# Fractional Partial Differential Equations associated with Lêvy Stable Process

Reem Abdullah Aljedhi[1] and Adem Kılıçman[1,2]

[1,2]Department of Mathematics and Institute for Mathematical Research, University Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia

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**Abstract**

In this study, we first present a time-fractional Lêvy diffusion equation of the exponential option pricing models of European option pricing and the risk-neutral parameter. Then, we modify a particular Lêvy-time fractional diffusion equation of European-style options. Introduce a more general model from the models based on the Lêvy-time fractional diffusion equation and review some recent findings regarding of the Europe option pricing of risk-neutral free.
**Keywords:** Price impact; Option pricing; liquidity, Lêvy process, fractional differential equations.

## 1 Introduction

One of the significant problem is finance to derive their value from financially traded assets that is the pricing of financial instruments, for example, stocks and also very interesting problem. In [8] were Among the first systematic solution for this problem, who proposed Black-Scholes (BS) model where the model rests on the assumption that the natural logarithm of the stock price $S_t$ follows as:

$$d(\ln S_t) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t \tag{1}$$

where $\mu > 0$ is the average compounded growth rate of the stock $S_t$, $dB_t$ is the increment of Brownian motion which is assumed to have the Normal or Gaussian distribution, and $\sigma \geq 0$ is the volatility of the returns from holding $S_t$.
It is well-known in the literature that when substituted the Lêvy process by the Brownian motion componential Equation (1) is, the pricing partial differential equation becomes a partial integro-differential equation, Ref. [9]. The partial integro-differential equations PIDEs are mentioned in order to the non-locality points produced by the jumps in the Lêvy process.
In this article, modify European-style options written on dynamic under a risk-neutral probability measure for the stock-price assets, that follow perfectly liquid market in the financial literature.
One of the method to solve PIDEs was The numerical methods (Cont, see.[10]) where proposed a finite-difference method for option pricing having jump-diffusion as well as exponential Lêvy models. On the other hand, other methods use the fast Fourier transform of European-style options (see. [11]). As the other option of these strategies, (see. [6]) propose methods from fractional calculus. This technique considered the less-studied issue of barrier options for finite moment having log-stable (FMLS) processes. The article is based on the following: Section 2 reviews the basic concepts of L $\hat{e}$ vy operations and applications in financial modeling. Section 3 introduces the concepts of fractional calculus and how to solve fractional differential equations. Reviews the main concepts of L $\hat{e}$ vy process . Section 4 introduce the main result. Finally, section 5 will conclude and discuss some applications.

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## 2 Fractional Diffusion Model and Option Pricing

In a fully liquid market, regardless of the trading size, the options trader cannot influence the price of the underlying asset in the trading of the underlying asset to duplicate the option. In the literature [Chen, at all (2014),[1]] studied this model, where $L_t^{\alpha,\beta}$ be a Lêvy $\alpha$–stable process with skew parameter $\beta$. Before viewing the idea of the research we will define $\alpha$–stable distribution, the distribution is said to be stable when location and scale parameters if it has the same distribution of any linear combination of two independent random variables with this distribution A random variable is said to be stable if its distribution is stable. The stable dissemination family is at times alluded to as the Lêvy alpha-stable distribution.

**Definition 1** *: Any random variable $X$ is s-stable if for each $n \in N$ with $X_1, X_2, \ldots, X_n$ is infinitely divisible copies of $X$ $X_1 + X_2 + \ldots + X_n = bX + c$ or some constants $b = b(n) > 0$ and $c = c(n) \in R^d$. It is called strictly stable for any $n \in N$ if $c(n) = 0$ .*

For an infinitely divisible random vector $X^{*t}$ define the alpha-stable as follows.

