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Article

Interval-valued vector optimization problems involving generalized approximate convexity

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Abstract: Interval-valued functions have been widely used to accommodate data inexactness in optimization and decision theory. In this paper, we study interval-valued vector optimization problems, and derive their relationships to interval variational inequality problems, of both Stampacchia and Minty types. Using the concept of interval approximate convexity, we establish necessary and sufficient optimality conditions for local strong quasi and approximate *LU*-efficient solutions to nonsmooth optimization problems with interval-valued multiobjective functions.

Keywords: Interval-valued vector optimization problems; generalized approximate *LU*-convexity; interval vector variational inequalities; *LU*-efficient solutions

MSC: 90C25; 90C29; 90C30; 90C46; 49J40

1. Introduction

In various real-life problems in engineering and economic, the occurrence of imprecision in the data which is taken from measurements or observations is inevitable. Therefore, in reason of simplicity and confidentiality, one has to consider an objective function taking values as real intervals. In the literature, numerous examples can be found where imprecision in real-life applications is modeled by the help of a mathematical tool [1–3]. Recently, many researchers studied interval-valued vector optimization problems. For instance, in order to characterize solutions of interval-valued programming problems, the Karush-Kuhn-Tucker optimality conditions were obtained in [4–6]. Wolfe and Mond-Weir-type duality were investigated for these problems in [7] where weak and strong duality results were provided.

On the other hand, a considerable and growing interest has been centered about studying the relationship between vector optimization problems and vector variational inequalities. In particular, many results providing optimality conditions in terms of vector variational inequalities were proven for vector-valued objective functions; see for instance [8–11] for the smooth case, [12–14] for the nonsmooth case and [15–19] for some more recent results. With respect to interval-valued objective functions, Zhang *et al* [6,20] studied optimality conditions under the assumption of LU-convexity introduced by Wu [4] as an extension of convexity for real-valued functions.

In this paper, we adapt the concepts of approximate convexity [21] and generalized approximate convexity [22] to vector-valued functions. Afterwards, we will use these concepts as a tool to establish optimality conditions for interval-valued vector optimization problems in terms of Stampacchia and Minty vector variational inequalities using two solutions types: local strong quasi and approximate *LU*-efficient solutions.

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The layout of this article is as follows: First, we recall in Section 2 some preliminary definitions. In Section 3, basic properties and arithmetic for intervals are introduced. In Section 4, necessary optimality conditions characterizing local strong quasi LU-efficient solutions for interval-valued vector optimization problems are established. In Section 5, sufficient and necessary optimality conditions are proved under generalized approximate convexity assumptions using the concept of approximate vector variational inequalities. Finally, we provide an example in Section 6 illustrating the main results, and conclude our work in Section 7.

2. Preliminaries

Throughout this paper, $\mathbb{R}^n$ be n-dimensional Euclidean space, $\mathbb{R}^n_+$ be its nonnegative orthant and X is a nonempty set in $\mathbb{R}^n$. For any $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$ in X, we say that $x < y$ if $x_i < y_i$ for all $i = 1, 2, \ldots, n$, $x \leq y$ if $x_i \leq y_i$ for all $i = 1, 2, \ldots, n$ except at least one index for which the inequality holds strict, and $x \leqq y$ if $x_i \leq y_i$ for all $i = 1, 2, \ldots, n$.

Definition 1 ([23]). *A function $\phi : X \to \mathbb{R}$ is said to be locally Lipschitz at $x_0 \in X$, if there are positive constants k and δ satisfying for all $x, y \in B(x_0, \delta)$*

$$\|\phi(x) - \phi(y)\| \leq k\|x - y\|.$$

It is said to be locally Lipschitz on X if it is so at each $x_0 \in X$.

Definition 2 ([23]). *Let $f : X \to \mathbb{R}^m$ be a vector valued function such that its components $f_i : X \to \mathbb{R}, i = 1, 2, \ldots, m$ are locally Lipschitz on X.*

(i) If $m = 1$, then Clarke's generalized subdifferential of f at $x \in X$ is defined as

$$\partial^C f(x) := \{y \in \mathbb{R}^n : f^\circ(x; v) \geq \langle y, v\rangle, \ \forall v \in \mathbb{R}^n\}.$$

where $f^\circ(x; v)$ is Clarke's generalized directional derivative of f along $v \in \mathbb{R}^n$ at $x \in X$, which is defined as

$$f^\circ(x, v) := \limsup_{\substack{t \downarrow 0 \\ y \to x}} \frac{f(y + \lambda v) - f(y)}{t}.$$

(ii) If $m > 1$, then Clarke's generalized Jacobian of f at $x \in X$ is the set of $m \times n$ matrices defined by

$$\partial^C f(x) := co\{\lim_{i \to +\infty} Jf(x^{(i)}) : x^{(i)} \to x, x^{(i)} \in S\},$$

where $Jf(x^{(i)})$ is the Jacobian of f at $x^{(i)}$, co is the convex hull, and S is the differentiability set of f.

