Closed Formulas for the Sums of Squares of Generalized Fibonacci Numbers

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Abstract. In this paper, closed forms of the summation formulas for generalized Fibonacci numbers are presented. As special cases, we give summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers.

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1. Introduction

Sequences have been fascinating topic for mathematicians for centuries. The Fibonacci and Lucas sequences are very well-known examples of second order recurrence sequences. The Fibonacci numbers are perhaps most famous for appearing in the rabbit breeding problem, introduced by Leonardo de Pisa in 1202 in his book called Liber Abaci. The Fibonacci sequences are a source of many nice and interesting identities. A similar interpretation exists for Lucas sequence.

The sequence of Fibonacci numbers \( \{F_n\} \) is defined by

\[
F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \quad F_0 = 0, \quad F_1 = 1.
\]

and the sequence of Lucas numbers \( \{L_n\} \) is defined by

\[
L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \quad L_0 = 2, \quad L_1 = 1.
\]

The Fibonacci numbers, Lucas numbers and their generalizations have many interesting properties and applications to almost every field. In 1965, Horadam [8] defined a generalization of Fibonacci sequence, that
is, he defined a second-order linear recurrence sequence \( \{W_n(W_0, W_1; r, s)\} \), or simply \( \{W_n\} \), as follows:

\[
W_n = rW_{n-1} + sW_{n-2}; \quad W_0 = a, \ W_1 = b, \quad (n \geq 2)
\]

where \( W_0, W_1 \) are arbitrary complex numbers and \( r, s \) are real numbers, see also Horadam [7], [9] and [10].

Now these generalized Fibonacci numbers \( \{W_n(a, b; r, s)\} \) are also called Horadam numbers. The sequence \( \{W_n\}_{n \geq 0} \) can be extended to negative subscripts by defining

\[
W_{-n} = \frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}
\]

for \( n = 1, 2, 3, \ldots \) when \( s \neq 0 \). Therefore, recurrence (1.1) holds for all integer \( n \).

For some specific values of \( a, b, r \) and \( s \), it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of \( r, s \) and initial values.

### Table 1. A few special case of generalized Fibonacci sequences.

<table>
<thead>
<tr>
<th>Name of sequence</th>
<th>Notation: ( W_n(a, b; r, s) )</th>
<th>OEIS: [17]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fibonacci</td>
<td>( F_n = W_n(0, 1; 1, 1) )</td>
<td>A000045</td>
</tr>
<tr>
<td>Lucas</td>
<td>( L_n = W_n(2, 1; 1, 1) )</td>
<td>A000032</td>
</tr>
<tr>
<td>Pell</td>
<td>( P_n = W_n(0, 1; 2, 1) )</td>
<td>A000129</td>
</tr>
<tr>
<td>Pell-Lucas</td>
<td>( Q_n = W_n(2, 2; 2, 1) )</td>
<td>A002203</td>
</tr>
<tr>
<td>Jacobsthal</td>
<td>( J_n = W_n(0, 1; 1, 2) )</td>
<td>A001045</td>
</tr>
<tr>
<td>Jacobsthal-Lucas</td>
<td>( j_n = W_n(2, 1; 1, 2) )</td>
<td>A014551</td>
</tr>
</tbody>
</table>

The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

\[
\sum_{i=1}^{n} F_i^2 = F_nF_{n+1}
\]

and

\[
\sum_{i=1}^{n} Q_i^2 = \frac{1}{2}(Q_nQ_{n+1} - 4).
\]

In this work, we derive expressions for sums of second powers of generalized Fibonacci numbers. We present some works on sum formulas of powers of the numbers in the following Table 2.

### Table 2. A few special study on sum formulas of second, third and arbitrary powers.

<table>
<thead>
<tr>
<th>Name of sequence</th>
<th>sums of second powers</th>
<th>sums of third powers</th>
<th>sums of powers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generalized Fibonacci</td>
<td>[1,2,6,11,12]</td>
<td>[5,18]</td>
<td>[3,4,13]</td>
</tr>
<tr>
<td>Generalized Tribonacci</td>
<td>[15]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Generalized Tetranacci</td>
<td>[14,16]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. **Summing Formulas of Generalized Fibonacci Numbers with Positive Subscripts**

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.
THEOREM 2.1. For \( n \geq 1 \) we have the following formulas: if \((s + 1)(r + s - 1)(r - s + 1) \neq 0\) then

(a):

\[
\sum_{i=1}^{n} W_{i}^{2} = \frac{(1 - s)W_{n+2}^{2} + (1 - s - r^2 - r^2s)W_{n+1}^{2} + 2rsW_{n+1}W_{n+2} + (s - 1)W_{2}^{2} + s^2 (s - 1) W_{0}^{2} - 2rsW_{1}W_{0}}{(s + 1)(r + s - 1)(r - s + 1)}.
\]