**Definition 2** *: A stable $X$ is called alpha-stable, whenever $X^{*t} = t^{\frac{1}{\alpha}} X + c$ or some constants $c = c(r) \in \mathcal{R}^d$ , $t > 0$, and $0 < \alpha \leq 2$. When $c(t) = 0$, for $t > 0$, then $X$ is called strictly alpha-stable .*

Now, consider the following dynamic under a risk neutral probability measure for the stock price $S_t$

$$dS_t = S_t\left((r-q)dt + \sigma dL_t^{\alpha,-1}\right) \tag{2}$$

for time $0 < t < T$, where index $\alpha$ of stability satisfies $1 < \alpha < 2$, and volatility $\sigma > 0$. When $\sigma = 0$, we will get original BS model. Moreover where $r$ and $q$ respectively denote deterministic parameters corresponding to the risk-free rate and dividend yield. We restrict our selves to the case where $\beta = -1$ to obtain finite moments of $S_t$ and negative skewness in the return density distribution. In particular for $n > 0$, then

$$E\left[\exp\left(n\sigma L_t^{\alpha,-1}\right)\right] = \exp\left(-tn^\alpha \sigma^\alpha \sec\left(\frac{\pi\alpha}{2}\right)\right) < +\infty.$$

The model in the equation (2) is known as Finite Moment Log Stable (FMLS) for short model. Under the risk–neutral measure the log price satisfies the following SDE:

$$d\left(\ln(S_t)\right) = (r - q - v)dt + \sigma dL_t^{\alpha,-1}, \tag{3}$$

where $v = -\frac{1}{2}\sigma^\alpha \sec\left(\frac{\pi\alpha}{2}\right)$ represents the convexity adjustment.
Let $u(t,x)$ be the price of the European call option with $x = x_t := \ln(S_t)$. [Chen et al. (2014),[1] ] In order to find FPDE let $u(t,x)$ satisfies under FMLS the following fractional PDEs

$$\frac{\partial u}{\partial t}(t,x) + (r-v)\frac{\partial u}{\partial x}(t,x) + v\frac{\partial^\alpha u}{\partial x^\alpha}(t,x) - ru(t,x) = 0$$

$$u(x,T) := \begin{cases} \max(e^x - K; 0) & \text{for European call option} \\ \max(K - e^x; 0) & \text{for European put option} \end{cases}$$

where $K$ is the strike price and,

$$\frac{\partial^\alpha u(t,x)}{\partial x^\alpha} = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\infty}^{x}\frac{u(t,u)}{(x-u)^\alpha}du$$

and $\Gamma(.)$ is the gamma function and defined by:

$$\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}dx.$$

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## 3 The model

In this research, we incorporate Let $L_t^{\alpha,\beta}$ be a Lêvy $\alpha$–stable process with skew parameter $\beta$. Consider the following dynamic under a risk-neutral probability measure for the stock price $S_t$, the goal is to consider a modified model to equation (2) that consists on an illiquid market with impact additional term that for $0 \leq t < T, 0 < \gamma < 1$ and $1 < \alpha \leq 2$,with boundary condition

$$d^\gamma S_t = S_t\left((r-q)dt^\gamma + \sigma dL_t^{\alpha,-1}\right) + \lambda(t,S_t)S_t d\beta_t^\gamma, S(0) = S_0 \tag{4}$$

where $\lambda(t,S_t) \geq 0$ is the price impact function of the trader and $\beta_t$ denotes the number of shares that the trader has in the stock at time $t$. The term $\lambda(t,S_t)d\beta_t$ represents the price impact of the investor's trading is additional term of Chen model (2). $\gamma = 2H$ and $H$ is Hurst number $0 < H \leq 1$. The Hurst number $H$ is a statistical measure which can be used to classify the time series. If $H = 0.5$ indicates a random series, and $H > 0.5$ indicates a trend reinforcing series. Similarly, the larger the $H$ value is considered the stronger trend. The Caputo fractional integral of f defined by the expression

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)}\int_{-\infty}^t f(u)(t-u)^{\gamma-1}du$$

and the Caputo fractional derivative of $u$ defined by the expression

$$\partial_t^\gamma u(x,t) = \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{dy}{(t-y)^{\gamma-1}}\frac{\partial u(x,y)}{\partial y}.$$

