Article

Deep Assessment Methodology Using Fractional Calculus on Mathematical Modeling and Prediction of Gross Domestic Product per Capita of Countries

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Abstract: In this study, by using Least Square Method, the dataset for the Gross Domestic Product per capita is modeled as a function satisfying the fractional differential equation. The function itself is assumed to be the finite summation of its previous values and the derivatives with unknown coefficients. Then, the prediction for the upcoming years is done by having an approach dividing the dataset into 4 regions corresponding to four different tasks. The mathematical model of the Gross Domestic Product (GDP) per capita of the countries (and union) which are Brazil, China, European Union (EU), India, Italy, Japan, UK, the USA, Spain, and Turkey is constructed with a new methodology called as the deep assessment method which comes from the expressing an arbitrary function modeling the dataset as the finite summation of its previous values and the derivatives with unknown coefficient. The method uses the fractional calculus properties combining with Least Square Method and is compared to Long short-term memory (LSTM) algorithm which is a special type of neural network used for time sequences in general.

Keywords: deep assessment; fractional calculus; least squares; modeling; GDP per capita; prediction

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1. Introduction

In the last quarter of the century, the data exchange with not only person to person but also, machine to machine has increased tremendously. Developments in technology and informatics parallel with data science lead the companies, institutions, universities and especially, the countries to give priority to evaluating produced data and predicting what can be forthcoming. The modeling of all technical, economic, social events and data has been the interest of scientists for many years [1-4]. There are many authors investigating on the modeling and predicting events, options, choices and data. Especially, there is a huge research interest in finding any relation among telecommunication, economic growth, and financial development [5-10]. In order to model a physical phenomenon or a mathematical study, there are many different kinds of approaches. One of them is to model the dependent variable satisfying differential equation with respect to the independent variable. In the study, modeling of the dataset is built on the fractional-order differential equation. The fractional calculus is a branch of mathematics that focuses on fractional-order differential and integral operators – this idea first offered at the end of the seventeenth century. Fractional calculus (FC) as a question) to Wilhelm Leibniz (1646-1716) is firstly arisen in 1695 by French Mathematician Marquis de L'Hopital (1661-1704) [11]. The main question of interest was what if the order of derivative were a real number instead of an integer. After that, the FC idea has been developed by many mathematicians and researchers throughout the eighteenth and nineteenth centuries. Now, there exist several definitions of the fractional-order derivative, including Grünwald-Letnikov, Riemann-Liouville, Weyl, Riesz, and the Caputo representation. In this study, the derivative with fractional-order ($0<\gamma<1$) and corresponding Laplace transform properties are utilized. The fractional derivative represents the intermediate states between two known states. For example, zero order-derivative of the function means the function itself and the first-order derivative is the first derivative of the function. Between these known states, there are infinite intermediate states [11].

In the last decade, using fractional operators which explain the events, situations or modes between two different stages or the phenomena with memory provide more accurate models in many branches of science and engineering including chemistry, biology, biomedical devices, nanotechnology, diffusion, diffraction, and economics [12-21]. In [22,23], the modeling and comparison of the countries in the sense of economics and its parameters are implemented. In [19,20], economic processes with memory are discussed and modeling are obtained by using the fractional calculus.

Besides the literature, in this study, a new mathematical model based on the fractional differential equation approach is developed for the modeling and prediction by using the properties of fractional calculus. Previously, proposed models in our studies work for only modeling. In our previous studies, the children's physical growth, subscriber's numbers of operators, Gross Domestic Product (GDP) per capita were modeled and compared with other modeling approaches such as Linear and Polynomial Models [24-26]. According to the results, proposed fractional models had better results compared to the results obtained from Linear and Polynomial Models [24-26]. Different from the previous research, the study works not only for modeling as it is done previously, but also for the prediction of next coming values. Here, the modeling is implemented with the similar mathematical tools developed in the previous study [4] with a different approach in which the finite numbers of previous values and the derivatives are taken into account. Then, the prediction is obtained by dividing the dataset into four regions which will be explained in the following section. Note that, the deep assessment method takes into account the previous values and the variation rates between each year (derivative) of dataset in order to model the data itself and to predict upcoming years. Combining the previous values with the variations weighted by the unknown coefficients lead to calling the method as "deep assessment".

In this study, the modeling, testing, and prediction are obtained for Gross Domestic Product (GDP) per capita of following countries (and union): Brazil, China, European Union, India, Italy, Japan, UK, the USA, Spain, and Turkey. Gross Domestic Product per capita is a measure of a country's total economic output divided by the number of the population of the country. In general, it is a very reasonable and good measurement of a country's living quality and standard [27]. Therefore, the modeling of Gross Domestic Product per capita is crucial and predicting GDP per capita is very essential not only for the researchers but also for companies, investors, manufacturers, and institutions.

The structure of the study is the following. Section 2 explains Formulation of the problem. After that, Section 3, namely the Solution of the problem, is devoted to explain how to obtain modeling, simulation, testing, and prediction results. Lastly, Section 4 highlights the conclusion of the study.