Remark 1. *The generalized gradient used by Mishra and Wang [14] to define new concepts of generalized convexity for studying relationships between vector critical points and solutions of vector optimization problems is the Cartesian product set $\partial f(x) := \partial^C f_1(x) \times \partial^C f_2(x) \times \ldots \times \partial^C f_m(x)$. It is worth mentioning that $\partial^C f(x)$ is contained but, in general, different from $\partial f(x)$. Observe that if all Clarke subdifferentials $\partial^C f_i(x)$ are a singleton except at most one, then $\partial^C f(x) = \partial f(x)$. We note also that $\partial(-f)(x) = -\partial f(x)$.*

Let $e = (e_1, \ldots, e_m) \in int(\mathbb{R}^m_+)$. Let us recall the notions of approximate convexity which are provided in [18,21,22] as follows:

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Definition 3. *A vector-valued function $f : X \to \mathbb{R}^m$ is called*

(i) approximate e-convex at $x_0 \in X$, if there exists $\delta > 0$ such that for all $x, y \in B(x_0, \delta)$

$$f(x) - f(y) \geq \langle \xi, \eta(x, y) \rangle - e\|x - y\|, \quad \text{for all } \xi \in \partial f(y);$$

(ii) approximate pseudo e-convex of type I at $x_0 \in X$ if there exists $\delta > 0$ such that for any $x, y \in B(x_0, \delta)$

$$f(x) - f(y) < -e\|x - y\| \quad \Rightarrow \quad \langle \xi, x - y \rangle < 0, \ \forall \xi \in \partial f(y);$$

(iii) approximate pseudo e-convex of type II at $x_0 \in X$ if there exists $\delta > 0$ such that for any $x, y \in B(x_0, \delta)$

$$f(x) - f(y) < 0 \quad \Rightarrow \quad \langle \xi, x - y \rangle + e\|x - y\| < 0, \ \forall \xi \in \partial f(y);$$

(iv) approximate quasi e-convex of type I at $x_0 \in X$ if there exists $\delta > 0$ such that for any $x, y \in B(x_0, \delta)$

$$\exists \xi \in \partial f(y) : \ \langle \xi, x - y \rangle - e\|x - y\| > 0 \quad \Rightarrow \quad f(x) > f(y);$$

(v) approximate quasi e-convex of type II at $x_0 \in X$ if there exists $\delta > 0$ such that for any $x, y \in B(x_0, \delta)$

$$\exists \xi \in \partial f(y) : \ \langle \xi, x - y \rangle > 0 \quad \Rightarrow \quad f(x) - f(y) > e\|x - y\|.$$

Remark 2. *The relationship between the above concepts of convexity can be summarized as follows.*

1. *If f is approximate pseudo (resp. quasi) e-convex of type II at $x_0 \in X$, then f is approximate pseudo (resp. quasi) e-convex of type I at x_0.*
2. *It is easy to see that any approximate e-convex function at x_0 is approximate pseudo e-convex function of type I and approximate quasi e-convex function of type I at x_0.*
3. *There is no relation between approximate pseudo e-convex functions of type II and approximate quasi e-convex functions of type II and approximate e-convex functions (see [18]).*

3. Interval-valued vector functions

We first recall some basic arithmetic operations on real intervals, which are the same for general sets. Let us denote by $\mathbb{IR}$ the class of all closed intervals in $\mathbb{R}$ and let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be in $\mathbb{IR}$. The sum and product are defined to be as follows:

$$A + B := \{a + b : a \in A, \ b \in B\} = [a^L + b^L, a^U + b^U],$$
$$A \times B := \{ab : a \in A, \ b \in B\} = [\min S, \max S],$$

where $S := \{a^L b^L, a^L b^U, a^U b^L, a^U b^U\}$. It is worth mentioning that any real number a can be regarded as a closed interval $A_a = [a, a]$ and for that the sum $a + B$ means $A_a + B$.

From the above operations, we can define the multiplication of an interval with a real number α as

$$\alpha A := \{\alpha a : a \in A\} = \begin{cases} [\alpha a^L, \alpha a^U], & \text{if } \alpha \geq 0, \\ [\alpha a^U, \alpha a^L], & \text{if } \alpha < 0. \end{cases}$$

Note the special case of $-A = \{-a : a \in A\} = [-a^U, -a^L]$. Henceforth, the difference of two sets is defined by

$$A - B := A + (-B) = [a^L - b^U, a^U - b^L].$$

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On the other hand, an order relation can be defined for intervals as follows: we write $A \leq_{LU} B$ if $a^L \leq b^L$ and $a^U \leq b^U$. Also, we say $A <_{LU} B$ if $A \leq_{LU} B$ and $A \neq B$; that is, we have either $a^L < b^L$ and $a^U \leq b^U$, $a^L \leq b^L$ and $a^U < b^U$, or $a^L < b^L$ and $a^U < b^U$.

Following the partial order defined above, A and B are said to be comparable if $A \leq_{LU} B$ or $A \geq_{LU} B$. Henceforth, these two intervals are not comparable means either $A \subseteq B$ or $A \supseteq B$. We call $A = (A_1, \ldots, A_n)$ an interval-valued vector if for each $k = 1, \ldots, n$ we have $A_k = [a_k^L, a_k^U]$ is a closed interval. We consider two interval-valued vectors $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ such that A_k and B_k are comparable for all $k = 1, \ldots, n$. We write

- $A \leq_{LU} B$ if $A_k \leq_{LU} B_k$ for all $k = 1, \ldots, n$;
- $A <_{LU} B$ if $A_k \leq_{LU} B_k$ for all $k = 1, \ldots, n$, and $A_j <_{LU} B_j$ for at least one j.

An interval-valued function $f : \mathbb{R}^n \to \mathbb{IR}$ is defined by $f(x) = [f^L(x), f^U(x)]$ for each $x \in \mathbb{R}^n$, where f^L and f^U are two real-valued functions on $\mathbb{R}^n$ satisfying $f^L(x) \leq f^U(x)$. If $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{IR}$ are m interval-valued functions, then we call $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{IR}^m$ an interval-valued vector function.

