A NEW GENERALIZATION OF FIBONACCI AND LUCAS TYPE SEDENIONS

CAN KIZILATEŞ AND SELIHAN KIRLAK

ABSTRACT. In this paper, by using the $q-$integer, we introduce a new generalization of Fibonacci and Lucas sedenions called $q-$Fibonacci and $q-$Lucas sedenions. We present some fundamental properties of these type of sedenions such as Binet formulas, exponential generating fuctions, summation formulas, Catalan's identity, Cassini's identity and d'Ocagne's identity.

1. INTRODUCTION

The set of Sedenions, denoted by $\mathbb{S}$, are $16-$dimensional algebra. Sedenions are noncommutative and nonassociative algebra over the set of real numbers, obtained by applying the Cayley–Dickson construction to the octonions. Like octonions, multiplication of sedenions are not neither commutative and associative. Sedenions appear in many areas of science, such as electromagnetic theory, linear gravity and the field of quantum mechanics [1–6].

A sedenion is defined by

$$p=\sum_{i=0}^{15} q_i \mathbf{e}_i,$$

where $q_0, q_1, ..., q_{15} \in \mathbb{R}$ and $\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_{15}$ are called unit sedenion such that $\mathbf{e}_0$ is the unit element and $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_{15}$ are imaginaries satisfying, for $i, j, k = 1, 2, \ldots, 15$ the following multiplication rules:

$$\mathbf{e}_0\mathbf{e}_i = \mathbf{e}_i\mathbf{e}_0 = \mathbf{e}_i, \quad (\mathbf{e}_i)^2 = -\mathbf{e}_0, \tag{1.1}$$

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i, \quad i \neq j, \tag{1.2}$$

$$\mathbf{e}_i(\mathbf{e}_j\mathbf{e}_k) = -(\mathbf{e}_i\mathbf{e}_j)\mathbf{e}_k, \quad i \neq j, \quad \mathbf{e}_i\mathbf{e}_j \neq \pm\mathbf{e}_k. \tag{1.3}$$

The addition of sedenions is defined as componentwise and for $p_1, p_2 \in \mathbb{S}$, the multiplication is defined as follows:

$$\begin{aligned} p_1p_2 &= \left(\sum_{i=0}^{15} a_i\mathbf{e}_i\right)\left(\sum_{j=0}^{15} b_j\mathbf{e}_j\right) \\ &= \sum_{i,j=0}^{15} a_ib_j(\mathbf{e}_i\mathbf{e}_j), \end{aligned}$$

where $\mathbf{e}_i\mathbf{e}_j$ satisfies the identities (1.1), (1.2) and (1.3). Interestingly, in [4] the authors defined and studied an efficient algorithm for the multiplication of sedenions.

Key words and phrases. Sedenion algebra, Horadam number, $q-$integer, Binet-Like formula, exponential generating function.

2000 *Math. Subject. Classification: 11B39, 11R52, 05A15, 05A30.*

For $a, b, p, q \in \mathbb{R}$, the Horadam numbers $h_n = h_n(a, b; p, q)$ are defined by the following recurrence relations

$$h_n = ph_{n-1} + qh_{n-2}, \qquad h_0 = a, \qquad h_1 = b.$$

Some special cases of the Horadam sequence $h_n(a, b; p, q)$ are as following table:

a	b	p	q	Sequence
0	1	1	1	Fibonacci sequence; F_n
2	1	1	1	Lucas sequence; L_n
0	1	2	1	Pell sequence; P_n
2	2	2	1	Pell-Lucas sequence; PL_n
0	1	k	1	$k-$Fibonacci sequence; $F_{k,n}$
2	1	k	1	$k-$Lucas sequence; $L_{k,n}$
0	1	1	2	Jacobsthal sequence; J_n
2	1	1	2	Jacobsthal-Lucas sequence; j_n
0	1	2	k	$k-$Pell sequence; $P_{k,n}$
2	2	2	k	$k-$Pell-Lucas sequence; $PL_{k,n}$

Table 1: Special cases of the Horadam sequence

The Binet formula of the Horadam number is given by, for $n \geq 0$,

$$h_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

where $A = b - a\beta$, $B = b - a\alpha$, α and β are the roots of the $x^2 - px - q = 0$. We note that the above numbers have been studied in the literature widely and extensively (see, for example, [7–10]). Now, we talk about some notations related to $q-$calculus. For $x \in \mathbb{N}_0$, we give the $q-$integer $[x]_q$

$$[x]_q = \frac{1 - q^x}{1 - q} = 1 + q + q^2 + ... + q^{x-1}. \tag{1.4}$$

From (1.4), for all $x, y \in \mathbb{Z}$, we can easily find that

$$[x + y]_q = [x]_q + q^x[y]_q.$$

Many of the papers on quantum ($q-$) calculus which has been provided by the mathematicians in the past for the purpose of application to the branches of mathematics such as combinatorics, differential equations as well as in other areas in physics, probability theory and so on. For further information, we specially refer to books in [11, 12].