2. Formulation of the Problem

In this section, the mathematical foundation is given. First, for the purpose of the study, it is reasonable idea to approximate a function $g(x)$ as the finite summation of previous values of the same function weighted with unknown coefficients $\boldsymbol{\alpha_k}$ and the summation of the derivatives of the previous values of the same function weighted with unknown coefficients $\boldsymbol{\beta_k}$ because, intuitively, the recent value of data, in general, is related and correlated with its previous values and the change rates. The purpose is to find the upcoming values of any dataset with a minimum error by employing the previously inherited features of the dataset. As a starting point, an arbitrary function is assumed to be approximately the finite summation of the previous values and the change rates weighted with some constant coefficients. In order to use the heritability of fractional calculus, this presupposition for modeling of the function itself and predicting for future values is done [6].

$$g(x) \cong \sum_{k=1}^{l} \alpha_k\, g(x-k) + \sum_{k=1}^{l} \beta_k\, g'(x-k) \tag{2.1}$$

In the study, $\boldsymbol{x}$ and $g(x)$ would be the time (year) and the GDP per capita, respectively. Here, g' is the first derivative of $g(x-k)$ with respect to $\boldsymbol{x}$. After assuming (2.1), the function $g(x)$ can be expanded as the summation of polynomials with unknown constant coefficients, $\boldsymbol{a_n}$ as given in (2.2.a)

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$$g(x) = \sum_{n=0}^{\infty} a_n\, x^n \tag{2.2.a}$$

Then, $g(x-k)$ becomes

$$g(x-k) = \sum_{n=0}^{\infty} a_n\,(x-k)^n \tag{2.2.b}$$

The final form of $g(x)$ is given as (2.3).

$$g(x) \cong \sum_{k=1}^{l} \alpha_k \sum_{n=0}^{\infty} a_n (x-k)^n + \sum_{k=1}^{l} \beta_k \sum_{n=0}^{\infty} a_n n (x-k)^{n-1} \tag{2.3}$$

After combining $\boldsymbol{\alpha_k a_n}$ as $\boldsymbol{a_{kn}}$, $\boldsymbol{\beta_k a_n}$ as $\boldsymbol{b_{kn}}$ and approximating (2.3), (2.4) is obtained. Here, $\sum_{n=0}^{\infty} a_n n(x-k)^{n-1}$ is truncated as $\sum_{n=0}^{M} a_n n(x-k)^{n-1}$.

$$g(x) \cong \sum_{k=1}^{l} \sum_{n=0}^{M} a_{kn}(x-k)^n + \sum_{k=1}^{l} \sum_{n=0}^{M} b_{kn} n (x-k)^{n-1} \tag{2.4}$$

The expression given in (2.5) is the definition of Caputo's fractional derivative [11]. Throughout the study, Caputo's description of the fractional derivative is employed.

$$\mathfrak{D}_x^{\gamma} g(x) = \frac{dg(x)}{dx^{\gamma}} = \frac{1}{\Gamma(1-\gamma)} \int_0^x \frac{g^{(1)}(k)dk}{(x-k)^{\gamma}} \quad (0 < \gamma < 1) \tag{2.5}$$

In (2.5), $\Gamma(1-\gamma)$ is the Gamma function, the derivative is taken with respect to x in the order of $\boldsymbol{\gamma}$ and $g^{(1)}$ corresponds to the first derivative with respect to x. Here, two expansions are done to express $g(x)$, approximately. The first one is to express the function as the finite summation of the previous values of the function. Second, expressing an arbitrary function $g(x)$ as the summation of polynomials known as Taylor Expansion.

Finally, the mathematical background is enough to go further in the proposed methodology. Now, it is time to express Deep Assessment Methodology by using fractional calculus for the modeling and prediction. Note that, the fractional derivative of $f(x)$ in the order of $\boldsymbol{\gamma}$ is assumed to be equal to (2.6).

$$\frac{d^{\gamma} f(x)}{dx^{\gamma}} \cong \sum_{k=1}^{l} \sum_{n=1}^{\infty} a_{kn} n\,(x-k)^{n-1} + \sum_{k=1}^{l} \sum_{n=1}^{\infty} b_{kn} n\,(x-k)^{n-2} \tag{2.6}$$

where, $f(x)$ stands for the GDP per capita of the countries and $\boldsymbol{x}$ corresponds to the time.

Here, the motivation is to find $\boldsymbol{a_{kn}}$ and $\boldsymbol{b_{kn}}$ given in (2.6). In order to find the unknowns, the differential equation needs to be solved. The strategy is as follows. First, it is required to take the Laplace transform which leads to having an algebraic equation instead of a differential equation. In other words, the Laplace transform is taken for (2.6) to reduce the differential equation to algebraic equation, then, by using inverse Laplace transform properties, the final form of $f(x)$ is obtained as (2.7a) [11].

$$f(x) \cong f(0) + \sum_{k=1}^{l} \sum_{n=1}^{\infty} a_{kn}\, C_{kn} + \sum_{k=1}^{l} \sum_{n=1}^{\infty} b_{kn}\, D_{kn} \tag{2.7a}$$

where,

$$C_{kn} \triangleq \frac{\Gamma(n+1)}{\Gamma(n+\gamma)} (x-k)^{n+\gamma-1}$$

$$D_{kn} \triangleq \frac{\Gamma(n+1)}{\Gamma(n+\gamma-1)} (x-k)^{n+\gamma-2}$$

In order to obtain the numerical calculation, the infinite summation of polynomials is approximated as a finite summation given in (2.7b).

$$f(x) \cong f(0) + \sum_{k=1}^{l} \sum_{n=1}^{M} a_{kn}\, C_{kn} + \sum_{k=1}^{l} \sum_{n=1}^{M} b_{kn}\, D_{kn} \tag{2.7b}$$

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Here, $\boldsymbol{f}(\mathbf{0}), \boldsymbol{a}_{kn}$, and $\boldsymbol{b}_{kn}$ are unknown coefficients need to be determined.

