 doi:10.20944/preprints202003.0145.v1

Gilbreath's sequences and proof of conditions for Gilbreath's conjecture

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Abstract

The conjecture attributed to Norman L. Gilbreath, but formulated by Francois Proth in the second half of the 1800s, concerns an interesting property of the ordered sequence of prime numbers P. Gilbreath's conjecture stated that, computing the absolute value of differences of consecutive primes on ordered sequence of prime numbers, and if this calculation is done for the terms in the new sequence and so on, every sequence will starts with 1. In this paper is defined the concept of Gilbreath's sequence, Gilbreath's triangle and Gilbreath's equation. On the basis of the results obtained from the proof of properties, an inductive proof is produced thanks to which it is possible to establish the necessary condition to state that the Gilbreath's conjecture is true.

1 Introduction to Gilbreath's conjecture

Let the ordered sequence $P = \{2, 3, 5, 7, 11, 13, 17, ...\} = \{p_1, ..., p_n, ...\}$ formed by prime numbers, and set $p_a^b = |p_{a+1}^{b-1} - p_a^{b-1}|$ where $b \geqslant 1$. Gilbreath conjectured that every term $p_1^b = 1$. In this notation, the elements of P should be indicated with $\{p_1^0, ..., p_n^0, ...\}$. For brevity, the superscript with $b = 0$ is omitted. It is likely that this conjecture is satisfied by many other sequences of integers, so it is necessary to define the general properties of all sequences that satisfy this conjecture.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 March 2020 doi:10.20944/preprints202003.0145.v1

2 Gilbreath's sequence

Definition 1. *Let a sequence* $S = \{s_1, s_2, s_3, \ldots, s_n\}$ *a sequence formed by integer number and* $s_a^b = |s_{a+1}^{b-1} - s_a^{b-1}|$. S *is defined a Gilbreath's sequence if* $s_1^b = 1 \forall b \geqslant 1$. *If* S *is a Gilbreath's sequence hence* $S \in \mathbb{G}_n$ *where* $\mathbb{G}_n$ *is the set of all the Gilbreath's sequences of length* n.

Let, for example, $S = \{2, 3, 5, 7, 11, 13, 17\}$ a sequence of length $n = 7$, the Gilbreath's triangle of S is defined by $s_a^b = |s_{a+1}^{b-1} - s_a^{b-1}|$. Hence:

$$\begin{array}{lllllllll} s_1 & s_2 & s_3 & s_4 & \ldots & s_{n-3} & s_{n-2} & s_{n-1} & s_n \\ s_1^1 & s_2^1 & s_3^1 & s_4^1 & \ldots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 & \\ \ldots & & & & & & & & \\ s_1^{n-2} & s_2^{n-2} & & & & & & & \\ s_1^{n-1} & & & & & & & & \end{array}$$

or

$$\begin{array}{llllll} 1 & 2 & 2 & 4 & 2 & 4 \\ 1 & 0 & 2 & 2 & 2 & \\ 1 & 2 & 0 & 0 & & \\ 1 & 2 & 0 & & & \\ 1 & 2 & & & & \\ 1 & & & & & \end{array}$$

The first term of every sequence is equal to 1, hence $S \in \mathbb{G}_7$.

Theorem 1. *Let a sequence* $S = \{s_1, ..., s_n\} \in \mathbb{G}_n$ *and a sequence* $S' = \{s_1, ..., s_{n-1}\}$, *then* $S' \in \mathbb{G}_{n-1}$.

Proof. The Gilbreath's triangle associated with S:

$$\begin{array}{lllllllll} s_1 & s_2 & s_3 & s_4 & \ldots & s_{n-3} & s_{n-2} & s_{n-1} & s_n \\ s_1^1 & s_2^1 & s_3^1 & s_4^1 & \ldots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 & \\ \ldots & & & & & & & & \\ s_1^{n-2} & s_2^{n-2} & & & & & & & \\ s_1^{n-1} & & & & & & & & \end{array}$$

where $s_1^1 = ... = s_1^{n-2} = s_1^{n-1} = 1$ as a consequence of $S \in \mathbb{G}_n$. Removing the last element of each sequence gives:

$$\begin{array}{llllllll} s_1 & s_2 & s_3 & s_4 & \ldots & s_{n-3} & s_{n-2} & s_{n-1} \\ s_1^1 & s_2^1 & s_3^1 & s_4^1 & \ldots & s_{n-3}^1 & s_{n-2}^1 & \\ \ldots & & & & & & & \\ s_1^{n-2} & & & & & & & \end{array}$$

which is the Gilbreath's triangle of S', $s_1^1 = ... = s_1^{n-2} = 1$ as a consequence of $S \in \mathbb{G}_n$, hence $S' \in \mathbb{G}_{n-1}$. □