Definition 4 ([20]). *An interval-valued function $f = [f^L, f^U] : X \to \mathbb{IR}$ is called locally Lipschitz at $x_0 \in X$ with respect to the Hausdorff metric if there exist $L > 0$ and $\delta > 0$ such that for any $x, y \in B(x_0, \delta)$ one has*

$$d_H(f(x), f(y)) \leq L\|x - y\|,$$

where $d_H(f(x), f(y))$ is the Hausdorff metric between $f(x)$ and $f(y)$ defined by

$$d_H(f(x), f(y)) = max\{| f(x)^L - f(y)^L |, | f(x)^U - f(y)^U |\}.$$

f is locally Lipschitz on X if it is so at any $x_0 \in X$.

Proposition 1 ([20]). *If $f = [f^L, f^U] : X \to \mathbb{IR}$ is locally Lipschitz on X, then both f^L and f^U are locally Lipschitz on X (as real-valued functions).*

Now, the following concept of generalized e-approximate LU-convexity for nonsmooth interval-valued functions can be introduced.

Definition 5. *Let $f = [f^L, f^U] : X \to \mathbb{IR}$ be a locally Lipschitz interval-valued function. f is e-approximate LU-convex (respectively e-approximate pseudo LU-convex, e-approximate quasi LU-convex) at $x_0 \in X$, if and only if both f^L and f^U are locally Lipschitz and approximate e-convex (respectively approximate pseudo e-convex, approximate quasi e-convex) at x_0.*

Remark 3. *The above definition is equivalent to saying that a locally Lipschitz function $f = [f^L, f^U] : X \to \mathbb{IR}$ is e-approximate LU-convex at $x_0 \in X$ if and only if there exists $\delta > 0$ such that for any $x, y \in B(x_0, \delta)$ one has*

$$f^L(x) - f^L(y) \geq \langle \xi^L, x - y \rangle - e\|x - y\|, \quad \textit{for all } \xi^L \in \partial f^L(y),$$

and

$$f^U(x) - f^U(y) \geq \langle \xi^U, x - y \rangle - e\|x - y\|, \quad \textit{for all } \xi^U \in \partial f^U(y).$$

Similar equivalence holds true for e-approximate pseudo (and quasi) LU-convexity.

To introduce the interval-valued vector optimization problem (IVOP), we consider in what follows an interval-valued multiobjective function $f = (f_1, \ldots, f_m)$ that is locally Lipschitz. Each component

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objective function $f_k = [f_k^L, f_k^U]$ is an interval-valued function defined on the nonempty open feasible set $X \subseteq \mathbb{R}^n$. Our optimization problem is written as:

$$\min_{x \in X} \; f(x) = (f_1(x), f_2(x), \ldots, f_m(x)). \tag{IVOP}$$

Assume we are given a vector $\overline{x}$ as a feasible solution to (IVOP).

Definition 6 ([5]). *The vector $\overline{x}$ is said to be*

i) an LU-efficient solution of (IVOP) if there is no $x \in X$ with $f(x) <_{LU} f(\overline{x})$.
ii) a strong LU-efficient solution of (IVOP) if there is no $x \in X$ with $f(x) \leq_{LU} f(\overline{x})$.

We extend the concepts of quasi LU-efficient solutions of vector optimization introduced in ([24]) to the concepts of local e-quasi LU-efficient and local strong e-quasi LU-efficient solutions of (IVOP).

Definition 7. *The vector $\overline{x}$ is said to be a local (strong) e-quasi LU-efficient solution to (IVOP) if there is $\delta > 0$ such that there is no $x \in B(\overline{x}, \delta)$ satisfying*

$$f(x) + e\|x - \overline{x}\| <_{LU} \;\; (\leq_{LU}) \; f(\overline{x}).$$

We introduce the following concept of approximate LU-efficient solutions to (IVOP), which is useful when no LU-efficient solution exists.

Definition 8. *The vector $\overline{x}$ is said to be an e-approximate LU-efficient solution of (IVOP) if there is no $\delta > 0$ such that, for all $x \in B(\overline{x}, \delta) \setminus \{\overline{x}\}$, one has*

$$f(x) + e\|x - \overline{x}\| \leq_{LU} f(\overline{x}).$$

4. Sufficient conditions for local strong e-quasi LU-efficient solutions

We consider the following vector variational inequalities of Stampacchia and Minty types:

$$\text{Find } \overline{x} \in X \text{ such that } \begin{cases} \langle \xi^L, x - \overline{x} \rangle_m > 0, \; \forall \xi^L \in \partial f^L(\overline{x}), \\ \langle \xi^U, x - \overline{x} \rangle_m > 0, \; \forall \xi^U \in \partial f^U(\overline{x}), \end{cases} \quad \forall x \in X. \tag{SVVI}$$

$$\text{Find } \overline{x} \in X \text{ such that } \begin{cases} \langle \zeta^L, x - \overline{x} \rangle_m > 0, \; \forall \zeta^L \in \partial f^L(x), \\ \langle \zeta^U, x - \overline{x} \rangle_m > 0, \; \forall \zeta^U \in \partial f^U(x). \end{cases} \quad \forall x \in X. \tag{MVVI}$$

Here $\langle \xi, x - \overline{x} \rangle_m = (\langle \xi_1, x - \overline{x} \rangle, \ldots, \langle \xi_m, x - \overline{x} \rangle)$.