(b):

\[
\sum_{i=1}^{n} W_{i+1}W_{i} = \frac{rW_{n+2}^{2} + rsW_{n+1}W_{n+2} + (1 - r^2 - s^2)W_{n+1}W_{n+2} - rW_{1}^{2} - rsW_{0}^{2} + s(-r^2 + s^2 - 1)W_{1}W_{0}}{(s + 1)(r + s - 1)(r - s + 1)}.
\]

Proof. Using the recurrence relation

\[ W_{n+2} = rW_{n+1} + sW_{n} \]

i.e.

\[ sW_{n} = W_{n+2} - rW_{n+1} \]

we obtain

\[
\begin{align*}
s^2W_{n}^{2} &= W_{n+2}^{2} + r^2W_{n+1}^{2} - 2rW_{n+2}W_{n+1} \\
s^2W_{n-1}^{2} &= W_{n+1}^{2} + r^2W_{n}^{2} - 2rW_{n+1}W_{n} \\
s^2W_{n-2}^{2} &= W_{n}^{2} + r^2W_{n-1}^{2} - 2rW_{n}W_{n-1} \\
s^2W_{n-3}^{2} &= W_{n-1}^{2} + r^2W_{n-2}^{2} - 2rW_{n-1}W_{n-2} \\
s^2W_{n-4}^{2} &= W_{n-2}^{2} + r^2W_{n-3}^{2} - 2rW_{n-2}W_{n-3} \\
&\vdots \\
s^2W_{3}^{2} &= W_{4}^{2} + r^2W_{3}^{2} - 2rW_{4}W_{3} \\
s^2W_{2}^{2} &= W_{3}^{2} + r^2W_{2}^{2} - 2rW_{3}W_{2}.
\end{align*}
\]

If we add the above equations by side by, we get

(2.1)

\[
\sum_{i=1}^{n} s^2W_{i}^{2} = \sum_{i=3}^{n+2} W_{i}^{2} + r^2 \sum_{i=2}^{n+1} W_{i}^{2} - 2r \sum_{i=2}^{n+1} W_{i+1}W_{i}
\]

Note that

\[
\begin{align*}
\sum_{i=3}^{n+2} W_{i}^{2} &= -W_{1}^{2} - W_{2}^{2} + W_{n+1}^{2} + W_{n+2}^{2} + \sum_{i=1}^{n} W_{i}^{2}, \\
\sum_{i=2}^{n+1} W_{i}^{2} &= -W_{1}^{2} + W_{n+1}^{2} + \sum_{i=1}^{n} W_{i}^{2}, \\
\sum_{i=2}^{n+1} W_{i+1}W_{i} &= -W_{2}W_{1} + W_{n+2}W_{n+1} + \sum_{i=1}^{n} W_{i+1}W_{i},
\end{align*}
\]
We put them into the (2.1) we obtain

\[ (2.2) \quad s^2 \sum_{i=1}^{n} W_i^2 = (-W_1^2 - W_2^2 + W_{n+1}^2 + W_{n+2}^2 + \sum_{i=1}^{n} W_i^2) + r^2 (-W_1^2 + W_{n+1}^2 + \sum_{i=1}^{n} W_i^2) \]

\[ -2r(-W_2W_1 + W_{n+2}W_{n+1} + \sum_{i=1}^{n} W_{i+1}W_i). \]

Next we calculate \( \sum_{i=1}^{n} W_{i+1}W_i \). Again, using the recurrence relation

\[ W_{n+2} = rW_{n+1} + sW_n \]

i.e.

\[ sW_n = W_{n+2} - rW_{n+1} \]

we obtain

\[ sW_{n+1}W_n = W_{n+2}W_{n+1} - rW_{n+1}^2 \]
\[ sW_nW_{n-1} = W_{n+1}W_n - rW_n^2 \]
\[ sW_{n-1}W_{n-2} = W_nW_{n-1} - rW_{n-1}^2 \]
\[ \vdots \]
\[ sW_3W_2 = W_4W_3 - rW_3^2 \]
\[ sW_2W_1 = W_3W_2 - rW_2^2. \]

If we add the above equations by side by, we get

\[ (2.3) \quad s \sum_{i=1}^{n} W_{i+1}W_i = \sum_{i=2}^{n+1} W_{i+1}W_i - r \sum_{i=2}^{n+1} W_i^2 \]

