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Suppressing the Gibbs effect on restored images

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Abstract: In the work, the problem is considered for eliminating mathematically a non-uniform rectilinear smearing of an image, for example, a picture obtained by a fixed camera of several cars moving at different speeds. The problem is described by a set of 1-dimensional integral equations (IEs) of a general type (not a convolution type) with a 2-dimensional point spread function (PSF) or one 2-dimensional IE with 4-dimensional PSF. IEs are solved by the quadrature/cubature method with the Tikhonov regularization. It is shown that in the case of non-uniform smearing, the use of a set of 1-dimensional IEs is preferably of one 2-dimensional IE. In the inverse problem (image restoration), the Gibbs effect (the effect of false waves) in the image can take place. It can be the edge and inner. The edge effect is well suppressed by the technique "diffusing the edges" (option 'diffusion'). In the case of an inner effect, it is difficult to eliminate it, but image smearing itself plays the role of diffusing and suppresses the Gibbs effect. Illustrative results are given.

Keywords: smeared image, non-uniform rectilinear smear, integral equations, edge and inner Gibbs effects, MatLab

Type of presentation: oral presentation.

Introduction. Consider one of the actual tasks of distorted images processing: the elimination of image smearing by mathematical methods [1–6]. Smearing may be due to the shift of the image registration device (photo camera, video camera, tracking device, etc.) or due to the movement of the subject (person, car, airplane) during the exposure. The problem of mathematical elimination of smearing consists of two tasks: a direct problem (modeling of smearing) and an inverse problem (elimination of smearing).

In this paper, we focus on the rarely considered image smearing, which is rectilinear, but it is non-uniform in speed when the camera or object moves during the exposure. Also, we focus on various types of integral equations used in the inverse problem. We pay special attention to the Gibbs effect (the effect of false waves), which often occurs in reconstructed images.

In many publications, a variant is considered for a uniform rectilinear image smearing [1, 4, 6, 7], as well as a variant of an arbitrary (non-uniform curvilinear) smearing by the method of "blind" deconvolution [8, p. 192], [9], but the intermediate version of an non-uniform rectilinear smearing is considered in less detail [2, 10, 11].

The purpose of this work is a comparative consideration of two variants for straight-line image smearing (uniform and non-uniform) and clarification of influence of the Gibbs effect on the quality of the reconstructed image.

Example: a smeared image of objects moving at different speeds, obtained by a fixed camera. Note that in [2], [10, p. 160–164], the case is considered when the camera (or object) moved rectilinearly at a certain speed $v(t)$ during the exposure, where t is time. In this paper and in [11], the case of smear $\Delta(x)$ is considered, where x is the spatial coordinate.

First, let us recall the well-known case of uniform smearing [4, 6, 7, 10].

Mathematical description of uniform rectilinear smearing of the image. Consider the direct and inverse problems.

The direct problem [6, 7, 11] of uniform rectilinear smearing is described by the integral

$$g_y(x) = \frac{1}{\Delta} \int_x^{x+\Delta} w_y(\xi)\, d\xi, \qquad (1)$$

where $\Delta = \text{const}$ is the length of smear; the x and ξ axes are directed along the smearing, and the y axis is perpendicular to the smearing (acts as a parameter); w_y is original unsmeared image, and g_y is calculated (modeled) smeared image in each y-row. To calculate g according to (1), we developed: the main program Autos.m, m-functions smearing.m [6] (for an arbitrary smear angle θ) and smear.m [11] (for $\theta = 0$). In the MatLab system there are m-functions fspecial.m and imfilter.m for modeling g [8].

The inverse problem. The inverse (more important and complex) problem can be solved by two approaches.

In the *first approach*, to eliminate the smearing, a set of 1-dimensional Fredholm integral equations of the first kind of convolutional type (for each value of y) is solved [6, 7]:

$$\int_{-\infty}^{\infty} h(x-\xi)\, w_y(\xi)\, d\xi = g_y(x), \quad -\infty < x < \infty, \tag{2}$$

where

$$h(x) = \begin{cases} 1/\Delta, & -\Delta \le x \le 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

IE (2) is obtained from relation (1); the x and ξ axes are directed along the smearing; h is the mathematical kernel of the IE, and physically and technically it is the point spread function (PSF) [3, 4, 6, 9]. The function h usually is differential function, or spatially invariant function, which means that the smearing is uniform and the length of smear Δ is the same at all points of the image ($\Delta = \text{const}$).