Several generalizations of the well-known sedenions such as Fibonacci sedenions, Lucas sedenions, $k-$Pell and $k-$Pell-Lucas sedenions, Jacobshtal and Jacobshtal-Lucas sedenions, and so on have been studied by several researchers. For example, in [13], the authors defined the Fibonaccci and Lucas sedenions

$$\hat{F}_n = \sum_{s=0}^{15} F_{n+s}\mathbf{e}_s,$$

and

$$\hat{L}_n = \sum_{s=0}^{15} L_{n+s}\mathbf{e}_s.$$

Then they also obtained the generating functions, Binet-Like formulas and some interesting identities related to Fibonacci and Lucas sedenions. For more detailed information, please refer to the closely related to the paper [13–16].

Akkus and Kizilaslan [17], defined a more general quaternion sequence by receiving components from complex sequences. Then, they gave some properties and identities related to these quaternions. In [18] Kızılateş defined the another generalization of hybrid numbers which called the $q-$Fibonacci hybrid numbers and $q-$Lucas hybrid numbers. Moreover, the author gave some important algebraic properties of these numbers.

Motivated by some of the above-cited recent papers, by the help of the $q-$integers, we define here a new family of sedenions called $q-$Fibonacci sedenions and $q-$Lucas sedenions. We obtain some special cases of the $q-$Fibonacci sedenions and $q-$Lucas sedenions studied by many researchers before. We get a number of results for $q-$Fibonacci sedenions and $q-$Lucas sedenions included Binet-Like formulas, exponential generating functions, summation formulas, Catalan's identities, Cassini's identities and d'Ocagne's identities.

2. $q-$Fibonacci Sedenions and $q-$Lucas Sedenions

In this part of the our paper, we will introduce the $q-$Fibonacci sedenions and the $q-$Lucas sedenions. Then we also give some algebraic properties of these sedenions. Throughout the paper, we take $n \in \mathbb{N}$ and $1-q \neq 0$.

Definition 2.1. *The $q-$Fibonacci sedenions and the $q-$Lucas sedenions are defined by the following:*

$$\mathbb{SF}_n(\alpha;q) = \sum_{s=0}^{15} \alpha^{n+s-1} [n+s]_q \, \mathbf{e}_s, \tag{2.1}$$

and

$$\mathbb{SL}_n(\alpha;q) = \sum_{s=0}^{15} \alpha^{n+s} \frac{[2(n+s)]_q}{[n+s]_q} \mathbf{e}_s. \tag{2.2}$$

Some special cases of $q-$Fibonacci sedenions $\mathbb{SF}_n(\alpha;q)$ or shortly $\mathbb{SF}_n$ and $q-$Lucas sedenions $\mathbb{SL}_n(\alpha;q)$ or shortly $\mathbb{SL}_n$ are as following table:

α	q	$q-$Fibonacci Sedenions	$q-$Lucas sedenions
$\frac{1+\sqrt{5}}{2}$	$\frac{-1}{\alpha^2}$	Fibonacci sedenion; SF_n	Lucas sedenion; SL_n
$1+\sqrt{2}$	$\frac{-1}{\alpha^2}$	Pell sedenion; SP_n	Pell-Lucas sedenion; SPL_n
$\frac{k+\sqrt{k^2+4}}{2}$	$\frac{-1}{\alpha^2}$	$k-$Fibonacci sedenion; $SF_{k,n}$	$k-$Lucas sedenion; $SL_{k,n}$
2	$\frac{-1}{2}$	Jacobsthal sedenion; SJ_n	Jacobsthal-Lucas sedenion; Sj_n
$1+\sqrt{1+k}$	$\frac{-k}{\alpha^2}$	$k-$Pell sedenion; $SP_{k,n}$	$k-$Pell-Lucas sedenion; $SPL_{k,n}$

Table 2: Special cases for $q-$Fibonacci Sedenions and $q-$Fibonacci sedenions

Theorem 2.2. *The Binet formulas for the $q-$Fibonacci sedenions $\mathbb{SF}_n$ and the $q-$Lucas sedenions $\mathbb{SL}_n$ are*

$$\mathbb{SF}_n = \frac{\alpha^n \tilde{\alpha} - (\alpha q)^n \tilde{\beta}}{\alpha(1-q)}, \tag{2.3}$$

and

$$\mathbb{SL}_n = \alpha^n \tilde{\alpha} + (\alpha q)^n \tilde{\beta}, \tag{2.4}$$

where $\tilde{\alpha} = \sum_{s=0}^{15} \alpha^s \mathbf{e}_s$ and $\tilde{\beta} = \sum_{s=0}^{15} \beta^s \mathbf{e}_s$.