Note that

\[ \sum_{i=2}^{n+1} W_{i+1}W_i = -W_2W_1 + W_{n+2}W_{n+1} + \sum_{i=1}^{n} W_{i+1}W_i, \]
\[ \sum_{i=2}^{n+1} W_i^2 = -W_1^2 + W_{n+1}^2 + \sum_{i=1}^{n} W_i^2. \]

If we put them into the (2.3) then it follows that

\[ (2.4) \quad s \sum_{i=1}^{n} W_{i+1}W_i = (-W_2W_1 + W_{n+2}W_{n+1} + \sum_{i=1}^{n} W_{i+1}W_i) - r(-W_1^2 + W_{n+1}^2 + \sum_{i=1}^{n} W_i^2). \]

Then, using

\[ W_2 = (rW_1 + sW_0) \]

and solving the system (2.2)-(2.4), the required results of (a) and (b) follow.

Taking \( r = s = 1 \) in Theorem 2.1 (a) and (b), we obtain the following proposition.

**Proposition 2.2.** If \( r = s = 1 \) then for \( n \geq 1 \) we have the following formulas:
(a): \[ \sum_{i=1}^{n} W_i^2 = \frac{1}{2} (-2 W_{n+1}^2 + 2 W_n W_{n+1} - 2 W_1 W_0) \].

(b): \[ \sum_{i=1}^{n} W_{i+1} W_i = \frac{1}{2} (W_{n+2}^2 + W_{n+1}^2 - W_{n+1} W_{n+2} - W_1^2 - W_0^2 - W_1 W_0) \].

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take \( W_n = F_n \) with \( F_0 = 0, F_1 = 1 \)).

**Corollary 2.3.** For \( n \geq 1 \), Fibonacci numbers have the following properties:

(a): \[ \sum_{i=1}^{n} F_i^2 = \frac{1}{2} (-2 F_{n+1}^2 + 2 F_n F_{n+1}) \].

(b): \[ \sum_{i=1}^{n} F_{i+1} F_i = \frac{1}{2} (F_{n+2}^2 + F_{n+1}^2 - F_{n+1} F_{n+2} - 1) \].

Taking \( W_n = L_n \) with \( L_0 = 2, L_1 = 1 \) in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

**Corollary 2.4.** For \( n \geq 1 \), Lucas numbers have the following properties:

(a): \[ \sum_{i=1}^{n} L_i^2 = \frac{1}{2} (-2 L_{n+1}^2 + 2 L_n L_{n+1} - 4) \].

(b): \[ \sum_{i=1}^{n} L_{i+1} L_i = \frac{1}{2} (L_{n+2}^2 + L_{n+1}^2 - L_{n+1} L_{n+2} - 7) \].

Taking \( r = 2, s = 1 \) in Theorem 2.1 (a) and (b), we obtain the following proposition.

**Proposition 2.5.** If \( r = 2, s = 1 \) then for \( n \geq 0 \) we have the following formulas:

(a): \[ \sum_{i=1}^{n} W_i^2 = \frac{1}{2} (-2 W_{n+1}^2 + W_{n+2} W_{n+1} - W_1 W_0) \].

(b): \[ \sum_{i=1}^{n} W_{i+1} W_i = \frac{1}{2} (W_{n+2}^2 + W_{n+1}^2 - 2 W_{n+2} W_{n+1} - W_1^2 - W_0^2 - 2 W_1 W_0) \].

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take \( W_n = P_n \) with \( P_0 = 0, P_1 = 1 \)).

**Corollary 2.6.** For \( n \geq 1 \), Pell numbers have the following properties:

(a): \[ \sum_{i=1}^{n} P_i^2 = \frac{1}{2} (-2 P_{n+1}^2 + P_{n+2} P_{n+1}) = \frac{1}{2} P_n P_{n+1} \].

(b): \[ \sum_{i=1}^{n} P_{i+1} P_i = \frac{1}{2} (P_{n+2}^2 + P_{n+1}^2 - 2 P_{n+2} P_{n+1} - 1) \].

Taking \( W_n = Q_n \) with \( Q_0 = 2, Q_1 = 2 \) in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

**Corollary 2.7.** For \( n \geq 1 \), Pell-Lucas numbers have the following properties:

(a): \[ \sum_{i=1}^{n} Q_i^2 = \frac{1}{2} (-2 Q_{n+1}^2 + Q_{n+2} Q_{n+1} - 4) = \frac{1}{2} (Q_n Q_{n+1} - 4) \].

(b): \[ \sum_{i=1}^{n} Q_{i+1} Q_i = \frac{1}{4} (Q_{n+2}^2 + Q_{n+1}^2 - 2 Q_{n+2} Q_{n+1} - 16) \].

