Gilbreath sequences and proof of conditions for Gilbreath conjecture

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Abstract

The conjecture attributed to Norman L. Gilbreath, but formulated by Francois Proth in the second half of the 1800s, concerns an interesting property of the ordered sequence of prime numbers P. Gilbreath conjecture stated that, if we compute the absolute values of differences of consecutive primes on ordered sequence of prime numbers, and if this calculation is repeated for the terms in the new sequence and so on, every sequence will start with 1. In this paper is defined the concept of Gilbreath sequence, Gilbreath triangle and Gilbreath equation. On the basis of the results obtained from the proof of properties, an inductive proof is produced thanks to which it is possible to establish the necessary condition to state that Gilbreath conjecture is true.

1 Introduction to Gilbreath conjecture

Let the ordered sequence $P = (2, 3, 5, 7, 11, 13, 17, ...) = (p_1, ..., p_n, ...)$ formed by prime numbers, and set

$$p_a^b = \begin{cases} p_{a+1} - p_a & \text{if } b = 1 \\ |p_{a+1}^{b-1} - p_a^{b-1}| & \text{otherwise} \end{cases} \tag{1}$$

where $b \in [1; n-1]$. Gilbreath conjectured that $p_1^b = 1$. It is likely that this conjecture is satisfied by many other sequences of integers, so it is necessary to define the general properties of all sequences that satisfy this conjecture.

2 Gilbreath sequence

Definition 1. *Let a finite sequence of integers* $S = (s_1, s_2, s_3, \ldots, s_n)$ *and*

$$s_a^b = \begin{cases} s_{a+1} - s_a & \text{if } b = 1 \\ |s_{a+1}^{b-1} - s_a^{b-1}| & \text{otherwise} \end{cases} \tag{2}$$

where $b \in [1; n-1]$. S *is defined Gilbreath sequence if* $s_1^b = 1 \forall b$.

Let, for example, $S = (2, 3, 5, 7, 11, 13, 17)$ a sequence of length $n = 7$, Gilbreath triangle of S is defined by (2). Hence

$$\begin{array}{llllllll} s_1 & s_2 & s_3 & s_4 & \ldots & s_{n-3} & s_{n-2} & s_{n-1} & s_n \\ s_1^1 & s_2^1 & s_3^1 & s_4^1 & \ldots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 \\ \ldots \\ s_1^{n-2} & s_2^{n-2} \\ s_1^{n-1} \end{array}$$

with values

$$\begin{array}{llllll} 1 & 2 & 2 & 4 & 2 & 4 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 \\ 1 \end{array}$$

We denote by $\mathbb{G}_n$ the set of all Gilbreath sequences of length n. In our example the first term of every sequence is equal to 1, hence $S \in \mathbb{G}_7$.

Lemma 1. *Let* $S = (s_1, ..., s_n) \in \mathbb{G}_n$ *and* $S' = (s_1, ..., s_{n-1})$ *finite sequences of integers, then* $S' \in \mathbb{G}_{n-1}$.

Proof. Gilbreath triangle of S

$$\begin{array}{llllllll} s_1 & s_2 & s_3 & s_4 & \ldots & s_{n-3} & s_{n-2} & s_{n-1} & s_n \\ s_1^1 & s_2^1 & s_3^1 & s_4^1 & \ldots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 \\ \ldots \\ s_1^{n-2} & s_2^{n-2} \\ s_1^{n-1} \end{array}$$

where $s_1^1 = ... = s_1^{n-2} = s_1^{n-1} = 1$ as a consequence of $S \in \mathbb{G}_n$. Removing the last element of each sequence gives

$$\begin{array}{llllllll} s_1 & s_2 & s_3 & s_4 & \ldots & s_{n-3} & s_{n-2} & s_{n-1} \end{array}$$

$$\begin{array}{lllllll} s_1^1 & s_2^1 & s_3^1 & s_4^1 & \dots & s_{n-3}^1 & s_{n-2}^1 \\ \dots \\ s_1^{n-2} \end{array}$$

which is Gilbreath triangle of S', $s_1^1 = \dots = s_1^{n-2} = 1$ as a consequence of $S \in \mathbb{G}_n$. Hence $S' \in \mathbb{G}_{n-1}$. □