3. The Solution of the Problem

3.1 Modeling and Simulation

In this part, the methodology for the modeling of the problem is given in detail. In order to predict the upcoming years, the problem has four regions as given in Figure 1. Dataset spans in Region 1, 2 and 3. Note that, there is no data for Region 4 where the prediction is aimed. Region 1 is called "before modeling region" which consists of historical data. Each of the coefficients $(x-k)^{n+\gamma-1}$ and derivative coming from previous values of GDP per capita for different values of $\boldsymbol{k}$ and multiplication by different weights as given in (2.7b) will add the contribution to the recent data. For modeling, the historical data is employed directly for the modeling of the data located in Region 2. Region 2 and 3 are named as modeling and testing, respectively. In the modeling region, the GDP per capita is tried to be modeled, and the unknown coefficients are found. Note that, the approach uses the previous $\boldsymbol{l}$ values $(\boldsymbol{P}_{i-1}, \boldsymbol{P}_{i-2}, \ldots, \boldsymbol{P}_{i-l}$ and corresponding $f(i-1), f(i-2), \ldots f(i-l))$ for arbitrary $\boldsymbol{P}_i$ located in Region 2. The third region consists of the data used to test for upcoming prediction Finally, Region 4 is called as "prediction region" where the aim is to find the GDP per capita values for time that the actual values have not known yet and implement prediction. The region division is required because there are parameters given in the previous section ((2.7b)) such as $\boldsymbol{M}, \boldsymbol{l}, \boldsymbol{\gamma}$ which need to be found before the prediction. In Region 2, the modeling is done to find the coefficients $\boldsymbol{a}_{kn}$ in (2.7b). For the modeling, Least Squares Method is employed, which is explained later in this section. After that, one of the purposes of the study is achieved. This is modeling of the data using the fractional approach. Then, the second purpose comes which is to predict the values of GDP per capita for the upcoming unknown years. In order to find optimum $\boldsymbol{M}, \boldsymbol{l}, \boldsymbol{\gamma}$ values for the prediction, Region 3, namely testing is needed. In the region, there is an iterative solution where the real discrete data is again known. For instance, in Region 3, it is required to find $f(m_1+1)$. Then, by using the proposed method employing the fractional calculus and Least Squares Method, $f(m_1+1)$ is obtained with a minimum error by optimizing $\boldsymbol{M}, \boldsymbol{l}, \boldsymbol{\gamma}$ values for $f(m_1+1)$ itself. Then, $f(m_1+1)$ is included the dataset for the next test which is done for $f(m_1+2)$. This continues up to $f(m)$. Then, with optimized $\boldsymbol{M}, \boldsymbol{l}, \boldsymbol{\gamma}$, the predicted $f(m_x)$ is found in Region 4.

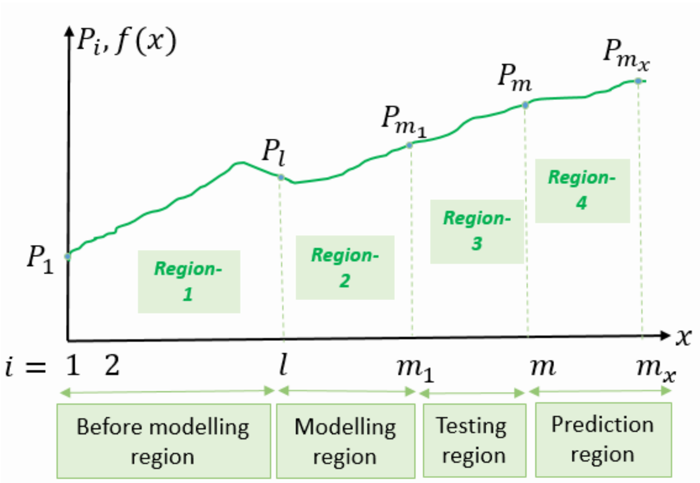

Figure 1: The regions of the dataset.

In order to model the known data, $f(x)$ representing the data optimally should be obtained. In other words, the unknowns $\boldsymbol{a}_{kn}$, $\boldsymbol{b}_{kn}$ and $\boldsymbol{f}(\mathbf{0})$ in (2.7b) or (3.1) should be determined. For this, the Least Squares Method is employed.

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$$f(i) = f(0) + \sum_{k=1}^{l}\sum_{n=1}^{M} a_{kn}\, C_{kn} + \sum_{k=1}^{l}\sum_{n=1}^{M} b_{kn}\, D_{kn} \tag{3.1}$$

In (3.2), the squares of total error $\boldsymbol{\epsilon_T}^2$ is given. The main purpose of the modeling region is to minimize $\boldsymbol{\epsilon_T}^2$ by a gradient-based approach which requires minimization of the square of the total error as the following.

$$\epsilon_T{}^2 = \sum_{i=l}^{m_1} \left(P_i - f(i)\right)^2 \tag{3.2}$$

$$\frac{\partial \epsilon_T{}^2}{\partial f(0)} = 0$$

and

$$\frac{\partial \epsilon_T{}^2}{\partial a_{rt}} = 0$$

where, $r = 1, 2, 3, \dots l$ and $t = 1, 2, 3, \dots M$.

This leads to having a system of linear algebraic equations (SLAE) as given in (3.3).

$$[A].[B] = [C] \tag{3.3}$$

where, [A], [B], and [C] is shown in (3.4), (3.5), and (3.6), respectively.

In order to find $f(x)$ the continuous curve modeling with a minimum error, the optimum fractional-order $\boldsymbol{\gamma}$ is inquired between (0, 1). Then, with optimum fractional-order $\boldsymbol{\gamma}$, the unknown coefficients are determined. In the study, the GDP per capita of Brazil, China, European Union, India, Italy, Japan, UK, the USA, Spain, and Turkey were used from 1960 until 2018 [28]. Dataset is shown in Table A1 and Table A2.

Among them, the year of 2018 is in Region 3 as testing in order to make a prediction for next years.

Here,

t (years): [1960, 1961, ..., 2018]

i (points): [1, 2, ..., 59]

P_i (value of i): $[P_1, P_2, \dots, P_{59}]$

P_i: It shows the actual GDP per capita of each country in each i^{th} year. For example, $\boldsymbol{P_2}$ is the GDP per capita of the country in 1961.

i: It stands for the number for each year. For example, $i = 1$ for 1960, $i = 3$ for 1962 and $i = 59$ for 2018.