If \( r = 1, s = 2 \) then \((s + 1)(r + s - 1)(r - s + 1) = 0\) so we can’t use Theorem 2.1 directly. Therefore we need another method to find \( \sum_{i=1}^{n} W_i^2 \) and \( \sum_{i=1}^{n} W_{i+1} W_i \) which is given in the following theorem.

**Theorem 2.8.** If \( r = 1, s = 2 \) then for \( n \geq 1 \) we have the following formulas:

(a): \[ \sum_{i=1}^{n} W_i^2 = \frac{1}{9} (W_{n+2}^2 - W_{n+1}^2 - 4 (W_0 + W_1) W_0 + (2W_0 - W_1)^2 n) \].
(b): $\sum_{i=1}^{n} W_{i+1} W_{i} = \frac{1}{30} (5W_{n+2}^2 + 4W_{n+1}^2 + (-9W_{i}^2 - 20W_{0}^2 - 20W_{1}W_{0}) - 4(W_{1} - 2W_{0})^2 n).$

Proof.

(a): The proof will be by induction on $n$. Before the proof, we recall some information on generalized Jacobsthal numbers. A generalized Jacobsthal sequence $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1)\}_{n \geq 0}$ is defined by the second-order recurrence relations

$$W_n = W_{n-1} + 2W_{n-2}; \quad W_0 = a, \quad W_1 = b, \quad (n \geq 2) \tag{2.5}$$

with the initial values $W_0, W_1$ not all being zero. The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{1}{2}W_{-(n-1)} + \frac{1}{2}W_{-(n-2)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (2.5) holds for all integer $n$. The first few generalized Jacobsthal numbers with positive subscript and negative subscript are given in the following Table 1.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$W_n$</th>
<th>$W_{-n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$W_0$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$W_1$</td>
<td>$-\frac{1}{2}W_0 + \frac{1}{2}W_1$</td>
</tr>
<tr>
<td>2</td>
<td>$2W_0 + W_1$</td>
<td>$\frac{3}{4}W_0 - \frac{1}{4}W_1$</td>
</tr>
<tr>
<td>3</td>
<td>$2W_0 + 3W_1$</td>
<td>$-\frac{5}{8}W_0 + \frac{3}{8}W_1$</td>
</tr>
<tr>
<td>4</td>
<td>$6W_0 + 5W_1$</td>
<td>$\frac{11}{16}W_0 - \frac{5}{16}W_1$</td>
</tr>
<tr>
<td>5</td>
<td>$10W_0 + 11W_1$</td>
<td>$-\frac{21}{32}W_0 + \frac{11}{32}W_1$</td>
</tr>
<tr>
<td>6</td>
<td>$22W_0 + 21W_1$</td>
<td>$\frac{43}{64}W_0 - \frac{21}{64}W_1$</td>
</tr>
</tbody>
</table>

Binet formula of generalized Jacobsthal sequence can be calculated using its characteristic equation which is given as

$$t^2 - t - 2 = 0.$$ 

The roots of characteristic equation are

$$\alpha = 2, \quad \beta = -1$$

and the roots satisfy the following

$$\alpha + \beta = 1, \quad \alpha\beta = -2, \quad \alpha - \beta = 3.$$ 

Using these roots and the recurrence relation, Binet formula can be given as

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} = \frac{A \times 2^n - B(-1)^n}{3} \tag{2.6}$$

where $A = W_1 - W_0\beta = W_1 + W_0$ and $B = W_1 - W_0\alpha = W_1 - 2W_0$. 
We now prove (a) by induction on \( n \). If \( n = 1 \) we see that the sum formula reduces to the relation
\[
W_1^2 = \frac{1}{9}(W_3^2 - W_2^2 + W_1^2 - 8W_1W_0).
\]
Since
\[
W_2 = 2W_0 + W_1,
\]
\[
W_3 = 2W_0 + 3W_1,
\]
(2.7) is true. Assume that the relation in (a) is true for \( n = m \), i.e.,
\[
\sum_{i=1}^{m} W_i^2 = \frac{1}{9}(W_{m+2}^2 - W_{m+1}^2 - 4(W_0 + W_1)W_0 + (2W_0 - W_1)^2 m).
\]
Then we get
\[
\sum_{i=1}^{m+1} W_i^2 = \sum_{i=1}^{m} W_i^2 + W_{m+1}^2
\]
\[
= \frac{1}{9}(W_{m+2}^2 + 8W_{m+1}^2) - 4(W_0 + W_1)W_0 + (2W_0 - W_1)^2 m
\]
\[
= \frac{1}{9}((W_{m+3}^2 - W_{m+2}^2 - 4(W_0 + W_1)W_0 + (2W_0 - W_1)^2 (m + 1))
\]
\[
= \frac{1}{9}(W_{(m+1)+2}^2 - W_{(m+1)+1}^2 - 4(W_0 + W_1)W_0 + (2W_0 - W_1)^2 (m + 1))
\]
where
\[
W_{m+2}^2 + 8W_{m+1}^2 - (2W_0 - W_1)^2 = W_{m+3}^2 - W_{m+2}^2.
\]
(2.8) can be proved by using Binet formula of \( W_n \). Hence, the relation in (a) holds also for \( n = m+1 \).