Theorem 2. *Let a sequence* $S = \{s_1, ..., s_n\} \in \mathbb{G}_n$ *and a sequence* $S' = \{s_1, ..., s_n, k\}$*, than* $S' \in \mathbb{G}_{n+1} \Leftrightarrow k \in \mathbb{K}_S$*.*

Proof. The Gilbreath's triangle associated with S:

$$\begin{array}{llllllll} s_1 & s_2 & s_3 & s_4 & \cdots & s_{n-3} & s_{n-2} & s_{n-1} & s_n \\ s_1^1 & s_2^1 & s_3^1 & s_4^1 & \cdots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 \\ \cdots \\ s_1^{n-2} & s_2^{n-2} \\ s_1^{n-1} \end{array}$$

where $s_1^1 = ... = s_1^{n-2} = s_1^{n-1} = 1$ as a consequence of $S \in \mathbb{G}_n$. The Gilbreath's triangle of S':

$$\begin{array}{llllllllll} s_1 & s_2 & s_3 & s_4 & \cdots & s_{n-3} & s_{n-2} & s_{n-1} & s_n & k \\ s_1^1 & s_2^1 & s_3^1 & s_4^1 & \cdots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 & |s_n - k| \\ \cdots \\ s_1^{n-2} & s_2^{n-2} & |s_3^{n-3} - |s_4^{n-4} - |... - |s_{n-1}^1 - |s_n - k||...||| \\ s_1^{n-1} & |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - |... - |s_{n-1}^1 - |s_n - k||...|||| \\ |s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - |... - |s_{n-1}^1 - |s_n - k||...||||| \end{array}$$

where $s_1^1 = ... = s_1^{n-2} = s_1^{n-1} = 1$ as a consequence of $S \in \mathbb{G}_n$ and if also $s_1^n = |s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - |... - |s_{n-1}^1 - |s_n - k||...||||| = 1$, then $S' \in \mathbb{G}_{n+1}$.

$$|s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - |... - |s_{n-1}^1 - |s_n - k||...||||| = 1 \qquad (1)$$

is defined as the Gilbreath's equation of S and $\mathbb{K}_S = \{k_1, ..., k_{2^n}\}$ is defined as the set of all solutions for k. □

Corollary 1. *Let a sequence* $S = \{s_1, ..., s_n\} \in \mathbb{G}_n$ *and a sequence* $S' = \{s_1, ..., s_n, k\}$*, then there are* 2^n *values of* k *that satisfy* $S' \in \mathbb{G}_{n+1}$*.*

Proof. The Gilbreath's equation is a 2^n degree equation, then there are 2^n value of k that satisfy the equation (1). The solutions are:

$$k_{1,...,2^n} = \pm s_1^{n-1} \pm s_2^{n-2} \pm s_3^{n-3} \pm s_4^{n-4} \pm ... \pm s_{n-1}^1 + s_n \pm 1 \qquad (2)$$

□

With respect to $\mathbb{K}_S$, the largest value that solves the equation is $\max \mathbb{K}_S = s_1^{n-1} + s_2^{n-2} + s_3^{n-3} + s_4^{n-4} + ... + s_{n-1}^1 + s_n + 1$ and the smallest value that solves the equation is $\min \mathbb{K}_S = -s_1^{n-1} - s_2^{n-2} - s_3^{n-3} - s_4^{n-4} - ... - s_{n-1}^1 + s_n - 1 = 2s_n - \max \mathbb{K}_S$. A remarkable relation is:

$$\max \mathbb{K}_S + \min \mathbb{K}_S = 2s_n \qquad (3)$$

Corollary 2. *Let a sequence* $S = \{s_1, ..., s_n\} \in \mathbb{G}_n$ *and a sequence* $S' = \{s_1, ..., s_n, s_n\}$*, than* $S' \in \mathbb{G}_{n+1}$.