Definition 2. *Let $S = (s_1, \dots, s_n) \in \mathbb{G}_n$ and $S' = (s_1, \dots, s_n, k)$ finite sequences of integers. We denote with $\mathbb{K}_S$ the set of solutions for k that satisfy $S' \in \mathbb{G}_{n+1}$.*

Gilbreath triangle of S is

$$\begin{array}{lllllllll} s_1 & s_2 & s_3 & s_4 & \dots & s_{n-3} & s_{n-2} & s_{n-1} & s_n \\ s_1^1 & s_2^1 & s_3^1 & s_4^1 & \dots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 \\ \dots \\ s_1^{n-2} & s_2^{n-2} \\ s_1^{n-1} \end{array}$$

where $s_1^1 = \dots = s_1^{n-2} = s_1^{n-1} = 1$ as a consequence of $S \in \mathbb{G}_n$. Gilbreath triangle of S' is

$$\begin{array}{llllllllll} s_1 & s_2 & s_3 & s_4 & \dots & s_{n-3} & s_{n-2} & s_{n-1} & s_n & k \\ s_1^1 & s_2^1 & s_3^1 & s_4^1 & \dots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 & |s_n - k| \\ \dots \\ s_1^{n-2} & s_2^{n-2} & |s_3^{n-3} - |s_4^{n-4} - |\dots - |s_{n-1}^1 - |s_n - k||\dots||| \\ s_1^{n-1} & |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - |\dots - |s_{n-1}^1 - |s_n - k||\dots|||| \\ |s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - |\dots - |s_{n-1}^1 - |s_n - k||\dots||||| \end{array}$$

where $s_1^1 = \dots = s_1^{n-2} = s_1^{n-1} = 1$ as a consequence of $S \in \mathbb{G}_n$ and if also $s_1^n = |s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - |\dots - |s_{n-1}^1 - |s_n - k||\dots||||| = 1$, then $S' \in \mathbb{G}_{n+1}$.

$$|s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - |\dots - |s_{n-1}^1 - |s_n - k||\dots||||| = 1 \qquad (3)$$

is defined as Gilbreath equation of S and $\mathbb{K}_S = (k_1, \dots, k_{2^n})$ is the set of all solutions for k.

Gilbreath equation is a 2^n degree equation, then there are 2^n value of k that satisfy the equation (3)

$$k_{1,\dots,2^n} = \pm s_1^{n-1} \pm s_2^{n-2} \pm s_3^{n-3} \pm s_4^{n-4} \pm \dots \pm s_{n-1}^1 + s_n \pm 1 \qquad (4)$$

With respect to $\mathbb{K}_S$, the largest value that solves (4) is $\max \mathbb{K}_S = s_1^{n-1} + s_2^{n-2} + s_3^{n-3} + s_4^{n-4} + \dots + s_{n-1}^1 + s_n + 1$ and the smallest value that solves (4) is $\min \mathbb{K}_S = -s_1^{n-1} - s_2^{n-2} - s_3^{n-3} - s_4^{n-4} - ... - s_{n-1}^1 + s_n - 1 = 2s_n - \max \mathbb{K}_S$. A remarkable relation is

$$\max \mathbb{K}_S + \min \mathbb{K}_S = 2s_n \tag{5}$$

Lemma 2. *Let $S = (s_1, ..., s_n) \in \mathbb{G}_n$ and $S' = (s_1, ..., s_n, s_n)$ finite sequences of integer, than $S' \in \mathbb{G}_{n+1}$.*

Proof. Gilbreath equations of S' with $k = s_n$ is $|s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - |... - |s_{n-1}^1|...|||| = 1$ which is true because $S \in \mathbb{G}_n$, hence $S' \in \mathbb{G}_{n+1}$. □