3.2. Testing

In order to find the optimized values of the unknowns for the prediction, the testing region is required. The predictions obtained in the test region (3rd region) $(m_1 < i < m)$ are also given in Table 1. For testing, the data up to $m_1 = 58$ have been taken into consideration in the operations. The $f(m_1 + 1)$ value was found from the obtained modeling. Then, the value is kept and the next step was started again for $(f(m_1 + 2))$. These operations are done until the last value of the test zone. In our case, $m = 59$.

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$$A=\begin{bmatrix}
m-t+1 & \sum_{i=1}^{k}C_{11} & \dots & \sum_{i=1}^{k}C_{1M} & \sum_{i=1}^{k}C_{21} & \dots & \sum_{i=1}^{k}C_{2M} & \dots & \sum_{i=1}^{k}C_{l1} & \dots & \sum_{i=1}^{k}C_{lM} & \dots & \sum_{i=1}^{k}D_{11} & \dots & \sum_{i=1}^{k}D_{1M} & \dots & \sum_{i=1}^{k}D_{lM}\\
\sum C_{11} & \sum_{i=1}^{k}C_{11}C_{11} & \dots & \sum_{i=1}^{k}C_{1M}C_{11} & \sum_{i=1}^{k}C_{21}C_{11} & \dots & \sum_{i=1}^{k}C_{2M}C_{11} & \dots & \sum_{i=1}^{k}C_{l1}C_{11} & \dots & \sum_{i=1}^{k}C_{lM}C_{11} & \cdots & \sum_{i=1}^{k}D_{11}C_{11} & \dots & \sum_{i=1}^{k}D_{1M}C_{11} & \dots & \sum_{i=1}^{k}D_{lM}C_{11}\\
\sum C_{12} & \sum_{i=1}^{k}C_{11}C_{12} & \dots & \sum_{i=1}^{k}C_{1M}C_{12} & \sum_{i=1}^{k}C_{21}C_{12} & \dots & \sum_{i=1}^{k}C_{2M}C_{12} & \dots & \sum_{i=1}^{k}C_{l1}C_{12} & \dots & \sum_{i=1}^{k}C_{lM}C_{12} & \cdots & \sum_{i=1}^{k}D_{11}C_{12} & \dots & \sum_{i=1}^{k}D_{1M}C_{12} & \dots & \sum_{i=1}^{k}D_{lM}C_{12}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\sum C_{lm} & \sum_{i=1}^{k}C_{11}C_{lM} & \dots & \sum_{i=1}^{k}C_{1M}c_{lM} & \sum_{i=1}^{k}C_{21}C_{lM} & \dots & \sum_{i=1}^{k}C_{2M}C_{lM} & \dots & \sum_{i=1}^{k}C_{l1}C_{lM} & \dots & \sum_{i=1}^{k}C_{lM}C_{lM} & \dots & \sum_{i=1}^{k}C_{11}C_{lM} & \dots & \sum_{i=1}^{k}D_{1M}C_{lM} & \dots & \sum_{i=1}^{k}D_{lM}C_{lM}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\
\sum D_{11} & \sum_{i=1}^{k}C_{11}D_{11} & \dots & \sum_{i=1}^{k}C_{1M}D_{11} & \sum_{i=1}^{k}C_{21}D_{11} & \dots & \sum_{i=1}^{k}C_{2M}D_{11} & \dots & \sum_{i=1}^{k}C_{l1}D_{11} & \dots & \sum_{i=1}^{k}C_{lM}D_{11} & \cdots & \sum_{i=1}^{k}D_{11}D_{11} & \dots & \sum_{i=1}^{k}D_{1M}D_{11} & \dots & \sum_{i=1}^{k}D_{lM}D_{11}\\
\sum_{i=1}^{k}D_{12} & \sum_{i=1}^{k}C_{11}D_{12} & \dots & \sum_{i=1}^{k}C_{1M}D_{12} & \sum_{i=1}^{k}C_{21}D_{12} & \dots & \sum_{i=1}^{k}C_{2M}D_{12} & \dots & \sum_{i=1}^{k}C_{l1}D_{12} & \dots & \sum_{i=1}^{k}C_{lM}D_{12} & \dots & \sum_{i=1}^{k}D_{11}D_{12} & \dots & \sum_{i=1}^{k}D_{1M}D_{12} & \dots & \sum_{i=1}^{k}D_{11}D_{12}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\sum_{i=1}^{k}D_{lM} & \sum_{i=1}^{k}C_{11}D_{lM} & \dots & \sum_{i=1}^{k}C_{1M}D_{lM} & \sum_{i=1}^{k}C_{21}D_{lM} & \dots & \sum_{i=1}^{k}C_{2M}D_{lM} & \dots & \sum_{i=1}^{k}C_{l1}D_{lM} & \dots & \sum_{i=1}^{k}C_{lM}D_{lM} & \dots & \sum_{i=1}^{k}D_{11}D_{lM} & \dots & \sum_{i=1}^{k}D_{1M}D_{lM} & \dots & \sum_{i=1}^{k}D_{lM}D_{lM}
\end{bmatrix}$$

(3.4)

$$[B]=[f(0)\quad a_{11}\quad a_{12}\quad \dots\quad a_{1M}\, a_{21}\quad a_{22}\quad \dots\, a_{2M}\quad a_{l1}\dots\quad a_{lM}\quad b_{11}\quad b_{12}\quad \cdots\, b_{1M}\quad b_{21}\quad \dots\, b_{2M}\quad \dots\quad b_{l1}b_{l2}\quad \dots\quad b_{lM}]^{T}$$