We first present sufficient conditions for local strong e-quasi LU-efficient solutions of (IVOP) using the e-approximate pseudo LU-convexity of type II assumption.

Theorem 1. *Suppose f is e-approximate pseudo LU-convex of type II at $\overline{x} \in X$. If $\overline{x}$ is a solution to (SVVI), then it is also a local strong e-quasi LU-efficient solution to (IVOP).*

Proof. Assume $\overline{x}$ fails to be a local strong e-quasi LU-efficient solution to (IVOP). Hence for any $\delta > 0$, there exists $x_0 \in B(\overline{x}, \delta) \cap X$ such that

$$f(x_0) - f(\overline{x}) \leq_{LU} -e\|x_0 - \overline{x}\|,$$

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which implies that $f_k(x_0) - f_k(\overline{x}) \leq_{LU} -e_k\|x_0 - \overline{x}\|$ for each $k = 1, \ldots, m$. Then

$$\begin{cases} f_k^L(x_0) - f_k^L(\overline{x}) \leq -e_k\|x_0 - \overline{x}\| < 0 \\ f_k^U(x_0) - f_k^U(\overline{x}) \leq -e_k\|x_0 - \overline{x}\| < 0 \end{cases} \tag{1}$$

are satisfied for all $k = 1, \ldots, m$.
Since $f_k = [f_k^L, f_k^U]$ is a locally Lipschitz and e-approximate pseudo LU-convex function of type II at $\overline{x}$ for all $k = 1, \ldots, m$, then f_k^L and f_k^U are both locally Lipschitz and e-approximate pseudo LU-convex of type II functions at $\overline{x}$. Then, there exists $\overline{\delta} > 0$ such that for each $x \in B(\overline{x}, \overline{\delta}) \cap X$ and $k = 1, \ldots, m$

$$\begin{cases} f_k^L(x) - f_k^L(\overline{x}) < 0 \Rightarrow \langle \xi_k^L, x - \overline{x} \rangle < -e_k\|x - \overline{x}\| \leq 0, \quad \forall \xi_k^L \in \partial f_k^L(\overline{x}) \\ f_k^U(x) - f_k^U(\overline{x}) < 0 \Rightarrow \langle \xi_k^U, x - \overline{x} \rangle < -e_k\|x - \overline{x}\| \leq 0, \quad \forall \xi_k^U \in \partial f_k^U(\overline{x}). \end{cases} \tag{2}$$

From (1) and (2), we deduce that there is $x_0 \in B(\overline{x}, \overline{\delta}) \cap X$ satisfying

$$\begin{cases} \langle \xi^L, x_0 - \overline{x} \rangle_m \leqq 0, \qquad \forall \xi^L \in \partial f^L(\overline{x}), \\ \langle \xi^U, x_0 - \overline{x} \rangle_m \leqq 0, \qquad \forall \xi^U \in \partial f^U(\overline{x}). \end{cases}$$

We conclude that $\overline{x}$ is not a solution of (SVVI). □

Remark 4. *We can obtain the same result of the above theorem using the e-approximate LU-convexity assumption (see Theorem 5.2 in [20]).*

Sufficient optimality conditions in terms of (MVVI) instead of (SVVI) requires approximate convexity assumptions to be imposed on $-f_k$ as shown in the next theorem.

Theorem 2. *Suppose each $-f_k$ is e-approximate LU-convex at $\overline{x}$ for $k = 1, \ldots, m$. If $\overline{x}$ is a solution to (MVVI), then it is a local strong e-quasi LU-efficient solution to (IVOP).*

Proof. Assume the vector $\overline{x}$ fails to be a local strong e-quasi LU-efficient solution to (IVOP). Hence for any $\delta > 0$ there is $x_0 \in B(\overline{x}, \delta) \cap X$ satisfying $f(x_0) - f(\overline{x}) \leq_{LU} -e\|x_0 - \overline{x}\|$, therefore $f_k(x_0) - f_k(\overline{x}) \leq_{LU} -e_k\|x_0 - \overline{x}\|$ for all $k = 1, \ldots, m$. Then

$$\begin{cases} f_k^L(x_0) - f_k^L(\overline{x}) \leq -e_k\|x_0 - \overline{x}\| \\ f_k^U(x_0) - f_k^U(\overline{x}) \leq -e_k\|x_0 - \overline{x}\| \end{cases} \tag{3}$$

are satisfied for each $k = 1, \ldots, m$.

Since $-f_k = [-f_k^L, -f_k^U]$ is a locally Lipschitz and e-approximate LU-convex function at $\overline{x}$ for all $k = 1, \ldots, m$, therefore, both $-f_k^L$ and $-f_k^U$ are locally Lipschitz and approximate e-convex functions at $\overline{x}$. Then, there is $\overline{\delta} > 0$ such that for each $x \in B(\overline{x}, \overline{\delta}) \cap X$ and $k = 1, \ldots, m$

$$\begin{cases} (-f_k^L)(\overline{x}) - (-f_k^L)(x) \geq \langle \zeta_k^L, x - \overline{x} \rangle - e\|\overline{x} - x\| \qquad \forall \zeta_k^L \in \partial(-f_k^L)(x), \\ (-f_k^U)(\overline{x}) - (-f_k^U)(x) \geq \langle \zeta_k^U, \overline{x} - x \rangle - e\|x - \overline{x}\| \qquad \forall \zeta_k^U \in \partial(-f_k^U)(x). \end{cases}$$

