Correct Expression of Material Derivative and Application to the Navier-Stokes Equation
—— The solution existence condition of Navier-Stokes equation

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The material derivative is important in continuum physics. This paper shows that the expression $\frac{d}{dt}=\frac{\partial}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})$, used in most literature and textbooks, is incorrect. This article presents correct expression of the material derivative, namely $\frac{d(:)}{dt}=\frac{\partial}{\partial t}(:)+\boldsymbol{v}\cdot[\boldsymbol{\nabla}(:)]$. As an application, the form-solution of the Navier-Stokes equation is proposed. The form-solution reveals that the solution existence condition of the Navier-Stokes equation is that " *The Navier-Stokes equation has a solution if and only if the determinant of flow velocity gradient is not zero, namely* $\det(\boldsymbol{\nabla}\boldsymbol{v})\neq 0$."

Keywords: material derivative, velocity gradient, tensor calculus, tensor determinant, Navier-Stokes equations, solution existence condition

In continuum physics, there are two ways of describing continuous media or flows, the Lagrangian description and the Eulerian description. In the Eulerian description, the material derivative with respect to time must be defined. For mass density $\rho(\boldsymbol{x},t)$, flow velocity $\boldsymbol{v}(\boldsymbol{x},t)=v_k\boldsymbol{e}_k$, and stress tensor $\boldsymbol{\sigma}=\sigma_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j$, the material derivatives are given by: $\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}+v_1\frac{\partial\rho}{\partial x^1}+v_2\frac{\partial\rho}{\partial x^2}+v_3\frac{\partial\rho}{\partial x^3}$,, $\frac{d\boldsymbol{v}}{dt}=\frac{\partial\boldsymbol{v}}{\partial t}+v_1\frac{\partial\boldsymbol{v}}{\partial x^1}+v_2\frac{\partial\boldsymbol{v}}{\partial x^2}+v_3\frac{\partial\boldsymbol{v}}{\partial x^3}$, and $\frac{d\boldsymbol{\sigma}}{dt}=\frac{\partial\boldsymbol{\sigma}}{\partial t}+v_1\frac{\partial\boldsymbol{\sigma}}{\partial x^1}+v_2\frac{\partial\boldsymbol{\sigma}}{\partial x^2}+v_3\frac{\partial\boldsymbol{\sigma}}{\partial x^3}$, respectively, where t is time, $\boldsymbol{x}=x_k\boldsymbol{e}_k$ is position, $\boldsymbol{e}_k$ is a base vector and $v_k=\frac{\partial x_k}{\partial t}$ is the flow velocity.

In most popular textbooks, handbooks and encyclopedia, such as Refs. 1–12, the above material derivatives are expressed as: $\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\rho$, $\frac{d\boldsymbol{v}}{dt}=\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$, and $\frac{d\boldsymbol{\sigma}}{dt}=\frac{\partial\boldsymbol{\sigma}}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$, where the gradient operator $\boldsymbol{\nabla}=\boldsymbol{e}_k\frac{\partial}{\partial x_k}$.

To simplify the aforementioned equations further, a differential operator $\frac{d}{dt}=\frac{\partial}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})$ is introduced. This operator is used by most fluid mechanics textbooks, including well-known graduate textbooks such as Landau & Lifshitz,[1, 2] Prandtl,[3, 4] Anderson,[5] Pope,[6] Cengel & Cimbala,[7], Kundu et al.[8], Woan [9], wikipedia [10, 11], Britannica [12] and even mathematical paper [13]. In the latest version of Landau & Lifshitz,[14] the material derivative of flow velocity is given in another form: $\frac{d\boldsymbol{v}}{dt}=\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}\nabla)\boldsymbol{v}$.

Besides the books mentioned above, there is a large amount of literature in fluid mechanics taking the same expressions as Refs. 1 and 14. This makes it difficult for readers to identify which expression for the material derivative is correct, causing great confusion to both students and scholars.

Although some authors, such as Lighthill,[15] Batchelor,[16] Frisch,[17], Nhan & Nam [18], Xie,[19] and Zhao [20] have used the correct expression, it seems that most fluid mechanics textbooks and academic literature have adopted an incorrect expression for the material derivative as in Ref. 1. Therefore, to revive the great influence of Landau in physics and fluid mechanics, we attempt to address this issue in this dedicated paper, where we revisit the material derivative to show why the operators $\frac{d}{dt}=\frac{\partial}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})$ and expression $\frac{d\boldsymbol{v}}{dt}=\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}\nabla)\boldsymbol{v}$ are incorrect, and derive a correct expression by using standard tensor calculus [21–24].