(3.5)

$$[C]=\left[\sum_{i=0}^{K}P_i\quad \sum_{i=0}^{K}P_iC_{11}\quad \sum_{i=0}^{K}P_iC_{12}\quad \dots\quad \sum_{i=0}^{K}P_iC_{lM}\,\sum_{i=0}^{K}P_iD_{11}\quad \sum_{i=0}^{K}P_iD_{12}\dots\quad \sum_{i=0}^{K}P_iD_{lM}\right]^{T} \tag{3.6}$$

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3.4. Prediction

The Region is called as "Prediction Region". Here, using Region 1, 2 and 3, the prediction for the upcoming years is obtained. After having modeling and testing regions, the unknowns in (3.1) have already found in an optimal manner. After testing, Region 4 is started. In the region, the first prediction $f(m+1)$ is found by using the coefficients and unknowns found by the testing region. After that, the first predicted value ($f(m+1)$) is included in Region 3 (testing) for the consecutive prediction $f(m+2)$. This procedure is reiterated and recycled up to $f(m_x)$.

The prediction results for 2019 are given in Table 2. For example, as of the end of 2019 ($f(m+2)$), Brazil, China, European Union, India, Italy, Japan, UK, the USA, Spain, and Turkey's GDP per capita values are expected as listed.

3.5. Algorithm

In this part, the algorithm of numerical calculation is given in detail. In Figure 2, the flowchart of the deep assessment algorithm is presented. The algorithm is explained as follows. The first step is to initialize the parameters ($\boldsymbol{l}, \boldsymbol{M}, \boldsymbol{x_1}, \boldsymbol{x_2}, \dots \boldsymbol{x_m}$ and $\boldsymbol{P_1}, \boldsymbol{P_2}, \dots\ \boldsymbol{P_m}$). Then, the counter variable $\boldsymbol{N}$ is introduced, which counts the number of prediction steps. The total number of required predicted steps is denoted as $\boldsymbol{n_0}$. As an initial value, the fractional-order $\boldsymbol{\gamma}$ is assigned 0, and the increment is 0.01 for each loop in order to find the optimized value. For each value of $\boldsymbol{\gamma}$ between 0 and 1, matrix A given as (3.4) is created, and then, the unknown coefficients given in (2.7b) are calculated. After that, using the actual data in Region 1 and Region 2, the modeling of data between $\boldsymbol{P_l}$ and $\boldsymbol{P_m}$ is actualized for Region 2. Then, the error defined in (3.2) is calculated. The value of the error is analyzed and compared to previously obtained values. If it is smaller than the previous one, the corresponding fractional-order value is memorized. At the end of Loop II, the optimal value of the fractional-order, which coincides with the optimal modeling is found and corresponding coefficients given in (2.7b) is determined. Then, the prediction for the next forthcoming value is made with (2.7b). After that, all the procedures starting from the increment of $\boldsymbol{N}$ is repeated so that the previously predicted value is added to the initial data for the next step prediction. This process is repeated up to the termination of Loop I. Finally, $\boldsymbol{n_0}$ the number of predictions are obtained. Keep in mind that, for the parameters $\boldsymbol{l}$ and $\boldsymbol{M}$, there exit two loops starting from 1 to $\boldsymbol{L_0}$, $\boldsymbol{M_0}$ searching the optimum values of the parameters in order to get the outcomes with a minimum error for the testing region, respectively.

3.6. Comparison

The comparison is done with LSTM Method. Conventional neural networks are insufficient for modeling the content of temporal data. Recursive neural networks (RNN) model sequential structure of data by feeding itself with the output of the previous time step. LSTMs are special types of RNNs that operate over sequences and are used in time series analysis [29]. An LSTM cell has four gates: input, forget, output and gate. With these gates, LSTMs optionally inherit the information from the previous time steps. Forget gate ($\boldsymbol{f}$), input gate ($\boldsymbol{i}$) and output gate ($\boldsymbol{o}$) are sigmoid functions ($\boldsymbol{\sigma}$) and they take values between 0 and 1. Gate $\boldsymbol{g}$ has hyperbolic tangent ($\boldsymbol{tanh}$) activation and is between -1 and 1. The Gate and forward propagation equations are listed below as (3.7) – (3.12). Here $\boldsymbol{c_t^l}$ and $\boldsymbol{h_t^l}$ refer to cell state and hidden state of layer $\boldsymbol{l}$ at time step $\boldsymbol{t}$, respectively. Each gate takes input from previous time step ($\boldsymbol{h_{t-1}^l}$) and previous layer ($\boldsymbol{h_t^{l-1}}$) and has its own set of learnable parameters $\boldsymbol{W}$'s and $\boldsymbol{b}$'s.

$$\boldsymbol{f_t} = \boldsymbol{\sigma}\left(\boldsymbol{W_f}[\boldsymbol{h_{t-1}^l}, \boldsymbol{h_t^{l-1}}] + \boldsymbol{b_f}\right) \tag{3.7}$$

$$\boldsymbol{i_t} = \boldsymbol{\sigma}\left(\boldsymbol{W_i}[\boldsymbol{h_{t-1}^l}, \boldsymbol{h_t^{l-1}}] + \boldsymbol{b_i}\right) \tag{3.8}$$

$$\boldsymbol{o_t} = \boldsymbol{\sigma}\left(\boldsymbol{W_o}[\boldsymbol{h_{t-1}^l}, \boldsymbol{h_t^{l-1}}] + \boldsymbol{b_o}\right) \tag{3.9}$$

$$\boldsymbol{g_t} = \boldsymbol{tanh}\left(\boldsymbol{W_g}[\boldsymbol{h_{t-1}^l}, \boldsymbol{h_t^{l-1}}] + \boldsymbol{b_g}\right) \tag{3.10}$$