Proof. Owing to (2.1) and (1.4), we find that

$$\begin{aligned}
\mathbb{SF}_n &= \sum_{s=0}^{15} \alpha^{n+s-1}(n+s)_q \mathbf{e}_s \\
&= \alpha^{n-1}(n)_q \mathbf{e}_0 + \alpha^n (n+1)_q \mathbf{e}_1 + ... + \alpha^{n+14}(n+15)_q \mathbf{e}_{15} \\
&= \alpha^{n-1}\left(\frac{1-q^n}{1-q}\right)\mathbf{e}_0 + \alpha^n\left(\frac{1-q^{n+1}}{1-q}\right)\mathbf{e}_1 + ... + \alpha^{n+14}\left(\frac{1-q^{n+15}}{1-q}\right)\mathbf{e}_{15} \\
&= \frac{1}{\alpha(1-q)}\left((\alpha^n - (\alpha q)^n)\mathbf{e}_0 + (\alpha^{n+1} - (\alpha q)^{n+1})\mathbf{e}_1 + ... + (\alpha^{n+15} - (\alpha q)^{n+15})\mathbf{e}_{15}\right) \\
&= \frac{1}{\alpha(1-q)}\left(\alpha^n(\mathbf{e}_0 + \alpha\mathbf{e}_1 + ... + \alpha^{15}\mathbf{e}_{15}) - (\alpha q)^n(\mathbf{e}_0 + (\alpha q)\mathbf{e}_1 + ... + (\alpha q)^{15}\mathbf{e}_{15}\right) \\
&= \frac{\alpha^n\tilde{\alpha} - (\alpha q)^n\tilde{\beta}}{\alpha(1-q)}.
\end{aligned}$$

Similarly, equality (2.4) can be obtained. □

Corollary 2.3. *The Binet formulas for the Fibonacci sedenions SF_n and the Lucas sedenions SL_n are*

$$SF_n = \frac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha - \beta}, \tag{2.5}$$

$$SL_n = \alpha^*\alpha^n + \beta^*\beta^n, \tag{2.6}$$

where $\alpha^* = \sum_{s=0}^{15} \alpha^s \mathbf{e}_s$ and $\beta^* = \sum_{s=0}^{15} \beta^s \mathbf{e}_s$, respectively.

Proof. This follows from substituting $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{-1}{\alpha}$ and $q = \frac{-1}{\alpha^2}$ in the Equation (2.3) and (2.4), respectively. □

Corollary 2.4. *The Binet formulas for the Pell sedenions SP_n and the Pell–Lucas sedenions SPL_n, are*

$$SP_n = \frac{r_1^* r_1^n - r_2^* r_2^n}{r_1 - r_2}, \tag{2.7}$$

$$SPL_n = r_1^* r_1^n + r_2^* r_2^n, \tag{2.8}$$

where $r_1^* = \sum_{s=0}^{15} r_1^s \mathbf{e}_s$ and $r_2^* = \sum_{s=0}^{15} r_2^s \mathbf{e}_s$, respectively.

Proof. This follows from substituting $\alpha = r_1 = 1 + \sqrt{2}$, $r_2 = \frac{-1}{r_1}$ and $q = \frac{-1}{\alpha^2}$ in the Equation (2.3) and (2.4), respectively. □

Corollary 2.5. *The Binet formulas for the Jacobsthal sedenions SJ_n and the Jacobsthal–Lucas sedenions Sj_n, are*

$$SJ_n = \frac{\alpha^* 2^n - \beta^*(-1)^n}{3}, \tag{2.9}$$

$$Sj_n = \alpha^* 2^n + \beta^*(-1)^n, \tag{2.10}$$

where $\alpha^* = \sum_{s=0}^{15} \alpha^s \mathbf{e}_s$ and $\beta^* = \sum_{s=0}^{15} \beta^s \mathbf{e}_s$, respectively.