In Landau & Lifshitz,[1] the material derivative is given as:

$$dx\frac{\partial\boldsymbol{v}}{\partial x}+dy\frac{\partial\boldsymbol{v}}{\partial y}+dz\frac{\partial\boldsymbol{v}}{\partial z}=(d\boldsymbol{r}\cdot\mathbf{grad})\boldsymbol{v}. \tag{1}$$

Although Landau's fluid mechanics is well known worldwide, the above expression is incorrect.

Proof: Since $\boldsymbol{v}\cdot\mathbf{grad}=\boldsymbol{v}\cdot\boldsymbol{\nabla}=(v_i\boldsymbol{e}_i)\cdot(\partial_j\boldsymbol{e}_j)=v_{i,j}\boldsymbol{e}_i\cdot\boldsymbol{e}_j=v_{i,j}\delta_{ij}=\text{div}\boldsymbol{v}$, and $(\boldsymbol{v}\cdot\mathbf{grad})\boldsymbol{v}=(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}=(v_{j,i}\boldsymbol{e}_i\cdot\boldsymbol{e}_j)(v_k\boldsymbol{e}_k)=\delta_{ij}v_{j,i}v_k\boldsymbol{e}_k=v_{i,i}v_k\boldsymbol{e}_k=(\text{div}\boldsymbol{v})\boldsymbol{v}$, where the divergence is $\text{div}\boldsymbol{v}=\frac{\partial v_1}{\partial x^1}+\frac{\partial v_2}{\partial x^2}+\frac{\partial v_3}{\partial x^3}$, so $(\text{div}\boldsymbol{v})\boldsymbol{v}=(\frac{\partial v_1}{\partial x^1}+\frac{\partial v_2}{\partial x^2}+\frac{\partial v_3}{\partial x^3})\boldsymbol{v}$. Thus:

$$(\boldsymbol{v}\cdot\mathbf{grad})\boldsymbol{v}\neq v_1\frac{\partial\boldsymbol{v}}{\partial x^1}+v_2\frac{\partial\boldsymbol{v}}{\partial x^2}+v_3\frac{\partial\boldsymbol{v}}{\partial x^3}. \tag{2}$$

In the latest version of Landau & Lifshitz,[14] for reasons which are unclear, the material derivative expression has been changed to $\frac{d\boldsymbol{v}}{dt}=\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}\nabla)\boldsymbol{v}$; this is also incorrect.

Proof: Since $\boldsymbol{v}\boldsymbol{\nabla}=(v_i\boldsymbol{e}_i)(\partial_j\boldsymbol{e}_j)=v_{i,j}\boldsymbol{e}_i\boldsymbol{e}_j$,

$$(\boldsymbol{v}\boldsymbol{\nabla})\boldsymbol{v}=(v_{j,i}\boldsymbol{e}_i\boldsymbol{e}_j)(v_k\boldsymbol{e}_k)=(v_{j,i}\boldsymbol{e}_i\boldsymbol{e}_j)(v_k\boldsymbol{e}_k),$$

which indicates that $\boldsymbol{v}\boldsymbol{\nabla}$ is a 3rd-order tensor rather than a vector. Thus:

$$(\boldsymbol{v}\boldsymbol{\nabla})\boldsymbol{v}\neq v_1\frac{\partial\boldsymbol{v}}{\partial x^1}+v_2\frac{\partial\boldsymbol{v}}{\partial x^2}+v_3\frac{\partial\boldsymbol{v}}{\partial x^3}. \tag{3}$$

To obtain the correct formulation of the material derivative, we perform some basic tensor calculus [21, 22]. The del operator is a vector differential operator, and defined by $\boldsymbol{\nabla} = \boldsymbol{e}_i \frac{\partial}{\partial x_i}$. An important note concerning the del operator $\boldsymbol{\nabla}$ is in order. Two types of gradients are used in continuum physics: left and right gradients. The left gradient is the usual gradient and the right gradient is the transpose of the forward gradient operator.