Proof. The Gilbreath's equations of S' with $k = s_n$ is:
$|s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - |... - |s_{n-1}^1|...||||| = 1$ which is true because $S \in \mathbb{G}_n$, hence $S' \in \mathbb{G}_{n+1}$. □

It is useful, for the following proofs to introduce the definition of two important Gilbreath's sequences. Let a sequence $S = \{s_1, ..., s_n\}$, from relation (2), any value of k cannot be greater than $\max \mathbb{K}_S$, so the sequence $\{s_1, ..., s_n, \max \mathbb{K}_S\}$ is the upper bound sequence for the sequence $\{s_1, ..., s_n\}$. The new sequence $S' = \{s_1, ..., s_n, \max \mathbb{K}_S, k\}$ will have the upper limit for $k = \max \mathbb{K}_{\{s_1,...,s_n,\max \mathbb{K}_S\}}$. Equally, let a sequence $S = \{s_1, ..., s_n\}$, from relation (2), any value of k cannot be smaller than $\min \mathbb{K}_S$ and the new sequence $S' = \{s_1, ..., s_n, \min \mathbb{K}_S, k\}$ will have the lower limit for $k = \min \mathbb{K}_{\{s_1,...,s_n,\min \mathbb{K}_S\}}$. From this example, it is now possible to introduce the definition of upper bound sequence and lower bound sequence.

Definition 2. *Let the sequence* $S = \{s_1, ..., s_n\} \in \mathbb{G}_n$

$$U_S := \{s_1, ..., s_n, \max \mathbb{K}_{\{s_1,...,s_n\}}, \max \mathbb{K}_{\{s_1,...,s_n,\max \mathbb{K}_{\{s_1,...,s_n\}}\}}, ...\} = \{u_{S_1}, ..., u_{S_m}, ...\} \quad (4)$$

is the upper bounds sequence for any S.

Definition 3. *Let the sequence* $S = \{s_1, ..., s_n\} \in \mathbb{G}_n$

$$L_S := \{s_1, ..., s_n, \min \mathbb{K}_{\{s_1,...,s_n\}}, \min \mathbb{K}_{\{s_1,...,s_n,\min \mathbb{K}_{\{s_1,...,s_n\}}\}}, ...\} = \{l_{S_1}, ..., l_{S_m}, ...\} \quad (5)$$

is the lower bounds sequence for any S.

From definition 2, definition 3, corollary 2 and (3)

$$S = \{s_1, ..., s_n\} \in \mathbb{G}_n \implies l_{\{s_1,...,s_m\}} \leqslant s_{m+1} \leqslant u_{\{s_1,...,s_m\}}, \text{ where } m \leqslant n-1 \quad (6)$$

Lemma 1. *Let a sequence* $S = \{s_1, ..., s_n\} \in \mathbb{G}_n$ *where* $s_1 \in 2\mathbb{Z}$*, then* $\{s_2, ..., s_n\} \subset (2\mathbb{Z} + 1)^{n-1}$

Proof. Let $S_1 = \{s_1\}$, where $s_1 \in 2\mathbb{Z}$. From theorem 2, corollary 1: $S_2 = \{s_1, k\} \in \mathbb{G}_2$ if $k = s_1 \pm 1 \in 2\mathbb{Z}+1$. Now let the sequence $S_3 = \{s_1, s_1 \pm 1, k\}$, from theorem 2, corollary 1, $S_3 \in \mathbb{G}_3$ if $k = |\pm 1| + (s_1 \pm 1) \pm 1 = 1 + s_1 \pm 1 \pm 1$.