It is useful, for the following proofs to introduce the definition of two Gilbreath sequences. Let a finite sequence of integers $S = (s_1, ..., s_n)$, from (4), any value of k cannot be greater than $\max \mathbb{K}_S$, so the sequence $(s_1, ..., s_n, \max \mathbb{K}_S)$ is the upper bound sequence for the sequence $(s_1, ..., s_n)$. The new sequence $S' = (s_1, ..., s_n, \max \mathbb{K}_S, k)$ will have the upper limit for $k = \max \mathbb{K}_{(s_1,...,s_n,\max \mathbb{K}_S)}$ ans so on. Equally, let a finite sequence of integers $S = (s_1, ..., s_n)$, from (4), any value of k cannot be smaller than $\min \mathbb{K}_S$ and the new sequence $S' = (s_1, ..., s_n, \min \mathbb{K}_S, k)$ will have the lower limit for $k = \min \mathbb{K}_{(s_1,...,s_n,\min \mathbb{K}_S)}$ and so on. From this it is now possible to introduce the definition of upper bound sequence and lower bound sequence.

Definition 3. *Let a finite sequence of integers $S = (s_1, ..., s_n) \in \mathbb{G}_n$, we denote with U_S the upper bounds sequence for S and with L_S the lower bounds sequence for S. Also, we introduce U'_S and L'_S as below.*

$$\begin{aligned}
U_S &:= \left(s_1, ..., s_n, \max \mathbb{K}_{(s_1,...,s_n)}, \max \mathbb{K}_{\left(s_1,...,s_n,\max \mathbb{K}_{(s_1,...,s_n)}\right)}, ...\right) = (u_{S_1}, ..., u_{S_m}, ...) \\
L_S &:= \left(s_1, ..., s_n, \min \mathbb{K}_{(s_1,...,s_n)}, \min \mathbb{K}_{\left(s_1,...,s_n,\min \mathbb{K}_{(s_1,...,s_n)}\right)}, ...\right) = (l_{S_1}, ..., l_{S_m}, ...) \\
U'_S &:= \left(\max \mathbb{K}_{(s_1,...,s_n)}, \max \mathbb{K}_{\left(s_1,...,s_n,\max \mathbb{K}_{(s_1,...,s_n)}\right)}, ...\right) = \left(u'_{S_1}, ..., u'_{S_m}, ...\right) \\
L'_S &:= \left(\min \mathbb{K}_{(s_1,...,s_n)}, \min \mathbb{K}_{\left(s_1,...,s_n,\min \mathbb{K}_{(s_1,...,s_n)}\right)}, ...\right) = \left(l'_{S_1}, ..., l'_{S_m}, ...\right)
\end{aligned} \tag{6}$$

From lemma 2 and definition 3

$$S = (s_1, ..., s_n) \in \mathbb{G}_n \implies l_{(s_1,...,s_m)} \leqslant s_{m+1} \leqslant u_{(s_1,...,s_m)}, \text{ where } m \in [1; n-1] \tag{7}$$

Lemma 3. *Let a finite sequence of integers* $S = (s_1, ..., s_n) \in \mathbb{G}_n$ *where* $s_1 \in 2\mathbb{Z}$*, then* $(s_2, ..., s_n) \subset (2\mathbb{Z}+1)^{n-1}$

Proof. Let $S_1 = (s_1)$, where $s_1 \in 2\mathbb{Z}$. From definition 2, $S_2 = (s_1, k) \in \mathbb{G}_2$ if $k = s_1 \pm 1 \in 2\mathbb{Z}+1$. Now let the sequence $S_3 = (s_1, s_1 \pm 1, k)$, from definition 2, $S_3 \in \mathbb{G}_3$ if $k = |\pm 1| + (s_1 \pm 1) \pm 1 = 1 + s_1 \pm 1 \pm 1$. From the previous step, $s_1 \pm 1 \in 2\mathbb{Z}+1$, hence $1 + s_1 \pm 1 \in 2\mathbb{Z}$ and $1 + s_1 \pm 1 \pm 1 \in 2\mathbb{Z}+1$. Iteratively, this can be proved for every element of S. Hence if $S \in \mathbb{G}_n$ and the first element of S is an even number, then all the other numbers of the sequence will be odd. □