To see the difference between the left and right of gradients, consider a vector function $\boldsymbol{A} = A_i(\boldsymbol{x})\boldsymbol{e}_i$. The left gradient of a vector $\boldsymbol{A}$ is $\boldsymbol{\nabla} \otimes \boldsymbol{A} \equiv \boldsymbol{e}_j \frac{\partial}{\partial x_j} \otimes (A_i \boldsymbol{e}_i) = \frac{\partial A_i}{\partial x_j} \boldsymbol{e}_j \otimes \boldsymbol{e}_i = A_{i,j} \boldsymbol{e}_j \otimes \boldsymbol{e}_i$, or written as $\boldsymbol{\nabla A} \equiv \boldsymbol{e}_j \frac{\partial}{\partial x_j}(A_i \boldsymbol{e}_i) = \frac{\partial A_i}{\partial x_j} \boldsymbol{e}_j \boldsymbol{e}_i = A_{i,j} \boldsymbol{e}_j \boldsymbol{e}_i$. The right gradient of a vector $\boldsymbol{A}$ is $\boldsymbol{A} \otimes \nabla \equiv (A_i \boldsymbol{e}_i) \otimes \boldsymbol{e}_j \frac{\partial}{\partial x_j} = \frac{\partial A_i}{\partial x_j} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = A_{i,j} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$, or written as $\boldsymbol{A\nabla} \equiv (A_i \boldsymbol{e}_i) \boldsymbol{e}_j \frac{\partial}{\partial x_j} = \frac{\partial A_i}{\partial x_j} \boldsymbol{e}_i \boldsymbol{e}_j = A_{i,j} \boldsymbol{e}_i \boldsymbol{e}_j$, where $A_{i,j} = \frac{\partial A_i}{x_j}$ [21–24].

The gradient of a scalar function is a vector, the divergence of a vector-valued function is a scalar $\boldsymbol{\nabla} \cdot \boldsymbol{A}$, and the gradient of a vector-valued function is a second-order tensor $\boldsymbol{\nabla A}$. Although the del operator has some of the properties of a vector, it does not have them all, because it is an operator. For instance, $\boldsymbol{\nabla} \cdot \boldsymbol{A}$ is a scalar (called the divergence of $\boldsymbol{A}$) whereas $\boldsymbol{A} \cdot \boldsymbol{\nabla}$ is a scalar differential operator, where $\boldsymbol{A}$ is a vector. Thus the del operator $\boldsymbol{\nabla}$ does not commute in this sense [21–24].

It worth to emphasise that the right gradient of a vector $\boldsymbol{A\nabla}$ is a more natural one, is often used in defining the deformation gradient tensor, displacement gradient tensor, and velocity gradient tensor. It is clear that $\boldsymbol{A\nabla} = (\boldsymbol{\nabla A})^T$.

With the above-mentioned understanding of tensor calculus, we material derivative expression for scalar, vector and the 2nd order tensor as follows:

1. Mass density $\rho = \rho(\boldsymbol{x}, t)$ is a scalar-valued function of $\boldsymbol{x}$ and t and its differential is $d\rho = \frac{\partial \rho}{\partial t} dt + \frac{\partial \rho}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial t} dt$. For a scalar $\frac{\partial \rho}{\partial \boldsymbol{x}} = \rho \boldsymbol{\nabla} = \boldsymbol{\nabla} \rho$, so $d\rho = \frac{\partial \rho}{\partial t} dt + (\rho \boldsymbol{\nabla}) \cdot \boldsymbol{v} dt = \frac{\partial \rho}{\partial t} dt + \boldsymbol{v} \cdot (\boldsymbol{\nabla} \rho) dt$, or, dividing both sides by dt,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\rho \boldsymbol{\nabla}) \cdot \boldsymbol{v} = \frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot (\boldsymbol{\nabla} \rho). \tag{4}$$

2. Flow velocity $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$ is a vector-valued function of $\boldsymbol{x}$ and t and its differential is $d\boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial t} dt + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial t} dt$. For a vector $\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} = \boldsymbol{v} \otimes \boldsymbol{\nabla} = \boldsymbol{v\nabla} = (\boldsymbol{\nabla v})^T$. Strictly speaking, the right gradient of $\boldsymbol{v}$ should be written as $\boldsymbol{v} \otimes \boldsymbol{\nabla}$; here, in order to agree as far as possible with conventional presentation, we write $\boldsymbol{v} \otimes \boldsymbol{\nabla} = \boldsymbol{v\nabla}$ and, similarly for the left gradient, $\boldsymbol{\nabla} \otimes \boldsymbol{v} = \boldsymbol{\nabla v}$. Hence $(\boldsymbol{v\nabla}) \cdot \boldsymbol{v} = \boldsymbol{v} \cdot (\boldsymbol{\nabla v})$ and $d\boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial t} dt + (\boldsymbol{v\nabla}) \cdot \boldsymbol{v} dt = \frac{\partial \rho}{\partial t} dt + \boldsymbol{v} \cdot (\boldsymbol{\nabla v}) dt$, or, dividing both sides by dt,