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which implies that

$$\begin{cases} f_k^L(x) - f_k^L(\overline{x}) \geq \langle \zeta_k^L, x - \overline{x} \rangle - e\|x - \overline{x}\| & \forall \zeta_k^L \in \partial(-f_k^L)(x), \\ f_k^U(x) - f_k^U(\overline{x}) \geq \langle \zeta_k^U, x - \overline{x} \rangle - e\|x - \overline{x}\| & \forall \zeta_k^U \in \partial(-f_k^U)(x). \end{cases} \tag{4}$$

Using (3), (4) and taking into account the fact that $\partial(-f)(x) = -\partial f(x)$, we obtain that there is $x_0 \in B(\overline{x}, \overline{\delta}) \cap X$ such that for all $\zeta_k^L \in \partial f_k^L(x_0)$ and $\zeta_k^U \in \partial f_k^U(x_0)$,

$$\begin{cases} \langle \zeta_k^L, x_0 - \overline{x} \rangle = \langle -\zeta_k^L, x_0 - \overline{x} \rangle \leq f_k^L(x_0) - f_k^L(\overline{x}) + e\|x_0 - \overline{x}\| \leq 0, \\ \langle \zeta_k^U, x_0 - \overline{x} \rangle = \langle -\zeta_k^U, x_0 - \overline{x} \rangle \leq f_k^U(x_0) - f_k^U(\overline{x}) + e\|x_0 - \overline{x}\| \leq 0. \end{cases}$$

Therefore, there is $x_0 \in B(\overline{x}, \overline{\delta}) \cap X$ satisfying

$$\begin{cases} \langle \zeta^L, x_0 - \overline{x} \rangle_m \leqq 0, & \forall \zeta^L \in \partial f^L(x_0), \\ \langle \zeta^U, x_0 - \overline{x} \rangle_m \leqq 0, & \forall \zeta^U \in \partial f^U(x_0). \end{cases}$$

We conclude that $\overline{x}$ does not solve (MVVI). □

The previous result still hold true if we replace the approximate convexity assumption by approximate pseudo convexity.

Theorem 3. *Assume each $-f_k$ is e-approximate pseudo LU-convex of type II at $\overline{x}$ for $k = 1, \ldots, m$. If $\overline{x}$ solves (MVVI), then $\overline{x}$ is a local strong e-quasi LU-efficient solution to (IVOP).*

Proof. The proof is similar to that of Theorem 2. □

5. Necessary and sufficient conditions for e-approximate LU-efficient solutions

We consider the following approximate vector variational inequalities of Stampacchia and Minty type as follows:

$$\text{Find } \overline{x} \in X \text{ such that there is no } \delta > 0 \text{ satisying} \tag{ASVVI}$$

$$\begin{cases} \langle \xi^L, x - \overline{x} \rangle_m \leqq -e\|x - \overline{x}\| \\ \langle \xi^U, x - \overline{x} \rangle_m \leqq -e\|x - \overline{x}\| \end{cases} \quad \forall x \in B(\overline{x}, \delta), \forall \xi^L \in \partial f^L(\overline{x}), \forall \xi^U \in \partial f^U(\overline{x}).$$

$$\text{Find } \overline{x} \in X \text{ such that there is no } \delta > 0 \text{ satisying} \tag{AMVVI}$$

$$\begin{cases} \langle \xi^L, x - \overline{x} \rangle_m \leqq -e\|x - \overline{x}\| \\ \langle \xi^U, x - \overline{x} \rangle_m \leqq -e\|x - \overline{x}\| \end{cases} \quad \forall x \in B(\overline{x}, \delta), \forall \xi^L \in \partial f^L(x), \forall \xi^U \in \partial f^U(x).$$

Hereafter, if the above definition is fulfilled for e, then we say that $\overline{x}$ is a solution for ASVVI (or AMVVI) with respect to e.

In the following theorem, we will see that solutions to (ASVVI) are also e-approximate LU-efficient solutions of (IVOP) when the interval-valued objective function satisfies the pseudo approximate convexity hypothesis.

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Theorem 4. *Suppose f is e-approximate pseudo LU-convex function of type II at $\overline{x}$. If $\overline{x}$ solves (ASVVI) w.r.t. e, then $\overline{x}$ is an e-approximate LU-efficient solution to (IVOP).*

Proof. Assume the vector $\overline{x}$ is an e-approximate LU-efficient solution of (IVOP). Hence, there exists $\overline{\delta} > 0$ such that, for all $x \in B(\overline{x}, \overline{\delta})$, we have $f(x) - f(\overline{x}) \leq_{LU} -e\|x - \overline{x}\|$, which implies that $f_k(x) - f_k(\overline{x}) \leq_{LU} -e_k\|x - \overline{x}\|$ for all $k = 1, \ldots, m$. Then

$$f_k^L(x) - f_k^L(\overline{x}) \leq -e_k\|x - \overline{x}\| < 0$$

and

$$f_k^U(x) - f_k^U(\overline{x}) \leq -e_k\|x - \overline{x}\| < 0$$

hold true for any $k = 1, \ldots, m$.