Proof. This follows from substituting $\alpha = 2$, $\beta = -1$ and $q = \frac{-1}{2}$ in the Equation (2.3) and (2.4), respectively. □

Theorem 2.6. *The exponential generating functions for the $q-$Fibonacci sedenions and $q-$Lucas sedenions are*

$$\sum_{n=0}^{\infty} \mathbb{SF}_n \frac{x^n}{n!} = \frac{\tilde{\alpha} e^{\alpha x} - \tilde{\beta} e^{\alpha q x}}{\alpha(1-q)}, \tag{2.11}$$

and

$$\sum_{n=0}^{\infty} \mathbb{SL}_n \frac{x^n}{n!} = \tilde{\alpha} e^{\alpha x} + \tilde{\beta} e^{\alpha q x}. \tag{2.12}$$

Proof. From the Binet formula for the $q-$Fibonacci sedenions, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbb{SF}_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \left(\frac{\alpha^n \tilde{\alpha} - (\alpha q)^n \tilde{\beta}}{\alpha(1-q)} \right) \frac{x^n}{n!} \\
&= \frac{\tilde{\alpha}}{\alpha(1-q)} \sum_{n=0}^{\infty} \frac{(\alpha x)^n}{n!} - \frac{\tilde{\beta}}{\alpha(1-q)} \sum_{n=0}^{\infty} \frac{(\alpha q x)^n}{n!} \\
&= \frac{\tilde{\alpha} e^{\alpha x} - \tilde{\beta} e^{\alpha q x}}{\alpha(1-q)}.
\end{aligned}$$

Equality (2.12) can be similarly derived. □

Corollary 2.7. *The exponential generating functions for the Fibonacci sedenions and the Lucas sedenions are*

$$\sum_{n=0}^{\infty} SF_n \frac{x^n}{n!} = \frac{\alpha^* e^{\alpha x} - \beta^* e^{\beta x}}{\alpha - \beta}, \tag{2.13}$$

and

$$\sum_{n=0}^{\infty} SL_n \frac{x^n}{n!} = \alpha^* e^{\alpha x} + \beta^* e^{\beta x}. \tag{2.14}$$

Proof. This follows from substituting $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{-1}{\alpha}$ and $q = \frac{-1}{\alpha^2}$ in the Equation (2.11) and (2.12), respectively. □

Corollary 2.8. *The exponential generating functions for the Pell sedenions and the Pell–Lucas sedenions are*

$$\sum_{n=0}^{\infty} SP_n \frac{x^n}{n!} = \frac{r_1^* e^{r_1 x} - r_2^* e^{r_2 x}}{r_1 - r_2}, \tag{2.15}$$

and

$$\sum_{n=0}^{\infty} SP_n \frac{x^n}{n!} = r_1^* e^{r_1 x} + r_2^* e^{r_2 x}. \tag{2.16}$$

Proof. This follows from substituting $\alpha = r_1 = 1 + \sqrt{2}$, $r_2 = \frac{-1}{r_1}$ and $q = \frac{-1}{\alpha^2}$ in the Equation (2.11) and (2.12), respectively. □

Corollary 2.9. *The exponential generating functions for the Jacobsthal sedenions and the Jacobsthal–Lucas sedenions are*

$$\sum_{n=0}^{\infty} SJ_n \frac{x^n}{n!} = \frac{\alpha^* e^{2x} - \beta^* e^{-x}}{3}, \tag{2.17}$$

and

$$\sum_{n=0}^{\infty} Sj_n \frac{x^n}{n!} = \alpha^* e^{2x} + \beta^* e^{-x}. \tag{2.18}$$

Proof. This follows from substituting $\alpha = 2$, $\beta = -1$ and $q = \frac{-1}{2}$ in the Equation (2.11) and (2.12), respectively. □

Theorem 2.10. *For $n, k \geq 0$, we have*

$$\sum_{i=0}^{n} \binom{n}{i} \left(-\alpha^2 q\right)^{n-i} \mathbb{SF}_{2i+k} = \begin{cases} \Delta^{\frac{n}{2}} \mathbb{SF}_{n+k} & \text{if } n \text{ is even}, \\ \Delta^{\frac{n-1}{2}} \mathbb{SL}_{n+k} & \text{if } n \text{ is odd}, \end{cases} \tag{2.19}$$

$$\sum_{i=0}^{n} \binom{n}{i} \left(-\alpha^2 q\right)^{n-i} \mathbb{SL}_{2i+k} = \begin{cases} \Delta^{\frac{n}{2}} \mathbb{SL}_{n+k} & \text{if } n \text{ is even}, \\ \Delta^{\frac{n+1}{2}} \mathbb{SF}_{n+k} & \text{if } n \text{ is odd}, \end{cases} \tag{2.20}$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i \left(-\alpha^2 q\right)^{n-i} \mathbb{SF}_{2i+k} = \left(-\alpha\, [2]_q\right)^n \mathbb{SF}_{n+k}, \tag{2.21}$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i \left(-\alpha^2 q\right)^{n-i} \mathbb{SL}_{2i+k} = \left(-\alpha\, [2]_q\right)^n \mathbb{SL}_{n+k}. \tag{2.22}$$

where $\Delta = (\alpha - \alpha q)^2$.