From the previous step, $s_1 \pm 1 \in 2\mathbb{Z}+1$, hence $1+s_1\pm1 \in 2\mathbb{Z}$ and $1+s_1\pm1\pm1 \in 2\mathbb{Z}+1$. Iteratively, this can be proved for every element of S. Hence if $S \in \mathbb{G}_n$ and the first element of S is an even number, then all the other numbers of the sequence must be odd. □

Lemma 2. *Let a sequence* $S = \{s_1, ..., s_n\} \in \mathbb{G}_n$ *where* $s_1 \in 2\mathbb{Z}+1$, *then* $\{s_2, ..., s_n\} \subset (2\mathbb{Z})^{n-1}$

Proof. Let $S_1 = \{s_1\}$, where $s_1 \in 2\mathbb{Z}+1$. From theorem 2, corollary 1: $S_2 = \{s_1, k\} \in \mathbb{G}_2$ if $k = s_1 \pm 1 \in 2\mathbb{Z}$. Now let the sequence $S_3 = \{s_1, s_1\pm1, k\}$, from theorem 2, corollary 1, $S_3 \in \mathbb{G}_3$ if $k = |\pm1|+(s_1 \pm 1)\pm1 = 1+s_1\pm1\pm1$. From the previous step, $s_1\pm1 \in 2\mathbb{Z}$, hence $1+s_1\pm1 \in 2\mathbb{Z}+1$ and $1+s_1\pm1\pm1 \in 2\mathbb{Z}$. Iteratively, this can be proved for every element of S. Hence if $S \in \mathbb{G}_n$ and the first element of S is an odd number, then all the other numbers of the sequence must be even. □

Definition 4. *Let* $A_1 = 2\mathbb{Z}$ *and* $A_2 = 2\mathbb{Z}+1$, *both sets are represented as* $A_{1,2} = 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right)$.

Lemma 3. *Let* $S = \{s_1, ..., s_n\} \in \mathbb{G}_n$ *where* $s_1 \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right)$, *then* $\{s_2, ..., s_n\} \subset \left[2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)\right]^{n-1}$

Proof. Is a direct conseguence of lemma 1 and lemma 2. □

Lemma 4. *Let a sequence* $S = \{s_1, ..., s_n, k\} \in \mathbb{G}_{n+1}$ *where* $s_1 \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right)$, *then* $\mathbb{K}_S = \left\{x \in]\min \mathbb{K}_S; \max \mathbb{K}_S[\wedge x \in 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)\right\}$

Proof. From theorem 2, corollary 1, there are 2^n values of k that satisfy $S \in \mathbb{G}_{n+1}$ and from definition 2 and definition 3, $\min \mathbb{K}_S \leqslant k \leqslant \max \mathbb{K}_S$. $\mathbb{K}_S$ is defined as the set of all solutions of k, hence it contais elements between $\min \mathbb{K}_S$ and $\max \mathbb{K}_S$. From lemma 3 it has already been shown that if $s_1 \in 2\mathbb{Z}$, than $s_a \in 2\mathbb{Z}+1, \forall a > 1$ and if $s_1 \in 2\mathbb{Z}+1$, than $s_a \in 2\mathbb{Z}, \forall a > 1$, the theorem is proved from theorem 2, definition 2, definition 3 and lemma 3. □

From lemma 4 is proved an important result regarding (2). (2) generates 2^n solutions for a sequence $S = \{s_1, ..., s_n, k\}$, but from lemma 43 it has been proved that these solutions are only even or only odd according to the nature of the sequence. Therefore, the number of distinct solutions generated by (2) is 2^{n-1} since half solutions between $\min \mathbb{K}_S$ and $\max \mathbb{K}_S$ are equally divided between even and odd: $\dim \mathbb{K}_S = 2^{n-1}$.

Theorem 3. *Let a sequence* $S = \{s_1, ..., s_n\} \in \mathbb{G}_n$ *and* $S' = \{s_1, ..., s_n, k\}$, *where* $s_1 \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right)$, *then* $k \in]\min \mathbb{K}_S; \max \mathbb{K}_S[\wedge k \in 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right) \Leftrightarrow S' \in \mathbb{G}_{n+1}$

Proof. From definition 2 and definition 3, $k \in]\min \mathbb{K}_S; \max \mathbb{K}_S[$ and from lemma 4, $k \in 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)$. Hence it is true $k \in]\min \mathbb{K}_S; \max \mathbb{K}_S[\wedge k \in 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right) \Rightarrow S' = \{s_1, ..., s_n, k\} \in \mathbb{G}_{n+1}$.
Suppose that $S' \in \mathbb{G}_{n+1}$ but $k \in]\min \mathbb{K}_S; \max \mathbb{K}_S[\wedge k \in 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)$ is false, so it is true $k \notin]\min \mathbb{K}_S; \max \mathbb{K}_S[\vee k \notin 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)$. From definition 2, definition 3 and lemma 4 it is not possible to have $S' \in \mathbb{G}_{n+1}$ if $k \geqslant \max \mathbb{K}_S \vee k \leqslant \min \mathbb{K}_S \vee k \notin 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)$. Hence it is also true $k \in]\min \mathbb{K}_S; \max \mathbb{K}_S[\wedge k \in 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right) \Leftarrow S' = \{s_1, ..., s_n, k\} \in \mathbb{G}_{n+1}$. □