Lemma 4. *Let a finite sequence of integers* $S = (s_1, ..., s_n) \in \mathbb{G}_n$ *where* $s_1 \in 2\mathbb{Z}+1$*, then* $(s_2, ..., s_n) \subset (2\mathbb{Z})^{n-1}$

Proof. Let $S_1 = (s_1)$, where $s_1 \in 2\mathbb{Z}+1$. From definition 2, $S_2 = (s_1, k) \in \mathbb{G}_2$ if $k = s_1 \pm 1 \in 2\mathbb{Z}$. Now let the sequence $S_3 = (s_1, s_1 \pm 1, k)$, from definition 2, $S_3 \in \mathbb{G}_3$ if $k = |\pm 1| + (s_1 \pm 1) \pm 1 = 1 + s_1 \pm 1 \pm 1$. From the previous step, $s_1 \pm 1 \in 2\mathbb{Z}$, hence $1 + s_1 \pm 1 \in 2\mathbb{Z}+1$ and $1 + s_1 \pm 1 \pm 1 \in 2\mathbb{Z}$. Iteratively, this can be proved for every element of S. Hence if $S \in \mathbb{G}_n$ and the first element of S is an odd number, then all the other numbers of the sequence will be even. □

Definition 4. *Let* $A_1 = 2\mathbb{Z}$ *and* $A_2 = 2\mathbb{Z}+1$*, we denote both sets with* $A_{1,2} = 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right)$.

Lemma 5. *Let a finite sequence of integers* $S = (s_1, ..., s_n) \in \mathbb{G}_n$ *where* $s_1 \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right)$*, then* $(s_2, ..., s_n) \subset \left[2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)\right]^{n-1}$

Proof. From lemma 3 and lemma 4. □

Lemma 6. *Let* $S = (s_1, ..., s_n) \in \mathbb{G}_n$ *and* $S' = (s_1, ..., s_n, k) \in \mathbb{G}_{n+1}$ *finite sequences of integers where* $s_1 \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right)$*, then*
$\mathbb{K}_S = \{x \in [\min \mathbb{K}_S, \max \mathbb{K}_S] \wedge x \in 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)\}$

Proof. From definition 2, there are 2^n values of k that satisfy $S \in \mathbb{G}_{n+1}$ and $\min \mathbb{K}_S \leqslant k \leqslant \max \mathbb{K}_S$. $\mathbb{K}_S$ is defined as the set of all solutions of k, hence it contais elements between $\min \mathbb{K}_S$ and $\max \mathbb{K}_S$. From lemma 5 it has already been shown that if $s_1 \in 2\mathbb{Z}$, than $s_n \in 2\mathbb{Z}+1, \forall n > 1$ and if $s_1 \in 2\mathbb{Z}+1$, than $s_n \in 2\mathbb{Z}, \forall n > 1$. □

From lemma 5 is proved an important result regarding (4). (4) generates 2^n solutions for a finite sequence $(s_1, ..., s_n, k) \in \mathbb{G}_{n+1}$ where $(s_1, ..., s_n) \in \mathbb{G}_n$, but from lemma 5 it has been proved that these solutions are only even or only odd according s_1. Therefore, the number of distinct solutions generated by (4) is 2^{n-1} since solutions are divided between even and odd: $\dim \mathbb{K}_S = 2^{n-1}$.

Theorem 1. *Let $S = (s_1, ..., s_n) \in \mathbb{G}_n$ and $S' = (s_1, ..., s_n, k)$ finite sequences of integers, where $s_1 \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right)$, then $k \in \mathbb{K}_S \Leftrightarrow S' \in \mathbb{G}_{n+1}$*