(b): We now prove (b) by induction on \( n \). If \( n = 1 \) we see that the sum formula reduces to the relation
\[
W_2W_1 = \frac{1}{36}(5W_3^2 + 4W_2^2 - 13W_1^2 - 36W_0^2 - 4W_1W_0).
\]
Since
\[
W_2 = 2W_0 + W_1,
\]
\[
W_3 = 2W_0 + 3W_1,
\]
(2.9) is true. Assume that the relation in (b) is true for \( n = m \), i.e.,
\[
\sum_{i=1}^{m} W_{i+1}W_i = \frac{1}{36}(5W_{m+2}^2 + 4W_{m+1}^2 + (-9W_1^2 - 20W_0^2 - 20W_1W_0) - 4(W_1 - W_0)^2 m).
\]
Then we get

$$\sum_{i=1}^{m+1} W_{i+1}W_i = W_{m+2}W_{m+1} + \sum_{i=1}^m W_{i+1}W_i$$

$$= \frac{1}{36}(5W_{m+2}^2 + 4W_{m+1}^2 + 36W_{m+2}W_{m+1} + (-9W_1^2 - 20W_0^2 - 20W_1W_0) - 4(W_1 - 2W_0)^2 m)$$

$$= \frac{1}{36}(5W_{m+2}^2 + 4W_{m+1}^2 + 36W_{m+2}W_{m+1} + 4(W_1 - 2W_0)^2 + (-9W_1^2 - 20W_0^2 - 20W_1W_0)$$

$$- 4(W_1 - 2W_0)^2 (m + 1))$$

$$= \frac{1}{36}(5W_{m+2}^2 + 4W_{m+1}^2 + (-9W_1^2 - 20W_0^2 - 20W_1W_0) - 4(W_1 - 2W_0)^2 (m + 1))$$

$$= \frac{1}{36}(5W_{m+2}^2 + 4W_{m+1}^2 + 4W_{(m+1)+1}^2 + (-9W_1^2 - 20W_0^2 - 20W_1W_0) - 4(W_1 - 2W_0)^2 (m + 1))$$

where

$$5W_{m+2}^2 + 4W_{m+1}^2 + 36W_{m+2}W_{m+1} + 4(W_1 - 2W_0)^2 = 5W_{m+3}^2 + 4W_{m+2}^2.$$  

(2.10) can be proved by using Binet formula of $W_n$. Hence, the relation in (b) holds also for $n = m + 1$.

From the last theorem we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

**Corollary 2.9.** For $n \geq 1$, Jacobsthal numbers have the following property:

(a): $\sum_{i=1}^{n} J_i^2 = \frac{1}{3}(J_{n+2}^2 - J_{n+1}^2 + n)$.

(b): $\sum_{i=1}^{n} J_{i+1}J_i = \frac{1}{36}(5J_{n+2}^2 + 4J_{n+1}^2 - 9 - 4n)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

**Corollary 2.10.** For $n \geq 1$, Jacobsthal-Lucas numbers have the following property:

(a): $\sum_{i=1}^{n} j_i^2 = \frac{1}{3}(j_{n+2}^2 - j_{n+1}^2 - 24 + 9n)$.

(b): $\sum_{i=1}^{n} j_{i+1}j_i = \frac{1}{36}(5j_{n+2}^2 + 4j_{n+1}^2 - 129 - 36n)$.