2.1 Notable upper and lower bound sequence

Definition 5. *Let the sequence* $S = \{s_1, ..., s_n\} \in \mathbb{G}_n$,

$$U'_S := \{\max \mathbb{K}_{\{s_1,...,s_n\}}, \max \mathbb{K}_{\{s_1,...,s_n,\max \mathbb{K}_{\{s_1,...,s_n\}}\}}, ...\} = \{u'_{S_1}, ..., u'_{S_m}, ...\} \tag{7}$$

Definition 6. *Let the sequence* $S = \{s_1, ..., s_n\} \in \mathbb{G}_n$,

$$L'_S := \{\min \mathbb{K}_{\{s_1,...,s_n\}}, \min \mathbb{K}_{\{s_1,...,s_n,\min \mathbb{K}_{\{s_1,...,s_n\}}\}}, ...\} = \{l'_{S_1}, ..., l'_{S_m}, ...\} \tag{8}$$

From definition 5 and definition 6 $U_S = \{S, U'_S\}$ and $L_S = \{S, L'_S\}$. Let $S = \{s_1\}$, $U'_S = \{s_1 + 1, s_1 + 3, ..., s_1 + 2^{n-1} - 1\}$ and $L'_S = \{s_1 - 1, s_1 - 3, ..., s_1 - 2^{n-1} + 1\}$.

No remarkable expression was found to be to analytically define the trend of U'_S and L'_S for a generic sequence S but it was observed that the exponential trend is preserved. However, this trend varies with the number of terms of U'_S and L'_S so it does not seem possible to establish what will be the $n+1$-th term of U'_S and L'_S given the previus n terms through an analytical formula. However, it is always possible use the recursive espression (2).

Let the sequence $S = \{s_1, ..., s_n\} \in \mathbb{G}_n$, using definition 2, definition 3, definition 5, definition 6, the (3) can be rewritten as:

$$u_{S_{n+m}} + l_{S_{n+m}} = u'_{S_m} + l'_{S_m} = 2s_n \tag{9}$$

equivalent to (6).

 doi:10.20944/preprints202003.0145.v1

If it is true that exponential trend is preserved, elements of U'_S can be written in the form $u'_{S_n} = \alpha e^{\beta n}$ or $\log u'_{S_n} = \log \alpha + \beta n$.

The best fit for a dataset $D = \{d_1, ..., d_n\}$ in a linear regression model is

$$\beta = \frac{n\sum_{i=1}^{n} i \log d_i - \sum_{i=1}^{n} i \sum_{i=1}^{n} \log d_i}{n\sum_{i=1}^{n} i^2 - \left(\sum_{i=1}^{n} i\right)^2} = \frac{12}{n(n^2-1)}\left(\sum_{i=1}^{n} i \log d_i - \frac{n+1}{2}\sum_{i=1}^{n} \log d_i\right) \tag{10}$$

$$\log \alpha = \frac{1}{n}\sum_{i=1}^{n} \log d_i - \frac{\beta(n+1)}{2} \tag{11}$$

hence

$$\alpha = e^{-\frac{\beta(n+1)}{2}}\left(\prod_{i=1}^{n} d_i\right)^{\frac{1}{n}} \tag{12}$$

and the coefficient of determination is

$$R^2 = 1 - \frac{\sum_{i=1}^{n}\left(d_i - \alpha e^{\beta i}\right)^2}{\sum_{i=1}^{n}\left(d_i - \overline{d}\right)^2} \tag{13}$$

Note that in D if $|d_a| < |d_{a+1}|$ and $d_a < 0$, it is not possible to calculate $\log d_b$, where $b > a$. To avoid this problem the transformation $d_i \to d_i + \frac{d_1}{2}\left(\frac{d_1}{|d_1|} - 1\right)$ is performed. In this way, if $d_1 > 0$, $d_i \to d_i$ and if $d_1 < 0$, $d_i \to d_i - d_1$. After that, the fitting curve will be $d_n = \alpha e^{\beta n} - \frac{d_1}{2}\left(\frac{d_1}{|d_1|} - 1\right)$.

