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Correct expression of material derivative in continuum physics

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The material derivative is important in continuum physics. It shows that $\frac{d}{dt}=\frac{\partial}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})$ is a wrong expression. The correct expression $\frac{d(:)}{dt}=\frac{\partial}{\partial t}(:)+\boldsymbol{v}\cdot[\boldsymbol{\nabla}(:)]$ is to be formulated.

Keywords: material derivative, continuum physics

INTRODUCTION

In fluid mechanics and continuum physics, there are two descriptions of continua media or flows. One is Lagrangian description and another is Eulerian description. In Eulerian description, material derivative with respect to time should be defined. For mass density $\rho(\boldsymbol{x},t)$, flow velocity $\boldsymbol{v}(\boldsymbol{x},t)=u_k\boldsymbol{e}_k$, and stress tensor $\boldsymbol{\sigma}=\sigma_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j$, their material derivatives are given by:

$$\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial x_k}\frac{\partial x_k}{\partial t}=\frac{\partial\rho}{\partial t}+v_1\frac{\partial\rho}{\partial x^1}+v_2\frac{\partial\rho}{\partial x^2}+v_3\frac{\partial\rho}{\partial x^3}, \tag{1}$$

$$\frac{d\boldsymbol{v}}{dt}=\frac{\partial\rho}{\partial t}+\frac{\partial\boldsymbol{v}}{\partial x_k}\frac{\partial x_k}{\partial t}=\frac{\partial\boldsymbol{v}}{\partial t}+v_1\frac{\partial\boldsymbol{v}}{\partial x^1}+v_2\frac{\partial\boldsymbol{v}}{\partial x^2}+v_3\frac{\partial\boldsymbol{v}}{\partial x^3}, \tag{2}$$

$$\frac{d\boldsymbol{\sigma}}{dt}=\frac{\partial\rho}{\partial t}+\frac{\partial\boldsymbol{\sigma}}{\partial x_k}\frac{\partial x_k}{\partial t}=\frac{\partial\boldsymbol{\sigma}}{\partial t}+v_1\frac{\partial\boldsymbol{\sigma}}{\partial x^1}+v_2\frac{\partial\boldsymbol{\sigma}}{\partial x^2}+v_3\frac{\partial\boldsymbol{\sigma}}{\partial x^3}, \tag{3}$$

respectively; where t is time, $\boldsymbol{x}=x_k\boldsymbol{e}_k$ is position coordinates, $\boldsymbol{e}_k$ is a base vector and $v_k=\frac{\partial x_k}{\partial t}$ is flow velocity.

In the most of textbooks such as famous one [1], the above material derivatives are usually expressed in following forms:

$$\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\rho, \tag{4}$$

$$\frac{d\boldsymbol{v}}{dt}=\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}, \tag{5}$$

$$\frac{d\boldsymbol{\sigma}}{dt}=\frac{\partial\boldsymbol{\sigma}}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}, \tag{6}$$

where the gradient operator $\boldsymbol{\nabla}=\boldsymbol{e}_k\frac{\partial}{\partial x_k}$.

To simplify Eqs.(1,2) and Eq.(3) more further, a differential operator is introduced as follows:

$$\frac{d}{dt}=\frac{\partial}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla}). \tag{7}$$

This operator is used by most textbook of fluid mechanics, such as well-known graduates textbook [1, 3, 7, 8].

In the latest version of Landau's fluid mechanics book [2], the material derivative of flow velocity is given in a another format:

$$\frac{d\boldsymbol{v}}{dt}=\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}\nabla)\boldsymbol{v}. \tag{8}$$

In summary, representative wrong expression of material derivative can be listed in Table I:

TABLE I: Material derivative expressions

Landau (1987) Kundu et al.(2012) Cengel et al (2010) Anderson (1995)	$\frac{d\boldsymbol{v}}{dt}=\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$
Landau (2017)	$\frac{d\boldsymbol{v}}{dt}=\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}\nabla)\boldsymbol{v}$

There are numerous fluid mechanics books took the same expression as Landau and Anderson [1–3]. It would be a confusion for readers on which expression is the right material derivative.

Although some scholars such as Lighthill [5], Frisch [6], Xie [9] and Zhao [10] have given a correct expression, however, it seems that most of fluid mechanics textbook have adopted a wrong expression of material derivation as Landau [1]. Therefore, to redeem the great influence of Landau in physics and fluid mechanics, it might be good to address this issue by a dedicated paper.