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v\nabla}) \cdot \boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot (\boldsymbol{\nabla v}). \tag{5}$$

3. The 2nd order stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{x}, t)$ is a tensor-valued function of $\boldsymbol{x}$ and t and its differential is $d\boldsymbol{\sigma} = \frac{\partial \boldsymbol{\sigma}}{\partial t} dt + \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial t} dt$. For an arbitrary tensor $\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{x}} = \boldsymbol{\sigma \nabla} \neq (\boldsymbol{\nabla \sigma})^T$, hence $d\boldsymbol{\sigma} = \frac{\partial \boldsymbol{\sigma}}{\partial t} dt + (\boldsymbol{\sigma \nabla}) \cdot \boldsymbol{v} dt$, or dividing both sides by dt leads to $\frac{d\boldsymbol{\sigma}}{dt} = \frac{\partial \boldsymbol{\sigma}}{\partial t} + (\boldsymbol{\sigma \nabla}) \cdot \boldsymbol{v}$. Since $(\boldsymbol{\sigma \nabla}) \cdot \boldsymbol{v} = \boldsymbol{v} \cdot (\boldsymbol{\nabla \sigma})$,

$$\frac{d\boldsymbol{\sigma}}{dt} = \frac{\partial \boldsymbol{\sigma}}{\partial t} + \boldsymbol{v} \cdot (\boldsymbol{\nabla \sigma}). \tag{6}$$

If an operator must be introduced for the material derivative, it should be in the following form:

$$\frac{d(:)}{dt} = \frac{\partial}{\partial t}(:) + \boldsymbol{v} \cdot [\boldsymbol{\nabla}(:)]. \tag{7}$$

As an application, the Navier-Stokes equations of incompressible flow can be expressed as follows:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot (\boldsymbol{\nabla v}) = -\frac{1}{\rho} \boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \boldsymbol{v}, \tag{8}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0. \tag{9}$$

The Eq. 8 is momentums equation and Eq. 9 is mass conservation equation. In which, $\boldsymbol{v}(\boldsymbol{x}, t)$ is flow velocity field, ρ is constant mass density, $p(\boldsymbol{x}, t)$ is flow pressure, ν is kinematical viscosity, t is time, $\boldsymbol{x} = x^k \boldsymbol{e}_k$ is position coordinates, $\boldsymbol{e}_k$ is a base vector and $\boldsymbol{v}$ is flow velocity, $\boldsymbol{\nabla} = \boldsymbol{e}_k \frac{\partial}{\partial x^k}$ is gradient operator, and $\boldsymbol{\nabla}^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$.

Applying the divergence operation to both sides of the momentum equation Eq.8 and use the mass conservation leads to a pressure equation: $\boldsymbol{\nabla}^2 \cdot (p\boldsymbol{1}) = -\rho \boldsymbol{\nabla} \cdot (\boldsymbol{v} \cdot \boldsymbol{\nabla v}) = -\rho\{(\boldsymbol{\nabla v})^2 + [\boldsymbol{\nabla}(\boldsymbol{v\nabla})] \cdot \boldsymbol{v}\}$, where $\boldsymbol{1} = \boldsymbol{e}_k \boldsymbol{e}_k$ is an identity tensor and $\boldsymbol{v\nabla} = (\boldsymbol{\nabla v})^T$.

The Navier - Stokes existence and smoothness problem is an open problem in mathematics [25], regardless of numerous abstract studies that have been done by mathematicians. Now the question is that, is it possible to propose a simple criteria on the solution existence of the Navier-Stokes equation without complicated mathematics.

In order to find some useful information from the Navier-Stokes equation, let's have a look at the meaning of $\boldsymbol{v} \cdot (\boldsymbol{\nabla v})$ in Eq. 1. This term is called convective acceleration that is caused by the flow velocity gradient. It is obvious that the convective acceleration $\boldsymbol{v} \cdot (\boldsymbol{\nabla v})$ is the central point of the Navier-Stokes equations. Without the convective acceleration, the solution existence would not be a problem at all. The understanding on the convective term is quite important for the study on the solution existence should be focus on $\boldsymbol{v} \cdot (\boldsymbol{\nabla v})$. Therefore, we attack the open problem from the $\boldsymbol{v} \cdot (\boldsymbol{\nabla v})$.