In this work, we consider trading strategies Written in the following form

$$d\beta_t^\gamma = \eta_t dt^\gamma + \zeta_t dL_t^{\alpha,-1} \tag{5}$$

for some processes $(\eta_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ to be determined endogenously and $\beta_0$ is the initial number of shares in the stock. The wealth process $(V_t)_{t\geq 0}$ corresponding to a self–financing strategy $(\theta_t,\beta_t)_{t\geq 0}$ for the trader is given by

$$V_t = \theta_t S_t^0 + \beta_t S_t = V_0 + \int_0^t \theta_u dS_u^0 + \int_0^t \beta_u dS_u.$$

To find the fractional partial differential equation satisfy our model in equation (4) we need method to solve fractional equation as method of literature (Demirci et. al. 2012) as example. In literature (Demirci et. al. 2012), they solved the fractional partial differential equation of the initial value problem in the sense of Caputo type FDE given by

$$D^\gamma x(t) = f(t,x(t)), x(0) = x_0$$

has a solution

$$x(t) = x_*\left(\frac{t^\gamma}{\Gamma(\gamma+1)}\right)$$

where $x_*(v)$ is a solution having integer order differential equation.
Also we need the method of the from literature ( Jumarie, [7]) of the equation,

$$dx = f(t)dt^\alpha, t \geq 0, x(0) = x_0$$

where $0 < \alpha \leq 1$, has a solution defined by the equality

$$\int_0^t f(\tau)d\tau^\alpha = \alpha\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau. \tag{6}$$

Furthermore, we will use in our model the following formula

$$d^\alpha x = \Gamma(1+\alpha)dx. \tag{7}$$

The general Fourier transform is defined by

$$\hat{f}(\xi) = \mathcal{F}[f(x)] = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\xi x}\hat{f}(\xi)d\xi \tag{8}$$

and the inverse Fourier transformation is defined by

$$f(x) = \int_{-\infty}^\infty \exp^{i\xi x} f(x)dx.$$

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### 3.1 Lêvy process

The distribution of Lêvy process is characterized by Lêvy-khintchine formula and considered characteristic function. Can characterize the Lêvy in a very compact way via the Lêvy-khintchine of the process. More definitely, a time-dependent random variable $X_t$ is a Lêvy process if and only if it has independent and stationary increments having log-characteristic function given by Lêvy-Khintchine theorem:

**Theorem 1 (Lêvy-Khintchine presentation theorem)** *Let $(X_t)_{t\geq 0}$ Lêvy process on $R$ with characteristic triplet $(m,\sigma,w)$, then $E[e^{izX_t}] = e^{t\Psi(z)}, t \in R$, with characteristic exponent of the Lêvy process*

$$\Psi(z) = im\xi + \frac{\sigma^2}{2}(i\xi)^2 + \int_{-\infty}^{\infty}(e^{i\xi x} - 1 - i\xi I_{|x|<1})W(dx) \tag{9}$$

where, $\int_R \min[1, x^2]W(dx) < \infty$, and $W = w(x)$ *Lêvy density, $m\ inR$, $\sigma \geq 0$.*

To accommodate how the Lêvy processes being incorporated in the derivatives pricing models, we recall the standard Black-Scholes framework and see how it was built by Gaussian shocks. To find the fair or arbitrage-free prices of a financial instrument whose value are deriven from the underlying share price $S_t$, it is also necessary to express the dynamics of $S_t$ under what is known as a neutral risk measure or the equivalent martingale scale. In the price, the European option may be expressed as the neutral condition for a risk as

$$V(t,S) = e^{-r(T-t)}E^Q[\max(S_T - k, 0)|\mathcal{F}_\sqcup]. \tag{10}$$

Fourier transform of European option can be written as (Du, [12])

$$\frac{\partial^\gamma \tilde{V}}{\partial t^\gamma} = \tilde{V}(t,S) + \Psi(\xi)\tilde{V}(t,S) - r\tilde{V}(t,S) \tag{11}$$

where

$$\Psi(\xi) = im\xi + \frac{\sigma^2}{2}(i\xi)^2 + \int_{-\infty}^{\infty}(e^{i\xi x} - 1 - i\xi I_{|x|<1})W(dx)$$