3. Summing Formulas of Generalized Fibonacci Numbers with Negative Subscripts

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

**Theorem 3.1.** For $n \geq 1$ we have the following formulas: If $(s + 1)(r + s - 1)(r - s + 1) \neq 0$ then

(a):

$$\sum_{i=1}^{n} W_{-i}^2 = \frac{(s - 1)W_{-n+1}^2 + (r^2 + r^2s + s - 1)W_{-n}^2 - 2rsW_{-n+1}W_{-n} + 2rsW_1W_0 + (1 - s)W_1^2 + (1 - s - r^2 - r^2s)W_0^2}{(s + 1)(r + s - 1)(r - s + 1)}.$$
\[ \sum_{i=1}^{n} W_{-i+1}W_{-i} = \frac{-rW_{-n+1}^2 - rs^2W_{-n}^2 + (r^2 + s^2 - 1)W_{-n+1}W_{-n} + (1 - r^2 - s^2)W_1W_0 + rW_1^2 + rs^2W_0^2}{(s+1)(r+s-1)(r-s+1)} \]

Proof. Using the recurrence relation

\[ W_{-n+2} = rW_{-n+1} + sW_{-n} \Rightarrow W_{-n} = -\frac{r}{s}W_{-n+1} + \frac{1}{s}W_{-n+2} \]

i.e.

\[ sW_{-n} = W_{-n+2} - rW_{-n+1} \]

we obtain

\[
\begin{align*}
    s^2W_{-n}^2 &= W_{-n+2}^2 + r^2W_{-n+1}^2 - 2rW_{-n+2}W_{-n+1} \\
    s^2W_{-n+1}^2 &= W_{-n+3}^2 + r^2W_{-n+2}^2 - 2rW_{-n+3}W_{-n+2} \\
    s^2W_{-n+2}^2 &= W_{-n+4}^2 + r^2W_{-n+3}^2 - 2rW_{-n+4}W_{-n+3} \\
    s^2W_{-n+3}^2 &= W_{-n+5}^2 + r^2W_{-n+4}^2 - 2rW_{-n+5}W_{-n+4} \\
    &\vdots \\
    s^2W_{-3}^2 &= W_{-1}^2 + r^2W_{-2}^2 - 2rW_{-1}W_{-2} \\
    s^2W_{-2}^2 &= W_0^2 + r^2W_{-1}^2 - 2rW_0W_{-1} \\
    s^2W_{-1}^2 &= W_1^2 + r^2W_0^2 - 2rW_1W_0.
\end{align*}
\]

If we add the above equations by side by, we get

\[
(3.1) \quad s^2 \sum_{i=1}^{n} W_{-i}^2 = (W_2^2 + W_0^2 - W_{-n+1}^2 - W_{-n}^2 + \sum_{i=1}^{n} W_{-i}^2) + r^2(W_0^2 - W_{-n}^2 + \sum_{i=1}^{n} W_{-i}^2) - 2r(W_1W_0 - W_{-n+1}W_{-n} + \sum_{i=1}^{n} W_{-i+1}W_{-i})
\]

Next we calculate \( \sum_{i=1}^{n} W_{-i+1}W_{-i} \). Again using the recurrence relation

\[ W_{-n+2} = rW_{-n+1} + sW_{-n} \Rightarrow W_{-n} = -\frac{r}{s}W_{-n+1} + \frac{1}{s}W_{-n+2} \]

i.e.

\[ sW_{-n} = W_{-n+2} - rW_{-n+1} \]

we obtain
Then, solving the system (3.1)-(3.2), the required results of (a) and (b) follow.

If we add the above equations by side by, we get

\[(3.2) \quad s \sum_{i=1}^{n} W_{-i+1}W_{-i} = (W_1W_0 - W_{-n+1}W_{-n} + \sum_{i=1}^{n} W_{-i+1}W_{-i}) - r(W_0^2 - W_{-n}^2 + \sum_{i=1}^{n} W_{-i}^2)\]

Then, solving the system (3.1)-(3.2), the required results of (a) and (b) follow.

Taking \(r = s = 1\) in Theorem 3.1 (a) and (b), we obtain the following proposition.

**Proposition 3.2.** If \(r = s = 1\) then for \(n \geq 1\) we have the following formulas:

(a): \(\sum_{i=1}^{n} W_{-i+1}W_{-i} = \frac{1}{2}(2W_{-n}^2 - 2W_{-n+1}W_{-n} + 2W_1W_0 - 2W_0^2)\).

(b): \(\sum_{i=1}^{n} W_{-i+1}W_{-i} = \frac{1}{2}(-W_{-n+1}^2 - W_{-n}^2 + W_{-n+1}W_{-n} - W_1W_0 + W_1^2 + W_0^2)\).

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take \(W_n = F_n\) with \(F_0 = 0, F_1 = 1\)).

**Corollary 3.3.** For \(n \geq 1\), Fibonacci numbers have the following properties.

(a): \(\sum_{i=1}^{n} F_{-i}^2 = \frac{1}{2}(2F_{-n}^2 - 2F_{-n+1}F_{-n})\).

(b): \(\sum_{i=1}^{n} F_{-i+1}F_{-i} = \frac{1}{2}(-F_{-n+1}^2 - F_{-n}^2 + F_{-n+1}F_{-n} + 1)\).