Example 1. Let the sequence $S = \{2, 3, 5, 7, 11, 13\}$ of length 6, the first 18 terms of the upper bound sequence are $U'_S = \{21, 47, 119, 297, 705, 1595, 3475, 7365, 15309, 31399, 63823, 128961, 259577, 521203, 1044907, 2092829, 4189253, 8382751\}$ and the first 18 terms of the lower bound sequence are $L'_S = \{5, -21, -93, -271, -679, -1569, -3449, -7339, -15283, -31373, -63797, -128935, -259551, -521177, -1044881, -2092803, -4189227, -8382725\}$.

According to (6) and to (9), $21 + 5 = 47 - 21 = 119 - 93 = 297 - 271 = ... = 8382751 - 8382725 = 26 = 2s_6$. Let begin fitting U'_S to $\alpha_{U'_S} e^{\beta_{U'_S} n}$. From (10), $\beta_{U'_S} = \frac{6}{2907}\left(\sum_{i=1}^{18} i \log u'_{S_i} - \frac{19}{2}\sum_{i=1}^{18} \log u'_{S_i}\right) \approx 0.75$,
from (12) $\alpha_{U'_S} = e^{-\frac{\beta(19)}{2}}\left(\prod_{i=1}^{18} u'_{S_i}\right)^{\frac{1}{18}} \approx 14.42$. The model fits the trend of u'_{S_n} with $R^2 \approx 0.92$ from (13). As regards L'_S, from (6), $l'_{S_n} = 2s_6 - \alpha_{U'_S} e^{\beta_{U'_S} n} \approx 26 - 14.42e^{0.75n} \approx -7.97e^{0.80n}$ with $R^2 \approx 0.99$.

 doi:10.20944/preprints202003.0145.v1

As explained above, the addition of a term to U'_S leads to new values of α and β, therefore this analysis can be carried out without pretending to evaluate the $n+1$-th element of a given U'_S of length n.

The numerical analysis of the values of the upper limit sequence was added only to show that no analytical formula has been found for the generations of the values of this sequence, with exception of (2).

3 Proof of conditions for $P = \{p1, ..., p_n\}$

Let $S = \{s_1, ..., s_n\} = \{f(n)\}$, from theorem 1 and 2, is true the following. The relationship $s_2 = s_1 \pm 1$ must be true, otherwise it would not be true that $s_1^1 = 1$, hence $f(2) = f(1) \pm 1$. As a consequence of theorem 3, for all elements subsequent to s_1, the absolute difference of two successive elements must be an integer multiple of 2 so as to maintain the absolute difference of two successive elements as an even value. So, if the first element in the sequence is even, the subsequent elements must be odd and if the first element is odd, the subsequent elements must be even.

As a consequence of theorem 2, solution of the Gilbreath's equation (2) and definition 2 and definition 3: each n-th element of a sequence S must be within the range between the upper and the lower sequences calculated on all the elements prior to the n-th ones. Hence, from theorem 2 and according to the solution of the Gilbreath's equation at (2), cannot exists a Gilbreath's sequence in which the n-th is larger than $\max \mathbb{K}_{\{s_1,...,s_{n-1}\}}$, since $\max \mathbb{K}_{\{s_1,...,s_{n-1}\}}$ is the maximum value that the n-th value can take according to (2). The same goes for $\min \mathbb{K}_{\{s_1,...,s_{n-1}\}}$, since it is the smallest value that the n-th value can take, according to (2). Hence:

$$l_{\{s_1,...,s_{n-1}\}_n} \leqslant f(n) \leqslant u_{\{s_1,...,s_{n-1}\}_n} \tag{14}$$

Following the results obtained in the previous paragraphs about Gilbreath's sequence and Gilbreath's equation, let proceed discussing the Gilbreath's conjecture. The results obtained so far will be used to establish if theorem (4) is true for an ordered sequence of prime numbers P.