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$$\boldsymbol{c}_t^l = \boldsymbol{f} \odot \boldsymbol{c}_{t-1}^l + \boldsymbol{i} \odot \boldsymbol{g} \quad (3.11)$$

$$\boldsymbol{h}_t^l = \boldsymbol{o} \odot \tanh(\boldsymbol{c}_t^l) \quad (3.12)$$

Here, $\odot$ is the hadamard product. Each LSTM neuron in a network may consist of one or more cells. In every time step, every cell updates its own cell state, $\boldsymbol{c}_t^l$. (3.11) describes how these cells get updated with forget gate and input gate; $\boldsymbol{f}$ gate decides how much of previous cell state that cell should remember while $\boldsymbol{i}$ gate decides how much it should consider the new input from the previous layer. Then, LSTM neuron updates its internal hidden state by multiplying output and squashed version of $\boldsymbol{c}_t^l$. An LSTM neuron gives outputs only its hidden state information to another LSTM neuron. Gate $\boldsymbol{o}$ and $\boldsymbol{c}_t$ are used internally in the computation of forward time steps [29]. To forecast time series and compare our proposed approach to neural networks, we employed a stacked LSTM model with 2 layers of LSTMs (each having 50 hidden units) and a linear prediction layer. LSTM model is trained with Adam optimizer. [30]

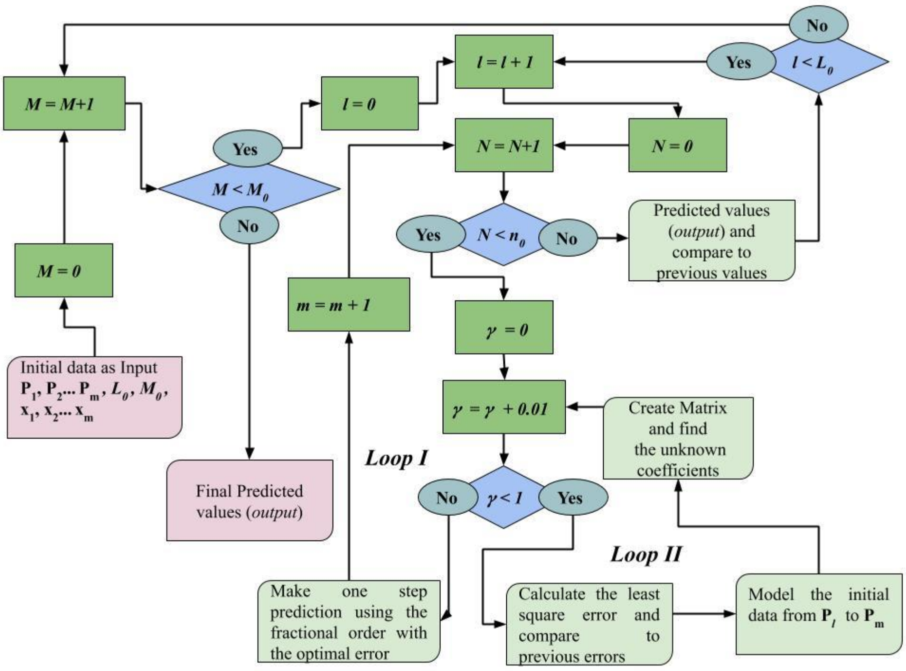

Figure 2: The algorithm for the prediction.

In order to achieve modeling, $\boldsymbol{l}$ value is investigated. For the modeling of the GDP per capita of each country, the required previous data l of past years used in the algorithm differs after optimization. Optimal values of $\boldsymbol{l}$ are listed for each country in Table 3. Optimized $\boldsymbol{M}$ values after processing test region can be seen again from Table 3.

Table 1 illustrates modeling and prediction results for deep assessment and prediction results for LSTM. Reported errors in this table are Mean Average Precision Error (MAPE) and calculated as given in (3.13), where k is the total number of samples, $v(i)$ is the actual value and $\tilde{v}(i)$ is the predicted value for i th sample. Column 2 and 3 refer to the optimal fractional-order values for test and prediction regions, respectively. Modeling error of Deep Assessment is shown in Column 4. Note that, unknown model coefficients in (2.7b) are optimized for the test prediction (2018) with minimum error. For instance, in our previous study [4], modeling error of Italy, Spain, Turkey and UK are obtained as 0.12%, 0.52%, 0.15% and 0.19% respectively for early Deep Assessment model on population data. The Fifth and sixth columns illustrate Deep Assessment and Deep Learning prediction errors for $m_i = 58$. In average, the proposed Deep Assessment model has 0.3861% MAPE whereas for LSTM model, average MAPE is 1.4523%. Deep Assessment outperforms LSTM for all countries. Both models perform relatively poor for the UK and minimum errors for 2018 test prediction are obtained on EU.

$$MAPE = \frac{1}{k}\sum_{i=1}^{k}\left|\frac{v(i) - \tilde{v}(i)}{v(i)}\right| \times 100 \tag{3.13}$$

Table 1: Modeling and test results (γ and MAPE values) of countries.

Country	γ test region	γ interpolation	Deep Assessment Region-2 ($l < i < m$)	Deep Assessment Region-3	Deep Learning Region-3 ($m_1 < i < m$)
US	0.39	0.18	0.7231%	0.1081%	0.8424%
UK	0.18	0.05	3.5407%	0.9129%	3.0508%
Brazil	0.18	0.32	5.0042%	0.1303%	0.4728%
China	0.97	0.5	16.5071%	0.7147%	1.6365%
India	0.96	0.99	21.2977%	0.3379%	0.7203%
Japan	0.57	1	15.4685%	0.3499%	1.1091%
EU	0.32	0.22	3.6834%	0.1044%	0.2522%
Italy	0.43	0.43	3.7622%	0.1048%	3.0796%
Spain	0.99	0.99	33.58%	0.0560%	1.5683%
Turkey	0.39	0.39	5.5235%	0.1167%	2.3691%

GDP per capita (current US dollars) prediction of all countries for year 2019 is illustrated in Table 2 for each method. For countries Brazil, China, India, Turkey, UK, and US, predictions obtained by two models are similar. On the other hand, Italy and Spain yields different results.