The problem of solving IE (2) is incorrect [12, 13]. We use the stable method of Tikhonov's regularization (TR) with the Fourier transform (FT) [4, 6, 7, 14]:

$$w_{\alpha y}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W_{\alpha y}(\omega)\, e^{-i\omega\xi}\, d\omega \ , \tag{4}$$

where

$$W_{\alpha y}(\omega) = \frac{H(-\omega)\, G_y(\omega)}{|H(\omega)|^2 + \alpha\, \omega^{2p}} \tag{5}$$

is the regularized Fourier spectrum, or FT of solution; $H(\omega) = F(h(x))$ and $G_y(\omega) = F(g_y(x))$ are Fourier spectra of functions $h(x)$ and $g_y(x)$, where F is the sign of FT; $\alpha > 0$ is regularization parameter; $p \ge 0$ is regularization order (usually $p = 1$ or 2). Several ways have been developed for choosing the regularization parameter α: the discrepancy principle, the method of training examples, the selection method, etc. [6, 10, 12, 14]. To calculate the reconstructed image using formulas (4)–(5), we developed the m-function desmearingf.m [6].

In the *second approach*, to eliminate the smearing (as well as defocusing), a 2-dimensional Fredholm integral equations of the first kind of convolution type is used (cf. (2)) [6, 7, 10, 11]:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-\xi, y-\eta)\, w(\xi,\eta)\, d\xi\, d\eta = g(x,y), \quad -\infty < x, y < \infty, \tag{6}$$

moreover, the x and ξ axes are directed horizontally, and y and η are directed vertically downward. PSF h is displayed on the plane (x, y) as a narrow strip [6, p. 112].

In this approach, the direct problem is calculated using the m-functions fspecial.m and imfilter.m [8]. And the solution of the 2-dimensional IE (6) (the inverse problem) by the TR method and the 2-dimensional FT is equal to $w_\alpha(x,y) = F^{-1}(W_\alpha(\omega_1,\omega_2))$, where F^{-1} is the inverse Fourier transform (IFT), or

$$w_\alpha(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_\alpha(\omega_1,\omega_2)\, e^{-i(\omega_1 x + \omega_2 y)}\, d\omega_1\, d\omega_2 \ . \tag{7}$$

In (7) $W_\alpha(\omega_1,\omega_2)$ is the regularized spectrum (2-dimensional FT) of the solution, equal to

$$W_{\alpha}(\omega_1,\omega_2)=\frac{H^*(\omega_1,\omega_2)G(\omega_1,\omega_2)}{|H(\omega_1,\omega_2)|^2+\alpha(\omega_1^2+\omega_2^2)^p}, \tag{8}$$

where $H(\omega_1,\omega_2)=F\big(h(x,y)\big)$, $G(\omega_1,\omega_2)=F\big(g(x,y)\big)$. The MatLab contains the m-function deconvreg.m [8] for solving IE (6) by the TR and FT methods according to (7)–(8).

We give the well-known formulas (1)–(8) in order to compare the various approaches below.

Mathematical description of non-uniform rectilinear smearing of the image. Taking into account the formulas (1)–(8), consider non-uniform rectilinear smearing of the image along the direction of the smearing. Let us consider *two approaches*.

The first (temporal) approach [2, 10]. In this approach, the speed of object (or camera) moving is assumed to be known as a function $v(t)$ of time $t\in[0,\tau]$, where τ is the exposure time. This approach was considered in detail in [15], and we will consider

The second (spatial) approach [11]. Let the dependence $\Delta=\Delta(x)$ of the smear Δ on the x coordinate, directed along the smearing, be determined from the smeared image by some way, for example, by the method of the "blind" deconvolution [9] or by the spectral method [7].

The direct problem. In this case, the PSF h is not differential, or spatially invariant, and the direct problem is written in the form (cf. (1)):

$$g_y(x)=\frac{1}{\Delta(x)}\int_x^{x+\Delta(x)} w_y(\xi)\,d\xi\,. \tag{9}$$

To calculate g according to (9), we developed the m-function smear_n.m [11].

