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On symmetrical deformation of toroidal shell with circular cross-section

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(Dated: May 7, 2020)

By introducing a variable transformation $\xi = \frac{1}{2}(\sin\theta + 1)$, the complicated deformation equation of toroidal shell is successfully transferred into a well-known equation, namely Heun's equation of ordinary differential equation, whose exact solution is obtained in terms of Heun's functions. The computation of the problem can be carried out by symbolic software that is able to with the Heun's function, such as Maple. The geometric study of the Gauss curvature shows that the internal portion of the toroidal shell has better bending capacity than the outer portion, which might be useful for the design of metamaterials with toroidal shell cells.

Keywords: toroidal shell, deformation, Gauss curvature, Heun's function, hypergeometric function, Maple

Among of most regular shell, such as circular cylindrical shell, conical shell, spherical shell and toroidal shell, the deformation of toroidal shell is one of upmost difficulty problem due to its complicated topology. Up to date, its exact solution has not been obtained yet.

Toroidal shell, in full or partial geometric form as shown in Figures 1 and 2, is widely used in structural engineering and have been extensively investigated [1-72].

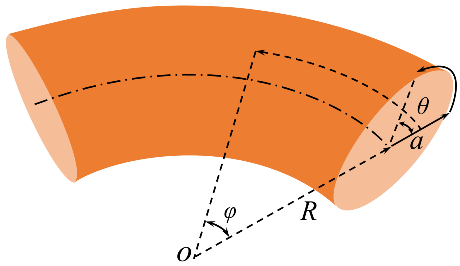

FIG. 1: Toroidal shell and cross-sectional view.

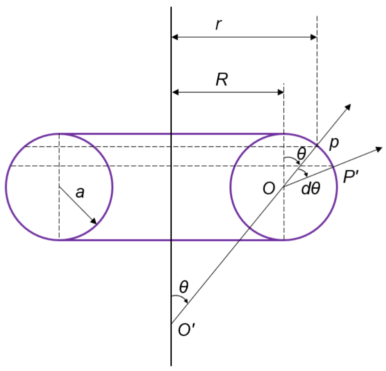

FIG. 2: Toroidal shell and geometry. The principal radius of curvature $R_1 = a$ and $R_2 = a + \frac{R}{\sin\theta}$; the principal curvature $K_1 = \frac{1}{a}$, $K_2 = \frac{\sin\theta}{R + a\sin\theta}$; the Gauss curvature $K = K_1 K_2 = \frac{\sin\theta}{a(R+a\sin\theta)}$.

The Gauss curvature K changes its sign as principal radius of curvature R_θ when the angle θ goes from 0 to 2π, it

 doi:10.20944/preprints202003.0156.v1

means that Gauss curvature has a turning point, namely $K = 0$ at $\theta = \pi$ as shown in Fig. 3. The exist of the turning point in a complete toroidal shell is one source of the difficulty to find a solution.

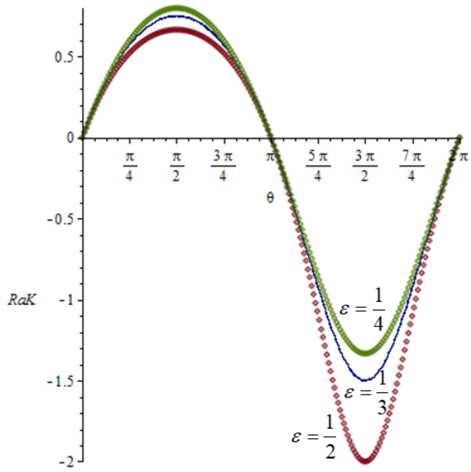

FIG. 3: The curves of $RaK = \frac{\sin\theta}{1+\varepsilon\sin\theta}$, where $\varepsilon = \frac{a}{R}$. The geometry of toroidal shell surface is elliptic in $\theta \in [0, \pi]$, parabolic at $\theta = 0$ and hyperbolic $\theta \in [0, -\pi]$.