Table 2: GDP per Capita Prediction of Countries for 2019 (US dollars)

Country	Deep Assessment	Deep Learning
Brazil	7932	8013
China	10312	10273
India	2154	1967
Italy	39028	35141
Japan	34421	37994
Spain	30385	35372
Turkey	8260	8920
US	65767	63844
UK	44897	44702
EU	40487	36487

Table 3: l and M values of corresponding countries.

Country	l	M
Brazil	24	3
China	11	3
India	3	2
Italy	20	4
Japan	4	3
Spain	2	3
Turkey	17	4
US	20	5
UK	25	2
EU	18	7

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4. Conclusion

The study is a first attempt to combine the data modeling and prediction using the fractional calculus and Least Square Method. In the approach, the data is expressed as the summation of the previous values and the derivatives. One of the two main aims of the study is to model the discrete dataset as a continuous curve with a minimum error to interpret not only for each year but also for any time instance. The second one which is an original and unprecedented approach is to predict the forthcoming years using fractional calculus properties and Least Square Method. As a summary, by using Least Square Method, the discrete data for GDP per capita is modeled as a function satisfying the fractional differential equation given in (2.6) and the function itself is assumed to be the finite summation of its previous values and the derivatives with unknown coefficients. Then, GDP per capita for 2018 is predicted and called as test prediction by using the dataset between 1960 and 2017, then the unknown coefficients and the parameters in (2.7b) determined with an optimum error. After that, with these findings, GDP per capita for 2019 is predicted. The mathematical model of the GDP per capita of the countries Brazil, China, European Union, India, Italy, Japan, UK, the USA, Spain, and Turkey is constructed with our deep assessment method using fractional calculus and compared to the LSTM algorithm which is a deep learning algorithm used for time sequences in general. The result of the proposed method is quite satisfactory and gives better results compared to the other methods including Linear, Polynomial and models which are also compared in [24-26] for only modeling. It is considered that better results can be obtained if the model includes other variables such as employment, literacy and population data, etc.

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Appendix A

Table A1: GDP per capita (US dollars) values of the countries.

i	Years	Brazil	China	EU	India	Italy
1	1960	210.1099	89.52054	890.4056	82.1886	804.4926
2	1961	205.0408	75.80584	959.71	85.3543	887.3367
3	1962	260.4257	70.90941	1037.326	89.88176	990.2602
4	1963	292.2521	74.31364	1135.194	101.1264	1126.019
5	1964	261.6666	85.49856	1245.499	115.5375	1222.545
6	1965	261.3544	98.48678	1346.058	119.3189	1304.454
7	1966	315.7972	104.3246	1448.551	89.99731	1402.442
8	1967	347.4931	96.58953	1546.804	96.33914	1533.693
9	1968	374.7868	91.47272	1602.06	99.87596	1651.939
10	1969	403.8843	100.1299	1762.472	107.6223	1813.388
11	1970	445.0231	113.163	1950.732	112.4345	2106.864
12	1971	504.7495	118.6546	2195.145	118.6032	2305.61
13	1972	586.2144	131.8836	2611.729	122.9819	2671.137
14	1973	775.2733	157.0904	3296.935	143.7787	3205.252
15	1974	1004.105	160.1401	3685.596	163.4781	3621.146
16	1975	1153.831	178.3418	4274.046	158.0362	4106.994
17	1976	1390.625	165.4055	4406.238	161.0921	4033.099
18	1977	1567.006	185.4228	4968.988	186.2135	4603.6
19	1978	1744.257	156.3964	6064.883	205.6934	5610.498
20	1979	1908.488	183.9832	7377.165	224.001	6990.286
21	1980	1947.276	194.8047	8384.718	266.5778	8456.919
22	1981	2132.883	197.0715	7391.077	270.4706	7622.833
23	1982	2226.767	203.3349	7093.702	274.1113	7556.523
24	1983	1570.54	225.4319	6859.966	291.2381	7832.575

(continued)

i	Years	Brazil	China	EU	India	Italy
25	1984	1578.926	250.714	6572.019	276.668	7739.715
26	1985	1648.082	294.4588	6775.647	296.4352	7990.687
27	1986	1941.491	281.9281	9265.924	310.4659	11315.02
28	1987	2087.308	251.812	11432.23	340.4168	14234.73
29	1988	2300.377	283.5377	12711.96	354.1493	15744.66
30	1989	2908.496	310.8819	12936.46	346.1129	16386.66
31	1990	3100.28	317.8847	15989.22	367.5566	20825.78
32	1991	3975.39	333.1421	16496.51	303.0556	21956.53
33	1992	2596.92	366.4607	17919.02	316.9539	23243.47
34	1993	2791.209	377.3898	16256.42	301.159	18738.76
35	1994	3500.611	473.4923	17194.12	346.103	19337.63
36	1995	4748.216	609.6567	19898.44	373.7665	20664.55
37	1996	5166.164	709.4138	20295.17	399.9501	23081.6
38	1997	5282.009	781.7442	19121.21	415.4938	21829.35
39	1998	5087.152	828.5805	19763.51	413.2989	22318.14
40	1999	3478.373	873.2871	19698.89	441.9988	21997.62
41	2000	3749.753	959.3725	18261.97	443.3142	20087.59
42	2001	3156.799	1053.108	18457.89	451.573	20483.22
43	2002	2829.283	1148.508	20055.33	470.9868	22270.14
44	2003	3070.91	1288.643	24310.25	546.7266	27465.68
45	2004	3637.462	1508.668	27960.05	627.7742	31259.72
46	2005	4790.437	1753.418	29115.63	714.861	32043.14
47	2006	5886.464	2099.229	30960.56	806.7533	33501.66
48	2007	7348.031	2693.97	35630.94	1028.335	37822.67
49	2008	8831.023	3468.304	38185.62	998.5223	40778.34
50	2009	8597.915	3832.236	34019.28	1101.961	37079.76
51	2010	11286.24	4550.454	33740.65	1357.564	36000.52
52	2011	13245.61	5618.132	36506.64	1458.104	38599.06
53	2012	12370.02	6316.919	34328.82	1443.88	35053.53
54	2013	12300.32	7050.646	35683.86	1449.606	35549.97
55	2014	12112.59	7651.366	36787.23	1573.881	35518.42
56	2015	8814.001	8033.388	32319.45	1605.605	30230.23
57	2016	8712.887	8078.79	32425.13	1729.268	30936.13
58	2017	9880.947	8759.042	33908	1981.269	32326.84
59	2018	8920.762	9770.847	36569.73	2009.979	34483.2