Proof. Let's first prove the equality (2.19). Similarly (2.20), (2.21) and (2.22) can be obtained. From Binet formula (2.11), we find that

$$\begin{aligned} \sum_{i=0}^{n} \binom{n}{i} \left(-\alpha^2 q\right)^{n-i} \mathbb{SF}_{2i+k} &= \sum_{i=0}^{n} \binom{n}{i} \left(-\alpha^2 q\right)^{n-i} \frac{\alpha^{2i+k}\tilde{\alpha} - (\alpha q)^{2i+k}\tilde{\beta}}{\alpha(1-q)} \\ &= \frac{1}{\alpha - \alpha q} \left(\alpha^2 - \alpha^2 q\right)^n \alpha^k \tilde{\alpha} - \left(\alpha^2 q^2 - \alpha^2 q\right)^n (\alpha q)^k \tilde{\beta} \\ &= \frac{\left(\alpha\sqrt{\Delta}\right)^n \alpha^k \tilde{\alpha} - (-\alpha q\sqrt{\Delta})^n (\alpha q)^k \tilde{\beta}}{\alpha - \alpha q}. \end{aligned} \tag{2.23}$$

If n is even in (2.23), we have

$$\begin{aligned} \sum_{i=0}^{n} \binom{n}{i} \left(-\alpha^2 q\right)^{n-i} \mathbb{SF}_{2i+k} &= \frac{\left(\alpha\sqrt{\Delta}\right)^n \alpha^k \tilde{\alpha} - (\alpha q\sqrt{\Delta})^n (\alpha q)^k \tilde{\beta}}{\alpha - \alpha q} \\ &= \sqrt{\Delta}^n \frac{\alpha^{n+k}\tilde{\alpha} - (\alpha q)^{n+k}\tilde{\beta}}{\alpha(1-q)} \\ &= \Delta^{\frac{n}{2}} \mathbb{SF}_{n+k}. \end{aligned}$$

If n is odd in (2.23), we obtain

$$\begin{aligned} \sum_{i=0}^{n} \binom{n}{i} \left(-\alpha^2 q\right)^{n-i} \mathbb{SF}_{2i+k} &= \frac{\left(\alpha\sqrt{\Delta}\right)^n \alpha^k \tilde{\alpha} + (\alpha q\sqrt{\Delta})^n (\alpha q)^k \tilde{\beta}}{\alpha - \alpha q} \\ &= \sqrt{\Delta}^{n-1} (\alpha^{n+k}\tilde{\alpha} + (\alpha q)^{n+k}\tilde{\beta}) \\ &= \Delta^{\frac{n-1}{2}} \mathbb{SL}_{n+k}. \end{aligned}$$

Thus the proof is finished. □

Corollary 2.11. *For* $n \geq 0$, *we have*

$$\sum_{n=0}^{m} \binom{m}{n} SF_{2n+k} = \begin{cases} 5^{\frac{m}{2}} SF_{m+k} & \text{if } m \text{ is even,} \\ 5^{\frac{m-1}{2}} SL_{m+k} & \text{if } m \text{ is odd,} \end{cases},$$

$$\sum_{n=0}^{m} \binom{m}{n} SL_{2n+k} = \begin{cases} 5^{\frac{m}{2}} SL_{m+k} & \text{if } m \text{ is even,} \\ 5^{\frac{m+1}{2}} SF_{m+k} & \text{if } m \text{ is odd,} \end{cases},$$

$$\sum_{n=0}^{m} \binom{m}{n} (-1)^n SF_{2n+k} = (-1)^m SF_{m+k},$$

and

$$\sum_{n=0}^{m} \binom{m}{n} (-1)^n SL_{2n+k} = (-1)^m SL_{m+k}.$$

Proof. This follows from substituting $\alpha = \frac{1+\sqrt{5}}{2}$ and $q = \frac{-1}{\alpha^2}$ in the Equation (2.19), (2.20), (2.21) and (2.22), respectively. □

Corollary 2.12. *For* $n \geq 0$, *we find that*

$$\sum_{n=0}^{m} \binom{m}{n} SP_{2n+k} = \begin{cases} 8^{\frac{m}{2}} SP_{m+k} & \text{if } m \text{ is even,} \\ 8^{\frac{m-1}{2}} SPL_{m+k} & \text{if } m \text{ is odd,} \end{cases},$$

$$\sum_{n=0}^{m} \binom{m}{n} SPL_{2n+k} = \begin{cases} 8^{\frac{m}{2}} SPL_{m+k} & \text{if } m \text{ is even,} \\ 8^{\frac{m+1}{2}} SP_{m+k} & \text{if } m \text{ is odd,} \end{cases},$$

$$\sum_{n=0}^{m} \binom{m}{n} (-1)^n SP_{2n+k} = (-1)^m SP_{m+k},$$

and

$$\sum_{n=0}^{m} \binom{m}{n} (-1)^n SPL_{2n+k} = (-1)^m SPL_{m+k}.$$

Proof. This follows from substituting $\alpha = r_1 = 1 + \sqrt{2}$ and $q = \frac{-1}{\alpha^2}$ in the Equation (2.19), (2.20), (2.21) and (2.22), respectively. □