Theorem 4. *For every n-th prime number, $n > 1$ it is true that $l_{\{p_1,...,p_{n-1}\}_n} \leqslant p_n \leqslant u_{\{p_1,...,p_{n-1}\}_n}$.*

Proof. By definition of L and U, $l_{\{p_1,\ldots,p_{n-1}\}_n} = \min \mathbb{K}_{\{p_1,\ldots,p_{n-1}\}}$ and $u_{\{p_1,\ldots,p_{n-1}\}_n} = \max \mathbb{K}_{\{p_1,\ldots,p_{n-1}\}}$, hence the statement becomes:

$$\min \mathbb{K}_{\{p_1,\ldots,p_{n-1}\}} \leqslant p_n \leqslant \max \mathbb{K}_{\{p_1,\ldots,p_{n-1}\}} \tag{15}$$

Let $S = \{p_1, p_2\} = \{2, 3\} \in \mathbb{G}_2$ a Gilbreath's sequence formed by the first two prime numbers. As $S \in \mathbb{G}_2$, from (3), it is true that $\min \mathbb{K}_{\{p_1,p_2\}} \leqslant p_2 \leqslant \max \mathbb{K}_{\{p_1,p_2\}}$, it is also true, for theorem 2 corollary 3, that $\{p_1, p_2, p_2\} \in \mathbb{G}_3$ and for every prime number is true that $p_n > p_{n-1}$. Since $\min \mathbb{K}_{\{p_1,p_2\}} \leqslant p_2$, it is certainly true that $\min \mathbb{K}_{\{p_1,p_2\}} \leqslant p_3$. The left inequality of (15) is proved for $n = 3$.
If $p_3 \leqslant \max \mathbb{K}_{\{p_1,p_2\}}$, then, subtracting the quantity $2p_2$ from both sides, $p_3 - 2p_2 \leqslant \max \mathbb{K}_{\{p_1,p_2\}} - 2p_2$. From Bertrand's postulate, $p_n < 2p_{n-1}$, hence $p_3 - 2p_2 < 0$. Replacing (3) in the previous inequation: $\min \mathbb{K}_{\{p_1,p_2\}} \leqslant \alpha$, where $\alpha > 0$. Hence, exist a value $\alpha > 0$ such that $\min \mathbb{K}_{\{p_1,p_2\}} \leqslant \alpha$. The value of $\min \mathbb{K}_{\{p_1,p_2\}}$ can be replaced using (2): $\min \mathbb{K}_{\{p_1,p_2\}} = -p_1^1 + p_2 - 1$ where $-p_1^1 - 1 < 0$ and $p_2 > 0$. If $\alpha = p_2$, the relation $\min \mathbb{K}_{\{p_1,p_2\}} \leqslant \alpha$, where $\alpha > 0$ is true, hence it is true that $p_3 \leqslant \max \mathbb{K}_{\{p_1,p_2\}}$. The right inequality of (15) is proved for $n = 3$.

At this point the proof can process showing that (15) is true for $n = 4$. Since, $p_4 > p_3$ and $\min \mathbb{K}_{\{p_1,p_2,p_3\}} \leqslant p_3 \leqslant \max \mathbb{K}_{\{p_1,p_2,p_3\}}$, it is true that $\min \mathbb{K}_{\{p_1,p_2,p_3\}} \leqslant p_4$. Again, the statement $p_4 \leqslant \max \mathbb{K}_{\{p_1,p_2,p_3\}}$ is equivalent to the statement $\min \mathbb{K}_{\{p_1,p_2,p_3\}} \leqslant \alpha$, where $\alpha > 0$. From the equation (2), $\min \mathbb{K}_{\{p_1,p_2,p_3\}} = -p_1^2 - p_2^1 + p_3 - 1$ where $-p_1^2 - p_2^1 - 1 < 0$ and $p_3 > 0$. If $\alpha = p_3$, the relation $\min \mathbb{K}_{\{p_1,p_2,p_3\}} \leqslant \alpha$, where $\alpha > 0$. Iteratively, this can be proved to verify (15) for every prime. □

Lemma 3 is already proved since $p_1 = 2$ is even and all other elements are odd: by definition of prime number, there are no even prime numbers except for 2. From theorem 4 and lemma 3 is proved theorem 3 for P.

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