and the indicator function of set $A$ where $I_A : A \subset X \to \{0,1\}$ and defined by

$$I_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

### 3.2 LS processes

Let $w(x) = w_{LS}(x)$ be Lêvy density function and given by

$$w_{LS} = \begin{cases} \frac{Dq}{|x|^{1+\alpha}} & \text{for } x < 0 \\ \frac{Dq}{x^{1+\alpha}} & \text{for } x > 0 \end{cases} \tag{12}$$

where $D > 0$, $q + p = 1$ and $\alpha \in (0,2)$. Then by using the equation (9) we obtain the characteristic exponent of an LS process in the parameters as follows: $\sigma, \alpha, \beta$ and $m$:

$$\Psi_{LS}(\xi) = i\xi m - \frac{1}{2}\sigma^\alpha |\xi|^\alpha + \left[1 - i\beta sign(\xi)\tan\left(\frac{\alpha\pi}{2}\right)\right]. \tag{13}$$

An equivalent form can be written as

$$\Psi_{LS}(\xi) = i\xi m - \frac{1}{4\cos(\frac{\alpha\pi}{2})}\sigma^\alpha[(1-\beta)(i\xi)^\alpha + (1+\beta)(-i\xi)^\alpha] \tag{14}$$

where $\beta = p - q$. If $\beta = -1$, then $p = 0$ and $q = 1$, that is (Alvaro Cartea et. al. [6])

$$\Psi_{LS}(\xi) = i\xi m - \frac{1}{4\cos(\frac{\alpha\pi}{2})}\sigma^\alpha\left[(2)(i\xi)^\alpha\right]. \tag{15}$$

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# 4 Main Results

Consider the Fractional differential lêvy equation is

$$d^{\gamma}S_t = S_t\left((r-q)dt^{\gamma} + \sigma dL_t^{\alpha,-1}\right) + \lambda(t,S_t)S_t(\eta_t dt^{\gamma} + \zeta_t dL_t^{\alpha,-1}). \tag{16}$$

That can be rewrite as where $\lambda_t = \lambda(t, S(t))$ and let $x_t = \ln S_t$

$$d^{\gamma}x_t = (r - q + \lambda_t\eta_t)dt^{\gamma} + (\sigma + \lambda_t\zeta_t)dL_t^{\alpha,-1}. \tag{17}$$

Nex, we derive revised and updated FPDEs for options which is written on assets and follows the L *hate* vy operations taht was mentioned in the previous section. In order To find the relation between the fractional price equations and LP process, then we will make use os Fourier transform, as in the next

$$\hat{f}(\xi) = \mathcal{F}[f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp^{-i\xi x}\hat{f}(\xi)d\xi \tag{18}$$

of the value of European style option price $\hat{\mathcal{U}}(\xi, T)$, written on $S_t$, and satisfies

$$\frac{\partial^{\gamma}\hat{\mathcal{U}}}{\partial t^{\gamma}} = r\hat{\mathcal{U}}_(\xi,t) + (-q + \lambda_u\eta_u)\, i\xi\hat{\mathcal{U}}(\xi,t) - \Psi(\xi)\hat{\mathcal{U}}(\xi,t). \tag{19}$$

Let $\zeta_T(v)$ denote the Fourier transform of the time value, where

$$\zeta_T(\xi,t)(i\xi)^{\gamma} = \frac{\partial^{\gamma}\hat{\mathcal{U}}}{\partial t^{\gamma}}$$

Let $\mathcal{U}(\xi, T)$ denotes the Fourier transform of a European-style option and then defined by

$$\zeta_T(\xi,t)(i\xi)^{\gamma} = r\hat{\mathcal{U}}_(\xi,t) + (-q + \lambda_u\eta_u)\, i\xi\hat{\mathcal{U}}(\xi,t) - \Psi(\xi)\hat{\mathcal{U}}(\xi,t). \tag{20}$$

with boundary condition $\mathcal{U}(\xi, T) = \Pi(\xi, T)$.