In this short article, we will revisit the material derivative and see where is wrong, and to derive a correct material derivative expression by standard tensor calculus.

L.D. LANDAU AND E.M. LIFSHITZ, FLUID MECHANICS (ENGLISH VERSION), 1987

In §2. Euler's equation of Landau's book [1] (page 3):

> S 2. Euler's equation
>
> $$dx\frac{\partial\boldsymbol{v}}{\partial x}+dy\frac{\partial\boldsymbol{v}}{\partial y}+dz\frac{\partial\boldsymbol{v}}{\partial z}=(d\boldsymbol{r}\cdot\mathbf{grad})\boldsymbol{v}$$
>
> Thus
>
> $$d\boldsymbol{v}=\frac{\partial\boldsymbol{v}}{\partial t}+(d\boldsymbol{r}\cdot\mathbf{grad})\boldsymbol{v}$$

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or, dividing both sides by dt,

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \mathbf{grad})\boldsymbol{v} \tag{2.2}$$

Although Landau's fluid mechanics is well-known in the world, however, the above expression (2.2) is wrong.

Proof:

Since $\boldsymbol{v} \cdot \mathbf{grad} = \boldsymbol{v} \cdot \boldsymbol{\nabla} = (v_i \boldsymbol{e}_i) \cdot (\partial_j \boldsymbol{e}_j) = v_{i,j} \boldsymbol{e}_i \cdot \boldsymbol{e}_j = v_{i,j}\delta_{ij} = div\boldsymbol{v}$,

$$(\boldsymbol{v} \cdot \mathbf{grad})\boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = (v_{j,i}\boldsymbol{e}_i \cdot \boldsymbol{e}_j)(v_k \boldsymbol{e}_k)$$

$$= \delta_{ij} v_{j,i} v_k \boldsymbol{e}_k = v_{i,i} v_k \boldsymbol{e}_k = (\mathrm{div}\boldsymbol{v})\boldsymbol{v}$$

where the divergence

$$\mathrm{div}\boldsymbol{v} = \frac{\partial v_1}{\partial x^1} + \frac{\partial v_2}{\partial x^2} + \frac{\partial v_3}{\partial x^3}$$

hence

$$(\mathrm{div}\boldsymbol{v})\boldsymbol{v} = (\frac{\partial v_1}{\partial x^1} + \frac{\partial v_2}{\partial x^2} + \frac{\partial v_3}{\partial x^3})\boldsymbol{v}$$

Thus

$$(\boldsymbol{v} \cdot \mathbf{grad})\boldsymbol{v} \neq v_1 \frac{\partial \boldsymbol{v}}{\partial x^1} + v_2 \frac{\partial \boldsymbol{v}}{\partial x^2} + v_3 \frac{\partial \boldsymbol{v}}{\partial x^3}$$

Therefore,

$$\frac{d\boldsymbol{v}}{dt} \neq \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \mathbf{grad})\boldsymbol{v}$$

L.D. LANDAU AND E.M. LIFSHITZ, FLUID MECHANICS (RUSSIAN VERSION), 2017

In the latest version of Landau's book [2], the materia derivative expression is changed for an unknown reason.

In §2. Euler's equation of Landau's book [2] (page 16-17):

$$dx\frac{\partial \boldsymbol{v}}{\partial x} + dy\frac{\partial \boldsymbol{v}}{\partial y} + dz\frac{\partial \boldsymbol{v}}{\partial z} = (d\boldsymbol{r}\nabla)\boldsymbol{v}$$

Thus

$$d\boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial t} + (d\boldsymbol{r}\nabla)\boldsymbol{v}$$

or, dividing both sides by dt,

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\nabla)\boldsymbol{v} \tag{2.2}$$

Proof:

Since $\boldsymbol{v}\boldsymbol{\nabla} = (v_i \boldsymbol{e}_i)(\partial_j \boldsymbol{e}_j) = v_{i,j}\boldsymbol{e}_i\boldsymbol{e}_j$, hence $(\boldsymbol{v}\boldsymbol{\nabla})\boldsymbol{v} = (v_{j,i}\boldsymbol{e}_i\boldsymbol{e}_j)(v_k\boldsymbol{e}_k) = (v_{j,i}\boldsymbol{e}_i\boldsymbol{e}_j)(v_k\boldsymbol{e}_k)$, which indicates $\boldsymbol{v}\boldsymbol{\nabla}$ is a 3rd-order tensor rather than a vector.