Since $f_k = [f_k^L, f_k^U]$ is a locally Lipschitz and e-approximate pseudo LU-convex function of type II at $\overline{x}$ for all $k = 1, \ldots, m$, then both f_k^L and f_k^U are locally Lipschitz and e-approximate pseudo LU-convex functions of type II at $\overline{x}$. Consequently, there exists $\delta > 0$ with $\delta < \overline{\delta}$, such that, for all $x \in B(\overline{x}, \delta)$ and $k = 1, \ldots, m$ one has

$$\begin{cases} \langle \xi_k^L, x - \overline{x} \rangle < -e_k\|x - \overline{x}\|, & \forall \xi_k^L \in \partial f_k^L(\overline{x}) \\ \langle \xi_k^U, x - \overline{x} \rangle < -e_k\|x - \overline{x}\|, & \forall \xi_k^U \in \partial f_k^U(\overline{x}). \end{cases} \tag{5}$$

From (5), there is $\delta > 0$ such that for all $x \in B(\overline{x}, \delta) \cap X$ one has

$$\begin{cases} \langle \xi^L, x - \overline{x} \rangle_m \leqq -e\|x - \overline{x}\|, & \forall \xi^L \in \partial f^L(\overline{x}), \\ \langle \xi^U, x - \overline{x} \rangle_m \leqq -e\|x - \overline{x}\|, & \forall \xi^U \in \partial f^U(\overline{x}). \end{cases}$$

We deduce that $\overline{x}$ cannot be a solution of (ASVVI) with respect to e. □

In the following theorem, we prove that every e-approximate LU-efficient solution to (IVOP) is still a solution of (ASVVI) w.r.t. e in the case of e-approximate quasi LU-convexity of type II of $-f$.

Theorem 5. *Suppose $-f$ is e-approximate quasi LU-convex function of type II at $\overline{x}$. If $\overline{x}$ is an e-approximate LU-efficient solution to (IVOP), then $\overline{x}$ solves (ASVVI) w.r.t. e.*

Proof. Assume that $\overline{x}$ is not a solution of (ASVVI) w.r.t. e. Hence, there is $\overline{\delta} > 0$ such that, for all $x \in B(\overline{x}, \overline{\delta})$, $\xi^L \in \partial f^L(\overline{x})$ and $\xi^U \in \partial f^U(\overline{x})$ one has

$$\begin{cases} \langle \xi^L, x - \overline{x} \rangle_m \leqq -e\|x - \overline{x}\|, & \forall \xi^L \in \partial f^L(\overline{x}), \\ \langle \xi^U, x - \overline{x} \rangle_m \leqq -e\|x - \overline{x}\|, & \forall \xi^U \in \partial f^U(\overline{x}). \end{cases}$$

Then,

$$\begin{cases} \langle \xi_k^L, x - \overline{x} \rangle \leq -e_k\|x - \overline{x}\| < 0, & \forall \xi_k^L \in \partial f_k^L(\overline{x}) \\ \langle \xi_k^U, x - \overline{x} \rangle \leq -e_k\|x - \overline{x}\| < 0, & \forall \xi_k^U \in \partial f_k^U(\overline{x}) \end{cases}$$

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hold true for all $k = 1, \ldots, m$. Consequently, from $\partial(-f)(x) = -\partial f(x)$, it follows that

$$\begin{cases} \langle -\xi_k^L, x - \overline{x} \rangle > 0, \quad \forall(-\xi_k^L) \in \partial(-f_k^L)(\overline{x}) \\ \langle -\xi_k^U, x - \overline{x} \rangle > 0, \quad \forall(-\xi_k^U) \in \partial(-f_k^U)(\overline{x}). \end{cases} \tag{6}$$

Since $-f_k = [-f_k^L, -f_k^U]$ is a locally Lipschitz and e-approximate quasi LU-convex function of type II at $\overline{x}$ for all $k = 1, \ldots, m$, therefore, both f_k^L and f_k^U are locally Lipschitz and e-approximate quasi LU-convex functions of type II at $\overline{x}$. Then, by (6) there is $\delta > 0$ with $\delta < \overline{\delta}$, such that, for each $x \in B(\overline{x}, \delta) \setminus \{\overline{x}\}$, one has

$$\begin{cases} (-f_k^L)(x) - (-f_k^L)(\overline{x}) > e_k \|x - \overline{x}\|, \\ (-f_k^U)(x) - (-f_k^U)(\overline{x}) > e_k \|x - \overline{x}\|. \end{cases}$$

This yields

$$\begin{cases} f_k^L(x) - f_k^L(\overline{x}) < -e_k \|x - \overline{x}\|, \\ f_k^U(x) - f_k^U(\overline{x}) < -e_k \|x - \overline{x}\|. \end{cases}$$

Therefore there is $\delta > 0$ satisfying for each $x \in B(\overline{x}, \delta) \setminus \{\overline{x}\}$,

$$f(x) - f(\overline{x}) \leq_{LU} -e\|x - \overline{x}\|.$$

This proves the theorem as $\overline{x}$ cannot be an e-approximate LU-efficient solution to (IVOP). □

A direct consequence of Theorem 4 and Theorem 5 is presented in the following corollary.

Corollary 1. *Suppose f is e-approximate pseudo LU-convex of type II at $\overline{x} \in X$ and $-f$ is e-approximate quasi LU-convex of type II at $\overline{x}$. Then, $\overline{x}$ is an e-approximate LU-efficient solution to (IVOP) if and only if $\overline{x}$ solves (ASVVI) w.r.t. e.*

The following theorem illustrates when a solution of (AMVVI) w.r.t. e is also an e-approximate LU-efficient solution to (IVOP).