Since the bending capacity of the shell is proportional to the Gauss curvature, the Fig.3 reveals that the internal part of toroidal shell ($\theta \in [\pi, 2\pi]$) is much stronger than outer part of the shell ($\theta \in [0, \pi]$). Therefore, for given amount of materials, to construct a high bending performance toroidal shell, the internal toroidal shell topology is the best choice. This secret mechanical performance might be useful to the design of metamaterials wuth toroidal shell cell.

To attack the problem, the high order and complicated governing equations of toroidal shell under symmetric loads is reduced to a single equation of lower order complex equation. The complex form governing equations of toroidal shell of revolution were formulated firstly by Reissner (1912)[1] and later finalized by Novozhilov (1959) [11] a follows:

$$(1 + \varepsilon \sin\theta)\frac{d^2V}{d\theta^2} - \varepsilon\cos\theta\frac{dV}{d\theta} + 2id^2\sin\theta V = P(\theta). \qquad (1)$$

Eq.1 is called Reissner-Novozhilov’ s equation of toroidal shell, in which, $P(\theta) = -2d^2(2d^2A + \frac{1}{2}i\varepsilon qa)\cos\theta$, $2d^2 = \frac{a^2}{Rh}\sqrt{12(1-\mu^2)}$, μ is the Poisson’s ratio, h is thickness, and q is distributed loads, A is an integration constant.

Beside the turning point issue, it is clear that the challenge is also comes from the variable coefficients of the differential equation in Eq.1.

Nevertheless, some kind solutions have been proposed successfully. The first exact series solution of toroidal equations was obtained by Wissler (1916), however, Wissler’s series solution has little practical value due to its slow convergency and not established a linkage with any special known functions as well.

To provide practical solution, thus a various of asymptotic solutions have been proposed [5, 11, 27], however, the turning point of the Gauss curvature makes all proposed asymptotic solutions invalid near the point. Up to date, no exact solution in terms of special functions has been obtained for symmetrical deformation of toroidal shell, except in the case of slender toroidal shell, whose displacement type closed-form solution is obtained by Sun (2011) [72].

To find an exact solution that can be expressed in a special function is still a open question eben after more than centenary development of the theory of toroidal shell. In this paper, we will shoulder this historical responsibility and propose an exact solution in terms of special functions. Once we obtain the solution in terms of well-known special functions, the convergency issue will be solved.

To solve the Eq.1, Wissler [3] introduced a variable transformation, $x = \sin\theta$, thus leads $dx = \cos\theta d\theta$, $\frac{dV}{d\theta} = \frac{dV}{dx}\frac{dx}{d\theta} = \cos\theta\frac{dV}{dx}$ and $\frac{d^2V}{d\theta^2} = \cos^2\theta\frac{d2V}{dx^2} - \sin\theta\frac{dV}{dx} = (1-x^2)\frac{d^2V}{dx2} - x\frac{dV}{dx}$, hence the Eq.1 is transferred into following

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form

$$(1-x^2)(1+\varepsilon x)\frac{d^2V}{dx^2}-(x+\varepsilon)\frac{dV}{dx}+2id^2xV=P(x), \tag{2}$$

where $P(x)=-2d^2(2d^2A+\frac{1}{2}i\varepsilon qa)\sqrt{1-x^2}$. The Eq.2 is a Fuchian type differential equation whose series solution was given by Wissler [3].

In order to establish Eq.2 with well-known equations, let's carry on and introducing another variable transformation,

$$\xi=\frac{1}{2}x+\frac{1}{2}=\frac{1}{2}(\sin\theta+1), \tag{3}$$

thus $\frac{dV}{dx}=\frac{1}{2}\frac{dV}{d\xi}$, $\frac{d^2V}{dx^2}=\frac{1}{4}\frac{d^2V}{d\xi^2}$, and $1-x^2=-4\xi(\xi-1)$, hence Eq.2 can be transferred into

$$\xi(\xi-1)[\xi-(\frac{1}{2}-\frac{1}{2\varepsilon})]\frac{d^2V}{d\xi^2}+\frac{1}{2\varepsilon}(\xi-\frac{1}{2}+\frac{\varepsilon}{2})\frac{dV}{d\xi}+(-\frac{4id^2}{\varepsilon}\xi+\frac{id^2}{\varepsilon})V=P(\xi), \tag{4}$$