Thus

$$(\boldsymbol{v}\boldsymbol{\nabla})\boldsymbol{v} \neq v_1 \frac{\partial \boldsymbol{v}}{\partial x^1} + v_2 \frac{\partial \boldsymbol{v}}{\partial x^2} + v_3 \frac{\partial \boldsymbol{v}}{\partial x^3}$$

Therefore

$$\frac{d\boldsymbol{v}}{dt} \neq \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\nabla)\boldsymbol{v}$$

FORMULATION OF MATERIAL DERIVATIVE

Let's to derive the correct expression of material derivatives.

1. Mass density $\rho = \rho(\boldsymbol{x}, t)$ is a scalar-valued function of bmx and t, its differential is

$$d\rho = \frac{\partial \rho}{\partial t}dt + \frac{\partial \rho}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial t}dt, \tag{9}$$

For scalar $\frac{\partial \rho}{\partial \boldsymbol{x}} = \rho\boldsymbol{\nabla} = \boldsymbol{\nabla}\rho$, hence

$$d\rho = \frac{\partial \rho}{\partial t}dt + (\rho\boldsymbol{\nabla}) \cdot \boldsymbol{v}dt = \frac{\partial \rho}{\partial t}dt + \boldsymbol{v} \cdot (\boldsymbol{\nabla}\rho)dt. \tag{10}$$

or, dividing both sides by dt,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\rho\boldsymbol{\nabla}) \cdot \boldsymbol{v} = \frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot (\boldsymbol{\nabla}\rho). \tag{11}$$

2. Flow velocity $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$ is a vector-valued function of bmx and t, its differential is

$$d\boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial t}dt + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial t}dt, \tag{12}$$

For vector $\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} = \boldsymbol{v}\boldsymbol{\nabla} = (\boldsymbol{\nabla}\boldsymbol{v})^T$ and $(\boldsymbol{v}\boldsymbol{\nabla}) \cdot \boldsymbol{v} = \boldsymbol{v} \cdot (\boldsymbol{\nabla}\boldsymbol{v})$, hence

$$d\boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial t}dt + (\boldsymbol{v}\boldsymbol{\nabla}) \cdot \boldsymbol{v}dt = \frac{\partial \rho}{\partial t}dt + \boldsymbol{v} \cdot (\boldsymbol{\nabla}\boldsymbol{v})dt. \tag{13}$$

or, dividing both sides by dt,

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\boldsymbol{\nabla}) \cdot \boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot (\boldsymbol{\nabla}\boldsymbol{v}). \tag{14}$$

3. The 2nd order stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{x}, t)$ is a tensor-valued function of bmx and t, its differential is

$$d\boldsymbol{\sigma} = \frac{\partial \boldsymbol{\sigma}}{\partial t}dt + \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial t}dt, \tag{15}$$

For an arbitrary tensor $\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{x}} = \boldsymbol{\sigma}\boldsymbol{\nabla} \neq (\boldsymbol{\nabla}\boldsymbol{\sigma})^T$, hence

$$d\boldsymbol{\sigma} = \frac{\partial \boldsymbol{\sigma}}{\partial t}dt + (\boldsymbol{\sigma}\boldsymbol{\nabla}) \cdot \boldsymbol{v}dt. \tag{16}$$

or, dividing both sides by dt,

$$\frac{d\boldsymbol{\sigma}}{dt} = \frac{\partial \boldsymbol{\sigma}}{\partial t} + (\boldsymbol{\sigma}\boldsymbol{\nabla}) \cdot \boldsymbol{v}. \tag{17}$$

Since $(\boldsymbol{\sigma}\boldsymbol{\nabla}) \cdot \boldsymbol{v} = \boldsymbol{v} \cdot (\boldsymbol{\nabla}\boldsymbol{\sigma})$, hence

$$\frac{d\boldsymbol{\sigma}}{dt} = \frac{\partial \boldsymbol{\sigma}}{\partial t} + \boldsymbol{v} \cdot (\boldsymbol{\nabla}\boldsymbol{\sigma}). \tag{18}$$

$\frac{d}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})$ IS A WRONG EXPRESSION