Taking \(W_n = L_n\) with \(L_0 = 2, L_1 = 1\) in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

**Corollary 3.4.** For \(n \geq 1\), Lucas numbers have the following properties.

(a): \(\sum_{i=1}^{n} L_{-i}^2 = \frac{1}{2}(2L_{-n}^2 - 2L_{-n+1}L_{-n} - 4)\).

(b): \(\sum_{i=1}^{n} L_{-i+1}L_{-i} = \frac{1}{2}(-L_{-n+1}^2 - L_{-n}^2 + L_{-n+1}L_{-n} + 3)\).

Taking \(r = 2, s = 1\) in Theorem 3.1 (a) and (b), we obtain the following proposition.

**Proposition 3.5.** If \(r = 2, s = 1\) then for \(n \geq 1\) we have the following formulas:

(a): \(\sum_{i=1}^{n} W_{-i}^2 = \frac{1}{2}(2W_{-n}^2 - W_{-n+1}W_{-n} - 2W_0^2 + W_1W_0)\).
(b): \[ \sum_{i=1}^{n} W_{i+1}W_{i} = \frac{1}{4}(-W_{n+1}^2 - W_n^2 + 2W_{n+1}W_{-n} + W_1^2 + W_0^2 - 2W_1W_0). \]

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers
(take \( W_n = P_n \) with \( P_0 = 0, P_1 = 1 \)).

**Corollary 3.6.** For \( n \geq 1 \), Pell numbers have the following properties.

(a): \[ \sum_{i=1}^{n} P_{i}^2 = \frac{1}{2}(2P_{n}^2 - P_{n+1}P_{-n}). \]

(b): \[ \sum_{i=1}^{n} P_{-i+1}P_{-i} = \frac{1}{4}(-P_{n+1}^2 - P_n^2 + 2P_{n+1}P_{-n} + 1). \]

Taking \( W_n = Q_n \) with \( Q_0 = 2, Q_1 = 2 \) in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

**Corollary 3.7.** For \( n \geq 1 \), Pell-Lucas numbers have the following properties.

(a): \[ \sum_{i=1}^{n} Q_{i}^2 = \frac{1}{2}(2Q_{n}^2 - Q_{n+1}Q_{-n} - 4). \]

(b): \[ \sum_{i=1}^{n} Q_{-i+1}Q_{-i} = \frac{1}{4}(-Q_{n+1}^2 - Q_{n}^2 + 2Q_{n+1}Q_{-n}). \]

If \( r = 1, s = 2 \) then \((s + 1)(r + s - 1)(r - s + 1) = 0 \) so we can’t use Theorem 3.1 directly. Therefore we need another method to find \( \sum_{i=1}^{n} W_{i+1}W_{i} \) and \( \sum_{i=1}^{n} W_{-i+1}W_{-i} \) which is given in the following theorem.

**Theorem 3.8.** If \( r = 1, s = 2 \) then for \( n \geq 1 \) we have the following formulas:

(a): \[ \sum_{i=1}^{n} W_{i}^2 = \frac{1}{9}(-W_{n+1}^2 + W_n^2 + (W_1^2 - W_0^2) + (W_1 - 2W_0)^2 n). \]

(b): \[ \sum_{i=1}^{n} W_{-i+1}W_{-i} = \frac{1}{2n}(2W_{n+1}^2 - 2W_{n+1}W_0 + (W_1 + 4W_0) (2W_1 - W_0) - 3(W_1 - 2W_0)^2 n). \]

Proof. (a) and (b) can be proved by mathematical induction.

(a): The proof will be by induction on \( n \). If \( n = 1 \) we find the sum formula reduces to the relation

\[ (3.3) \quad W_{-1}^2 = \frac{1}{9}(2W_0^2 - 4W_0W_1 + 2W_1^2 + W_0^2) \]

Since

\[ W_{-1} = (-\frac{1}{2}W_0 + \frac{1}{2}W_1) \]

(3.3) is true. Assume that the relation in (a) is true for \( n = m \), i.e.

\[ \sum_{i=1}^{m} W_{i}^2 = \frac{1}{9}(-W_{m+1}^2 + W_m^2 + (W_1^2 - W_0^2) + (W_1 - 2W_0)^2 m). \]
Then we get

$$
\sum_{i=1}^{m+1} W_{2i} = W_{2(m+1)} + \sum_{i=1}^{m} W_{2i} = W_{2m-1} + \frac{1}{9}(-W_{2m+1} + W_{2m} + (W_{1}^2 - W_{0}^2) + (W_{1} - 2W_{0})^2 m)
$$

where

$$
\sum_{i=1}^{m+1} W_{2i} = \sum_{i=1}^{m} W_{2i} = W_{2m-1} + \frac{1}{9}(-W_{2m+1} + W_{2m} + 9W_{2m-1} - (W_{1} - 2W_{0})^2 + (W_{2}^2 - W_{0}^2) + (W_{1} - 2W_{0})^2 (m + 1))
$$

(3.4) can be proved by using Binet formula of $W_n$. Hence, the relation in (a) holds also for $n = m+1$.