Assuming the determinant of the velocity gradient is not zero, namely $\det \boldsymbol{\nabla v} \neq 0$, the form-solution of the Navier-Stokes momentum equation in Eq.(8) can be expressed as follows

$$\boldsymbol{v} = \left[\nu \boldsymbol{\nabla}^2 \boldsymbol{v} - \frac{1}{\rho} \boldsymbol{\nabla} p - \frac{\partial \boldsymbol{v}}{\partial t}\right] \cdot (\boldsymbol{\nabla v})^{-1}, \tag{10}$$

equivalently

$$\boldsymbol{v} = (\boldsymbol{\nabla v})^{-T} \cdot \left[\nu \boldsymbol{\nabla}^2 \boldsymbol{v} - \frac{1}{\rho} \boldsymbol{\nabla} p - \frac{\partial \boldsymbol{v}}{\partial t}\right], \tag{11}$$

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equivalently

$$\boldsymbol{v} = \left[\boldsymbol{\nabla}\cdot(\nu\boldsymbol{\nabla v} - \frac{p}{\rho}\mathbf{1}) - \frac{\partial \boldsymbol{v}}{\partial t}\right]\cdot(\boldsymbol{\nabla v})^{-1}, \quad (12)$$

equivalently

$$\boldsymbol{v} = (\boldsymbol{\nabla v})^{-T}\cdot\left[\boldsymbol{\nabla}\cdot(\nu\boldsymbol{\nabla v} - \frac{p}{\rho}\mathbf{1}) - \frac{\partial \boldsymbol{v}}{\partial t}\right]. \quad (13)$$

The new formats of Naver-Stokes equations in Eqs. 10, 11, Eq.(12) and Eq.(13) have never been seen in literature. They are formulated for the first time by Bo-Hua Sun [24]. Those form-solutions provide a rich information on the solution existence.

Therefore we have a conjecture as follows:

Conjecture 1 *The 3D Navier-Stokes equation has a solution if and only if the determinant of flow velocity gradient is not zero, namely*

$$\det(\boldsymbol{\nabla v}) \neq 0.$$

Although we have successfully split the velocity field $\boldsymbol{v}$ from the convective term $\boldsymbol{v}\cdot(\boldsymbol{\nabla v})$, the calculation of the inverse of the velocity gradient $(\boldsymbol{\nabla v})^{-1}$ is still great challenge.

According to the Cayley-Hamilton theorem [22, 23, 26, 27], for the 2nd order tensor $\boldsymbol{\nabla v}$, the following polynomial holds:

$$(\boldsymbol{\nabla v})^3 - I_1(\boldsymbol{\nabla v})^2 + I_2\boldsymbol{\nabla v} - I_3\mathbf{1} = \mathbf{0}, \quad (14)$$

where $I_1 = tr(\boldsymbol{\nabla v}) = \boldsymbol{\nabla}\cdot\boldsymbol{v}$, $I_2 = \frac{1}{2}[(tr\boldsymbol{\nabla v})^2 - tr(\boldsymbol{\nabla v})^2]$ and $I_3 = \det(\boldsymbol{\nabla v})$. Hence, for the case of $\det(\boldsymbol{\nabla v}) \neq 0$, we have:

$$(\boldsymbol{\nabla v})^{-1} = \frac{(\boldsymbol{\nabla v})^2 - I_1\boldsymbol{\nabla v} + I_2\mathbf{1}}{\det(\boldsymbol{\nabla v})}. \quad (15)$$

For incompressible flow, the divergence of velocity gradient is zero, namely, $I_1 = tr(\boldsymbol{\nabla v}) = \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$, thus $I_2 = \frac{1}{2}[(tr\boldsymbol{\nabla v})^2 - tr(\boldsymbol{\nabla v})^2] = -\frac{1}{2}tr(\boldsymbol{\nabla v})^2$. Therefore, the inverse of the velocity gradient for incompressible flow takes a simpler form:

$$(\boldsymbol{\nabla v})^{-1} = \frac{(\boldsymbol{\nabla v})^2 - \frac{1}{2}\mathbf{1}tr(\boldsymbol{\nabla v})^2}{\det(\boldsymbol{\nabla v})}. \quad (16)$$

