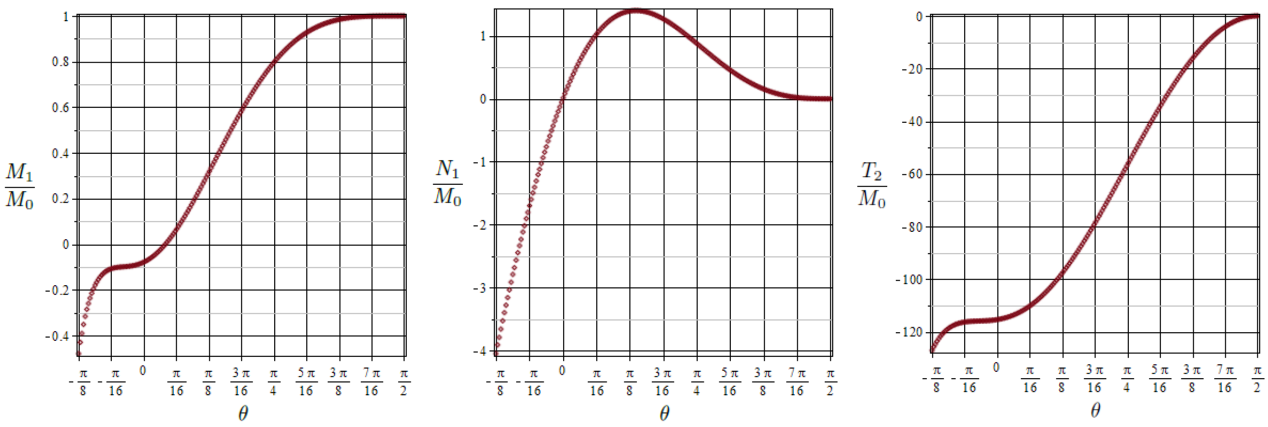

FIG. 4: Case $\varepsilon = 0$: The resultant bending moment M_1, the resultant shear force N_1 and the resultant middle-surface force T_2 vs θ.

where $V_1 = HeunG(-9.5, -41.307i, 5.937+6.4173i, -6.937-6.417i, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\sin\theta + \frac{1}{2})$ and $V_2 = HeunG(-9.5, -2.875-41.307i, 6.437+6.417i, -6.437-6.417i, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\sin\theta+\frac{1}{2})$. The constants of integration C_1 and C_2 can be determined by the boundary, $C_1 = 0$ and $C_2 = 2.9109M_0$. All other quantities can be obtained accordingly. Due to length of their expressions, we only give T_1 as follows

$$T_1 = -5.8218\sqrt{2}M_0 \frac{\cos\theta}{1+\frac{1}{2}\sin\theta} Im(V_2). \tag{25}$$

For the case of $\varepsilon = \frac{1}{20}$, the resultant bending moment M_1, the resultant shear force N_1 and the resultant middle-surface force T_1 vs θ are plotted in Fig.5 below.

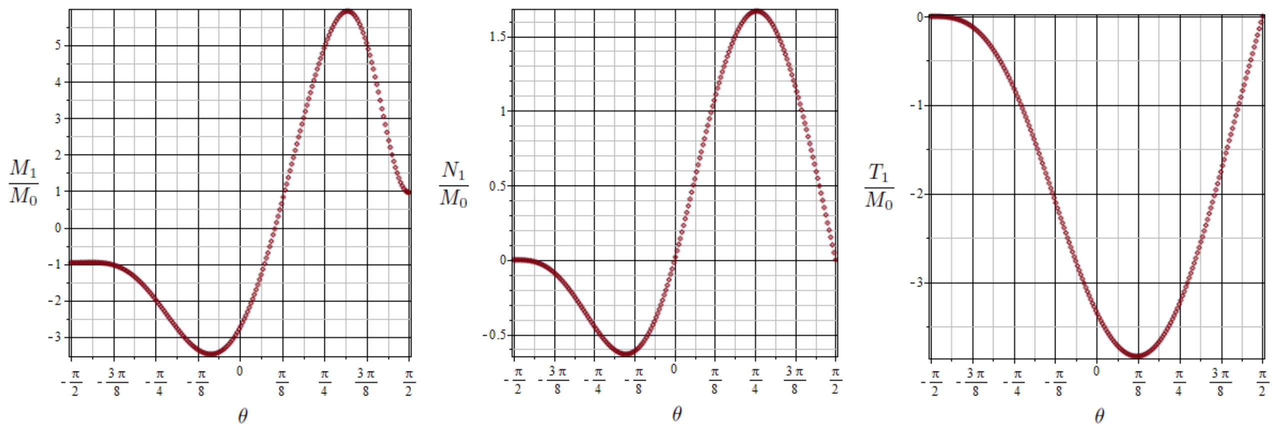

FIG. 5: Case $\varepsilon = \frac{1}{20}$: The resultant bending moment M_1, the resultant shear force N_1 and the resultant middle-surface force T_1 vs θ.