Proof. Prove the right implication first. From definition 2, $k \in [\min \mathbb{K}_S, \max \mathbb{K}_S]$ and from lemma 5, $k \in 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)$. Hence $k \in \mathbb{K}_S \Rightarrow S' \in \mathbb{G}_{n+1}$. Prove the left implication by contraddiction. Suppose that $S' \in \mathbb{G}_{n+1}$ but $k \notin \mathbb{K}_S$, so $k \in \left\{x \notin [\min \mathbb{K}_S, \max \mathbb{K}_S] \vee x \notin 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)\right\}$. From definition 2 and lemma 5 it is not possible to have $S' \in \mathbb{G}_{n+1}$ if $k > \max \mathbb{K}_S \vee k < \min \mathbb{K}_S \vee k \notin 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)$. Hence it is also true $k \in \mathbb{K}_S \Leftarrow S' \in \mathbb{G}_{n+1}$. □

2.1 Notable upper and lower bound sequence

From definition 3, $U_S = (S, U'_S)$ and $L_S = (S, L'_S)$. Let $S = (s_1)$, $U'_S = (s_1 + 1, s_1 + 3, ..., s_1 + 2^{n-1} - 1)$ and $L'_S = (s_1 - 1, s_1 - 3, ..., s_1 - 2^{n-1} + 1)$.

No remarkable expression was found to analytically define the trend of U'_S and L'_S for a generic sequence S but it was observed that the exponential trend is preserved. However, this trend varies with the number of terms of U'_S and L'_S so it does not seem possible to establish what will be the $n+1$-th term of U'_S and L'_S given the previus n terms through an analytical formula. However, it is always possible use the recursive espression (4).

Let the sequence $S = (s_1, ..., s_n) \in \mathbb{G}_n$, using definition 3, the (5) can be rewritten as:

$$u_{S_{n+m}} + l_{S_{n+m}} = u'_{S_m} + l'_{S_m} = 2s_n \tag{8}$$

equivalent to (7).

If it is true that exponential trend is preserved, elements of U'_S can be written in the form $u'_{S_n} = \alpha e^{\beta n}$ or $\log u'_{S_n} = \log \alpha + \beta n$.

The best fit for a dataset $D = (d_1, ..., d_n)$ in a linear regression model is

$$\beta = \frac{n \sum_{i=1}^n i \log d_i - \sum_{i=1}^n i \sum_{i=1}^n \log d_i}{n \sum_{i=1}^n i^2 - \left(\sum_{i=1}^n i\right)^2} = \frac{12}{n(n^2-1)} \left(\sum_{i=1}^n i \log d_i - \frac{n+1}{2} \sum_{i=1}^n \log d_i \right) \tag{9}$$

$$\log\alpha = \frac{1}{n}\sum_{i=1}^{n}\log d_i - \frac{\beta(n+1)}{2} \tag{10}$$

hence

$$\alpha = e^{-\frac{\beta(n+1)}{2}}\left(\prod_{i=1}^{n} d_i\right)^{\frac{1}{n}} \tag{11}$$

and the coefficient of determination is

$$R^2 = 1 - \frac{\sum_{i=1}^{n}\left(d_i - \alpha e^{\beta i}\right)^2}{\sum_{i=1}^{n}\left(d_i - \overline{d}\right)^2} \tag{12}$$

Note that in D if $|d_a| < |d_{a+1}|$ and $d_a < 0$, it is not possible to calculate $\log d_b$, where $b > a$. To avoid this problem the transformation $d_i \to d_i + \frac{d_1}{2}\left(\frac{d_1}{|d_1|} - 1\right)$ is performed. In this way, if $d_1 > 0$, $d_i \to d_i$ and if $d_1 < 0$, $d_i \to d_i - d_1$. After that, the fitting curve will be $d_n = \alpha e^{\beta n} - \frac{d_1}{2}\left(\frac{d_1}{|d_1|} - 1\right)$.

Example 1. Let a finite sequence $S = (2, 3, 5, 7, 11, 13)$ of length 6, the first 18 terms of the upper bound sequence are

$U'_S = (21, 47, 119, 297, 705, 1595, 3475, 7365, 15309, 31399, 63823, 128961, 259577, 521203, 1044907, 2092829, 4189253, 8382751)$

The first 18 terms of the lower bound sequence are

$L'_S = (5, -21, -93, -271, -679, -1569, -3449, -7339, -15283, -31373, -63797, -128935, -259551, -521177, -1044881, -2092803, -4189227, -8382725)$.