Therefore, the incompressible flow velocity field in Eq.(12) is then reduced to the following form:

$$\boldsymbol{v} = \frac{\left[\nu\boldsymbol{\nabla}^2\boldsymbol{v} - \frac{1}{\rho}\boldsymbol{\nabla}p - \frac{\partial\boldsymbol{v}}{\partial t}\right]\cdot[(\boldsymbol{\nabla v})^2 - \frac{1}{2}\mathbf{1}tr(\boldsymbol{\nabla v})^2]}{\det(\boldsymbol{\nabla v})}, \quad (17)$$

where $\boldsymbol{\nabla v} = v_{j,i}\boldsymbol{e}_i\boldsymbol{e}_j$, $(\boldsymbol{\nabla v})^2 = \boldsymbol{\nabla v}\cdot\boldsymbol{\nabla v} = v_{j,k}v_{k,i}\boldsymbol{e}_i\boldsymbol{e}_j$, $tr(\boldsymbol{\nabla v})^2 = \mathbf{1} : (\boldsymbol{\nabla v})^2 = v_{i,k}v_{k,i}$ and $\det(\boldsymbol{\nabla v}) = \varepsilon_{ijk}v_{1,i}v_{2,j}v_{3,k}$, and ε_{ijk} is permutation symbol.

For steady flow, $\frac{\partial\boldsymbol{v}}{\partial t} = \mathbf{0}$, the Eq. 18 is reduced to

$$\boldsymbol{v} = \frac{\left[\nu\boldsymbol{\nabla}^2\boldsymbol{v} - \frac{1}{\rho}\boldsymbol{\nabla}p\right]\cdot[(\boldsymbol{\nabla v})^2 - \frac{1}{2}\mathbf{1}tr(\boldsymbol{\nabla v})^2]}{\det(\boldsymbol{\nabla v})}, \quad (18)$$

The great challenge to find solution of the Navier-Stokes equation are all from the existence of the velocity gradient $\boldsymbol{\nabla v}$. The form solution in Eq.(18) reveals that the difficulty of finding a solution for N-S equations is because of existence of the nonlinear term $\boldsymbol{v}\cdot(\boldsymbol{\nabla v})$, or in other words, due to the existence of the velocity field gradient $\boldsymbol{\nabla v}$. Accordingly, the solution of the Navier-Stokes equation will be blowup as $\det\boldsymbol{\nabla v}$ tends to an infinitesimal, and has no solution when $\det\boldsymbol{\nabla v} = 0$.

The 2D N-S equations can be written as follows:

$$\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x_1} + v_2\frac{\partial v_1}{\partial x_2} = \nu(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2}) - \frac{1}{\rho}\frac{\partial p}{\partial x_1}, \quad (19)$$

$$\frac{\partial v_2}{\partial t} + v_1\frac{\partial v_2}{\partial x_1} + v_2\frac{\partial v_2}{\partial x_2} = \nu(\frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2}) - \frac{1}{\rho}\frac{\partial p}{\partial x_1}. \quad (20)$$

where $v_{i,j} = \frac{\partial v_i}{\partial x_j}$. The above equations can be expressed in matrix format

$$\frac{\partial}{\partial t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (21)$$

$$= -\begin{pmatrix} \frac{1}{\rho}\frac{\partial p}{\partial x_1} \\ \frac{1}{\rho}\frac{\partial p}{\partial x_2} \end{pmatrix} + \nu\nabla^2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad (22)$$

where the 2D Laplace operator $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2}$.

The 2D velocity gradient is

$$\boldsymbol{\nabla v} = \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix}, \quad (23)$$

its determinant is

$$\det(\boldsymbol{\nabla v}) = v_{1,1}v_{2,2} - v_{1,2}v_{2,1}, \quad (24)$$

and the inverse of the 2D velocity gradient is thus

$$(\boldsymbol{\nabla}v)^{-1} = \frac{1}{\det(\boldsymbol{\nabla}v)}\begin{pmatrix} v_{2,2} & -v_{1,2} \\ -v_{2,1} & v_{1,1} \end{pmatrix} = \frac{\begin{pmatrix} v_{2,2} & -v_{1,2} \\ -v_{2,1} & v_{1,1} \end{pmatrix}}{v_{1,1}v_{2,2} - v_{1,2}v_{2,1}}. \quad (25)$$

Therefore, from Eq. 21, the 2D flow velocity is given by

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{\begin{pmatrix} v_{2,2} & -v_{1,2} \\ -v_{2,1} & v_{1,1} \end{pmatrix}}{v_{1,1}v_{2,2} - v_{1,2}v_{2,1}}\left[(\nu\nabla^2 - \frac{\partial}{\partial t})\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \frac{1}{\rho}\begin{pmatrix} p_{,1} \\ p_{,2} \end{pmatrix}\right]. \quad (26)$$

For the 2D steady flow, Eq.26 is reduced to

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{\begin{pmatrix} v_{2,2} & -v_{1,2} \\ -v_{2,1} & v_{1,1} \end{pmatrix}}{v_{1,1}v_{2,2} - v_{1,2}v_{2,1}}\left[\nu\nabla^2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \frac{1}{\rho}\begin{pmatrix} p_{,1} \\ p_{,2} \end{pmatrix}\right]. \quad (27)$$

Hence, we have a similar conjecture for 2D flow as follows.