Theorem 6. *Suppose $-f$ is e-approximate pseudo LU-convex function of type II at $\overline{x}$. If $\overline{x}$ solves (AMVVI) w.r.t. e, then $\overline{x}$ is an e-approximate LU-efficient solution to (IVOP).*

Proof. Assume that $\overline{x}$ is not an e-approximate LU-efficient solution to (IVOP). Thus, there exists $\overline{\delta} > 0$ such that, for all $x \in B(\overline{x}, \overline{\delta})$, we have $f(x) - f(\overline{x}) \leq_{LU} -e\|x - \overline{x}\|$, which implies that $f_k(x) - f_k(\overline{x}) \leq_{LU} -e_k\|x - \overline{x}\|$ for each $k = 1, \ldots, m$. Then

$$f_k^L(x) - f_k^L(\overline{x}) \leq -e_k\|x - \overline{x}\| < 0$$

and

$$f_k^U(x) - f_k^U(\overline{x}) \leq e_k\|x - \overline{x}\| < 0,$$

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are satisfied for each $k = 1, \dots, m$. Then

$$\begin{cases} -f_k^L(\overline{x}) - (-f_k^L)(x) < 0 \\ -f_k^U(\overline{x}) - (-f_k^U)(x) < 0. \end{cases} \tag{7}$$

Since $-f_k = [-f_k^L, -f_k^U]$ is a locally Lipschitz and e-approximate pseudo LU-convex function of type II at $\overline{x}$ for all $k = 1, \dots, m$, therefore, both $-f_k^L$ and $-f_k^U$ are all locally Lipschitz and e-approximate pseudo LU-convex functions of type II at $\overline{x}$. Then, by (7) there exists $\delta > 0$ with $\delta < \overline{\delta}$, such that, for all $x \in B(\overline{x}, \delta)$,

$$\begin{cases} \langle \zeta_k^L, \overline{x} - x \rangle < -e_k \|\overline{x} - x\| & \forall \zeta_k^L \in \partial(-f_k^L)(x), \\ \langle \zeta_k^U, \overline{x} - x \rangle < -e_k \|\overline{x} - x\| & \forall \zeta_k^U \in \partial(-f_k^U)(x). \end{cases} \tag{8}$$

Using (8) and taking into account the fact that $\partial(-f)(x) = -\partial f(x)$ for all $x \in X$, we obtain

$$\begin{cases} \langle \zeta_k^L, x - \overline{x} \rangle = \langle -\zeta_k^L, \overline{x} - x \rangle \leq -e_k \|x - \overline{x}\|, & \forall \zeta_k^L \in \partial f_k^L(\overline{x}) \\ \langle \zeta_k^U, x - \overline{x} \rangle = \langle -\zeta_k^U, \overline{x} - x \rangle \leq -e_k \|x - \overline{x}\|, & \forall \zeta_k^U \in \partial f_k^U(\overline{x}). \end{cases}$$

Therefore, there exists $\delta > 0$ such that for any $x \in B(\overline{x}, \delta) \cap X$ we have

$$\begin{cases} \langle \zeta^L, x - \overline{x} \rangle_m \leqq -e\|x - \overline{x}\|, & \forall \zeta^L \in \partial f^L(\overline{x}), \\ \langle \zeta^U, x - \overline{x} \rangle_m \leqq -e\|x - \overline{x}\|, & \forall \zeta^U \in \partial f^U(\overline{x}). \end{cases}$$

This establishes that $\overline{x}$ is not a solution of (AMVVI) w.r.t. e. □

The next result specifies when an e-approximate LU-efficient solution to (IVOP) is also a solution of (AMVVI) w.r.t. e.

Theorem 7. *Suppose f is e-approximate quasi LU-convex function of type II at $\overline{x}$. If $\overline{x}$ is an e-approximate LU-efficient solution to (IVOP), then $\overline{x}$ solves (AMVVI) w.r.t. e.*

Proof. Assume that $\overline{x}$ is not a solution of (ASVVI) w.r.t. e. Then there is $\overline{\delta} > 0$ satisfying for each $\xi^L \in \partial f^L(\overline{x})$, $\xi^U \in \partial f^U(\overline{x})$ and $x \in B(\overline{x}, \overline{\delta})$, we have

$$\begin{cases} \langle \xi^L, x - \overline{x} \rangle_m \leqq -e\|x - \overline{x}\|, \\ \langle \xi^U, x - \overline{x} \rangle_m \leqq -e\|x - \overline{x}\|. \end{cases}$$

Hence

$$\begin{cases} \langle \xi_k^L, x - \overline{x} \rangle \leq -e_k\|x - \overline{x}\| < 0, & \forall \xi_k^L \in \partial f_k^L(\overline{x}) \\ \langle \xi_k^U, x - \overline{x} \rangle \leq -e_k\|x - \overline{x}\| < 0, & \forall \xi_k^U \in \partial f_k^U(\overline{x}). \end{cases}$$

are satisfied for all $k = 1, \dots, m$. Consequently, from $\partial(-f)(x) = -\partial f(x)$ we deduce that

$$\begin{cases} \langle -\xi_k^L, x - \overline{x} \rangle > 0, & \forall (-\xi_k^L) \in \partial(-f_k^L)(\overline{x}) \\ \langle -\xi_k^U, x - \overline{x} \rangle > 0, & \forall (-\xi_k^U) \in \partial(-f_k^U)(\overline{x}) \end{cases} \tag{9}$$

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Since $-f_k = [-f_k^L, -f_k^U]$ is a locally Lipschitz and e-approximate quasi LU-convex function of type II at $\overline{x}$ for all $k = 1, \ldots, m$, then both $-f_k^L$ and $-f_k^U$ are all locally Lipschitz and e-approximate quasi LU-convex functions of type II at $\overline{x}$. It follows from (9) that there exists $\delta > 0$ with $\delta < \overline{\delta}$ such that for all $x \in B(\overline{x}, \delta)$,

$$\begin{cases} (-f_k^L)(x) - (-f_k^L)(\overline{x}) > e_k\|x - \overline{x}\|, \\ (-f_k^U)(x) - (-f_k^U)(\overline{x}) > e_k\|x - \overline{x}\|. \end{cases}$$