The inverse problem. *The inverse problem* in the case of the *second approach* is written in the form of a set of 1-dimensional Fredholm integral equations of the first kind of general type (not convolutional type) for each value y [6, c. 125]:

$$Aw_y\equiv\int_a^b h(x,\xi)\,w_y(\xi)\,d\xi=g_y(x),\quad c\le x\le d\,, \tag{10}$$

where A is integral operator; $[a,b]$ and $[c,d]$ are limits for ξ and x. PSF h is written as

$$h(x,\xi)=\begin{cases}1/\Delta(x), & x\le\xi\le x+\Delta(x),\\ 0, & \text{otherwise.}\end{cases} \tag{11}$$

To solve the IE (10), one cannot apply the FT, since the IE (10) is not a convolutional type equation, but the *quadrature method* can be applied, which reduces the IE (10) to a system of linear algebraic equations (SLAE) for each y [6, p. 126]:

$$Aw_y=g_y\,, \tag{12}$$

where A is a matrix, associated with h (the same for all y-rows), w_y is the desired vector, g_y is the right-hand side of the SLAE. A stable solution of SLAE (12) is provided by the Tikhonov regularization method [6, p. 126]:

$$(\alpha I+A^TA)\,w_{y\alpha}=A^Tg_y\,, \tag{13}$$

where $\alpha>0$ is the regularization parameter, I is the identity matrix, A^T is the transposed matrix, and $w_{y\alpha}$ is the regularized solution in the y-row, equal to

$$w_{y\alpha}=(\alpha I+A^TA)^{-1}A^Tg_y\,. \tag{14}$$

For the computer realization of the formulas (10)–(14), we developed the m-function desmearq_n.m [11]. Note that the quadrature method with Tikhonov's regularization (12)–(14) can be used also to solve IE of convolution type (2) with PSF (3), i.e. for uniform smearing. For this, the m-function desmearq.m has been developed.

The *inverse problem* in the framework of the *second approach* can be written also in the form of a 2-dimensional Fredholm IE of the first kind of general type [14] (cf. (6), (12)):

$$Aw \equiv \int_a^b \int_c^d h(x,\xi,y,\eta)\, w(\xi,\eta)\, d\xi\, d\eta = g(x,y), \quad a \le x \le b, \quad c \le y \le d\,. \tag{15}$$

Equation (15) can be solved by *the quadrature method* (more precisely, *cubature*) (cf. [14, p. 167]). According to this method, each of the integrals in (15) is replaced by a finite sum on discrete grids of nodes in x, ξ, y, η and we obtain a SLAE with a 4-dimensional matrix A and a 2-dimensional right-hand side g. To solve such a SLAE, it is necessary to transform the 4-dimensional matrix A into a 2-dimensional one, 2-dimensional right-hand side g to transform into a 1-dimensional one, and to transform the resulting 1-dimensional solution w into 2-dimensional one. Although solving a 2-dimensional IE by the cubature method took place [14, p. 167–169], nevertheless, this is a cumbersome method and its application to restore a non-uniform smeared image is difficult.

As a result, it should be recognized that the most effective methodology in the case of non-uniform smearing is the methodology (10)–(14), based on line-by-line image processing by solving for each y 1-dimensional IE (10) and SLAE (12) with 2-dimensional matrix.

Illustrative example. The following *numerical example* was solved. Fig. 1 shows color (RGB) and grayscale (gray) initial image I of three cars, file Autos.png. For further processing, we selected image I (143×1307 pixels).