Now substitute the equation (15) in equation (19) and taking the inverse Fourier transform we reach to

$$\zeta_T(\xi,t)(i\xi)^{\gamma} = r\hat{\mathcal{U}}(\xi,t) + (r - q + \lambda_u\eta_u)\, i\xi\hat{\mathcal{U}}(\xi,t) - \left[-\frac{1}{4}\sec\left(\frac{\alpha\pi}{2}\right)\sigma^{\alpha}(2)(i\xi)^{\alpha}\right]\hat{\mathcal{U}}(\xi,t)$$

then taking the inverse Fourier transform delivered to

$$\frac{\partial^{\gamma}\mathcal{U}}{\partial t^{\gamma}}(x,t) + (r - q + \lambda_u\eta_u)\frac{\partial\mathcal{U}}{\partial x}(x,t) + \frac{1}{2}\sec\left(\frac{\alpha\pi}{2}\right)\sigma^{\alpha}\frac{\partial^{\alpha}\mathcal{U}}{\partial x^{\alpha}}(x,t) = r\mathcal{U}(x,t). \tag{21}$$

To prove equation (17) satisfies the equation (19).

First we can find the solution of equation (17). Rewrite the equation (17) in the form, where $x_T = \ln(S_T)$

$$dx_T = \frac{1}{\Gamma(1+\gamma)}\left[(r - q + \lambda_t\eta_t)dt^{\gamma} + (\sigma + \lambda_t\zeta_t)dL_t^{\alpha,-1}\right].$$

Take the integral for the above equation and using method (6) we get

$$x_T = \frac{\gamma}{\Gamma(1+\gamma)}\int_t^T (t-\tau)^{\gamma-1}[(r - q + \lambda\eta)d\tau + dL_u^{\alpha,-1}].$$

So

$$S_t = S_t \exp\left[\frac{(T-t)^{\gamma}}{\Gamma(1+\gamma)}((r - q + \lambda_t\eta_t) + \int_t^T dL_u^{\alpha,-1})\right]. \tag{22}$$

By the same way and using method of ( Demirci and Ozalp (2012)) the equation (19) has a solution

$$\hat{\mathcal{U}}(\xi,t) = \exp\left[r - i\xi(r - q + \lambda_t\eta_t + \psi(-\xi))\frac{(T-t)^{\gamma}}{\Gamma(1+\gamma)}\right].$$

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To prove equation (17) satisfies the equation (19), start with

$$\mathcal{U}(x,t) = e^{\left[\frac{-r(T-t)^{\gamma}}{\Gamma(1+\gamma)}\right]} E^{Q}(\Pi(x_T, T))$$

using inverse Fourier of $\Pi(x_T, T)$, thus

$$\mathcal{U}(x,t) = \frac{1}{2\pi} e^{\left[\frac{-r(T-t)^{\gamma}}{\Gamma(1+\gamma)}\right]} \int_{i\xi+R} E^{Q}(e^{(i\xi x_T)})\hat{\Pi}(\xi, T) d\xi$$

from solution 22 we get

$$\mathcal{U}(x,t) = \frac{1}{2\pi} e^{\left[\frac{-r(T-t)^{\gamma}}{\Gamma(1+\gamma)}\right]} \int_{i\xi+R} e^{\left[i\xi(r-q+\lambda_t\eta_t+\psi(-\xi))\frac{(T-t)^{\gamma}}{\Gamma(1+\gamma)}\right]}\hat{\Pi}(\xi, T) d\xi. \qquad (23)$$

That is

$$\hat{\mathcal{U}}(\xi,t) = \exp\left[-r - i\xi(r-q+\lambda_t\eta_t+\psi(-\xi))\frac{(T-t)^{\gamma}}{\Gamma(1+\gamma)}\right]\hat{\Pi}(\xi, T)$$

is a solution of the equation (19).

# 5 Conclusion

In this paper, we modified the particular Lêvy-time fractional diffusion equation, and apply to the price of fractional financial derivatives of European-style options such that we demonstrate as a fractional partial differential equation (FPDE). A more general class of models based on the fractional diffusion equation L $\hat{e}$ vy was also presented and demonstrated to be a solution to the European option pricing and applied to the risk-free parameter.

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