Note that $(S, U'_S) \in \mathbb{G}_{24}$ and $(S, L'_S) \in \mathbb{G}_{24}$. According to (5), $21 + 5 = 47 - 21 = 119 - 93 = 297 - 271 = ... = 8382751 - 8382725 = 26 = 2s_6$. Let fitting U'_S to $\alpha_{U'_S} e^{\beta_{U'_S} n}$. From (9), $\beta_{U'_S} = \frac{6}{2907}\left(\sum_{i=1}^{18} i\log u'_{S_i} - \frac{19}{2}\sum_{i=1}^{18}\log u'_{S_i}\right) \approx 0.75$,
from (11) $\alpha_{U'_S} = e^{-\frac{\beta(19)}{2}}\left(\prod_{i=1}^{18} u'_{S_i}\right)^{\frac{1}{18}} \approx 14.42$. The model fits the trend of u'_{S_n} with $R^2 \approx 0.92$ from (12). As regards L'_S, from (8), $l'_{S_n} = 2s_6 - \alpha_{U'_S} e^{\beta_{U'_S} n} \approx 26 - 14.42e^{0.75n} \approx -7.97e^{0.80n}$ with $R^2 \approx 0.99$.

As explained above, the addition of a term to U'_S leads to new values of α and β, therefore this analysis is carried out without pretending to evaluate the $n+1$-th element of a given U'_S of length n.

The numerical analysis of the values of the upper limit sequence was added only to show that no analytical formula has been found for the generations of the values of this sequence, with exception of (4).

3 Proof of theorem 1 for $P = (p1, ..., p_n)$

Let a finite sequence of integers $S = (s_1, ..., s_n)$, from theorem 1 is true the following. The relationship $s_2 = s_1 \pm 1$ must be true, otherwise it would not be true that $s_1^1 = 1$. As a consequence of theorem 1, for all elements subsequent to s_1, the absolute difference of two successive elements must be an integer multiple of 2 so as to maintain the absolute difference of two successive elements as an even value. So, if the first element in the sequence is even, the subsequent elements must be odd and if the first element is odd, the subsequent elements must be even.

As a consequence of definition 2, solution of Gilbreath equation (4) and definition 3, each n-th element of a sequence S must be within the range between the upper and the lower sequences calculated on all the elements prior to the n-th ones. Hence, from definition 2 and according to the solution of Gilbreath equation (4), cannot exists Gilbreath sequence in which the n-th is larger than $\max \mathbb{K}_{(s_1,...,s_{n-1})}$, since $\max \mathbb{K}_{(s_1,...,s_{n-1})}$ is the maximum value that the n-th value can take according to (4). The same goes for $\min \mathbb{K}_{(s_1,...,s_{n-1})}$, since it is the smallest value that the n-th value can take. Hence:

$$l_{(s_1,...,s_{n-1})_n} \leqslant s_n \leqslant u_{(s_1,...,s_{n-1})_n} \tag{13}$$

Following the results obtained in the previous paragraphs about Gilbreath sequence and Gilbreath equation, let proceed discussing Gilbreath conjecture. The results obtained so far will be used to establish if theorem 1 is true for the ordered sequence of prime numbers P.

Theorem 2. *For every n-th prime number, $n > 1$, it is true that $l_{(p_1,...,p_{n-1})_n} \leqslant p_n \leqslant u_{(p_1,...,p_{n-1})_n}$.*

Proof. By definition of L and U, $l_{(p_1,...,p_{n-1})_n} = \min \mathbb{K}_{(p_1,...,p_{n-1})}$ and $u_{(p_1,...,p_{n-1})_n} = \max \mathbb{K}_{(p_1,...,p_{n-1})}$, hence (13) becomes

$$\min \mathbb{K}_{(p_1,...,p_{n-1})} \leqslant p_n \leqslant \max \mathbb{K}_{(p_1,...,p_{n-1})} \tag{14}$$