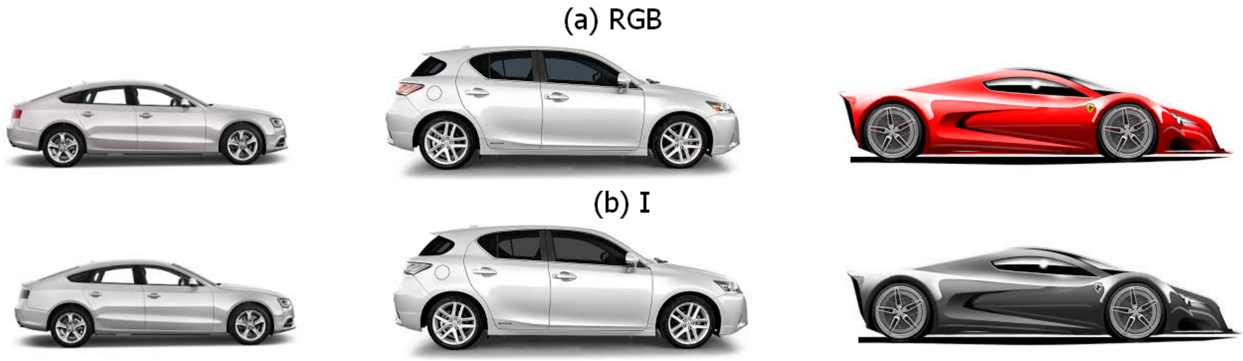

Fig. 1. Initial (undistorted) images of stationary cars

Direct and inverse problems of uniform smearing with diffusing the image edges. Consider the case when cars move, and move with the same speeds and therefore give the same smears on the image: $\Delta = \text{const} = 20$ pixels (Fig. 2a).

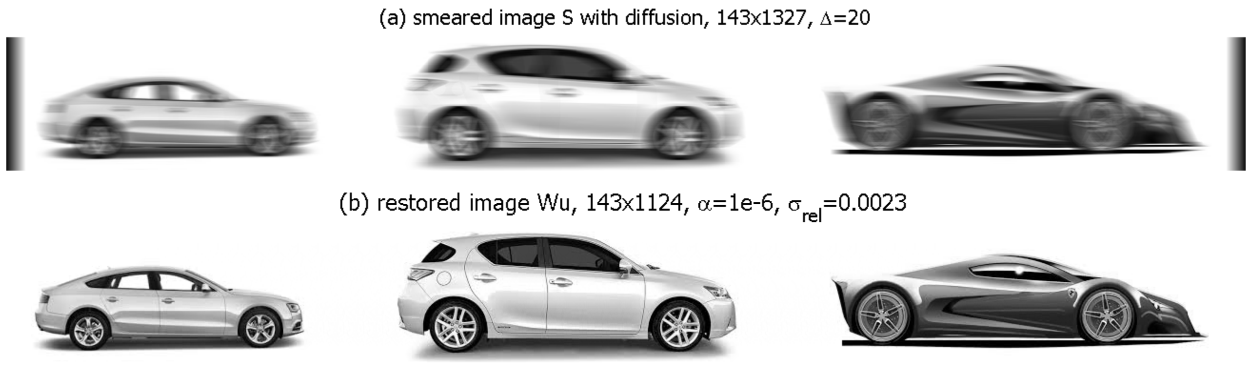

Fig. 2. Direct and inverse problems of uniform smearing of the image with diffusing the edges (option 'diffusion'). a – smeared image 143 × 1327 ($\Delta = 20$ pixels); b – image 143 × 1124 restored by the TR method ($\alpha = 10^{-6}$), $\sigma_{\text{rel}} = 0.0023$.

When solving the inverse problem, false waves often appear on the reconstructed image – *the Gibbs effect* (distortions of the "ringing" type) [11]. It is caused by a sharp difference of intensities on the image and can be edge and inner one.

To suppress the Gibbs edge effect, we proposed [16] an artificial diffusing the image edges in a direct problem. Fig. 2a shows a smeared image according to (1) with diffuse edges by the m-function smear.m, and Fig. 2b shows the result of restorating a car image by the quadrature method with Tikhonov regularization according to (12)–(14) by the m-function desmearq.m.

To numerically estimate the restoration quality, we propose the following formula for the quantitative calculation of the relative error in the form of the standard deviation of the calculated image $\tilde{\overline{w}}$ from the exact image $\overline{w}$ [7]:

$$\sigma_{\text{rel}} = \frac{\| \tilde{\overline{w}} - \overline{w} \|_{L_2}}{\| \overline{w} \|_{L_2}} = \frac{\sqrt{\sum_{j=1}^{M}\sum_{i=1}^{N}\left(\tilde{\overline{w}}_{ji} - \overline{w}_{ji}\right)^2}}{\sqrt{\sum_{j=1}^{M}\sum_{i=1}^{N}\overline{w}_{ji}^2}}, \tag{16}$$

where M is the number of rows and N is the number of columns in the image. Such an expression for the image error can be used only in the case of model image processing when $\overline{w}$ is known (image on Fig. 1b). Image error in Fig. 2b found equal $\sigma_{\text{rel}} = 0.0023$, i.e. the image was restored well and this was facilitated by diffusing the image edges for suppressing the Gibbs edge effect.