Table A2: GDP per capita (US dollars) values of the countries.

i	Years	Japan	Spain	UK	US	Turkey
1	1960	478.9953	396.3923	1397.595	3007.123	509.4239
2	1961	563.5868	450.0533	1472.386	3066.563	283.8283
3	1962	633.6403	520.2061	1525.776	3243.843	309.4467
4	1963	717.8669	609.4874	1613.457	3374.515	350.6629
5	1964	835.6573	675.2416	1748.288	3573.941	369.5834
6	1965	919.7767	774.7616	1873.568	3827.527	386.3581
7	1966	1058.504	889.6599	1986.747	4146.317	444.5494
8	1967	1228.909	968.3068	2058.782	4336.427	481.6937
9	1968	1450.62	950.5457	1951.759	4695.923	526.2135
10	1969	1669.098	1077.679	2100.668	5032.145	571.6178
11	1970	2037.56	1212.289	2347.544	5234.297	489.9303
12	1971	2272.078	1362.166	2649.802	5609.383	455.1049
13	1972	2967.042	1708.809	3030.433	6094.018	558.421
14	1973	3997.841	2247.553	3426.276	6726.359	686.4899
15	1974	4353.824	2749.925	3665.863	7225.691	927.7991
16	1975	4659.12	3209.837	4299.746	7801.457	1136.375
17	1976	5197.807	3279.313	4138.168	8592.254	1275.956
18	1977	6335.788	3627.591	4681.44	9452.577	1427.372
19	1978	8821.843	4356.439	5976.938	10564.95	1549.644

(continued)

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i	Years	Japan	Spain	UK	US	Turkey
20	1979	9105.136	5770.215	7804.762	11674.19	2079.22
21	1980	9465.38	6208.578	10032.06	12574.79	1564.247
22	1981	10361.32	5371.166	9599.306	13976.11	1579.074
23	1982	9578.114	5159.709	9146.077	14433.79	1402.406
24	1983	10425.41	4478.5	8691.519	15543.89	1310.256
25	1984	10984.87	4489.989	8179.194	17121.23	1246.825
26	1985	11584.65	4699.656	8652.217	18236.83	1368.401
27	1986	17111.85	6513.503	10611.11	19071.23	1510.677
28	1987	20745.25	8239.614	13118.59	20038.94	1705.895
29	1988	25051.85	9703.124	15987.17	21417.01	1745.365
30	1989	24813.3	10681.97	16239.28	22857.15	2021.859
31	1990	25359.35	13804.88	19095.47	23888.6	2794.35
32	1991	28925.04	14811.9	19900.73	24342.26	2735.708
33	1992	31464.55	16112.19	20487.17	25418.99	2842.37
34	1993	35765.91	13339.91	18389.02	26387.29	3180.188
35	1994	39268.57	13415.29	19709.24	27694.85	2270.338
36	1995	43440.37	15471.96	23123.18	28690.88	2897.866
37	1996	38436.93	16109.08	24332.7	29967.71	3053.947
38	1997	35021.72	14730.8	26734.56	31459.14	3144.386
39	1998	31902.77	15394.35	28214.27	32853.68	4496.497
40	1999	36026.56	15715.33	28669.54	34513.56	4108.123
41	2000	38532.04	14713.07	28149.87	36334.91	4316.549
42	2001	33846.47	15355.7	27744.51	37133.24	3119.566
43	2002	32289.35	17025.53	30056.59	38023.16	3659.94
44	2003	34808.39	21463.44	34419.15	39496.49	4718.2
45	2004	37688.72	24861.28	40290.31	41712.8	6040.608
46	2005	37217.65	26419.3	42030.29	44114.75	7384.252
47	2006	35433.99	28365.31	44599.7	46298.73	8035.377
48	2007	35275.23	32549.97	50566.83	47975.97	9711.874
49	2008	39339.3	35366.26	47287	48382.56	10854.17
50	2009	40855.18	32042.47	38713.14	47099.98	9038.52
51	2010	44507.68	30502.72	39435.84	48466.82	10672.39
52	2011	48168	31636.45	42038.5	49883.11	11335.51
53	2012	48603.48	28324.43	42462.71	51603.5	11707.26
54	2013	40454.45	29059.55	43444.56	53106.91	12519.39
55	2014	38109.41	29461.55	47417.64	55032.96	12095.85
56	2015	34524.47	25732.02	44966.1	56803.47	10948.72
57	2016	38794.33	26505.62	41074.17	57904.2	10820.63
58	2017	38331.98	28100.85	40361.42	59927.93	10513.65
59	2018	39289.96	30370.89	42943.9	62794.59	9370.176

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