In all literature and textbooks, there is a common used operator $\frac{d}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})$, from the above calculation, it is easy to prove that the operator is a wrong one due to the fact that:

$$\boldsymbol{v} \cdot \boldsymbol{\nabla} = (\boldsymbol{e}_i v_i) \cdot (\frac{\partial}{\partial x^j}\boldsymbol{e}_j) = \frac{\partial v_i}{\partial x^j}(\boldsymbol{e}_i \cdot \boldsymbol{e}_j) = \frac{\partial v_i}{\partial x^j}\delta_{ij}$$

$$= \frac{\partial v_i}{\partial x^i} = \frac{\partial v_1}{\partial x^1} + \frac{\partial v_2}{\partial x^2} + \frac{\partial v_3}{\partial x^3} = \mathrm{div}\boldsymbol{v} = \boldsymbol{\nabla} \cdot \boldsymbol{v}$$

Thus

$$(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} \neq v_1\frac{\partial \boldsymbol{v}}{\partial x^1} + v_2\frac{\partial \boldsymbol{v}}{\partial x^2} + v_3\frac{\partial \boldsymbol{v}}{\partial x^3}$$

Therefore, $\frac{d}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})$ is a wrong expression for material derivative.

$\frac{d(:)}{dt} = \frac{\partial}{\partial t}(:) + \boldsymbol{v} \cdot [\boldsymbol{\nabla}(:)]$ IS A CORRECT MATERIAL DERIVATIVE EXPRESSION

If you must introduce an operator for material derivative, the correct one must be in following form:

$$\frac{d(:)}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot [\boldsymbol{\nabla}(:)]. \tag{19}$$

For a scalar function $f(\boldsymbol{x}, t)$, we have

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \boldsymbol{v} \cdot (\boldsymbol{\nabla} f). \tag{20}$$

If f is considered as a distributed function of the gas molecules in their phase space, the above derivative is the Eq.(3.2) in Physical Kinetics by E.M. Lifshitz and L.P. Pitaevskii [4].

For a vector function $\boldsymbol{u}(\boldsymbol{x}, t)$, we have

$$\frac{d\boldsymbol{u}}{dt} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{v} \cdot (\boldsymbol{\nabla}\boldsymbol{u}). \tag{21}$$

For a tensor function $\boldsymbol{A}(\boldsymbol{x}, t)$, we have

$$\frac{d\boldsymbol{A}}{dt} = \frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{v} \cdot (\boldsymbol{\nabla}\boldsymbol{A}). \tag{22}$$

APPLICATION TO NAVIER-STOKES MOMENTUM EQUATIONS

With the material derivative operator $\frac{d(:)}{dt} = \frac{\partial}{\partial t}(:) + \boldsymbol{v} \cdot [\boldsymbol{\nabla}(:)]$, the Navier-Stokes momentum equation is given by

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot (\boldsymbol{\nabla}\boldsymbol{v}) = \nu\boldsymbol{\nabla}^2\boldsymbol{v} - \frac{1}{\rho}\boldsymbol{\nabla}p, \tag{23}$$

where p is flow pressure and ν is kinematical viscosity.

Assuming $\det \boldsymbol{\nabla}\boldsymbol{v} \neq 0$, the Navier-Stokes momentum equation in Eq.(23) can be rewritten as follows

$$\boldsymbol{v} = \left(\nu\boldsymbol{\nabla}^2\boldsymbol{v} - \frac{1}{\rho}\boldsymbol{\nabla}p - \frac{\partial \boldsymbol{v}}{\partial t}\right) \cdot (\boldsymbol{\nabla}\boldsymbol{v})^{-1}. \tag{24}$$

If the wrong material derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})$ is used in the Navier-Stokes momentum equation, the Eq.(24) can not be obtained.

DISCUSSION AND CONCLUSION

The mistake in the definition of the convectional operator $\frac{d}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})$ is mainly due to the wrong doing of $\boldsymbol{v} \cdot \boldsymbol{\nabla}$, where $\boldsymbol{\nabla}$ is a gradient operator rather than a vector. $\boldsymbol{v} \cdot \boldsymbol{\nabla}$ will have to obey a differential operation of $\boldsymbol{\nabla}$, namely

$$\boldsymbol{v} \cdot \boldsymbol{\nabla} = (v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3) \cdot (\boldsymbol{e}_1\frac{\partial}{\partial x^1} + \boldsymbol{e}_2\frac{\partial}{\partial x^2} + \boldsymbol{e}_3\frac{\partial}{\partial x^3})$$

$$\neq v_1\frac{\partial}{\partial x^1} + v_2\frac{\partial}{\partial x^2} + v_3\frac{\partial}{\partial x^3}$$

In conclusion, the correct material derivative operator is defined as:

$$\frac{d(:)}{dt} = \frac{\partial}{\partial t}(:) + \boldsymbol{v} \cdot [\boldsymbol{\nabla}(:)]$$

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