Corollary 2.13. *For* $n \geq 0$, *we obtain*

$$\sum_{n=0}^{m} \binom{m}{n} SJ_{2n+k} = \begin{cases} 9^{\frac{m}{2}} SJ_{m+k} & \text{if } m \text{ is even,} \\ 9^{\frac{m-1}{2}} Sj_{m+k} & \text{if } m \text{ is odd,} \end{cases},$$

$$\sum_{n=0}^{m} \binom{m}{n} Sj_{2n+k} = \begin{cases} 9^{\frac{m}{2}} Sj_{m+k} & \text{if } m \text{ is even,} \\ 9^{\frac{m+1}{2}} SJ_{m+k} & \text{if } m \text{ is odd,} \end{cases},$$

$$\sum_{n=0}^{m} \binom{m}{n} (-1)^n 2^{m-n} SJ_{2n+k} = (-1)^m SJ_{m+k},$$

and

$$\sum_{n=0}^{m} \binom{m}{n} (-1)^n 2^{m-n} Sj_{2n+k} = (-1)^m Sj_{m+k}.$$

Proof. This follows from substituting $\alpha = 2$ and $q = \frac{-1}{2}$ in the Equation (2.19), (2.20), (2.21) and (2.22), respectively. □

Theorem 2.14. *For $n \geq 0$, we get*

$$\sum_{i=0}^{n} \binom{n}{i} (\alpha + \alpha q)^i \left(-\alpha^2 q\right)^{n-i} \mathbb{SF}_i = \mathbb{SF}_{2n}, \tag{2.24}$$

$$\sum_{i=0}^{n} \binom{n}{i} (\alpha + \alpha q)^i \left(-\alpha^2 q\right)^{n-i} \mathbb{SL}_i = \mathbb{SL}_{2n}. \tag{2.25}$$

Proof. Using the Binet formula for the the $q-$Fibonacci sedenions (2.3), we have

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} (\alpha + \alpha q)^i \left(-\alpha^2 q\right)^{n-i} \mathbb{SF}_i \\
&= \sum_{i=0}^{n} \binom{n}{i} (\alpha + \alpha q)^i \left(-\alpha^2 q\right)^{n-i} \frac{\alpha^i \tilde{\alpha} - (\alpha q)^i \tilde{\beta}}{\alpha(1-q)} \\
&= \sum_{i=0}^{n} \binom{n}{i} \frac{(\alpha + \alpha q)^i \left(-\alpha^2 q\right)^{n-i} \alpha^i \tilde{\alpha}}{\alpha(1-q)} \\
&\quad - \sum_{i=0}^{n} \binom{n}{i} \frac{(\alpha + \alpha q)^i \left(-\alpha^2 q\right)^{n-i} (\alpha q)^i \tilde{\beta}}{\alpha(1-q)} \\
&= \frac{(\alpha^2(1+q) - \alpha^2 q)^n \tilde{\alpha}}{\alpha(1-q)} \\
&\quad - \frac{(\alpha^2 q(1+q) - \alpha^2 q)^n \tilde{\beta}}{\alpha(1-q)} \\
&= \mathbb{SF}_{2n}.
\end{aligned}$$

Equality (2.25) can be similarly derived. □

Theorem 2.15. *For $n \geq 0$, we get*

$$\sum_{n=0}^{m} \binom{m}{n} SF_n = SF_{2m},$$

and

$$\sum_{n=0}^{m} \binom{m}{n} SL_n = SL_{2m}.$$

Proof. This follows from substituting $\alpha = \frac{1+\sqrt{5}}{2}$ and $q = \frac{-1}{\alpha^2}$ in the Equation (2.24), (2.25). □

Corollary 2.16. *For $n \geq 0$, we obtain*

$$\sum_{n=0}^{m} \binom{m}{n} SP_n = SP_{2m},$$

and

$$\sum_{n=0}^{m} \binom{m}{n} SPL_n = SPL_{2m}.$$

Proof. This follows from substituting $\alpha = r_1 = 1 + \sqrt{2}$ and $q = \frac{-1}{\alpha^2}$ in the Equation (2.24), (2.25). □

Corollary 2.17. *For $n \geq 0$, we have*

$$\sum_{n=0}^{m} \binom{m}{n} 2^{m-n} SJ_n = SJ_{2n},$$

ve

$$\sum_{n=0}^{m} \binom{m}{n} 2^{m-n} Sj_n = Sj_{2n}.$$

Proof. This follows from substituting $\alpha = 2$ and $q = \frac{-1}{2}$ in the Equation (2.24), (2.25). □