This implies that

$$\begin{cases} f_k^L(x) - f_k^L(\overline{x}) < -e_k\|x - \overline{x}\|, \\ f_k^U(x) - f_k^U(\overline{x}) < -e_k\|x - \overline{x}\|. \end{cases}$$

Thus there is $\delta > 0$ satisfying for each $x \in B(\overline{x}, \delta) \setminus \{\overline{x}\}$,

$$f(x) - f(\overline{x}) \leq_{LU} -e\|x - \overline{x}\|.$$

We conclude that $\overline{x}$ cannot be an e-approximate LU-efficient solution to (IVOP). $\square$

The following corollary is a direct consequence of Theorems 6 and 7.

Corollary 2. *Suppose f is e-approximate pseudo LU-convex of type II at $\overline{x}$ and $-f$ is e-approximate quasi LU-convex of type II at $\overline{x}$. Then, $\overline{x}$ is an e-approximate LU-efficient solution to (IVOP) if and only if $\overline{x}$ solves (AMVVI) w.r.t. e.*

Remark 5.
i) *We can show that similar results of this section can be obtained when using e-approximate LU-convexity assumptions.*
ii) *As the interval-valued vector optimization problems is more general than vector optimization problems, the results of this section represent a generalization of some results obtained in [17,18].*

6. Numerical example

Consider the following example of (IVOP):

$$\begin{aligned} \min f(x) = (f_1(x), f_2(x)) = ([f_1^L(x), f_1^U(x)], [f_2^L(x), f_2^U(x)]) \\ \text{such that} \quad x \in X = [-1, 1], \end{aligned}$$

where

$$f_1^L(x) = \begin{cases} 2x - x^2 & x \geq 0 \\ 3x & x < 0 \end{cases}, \quad f_1^U(x) = \begin{cases} x^3 + x & x \geq 0 \\ 2x & x < 0 \end{cases}$$
$$f_2^L(x) = \begin{cases} x^3 + x & x \geq 0 \\ 3x & x < 0 \end{cases}, \quad f_2^U(x) = \begin{cases} 2x^3 + x & x \geq 0 \\ 1.5x & x < 0 \end{cases}.$$

Let $e = (1, 1)^T$. Observe that f is an e-approximate pseudo LU-convex function of type II at $\overline{x} = 0$. It is also easy to check that for any $\delta > 0$ and $x \in (0, \delta) \cap X$, the following inequalities are not satisfied

$$(\langle \xi_1^L, x - \overline{x} \rangle, \langle \xi_2^L, x - \overline{x} \rangle)^T + e\|x - \overline{x}\| = (\xi_1^L x, \xi_2^L x)^T + (|x|, |x|)^T < 0,$$

$$(\langle \xi_1^U, x-\overline{x}\rangle, \langle \xi_2^U, x-\overline{x}\rangle)^T + e\|x-\overline{x}\| = (\xi_1^U x, \xi_2^U x)^T + (|x|,|x|)^T < 0,$$

where

$$\xi_1^L \in \partial f_1^L(0) = [2,3], \quad \xi_2^L \in \partial f_2^L(0) = [1,3],$$
$$\xi_1^U \in \partial f_1^U(0) = [1,2], \quad \xi_2^U \in \partial f_2^U(0) = [1,1.5].$$

Thus, there does not exist $\delta > 0$ such that, for all $x \in (-\delta,\delta) \cap X, x \neq \overline{x}$, $\xi^L \in \partial f^L(\overline{x})$ and $\xi^U \in \partial f^U(\overline{x})$ one has

$$\begin{cases} \langle \xi^L, x-\overline{x}\rangle_2 \leqq -e\|x-\overline{x}\|, \\ \langle \xi^U, x-\overline{x}\rangle_2 \leqq -e\|x-\overline{x}\|. \end{cases}$$

Therefore the point $\overline{x} = 0$ solves (ASVVI).

Now, since f is e-approximate pseudo LU-convex of type II at $\overline{x} = 0$, then by Theorem 4, $\overline{x} = 0$ should be an e-approximate LU-efficient solution to (IVOP). Indeed, for any $\delta > 0$ and $x \in (0,\delta) \cap X$, the following inequalities are not satisfied

$$f_1(x) + \|x-\overline{x}\| = [3x - x^2, x^3 + 2x] <_{LU} f_1(0) = [0,0],$$

$$f_2(x) + \|x-\overline{x}\| = [x^3 + 2x, 2x^3 + 2x] <_{LU} f_2(0) = [0,0].$$

Hence, we deduce that there exists no $\delta > 0$ such that, for all $x \in (-\delta,\delta) \cap X, x \neq \overline{x}$, one has

$$(f_1(x), f_2(x))^T + e\|x-\overline{x}\| \leq_{LU} (f_1(0), f_2(0))^T.$$

7. Conclusion

In this paper, we have introduced new optimality conditions for a vector optimization problem with interval-valued vector functions using the concept of local strong e-quasi efficiency and e-approximate efficiency hypotheses. We have established the relationships between this problem and vector variational inequality problems under the hypotheses of e-approximate LU-convexity or generalized e-approximate LU-convexity. Hence, our presented results extend and improve the corresponding main results obtained in [17,18,20].

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