where $P(\xi)=\frac{2d^2}{\varepsilon}(2d^2A+\frac{1}{2}i\varepsilon qa)\sqrt{\xi(1-\xi)}$; and or in another popular format as follows

$$\frac{d^2V}{d\xi^2}+(\frac{\frac{1}{2}}{\xi}+\frac{\frac{1}{2}}{\xi-1}+\frac{-1}{\xi-\frac{\varepsilon-1}{2\varepsilon}})\frac{dV}{d\xi}+\frac{-\frac{2id^2}{\varepsilon}\xi+\frac{id^2}{\varepsilon}}{\xi(\xi-1)(\xi-\frac{\varepsilon-1}{2\epsilon})}V=\frac{2d^2}{\varepsilon}\frac{2d^2A+\frac{1}{2}i\varepsilon aq}{\xi(\xi-1)(\xi-\frac{\varepsilon-1}{2\varepsilon})}. \tag{5}$$

The Eq.4 and/or 5 is a Fuchian type differential equation that has been studied by Heun (1889) [73–75]. Eq.4 is called general Heun's equation, whose solutions can be represented by the Heun's functions. It is clear that the numerical advantage of solution being expressed by Heun's functions is that software package able to work with the Heun functions, such as MAPLE [76], can be used for calculations, which by the way will sort out the convergency issue of the solution.

The Heun functions (named after Karl Heun:1859-1929) are unique local Frobenius solutions of a second-order linear ordinary differential equation of the Fuchsian type which in the general case have 4 regular singular points. Heun's equation is an extension of the ${}_2F_1$ hypergeometric equation in that it is a second-order Fuchsian equation with four regular singular points. The ${}_2F_1$ equation has three regular singularities. The HeunG function, thus, contains as particular cases all the functions of the hypergeometric ${}_2F_1$ class [76].

The Heun functions generalize the hypergeometric function, the Láme function, Mathieu function and the spheroidal wave functions. Because of the wide range of their applications, they can be considered as the 21st century successors of the hypergeometric functions. [75].

The exact solution of Eq.4 is summation of homogenous solution $V^h(x)$ and particular solution $V^p(x)$, namely, $V=V^h+V^p$, both of them can be expressed by Heun functions. The homogenous solution can be given as follows

$$V^h(x)=C_1y_1(x)+C_2(x+1)^{\frac{1}{2}}y_2(x), \tag{6}$$

where

$$y_1(x)=\mathrm{HeunG}(\frac{\varepsilon-1}{2\varepsilon},-\frac{id^2}{\varepsilon},-\frac{1}{2}\frac{\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}+\varepsilon}{\varepsilon},\frac{1}{2}\frac{8id^2+\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}-\varepsilon}{\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}+2\varepsilon},\frac{1}{2},\frac{1}{2},\frac{1}{2}(x+1)), \tag{7}$$

$$y_2(x)=\mathrm{HeunG}(\frac{\varepsilon-1}{2\varepsilon},-\frac{1}{8}\frac{8id^2+3\varepsilon+1}{\varepsilon},-\frac{1}{2}\frac{\sqrt{8id^2+\varepsilon}}{\sqrt{\varepsilon}},\frac{1}{2}\frac{8id^2+2\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}+\varepsilon}{\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}+2\varepsilon},\frac{3}{2},\frac{1}{2},\frac{1}{2}(x+1)). \tag{8}$$

where the general Heun's function HeunG can be easily computed by well-know symbolic software package Maple [76].

Substitute the Wissler's transformation $x=\sin\theta$ into the above solutions, we have

$$V^h(\theta)=C_1y_1(\theta)+C_2(\sin\theta+1)^{\frac{1}{2}}y_2(\theta), \tag{9}$$

where

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$$y_1(\theta) = \mathrm{HeunG}(\frac{\varepsilon-1}{2\varepsilon}, -\frac{id^2}{\varepsilon}, -\frac{1}{2}\frac{\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}+\varepsilon}{\varepsilon}, \frac{1}{2}\frac{8id^2+\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}-\varepsilon}{\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}+2\varepsilon}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}(\sin\theta+1)), \tag{10}$$

$$y_2(\theta) = \mathrm{HeunG}(\frac{\varepsilon-1}{2\varepsilon}, -\frac{1}{8}\frac{8id^2+3\varepsilon+1}{\varepsilon}, -\frac{1}{2}\frac{\sqrt{8id^2+\varepsilon}}{\sqrt{\varepsilon}}, \frac{1}{2}\frac{8id^2+2\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}+\varepsilon}{\sqrt{\varepsilon}\sqrt{8id^2+\varepsilon}+2\varepsilon}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}(\sin\theta+1)). \tag{11}$$