Theorem 2.18. *(Catalan's Identity). For positive integers n and r, with $n \geq r$, then the following identity is true:*

$$\mathbb{SF}_{n+r}\mathbb{SF}_{n-r} - \mathbb{SF}_n^2 = \frac{\alpha^{2n-2}q^n(1-q^r)(\tilde{\beta}\tilde{\alpha} - q^{-r}\tilde{\alpha}\tilde{\beta})}{(1-q)^2}, \tag{2.26}$$

$$\mathbb{SL}_{n+r}\mathbb{SL}_{n-r} - \mathbb{SL}_n^2 = \alpha^{2n}q^{n-r}(1-q^r)\left(\tilde{\alpha}\tilde{\beta} - q^r\tilde{\beta}\tilde{\alpha}\right). \tag{2.27}$$

Proof. From the Binet formula of the $q-$Fibonacci sedenions, we have the LHS of the equality (2.26),

$$\begin{aligned} &\mathbb{SF}_{n+r}\mathbb{SF}_{n-r} - \mathbb{SF}_n^2 \\ = &\left(\frac{\alpha^{n+r}\tilde{\alpha} - (\alpha q)^{n+r}\tilde{\beta}}{\alpha(1-q)}\right)\left(\frac{\alpha^{n-r}\tilde{\alpha} - (\alpha q)^{n-r}\tilde{\beta}}{\alpha(1-q)}\right) \\ &- \left(\frac{\alpha^{n}\tilde{\alpha} - (\alpha q)^{n}\tilde{\beta}}{\alpha(1-q)}\right)^2. \end{aligned}$$

After some calculations, we get

$$\mathbb{SF}_{n+r}\mathbb{SF}_{n-r} - \mathbb{SF}_n^2 = \frac{\alpha^{2n-2}q^n(1-q^r)(\tilde{\beta}\tilde{\alpha} - q^{-r}\tilde{\alpha}\tilde{\beta})}{(1-q)^2}.$$

The result (2.27) can be similarly obtained. □

Theorem 2.19. *(Cassini's Identity). For $n \geq 1$, the following equality holds:*

$$\mathbb{SF}_{n+1}\mathbb{SF}_{n-1} - \mathbb{SF}_n^2 = \frac{\alpha^{2n-2}q^n(\tilde{\beta}\tilde{\alpha} - q^{-1}\tilde{\alpha}\tilde{\beta})}{(1-q)}, \tag{2.28}$$

$$\mathbb{SL}_{n+1}\mathbb{SL}_{n-1} - \mathbb{SL}_n^2 = \alpha^{2n}q^{n-1}(1-q)\left(\tilde{\alpha}\tilde{\beta} - q\tilde{\beta}\tilde{\alpha}\right) \tag{2.29}$$

Proof. If we take $r = 1$, in (2.26) and (2.27), we obtain the assertions of the theorem. □

Theorem 2.20. *(d'Ocagne's Identity). Suppose that n is a non-negative integer number and m natural number. If $m > n+1$, then the expression of the d'Ocagne's identities are given by the following:*

$$\mathbb{SF}_m\mathbb{SF}_{n+1} - \mathbb{SF}_{m+1}\mathbb{SF}_n = \frac{\alpha^{m+n-1}(q^n\tilde{\alpha}\tilde{\beta} - q^m\tilde{\beta}\tilde{\alpha})}{(1-q)}, \tag{2.30}$$

$$\mathbb{SL}_m\mathbb{SL}_{n+1} - \mathbb{SL}_{m+1}\mathbb{SL}_n = \alpha^{m+n+1}(q-1)(q^n\tilde{\alpha}\tilde{\beta} - q^m\tilde{\beta}\tilde{\alpha}). \tag{2.31}$$

Proof. Using the Binet formula of the $q-$Fibonacci sedenions, we obtain

$$\begin{aligned}&\mathbb{SF}_m\mathbb{SF}_{n+1}-\mathbb{SF}_{m+1}\mathbb{SF}_n\\ =\ &\frac{\alpha^m\tilde{\alpha}-(\alpha q)^m\tilde{\beta}}{\alpha(1-q)}\frac{\alpha^{n+1}\tilde{\alpha}-(\alpha q)^{n+1}\tilde{\beta}}{\alpha(1-q)}\\ &-\frac{\alpha^{m+1}\tilde{\alpha}-(\alpha q)^{m+1}\tilde{\beta}}{\alpha(1-q)}\frac{\alpha^n\tilde{\alpha}-(\alpha q)^n\tilde{\beta}}{\alpha(1-q)}.\end{aligned}$$