Let $S = (p_1, p_2) = (2, 3) \in \mathbb{G}_2$ Gilbreath sequence formed by the first two prime numbers. As $S \in \mathbb{G}_2$, from (5), $\min \mathbb{K}_{(p_1,p_2)} \leqslant p_2 \leqslant \max \mathbb{K}_{(p_1,p_2)}$ and from lemma 2, $(p_1, p_2, p_2) \in \mathbb{G}_3$. By definition of P, $p_n > p_{n-1}$. Since $\min \mathbb{K}_{(p_1,p_2)} \leqslant p_2$, it is certainly true that $\min \mathbb{K}_{(p_1,p_2)} \leqslant p_3$. The left inequality of (14) is proved for $n = 3$. If $p_3 \leqslant \max \mathbb{K}_{(p_1,p_2)}$, then, subtracting the quantity $2p_2$ from both sides gets

$$p_3 - 2p_2 \leqslant \max \mathbb{K}_{(p_1,p_2)} - 2p_2 \tag{15}$$

Let Bertrand postulate

$$p_n < 2p_{n-1} \tag{16}$$

Replacing (5) and (16) in (15) gets

$$\min \mathbb{K}_{(p_1,p_2)} \leqslant \alpha,\ \alpha \in \mathbb{N} \tag{17}$$

Hence, exist a value $\alpha \in \mathbb{N}$ such that $\min \mathbb{K}_{(p_1,p_2)} \leqslant \alpha$. $\min \mathbb{K}_{(p_1,p_2)}$ can be written using (4) as $\min \mathbb{K}_{(p_1,p_2)} = -p_1^1 + p_2 - 1$ where $-p_1^1 - 1 < 0$ and $p_2 > 0$. If $\alpha = p_2$, (17) is proven. Hence the right inequality of (14) is proved for $n = 3$ and $(p_1, p_2, p_3) \in \mathbb{G}_3$.

At this point the proof can process showing that (14) is true for $n = 4$. Since, $p_4 > p_3$ and $\min \mathbb{K}_{(p_1,p_2,p_3)} \leqslant p_3 \leqslant \max \mathbb{K}_{(p_1,p_2,p_3)}$, it is true that $\min \mathbb{K}_{(p_1,p_2,p_3)} \leqslant p_4$. Again, $p_4 \leqslant \max \mathbb{K}_{(p_1,p_2,p_3)}$ is equivalent to

$$\min \mathbb{K}_{(p_1,p_2,p_3)} \leqslant \alpha,\ \alpha \in \mathbb{N} \tag{18}$$

From the equation (4), $\min \mathbb{K}_{(p_1,p_2,p_3)} = -p_1^2 - p_2^1 + p_3 - 1$ where $-p_1^2 - p_2^1 - 1 < 0$ and $p_3 > 0$. If $\alpha = p_3$ (18) is proved for $n = 4$ and $(p_1, p_2, p_3, p_4) \in \mathbb{G}_4$. Iteratively, this can be proved to verify (14) for every prime. $\square$

Lemma 3 is already proved for P since p_1 is even and all other elements are odd: by definition of prime number, there are no even prime numbers except for 2. Theorem 2 prove theorem 1 in the case of P.

4 Conclusions

Lemma 1, 2, 3, 4, 5 and 6, about foundamental properties of Gilbreath sequence, are used to prove theorem 1. This theorem summarizes previous results stating that if and only if the first element of a finite sequence of integers $(s_1, ..., s_n) \in \mathbb{G}_n$ is even, any odd integer k between U_S and L_S makes the sequence $(s_1, ..., s_n, k) \in \mathbb{G}_{n+1}$ and if and only if the first element of a finite sequence of integers $(s_1, ..., s_n) \in \mathbb{G}_n$ is odd, any even integer k between U_S and L_S makes the sequence $(s_1, ..., s_n, k) \in \mathbb{G}_{n+1}$. The most important result of this paper is contained in theorem 2 in which theorem 1 is proved for the particular case of the ordered sequence of prime numbers P.

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