After determination of the auxiliary function $V(\theta)$, all other quantities can be expressed as by the function $V(\theta)$, such as middle surface resultant force T_1, T_2

$$T_1 = -\frac{\varepsilon\cos\theta}{2d^2(1+\varepsilon\sin\theta)}Im(V) + \frac{qa}{2}\frac{2+\varepsilon\sin\theta}{1+\varepsilon\sin\theta} - \varepsilon A\frac{\varepsilon+\sin\theta}{(1+\varepsilon\sin\theta)^2}, \tag{12}$$

$$T_2 = -\frac{1}{2d^2}Im\left[\frac{d}{d\theta}(\frac{V}{1+\varepsilon\sin\theta})\right] + \frac{qa}{2} + \varepsilon A\frac{\varepsilon+\sin\theta}{(1+\varepsilon\sin\theta)^2}, \tag{13}$$

resultant moments M_1

$$M_1 = -\frac{h}{2d^2\sqrt{12(1-\mu)}}\frac{\mu\varepsilon\cos\theta}{(1+\varepsilon\sin\theta)^2}Re(V) - \frac{h}{2d^2\sqrt{12(1-\mu)}}Re\left[\frac{d}{d\theta}(\frac{V}{1+\varepsilon\sin\theta})\right]. \tag{14}$$

and resultant shear force N_1

$$N_1 = -\frac{h}{a\sqrt{12(1-\mu^2)}}\frac{\sin\theta Im(V) + 2d^2A\cos\theta}{(1+\varepsilon\sin\theta)^2}, \tag{15}$$

the angle of rotation of the tangent to the meridian ϑ,

$$\vartheta = -\frac{1}{Eh}\frac{Re(V)}{\varepsilon(1+\varepsilon\sin\theta)}, \tag{16}$$

the displacement component $\triangle_z$ of an arbitrary point on the meridian in the direction of the axis of the shell

$$\triangle_z = -a\int_0^\theta \vartheta\cos\theta d\theta + C, \tag{17}$$

and the displacement component $\triangle_z$ in the direction perpendicular to the axis

$$\triangle_x = \frac{R}{Eh}(1+\varepsilon\sin\theta)(T_2-\mu T_1). \tag{18}$$

It must point out here that although we can find the particular solution by Maple, however, its analytical expression can not be obtained due to the fact that integration of the Heun's function can not be expressed in any special functions, which unfortunately decreases the value of the complex form of toroidal shell. Nevertheless, approximation and or numerical treatment of the particular solution can still be possible because it have been obtained and represented by the Heun's function even in its integration form.

This research has confirmed that the deformation of all regular shell structure, such cylindrical shell, conical shell, spherical shell and toiroidal shell can be solved by hypergeometric functions. This supports the doctrine of Zurich school of shell theory [1, 2, 11], which predicted that bending deformation of all regular shells can be expressed by the hypergeometric functions.

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Appendix: Heun's function $HeunG$

Heun's of differential equation

$$\frac{d^2}{dz^2}y(z) + \frac{((\alpha+\beta+1)z^2 + ((-\delta-\gamma)c - \alpha + \delta - \beta - 1)z + \gamma\, c)\frac{d}{dz}y(z)}{z(z-1)(z-c)} + \frac{(z\alpha\,\beta - p)y(z)}{z(z-1)(z-c)} = 0$$

its solution

$$y(z) = C_1 HeunG(c,p,\alpha,\beta,\gamma,\delta,z) + C_2 z^{1-\gamma} HeunG(c, p-(\gamma-1)((c-1)\delta+\alpha+\beta-\gamma+1), \beta-\gamma+1, \alpha-\gamma+1, 2-\gamma, \delta, z)$$