Conjecture 2 *The 2D Navier-Stokes equation has a solution if and only if the determinant of flow velocity gradient is not zero, namely* $\det(\boldsymbol{\nabla v}) \neq 0$*, or equivalently*

$$v_{1,1}v_{2,2} - v_{1,2}v_{2,1} \neq 0.$$

Concerning the geometrical and physical meaning of the conjectures, let's try to give a basic interpretation. Assume that $d^2\boldsymbol{X}$ is small surface area of a moving fluid element with volume $d^3\boldsymbol{X}$, they become to $d^2\boldsymbol{x}$ and $d^3\boldsymbol{x}$ due to the velocity gradient $\det(\boldsymbol{\nabla v})$, respectively. Their relations are

$$d^2\boldsymbol{x} = \det(\boldsymbol{\nabla v})(\boldsymbol{\nabla} v)^{-T} \cdot d^2\boldsymbol{X}, \tag{28}$$

and the volume induced by the velocity gradient $\boldsymbol{\nabla v}$ is

$$d^3\boldsymbol{x} = \det(\boldsymbol{\nabla v})d^3\boldsymbol{X}. \tag{29}$$

From the relations in Eqs.28 and 29, both the surface area $d^2\boldsymbol{x}$ and volume $d^3\boldsymbol{x}$ induced by the velocity gradient $\boldsymbol{\nabla v}$ will shrink to a point as $\det(\boldsymbol{\nabla} v) \to 0$. If we image the finite surface area ($d^2\boldsymbol{x} \neq \boldsymbol{0}$) as a window that flow can go through, it means that, if $d^2\boldsymbol{x} = \boldsymbol{0}$, the window is closed and no flow can go through it.

Denoting the momentum flux density tensor in a viscous fluid $\boldsymbol{\Pi} = p\boldsymbol{I} + \rho\boldsymbol{v} \otimes \boldsymbol{v} - \mu\boldsymbol{\nabla v}$, and μ dynamical viscosity, according to Landau and Lifshitz [14], the local form of the equation of motion of the viscous fluid is $\frac{\partial \rho \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} = \boldsymbol{0}$, which can be rewritten in integral form $\int \left(\frac{\partial \rho \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}\right) d^3\boldsymbol{X} = \boldsymbol{0}$, and further simplified to

$$\frac{\partial}{\partial t}\int \rho \boldsymbol{v} d^3\boldsymbol{X} + \oint \boldsymbol{\Pi} \cdot d^2\boldsymbol{X} = \boldsymbol{0}, \tag{30}$$

by Green's formula.

Using Eqs.28 and 29, we can get an induced form of Eq.30 by the velocity gradient $\boldsymbol{\nabla v}$ as follows

$$\frac{\partial}{\partial t}\int \frac{\rho \boldsymbol{v} d^3\boldsymbol{x}}{\det(\boldsymbol{\nabla v})} + \oint \frac{\boldsymbol{\Pi} \cdot (\boldsymbol{\nabla} v)^T \cdot d^2\boldsymbol{x}}{\det(\boldsymbol{\nabla v})} = \boldsymbol{0}. \tag{31}$$

The Eq.31 will be invalid as $\det(\boldsymbol{\nabla} v) \to 0$. These might be viewed as another interpretation of the conjectures on solution existence.