Thus

$$\mathbb{SF}_m\mathbb{SF}_{n+1}-\mathbb{SF}_{m+1}\mathbb{SF}_n=\frac{\alpha^{m+n-1}(q^n\tilde{\alpha}\tilde{\beta}-q^m\tilde{\beta}\tilde{\alpha})}{(1-q)}.$$

Equality (2.31) can be similarly derived. □

3. Conclusion and Discussion

In our present research, we have defined and examined systematically $q-$Fibonacci sedenions and $q-$Lucas sedenions which are defined by means of the $q-$integer. We have obtained several quirky properties of $q-$Fibonacci sedenions and $q-$Lucas sedenions such as Binet-Like formulas, exponential generating functions, summation formulas, Cassini's identities, Catalan's identities and d'Ocagne's identities. According to the special cases of α and q, all the results given in Section 2 are applicable to all Fibonacci-type sedenions and Lucas-type sedenions mentioned in Table 2. On the other hand, a trigintaduonion is defined by

$$t=a_0+\sum_{i=0}^{31}a_i\mathbf{e}_i,$$

where $\{a_i\}$, $i=0,1,\ldots,31$ are real numbers and $\{\mathbf{e}_i\}$, $i=0,1,\ldots,31$ are imaginary units. Detailed information about the trigintaduonion have been presented in the literature. (for example, see [19]). Indeed, for the interested readers of this work, results presented here have the potential to motivate further researches of the subject of the $q-$Fibonacci Trigintaduonions and $q-$Lucas Trigintaduonions.

References

[1] Baez, J.C., The octonions. Bull. Am. Math. Soc. 39, 145–205 (2002).

[2] Carmody, K., Circular and Hyperbolic Quaternions, Octonions and Sedenions, Appl. Math.Comput., 28, 47–72, (1988).

[3] Cawagas, R. E., On the structure and zero divisors of the Cayley-Dickson sedenion algebra, Discuss. Math. Gen. Algebra Appl. 24, 251–265, (2004).

[4] Cariow, A. and Cariowa, G., An algorithm for fast multiplication of sedenions, Inform. Process. Lett.113, 324–331, (2013).

[5] V. L. Mironov and S. V. Mironov, Associative space-time sedenions and their application in relativistic quantum mechanics and field theory, Appl. Math. 6(1), 46–56, (2015).

[6] Imaeda, K. and Imaeda, M., Sedenions: algebra and analysis, Appl. Math. Comput. 115, 77–88, (2000).

[7] Horadam, A.F., Basic properties of a certain generalized sequence of numbers, Fibonacci Quart., 3, 161–176, (1965).

[8] Horadam,A.F., Generating functions for powers of a certain generalized sequence of numbers, Duke Math. J., 32, 437–446, (1965).

[9] Horzum,T., Kocer, E.G., On Some Properties of Horadam Polynomials, Int. Math. Forum, 4, 25, 1243–1252, (2009).
[10] Cimen, C.B., Ipek, A. On Jacobsthal and Jacobsthal–Lucas Octonions, Mediterr. J. Math., 14:37 (2017).
[11] Kac V, Cheung P. Quantum Calculus Universitext, Springer-Verlag, New York, (2002).
[12] Andrews GE, Askey R, Roy R. Special functions. Cambridge Univ. Press, New York, (1999).
[13] Bilgici, G., Tokeser, U. and Unal, Z., Fibonacci and Lucas Sedenions, J. Integer Seq., Vol. 20, Article 17.1.8, (2017).
[14] Çimen, C., Ipek, A., On Jacobsthal and Jacobsthal-Lucas Sedenios and Several Identities Involving These Numbers. Math. Aeterna, 7, 447–454, (2017).
[15] Catarino, P. $k-$Pell, k-Pell-Lucas and modified $k-$Pell sedenions. Asian-Eur. J. Math., 12 (1), (2019).
[16] Çimen, C., Ipek, A., Çimen, E., On Horadam Sedenions, Journal of Science and Arts, 4, (49), 887–898, (2019).
[17] Akkus, I., Kizilaslan, G., Quaternions: Quantum calculus approach with applications, Kuwait Journal of Science, 46 (4), 1–13, (2019).
[18] Kızılateş C., A new generalization of Fibonacci hybrid and Lucas hybrid numbers, Chaos Solitons Fractals, 130, (2020).
[19] Cawagas, R. E., Carrascal, A. S., Bautista, L. A., Maria, J. P. Sta., Urrutia J. D., Nobles, B., The Basic Subalgebra Structure Of The Cayley-Dickson Algebra Of Dimension 32 (Trigintaduonions), arXiv:0907.2047v3, (2009).

BULENT ECEVIT UNIVERSITY, DEPARTMENT OF MATHEMATICS, ZONGULDAK, 67100, TURKEY

E-mail address: cankizilates@gmail.com and selihankirlak@gmail.com