(b): We now prove (b) by induction on $n$. If $n = 1$ we see that the sum formula reduces to the

$$
W_{0}W_{-1} = \frac{1}{27}(-W_{1}^2 - 18W_{0}^2 + 4W_{2}^2 + 19W_{0}W_{1} - 7W_{0}W_{-1}).
$$

Since

$$
W_{-1} = (-\frac{1}{2}W_{0} + \frac{1}{2}W_{1}),
$$

(3.5) is true. Assume that the relation in (b) is true for $n = m$ i.e.,

$$
\sum_{i=1}^{m} W_{-i+1}W_{-i} = \frac{1}{27}(-2W_{-m+1}^2 + 4W_{-m} - 7W_{-m+1}W_{-m} + (W_{1} + 4W_{0})(2W_{1} - W_{0}) - 3(W_{1} - 2W_{0})^2 m).
$$
Then we get
\[
\sum_{i=1}^{m+1} W_{-i+1}W_{-i} = W_{-(m+1)+1}W_{-(m+1)} + \sum_{i=1}^{m} W_{-i+1}W_{-i}
\]
\[
= W_{-m}W_{-m-1} + \frac{1}{27}(-2W_{-m+1}^2 + 4W_{-m}^2 - 7W_{-m+1}W_{-m} + (W_1 + 4W_0)(2W_1 - W_0) - 3(W_1 - 2W_0)^2m + 1)
\]
\[
= \frac{1}{27}(-2W_{-m+1}^2 + 4W_{-m}^2 - 7W_{-m+1}W_{-m} + 27W_{-m}W_{-m-1} + (W_1 + 4W_0)(2W_1 - W_0) - 3(W_1 - 2W_0)^2m + 1)
\]
\[
= \frac{1}{27}(-2W_{-m}^2 + 4W_{-m}^2 - 7W_{-m}W_{-m-1} + (W_1 + 4W_0)(2W_1 - W_0) - 3(W_1 - 2W_0)^2m + 1)
\]
\[
= \frac{1}{27}(-2W_{-(m+1)+1} + 4W_{-(m+1)} - 7W_{-(m+1)+1}W_{-(m+1)} + (W_1 + 4W_0)(2W_1 - W_0) - 3(W_1 - 2W_0)^2m + 1)
\]

where
\[(3.6) -2W_{-m+1}^2 + 4W_{-m}^2 - 7W_{-m+1}W_{-m} + 27W_{-m}W_{-m-1} + 3(W_1 - 2W_0)^2 = -2W_{-m}^2 + 4W_{-m}^2 - 7W_{-m}W_{-m-1} \]

(3.6) can be proved by using Binet formula of \(W_n\). Hence, the relation in (b) holds also for \(n = m+1\).

From the last theorem, we have the following corollary which gives sum formula of Jacobsthal numbers
(take \(W_n = J_n\) with \(J_0 = 0, J_1 = 1\).

**Corollary 3.9.** For \(n \geq 1\), Jacobsthal numbers have the following property:
(a): \[\sum_{i=1}^{n} J_{-i+1}J_{-i} = \frac{1}{8}(-J_{-n+1}^2 - J_{-n}^2 + 1 + n).\]
(b): \[\sum_{i=1}^{n} J_{-i+1}J_{-i} = \frac{1}{27}(-2J_{-n+1}^2 + 4J_{-n}^2 - 7J_{-n+1}J_{-n} + 2 - 3n).\]

Taking \(W_n = j_n\) with \(j_0 = 2, j_1 = 1\) in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

**Corollary 3.10.** For \(n \geq 1\), Jacobsthal-Lucas numbers have the following property:
(a): \[\sum_{i=1}^{n} J_{-i+1}J_{-i} = \frac{1}{8}(-J_{-n+1}^2 - J_{-n}^2 + 3 + 9n).\]
(b): \[\sum_{i=1}^{n} J_{-i+1}J_{-i} = \frac{1}{27}(-2J_{-n+1}^2 + 4J_{-n}^2 - 7J_{-n+1}j_{-n} + 27n).\]

4. Conclusion

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear
recurrence sequences, too. We have written sum identities in terms of the generalized Fibonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

References