Direct and inverse problems of non-uniform image smearing. The next step is *non-uniform image smearing*. We believe that cars move with different speeds and therefore they have different smears on the image, namely, $\Delta = 15$ pixels for the left car, $\Delta = 20$ pixels for the middle car and $\Delta = 25$ pixels for the right car.

As a result, smear $\Delta(x)$ is a piecewise constant function:

if (i<=360) Δ=15; elseif (i<=820) Δ=20; else Δ=25; (17)

where i = 1 ... 1307 is the number of the discrete reference along *x*.

Fig. 3a shows a non-uniformly smeared image according to (9) and (17) with diffusing the edges for suppressing the Gibbs edge effect using m-function smear_n.

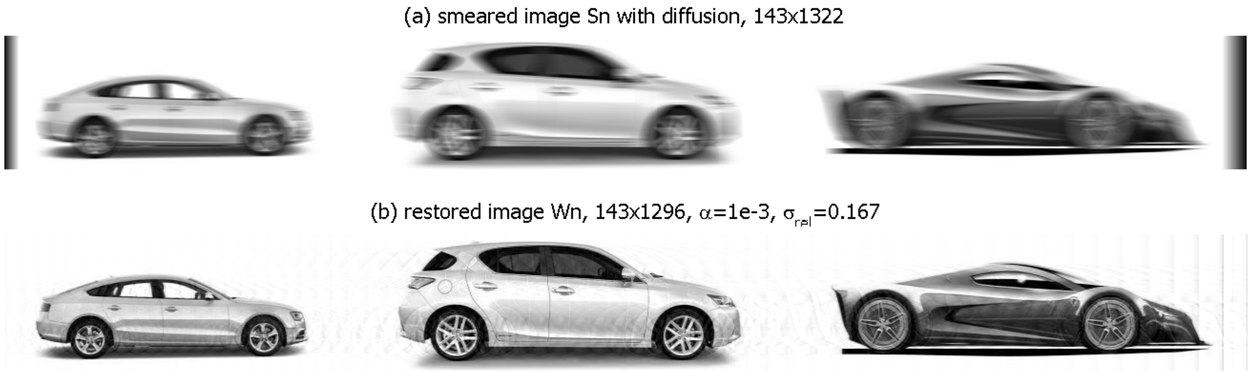

Fig. 3. Direct and inverse problems of non-uniform image smearing with diffused edges ('diffusion'). a – smeared image 143 × 1322 ($\Delta \neq$ const); b – restored image 143 × 1296 by the TR method ($\alpha = 10^{-3}$), $\sigma_{\text{rel}} = 0.167$.

Fig. 3b shows the result of image restoration by the Tikhonov regularization method according to (14) using developed m-function desmearq_n.m. The image is restored well, but with a small inner Gibbs effect between the cars on Fig. 2b. It is difficult to eliminate the inner Gibbs effect as well as the edge effect, but the image smearing itself, which reduces the intensity difference, plays the role of diffusing and suppresses the Gibbs effect.

Conclusion. The technique is described for restorating smeared images in the case when the smearing is non-uniform rectilinear, namely piecewise uniform one (the example: cars on a road). In this case, it is necessary to solve a set of one-dimensional integral equations (the first approach) or one two-dimensional IE (the second approach). In both approaches, the equations are not convolution type IEs, so they are solved by the quadrature/cubature method with Tikhonov's regularization. It is shown that the first approach is more preferable than the second approach. Numerical examples have confirmed this.

It was also shown that to improve the image restoration quality, the diffusing of image edges should be used to suppress the edge Gibbs effect (false-wave effect). The Gibbs effect can also be inner. In this case, it is suppressed by image smearing, which reduces the intensity difference.

The technique can be used in practice to restore group images of several objects (people, planes, cars) moving with different speeds and therefore receiving different smears Δ on the image during the exposure by a fixed camera.

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