In conclusion, the incorrect expression of material derivative operator $\frac{d}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})$ is mainly because of the miscalculation of $\boldsymbol{v} \cdot \boldsymbol{\nabla}$; now we know that $\boldsymbol{\nabla}$ is a gradient operator rather than a vector, noting $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$, so that: $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = [(\boldsymbol{e}_i v_i) \cdot (\boldsymbol{e}_j \partial_j)]\boldsymbol{v} = [v_{i,j}\boldsymbol{e}_i \cdot \boldsymbol{e}_j]\boldsymbol{v} = [v_{i,j}\delta_{ij}]\boldsymbol{v} = v_{i,i}\boldsymbol{v} = (\frac{\partial v_1}{\partial x^1} + \frac{\partial v_2}{\partial x^2} + \frac{\partial v_3}{\partial x^3})\boldsymbol{v}$, while $\boldsymbol{v} \cdot (\boldsymbol{\nabla v}) = v_1\frac{\partial}{\partial x^1} + v_2\frac{\partial}{\partial x^2} + v_3\frac{\partial}{\partial x^3}$, therefore $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} \neq \boldsymbol{v} \cdot (\boldsymbol{\nabla v})$. The correct material derivative operator should be defined as: $\frac{d(:)}{dt} = \frac{\partial}{\partial t}(:) + \boldsymbol{v} \cdot [\boldsymbol{\nabla}(:)]$.

By taking into account of the importance of the convective term $\boldsymbol{v} \cdot \boldsymbol{\nabla v}$, the conjectures on solution existence condition of Navier-Stokes equation have been proposed, which state that " *The Navier-Stokes equation has a solution if and only if the determinant of flow velocity gradient is not zero, namely* $\det(\boldsymbol{\nabla v}) \neq 0$." [24]. To be honest, this study on the solution existence is still in a very basic stage. From future perspective, the mathematicians should be invited for comprehensive investigation and proof of the conjectures.

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[1] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (English version) (1987).

[2] E. M. Lifshitz and L. P. Pitaevskii, *Physical Kinetics* (Elsevier (Singapore) Ltd, 2008)

[3] L. Prandtl, *Führer durch die Strömungslehre*, Herbert Oertel (ed.) (Springer Vieweg, 2012).

[4] L. Prandtl, *Prandtl's essentials of fluid mechanics*, Herbert Oertel (ed.) (Springer-Verlag, 2004).

[5] J. D. Anderson, *Computational Fluid Dynamics–The Basics with Applications* (The McGraw-Hill Companies, Inc., 1995).

[6] S. B. Pope, *Turbulent Flows* (Cambridge University Press, 2000).

[7] Y. A. Cengel and J. M. Cimbala, *Fluid Mechanics: Fundamentals and Applications* (McGraw-Hill Education, Inc., 2020).

[8] P. K. Kundu, I. M. Cohen and D. R. Dowling, *Fluid Mechanics* (Elsevier, 2012).

[9] G. Woan, *The Cambridge Handbook of Physics Formulas* (Cambridge University Press, 2000)

[10] https://en.wikipedia.org/wiki/Gauge_covariant_derivative

[11] https://en.wikipedia.org/wiki/Navier_Stokes_equations

[12] https://www.britannica.com/science/Navier-Stokes-equation

[13] T. Tao, Finite time blowup for an averaged three-dimensional Navier-Stokes equation, J. of the American Math. Society, 29, 3:601-674 (2016).

[14] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Russian version) (2017).

[15] J. Lighthill, *Waves in Fluids* (Cambridge University Press, 1978)

[16] G. K. Batchelor, *An Introduction to Fluid Dynamics*

(Cambridge University Press, 2007)
[17] U. Frisch, *Turbulence* (Cambridge University Press, 1995)
[18] P.T. Nhan and M.D. Nam, *Understanding Viscoelasticity: An Introduction to Rheology* (Springer-Verlag Berlin Heidelberg, 2013).
[19] X. C. Xie, *Modern Tensor Analysis and Its Applications in Continuum Mechanics* (Fudan University Press, 2014)
[20] Y. P. Zhao, *Lectures on Mechanics* (Science Press, China, 2018)
[21] J.N. Reddy, *An Introduction to Continuum Mechanics* (Cambridge University Press NY, 2013).
[22] C. Truesdell, *A First Course in Rational Continuum Mechanics* (Academic Press, New York, 1977).
[23] Bo-Hua Sun, A new additive decomposition of velocity gradient, Physics of Fluids (Letter), 31, 061702 (2019).
[24] B.H. Sun, Correct expression of material derivative in continuum physics. Preprints 2020, 2020030030 (doi: 10.20944/preprints202003.0030.v3).
[25] C.L. Fefferman, Existence and smoothness of the Navier-Stokes equation, The millennium prize problems, Clay Math. Institute, pp. 57-67(2006).
[26] W.R. Hamilton, Lectures on Quaternions. Dublin(1853).
[27] A. Cayley, A Memoir on the Theory of Matrices. Philos. Trans. 148.(1858).