With slight change of the radius ration from 0 to $\frac{1}{20}$, the solution curves in the above figures are much improved, which reveals that the radius ration ε is a vital parameter for .

Comparison study of two case

To evaluate the validity of of slender model od toroida shell, some kind of comparison studies should be carried. For instance, the bending moments of the shells M_1 are plotted in Fig.6. The comparisons strongly indicate the slender

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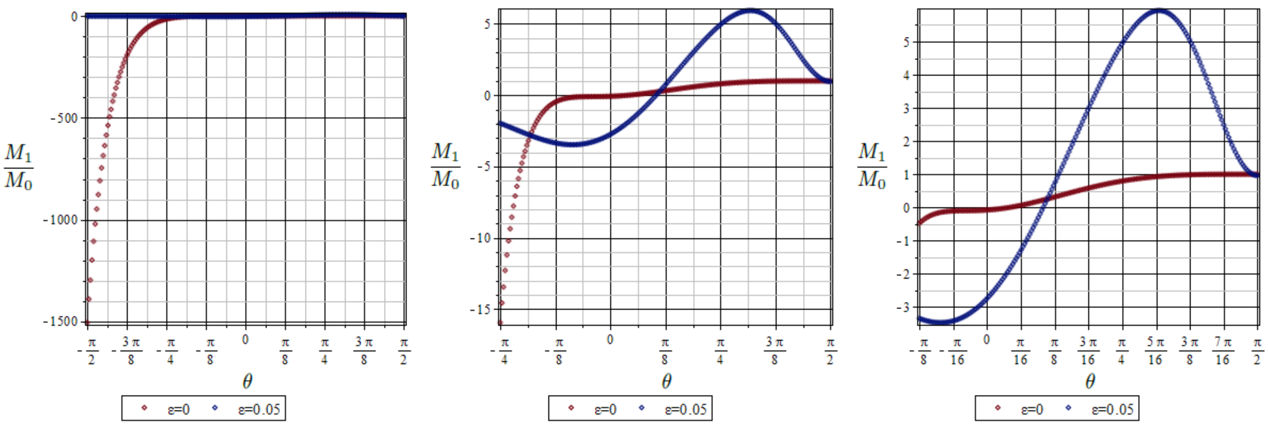

FIG. 6: Comparison study for the resultant bending moment M_1 as ε varies

model is questionable and must be use with care. In other words, the deformation feature of the toirdal shells are quite sensitive to the radius ratio ε.

This geometrical-mechanics sensitively can be utilised in the design of the shells, ie., the desired mechanics of the shells can be archived by slightly change of the radius ratio ε.

SUMMARY AND CONCLUSIONS

We have successfully obtained exact solution of complex form equation for symmetrical deformation of toroidal shells. The research has confirmed that the deformation of all regular shell structure, such cylindrical shell, conical shell, spherical shell and toiroidal shell can be solved by hypergeometric functions. This supports the doctrine of Zurich school of shell theory [1, 2, 7], which predicted that bending deformation of all regular shells can be expressed by the hypergeometric functions. Through numerical comparison study, the mechanics of toroidal shells is quite sensitive to the radius ratio $\varepsilon = \frac{a}{R}$. By slightly adjustment of the ratio might get a desired high performance shell structure.

It must point out here that although we can find the particular solution by Maple, however, its analytical expression can not be obtained due to the fact that integration of the Heun's function can not be expressed in any special functions, which unfortunately decreases the value of the complex form of toroidal shells. Nevertheless, approximation and or numerical treatment of the particular solution can still be possible because it has been obtained and represented by the Heun's function even in its integration form.

Besides the integration difficulties, the complex form governing equations of toroidal shells and even general shells have some disadvantages, such as they are not able to deal with dynamical and buckling problems. This definitely limits the use of the complex form modelling. To avoid the issues of integration difficulty and others, the displacement type equations must be used as Sun [17], where the closed-form solution for displacement type equation of slender toroidal shells has been obtained. However, the displacement solution for the arbitrary toroidal shells is still an open problem, which therefore is the task for the future.

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