Closed Formulas for the Sums of Squares of Generalized Fibonacci Numbers

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Abstract. In this paper, closed forms of the summation formulas for generalized Fibonacci numbers are presented. As special cases, we give summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers.

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1. Introduction

Sequences have been fascinating topic for mathematicians for centuries. The Fibonacci and Lucas sequences are very well-known examples of second order recurrence sequences. The Fibonacci numbers are perhaps most famous for appearing in the rabbit breeding problem, introduced by Leonardo de Pisa in 1202 in his book called Liber Abaci. The Fibonacci sequences are a source of many nice and interesting identities. A similar interpretation exists for Lucas sequence.

The sequence of Fibonacci numbers $\{F_n\}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \qquad F_0 = 0, \; F_1 = 1.$$

and the sequence of Lucas numbers $\{L_n\}$ is defined by

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \qquad L_0 = 2, \; L_1 = 1.$$

The Fibonacci numbers, Lucas numbers and their generalizations have many interesting properties and applications to almost every field. In 1965, Horadam [8] defined a generalization of Fibonacci sequence, that is, he defined a second-order linear recurrence sequence $\{W_n(W_0, W_1; r, s)\}$, or simply $\{W_n\}$, as follows:

$$W_n = rW_{n-1} + sW_{n-2}; \quad W_0 = a, \; W_1 = b, \quad (n \geq 2) \tag{1.1}$$

where W_0, W_1 are arbitrary complex numbers and r, s are real numbers, see also Horadam [7], [9] and [10]. Now these generalized Fibonacci numbers $\{W_n(a, b; r, s)\}$ are also called Horadam numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

For some specific values of a, b, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values.

Table 1. A few special case of generalized Fibonacci sequences.

Name of sequence	Notation: $W_n(a, b; r, s)$	OEIS: [17]
Fibonacci	$F_n = W_n(0, 1; 1, 1)$	A000045
Lucas	$L_n = W_n(2, 1; 1, 1)$	A000032
Pell	$P_n = W_n(0, 1; 2, 1)$	A000129
Pell-Lucas	$Q_n = W_n(2, 2; 2, 1)$	A002203
Jacobsthal	$J_n = W_n(0, 1; 1, 2)$	A001045
Jacobsthal-Lucas	$j_n = W_n(2, 1; 1, 2)$	A014551

The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$$

and

$$\sum_{i=1}^{n} Q_i^2 = \frac{1}{2}(Q_n Q_{n+1} - 4).$$

In this work, we derive expressions for sums of second powers of generalized Fibonacci numbers. We present some works on sum formulas of powers of the numbers in the following Table 2.

Table 2. A few special study on sum formulas of second, third and arbitrary powers.

Name of sequence	sums of second powers	sums of third powers	sums of powers
Generalized Fibonacci	[1,2,6,11,12]	[5,18]	[3,4,13]
Generalized Tribonacci	[15]		
Generalized Tetranacci	[14,16]		

2. Summing Formulas of Generalized Fibonacci Numbers with Positive Subscripts

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

THEOREM 2.1. *For $n \geq 1$ we have the following formulas: if $(s+1)(r+s-1)(r-s+1) \neq 0$ then*

(a):

$$\sum_{i=1}^{n} W_i^2 = \frac{(1-s)W_{n+2}^2 + (1-s-r^2-r^2s)W_{n+1}^2 + 2rsW_{n+1}W_{n+2} + (s-1)W_1^2 + s^2(s-1)W_0^2 - 2rsW_1W_0}{(s+1)(r+s-1)(r-s+1)}.$$

(b):

$$\sum_{i=1}^{n} W_{i+1}W_i = \frac{rW_{n+2}^2 + rs^2W_{n+1}^2 + (1-r^2-s^2)W_{n+1}W_{n+2} - rW_1^2 - rs^2W_0^2 + s(-r^2+s^2-1)W_1W_0}{(s+1)(r+s-1)(r-s+1)}.$$

Proof. Using the recurrence relation

$$W_{n+2} = rW_{n+1} + sW_n$$

i.e.

$$sW_n = W_{n+2} - rW_{n+1}$$

we obtain

$$\begin{aligned}
s^2W_n^2 &= W_{n+2}^2 + r^2W_{n+1}^2 - 2rW_{n+2}W_{n+1} \\
s^2W_{n-1}^2 &= W_{n+1}^2 + r^2W_n^2 - 2rW_{n+1}W_n \\
s^2W_{n-2}^2 &= W_n^2 + r^2W_{n-1}^2 - 2rW_nW_{n-1} \\
s^2W_{n-3}^2 &= W_{n-1}^2 + r^2W_{n-2}^2 - 2rW_{n-1}W_{n-2} \\
s^2W_{n-4}^2 &= W_{n-2}^2 + r^2W_{n-3}^2 - 2rW_{n-2}W_{n-3} \\
&\vdots \\
s^2W_2^2 &= W_4^2 + r^2W_3^2 - 2rW_4W_3 \\
s^2W_1^2 &= W_3^2 + r^2W_2^2 - 2rW_3W_2.
\end{aligned}$$

If we add the above equations by side by, we get

$$s^2\sum_{i=1}^{n} W_i^2 = \sum_{i=3}^{n+2} W_i^2 + r^2\sum_{i=2}^{n+1} W_i^2 - 2r\sum_{i=2}^{n+1} W_{i+1}W_i \tag{2.1}$$

Note that

$$\begin{aligned}
\sum_{i=3}^{n+2} W_i^2 &= -W_1^2 - W_2^2 + W_{n+1}^2 + W_{n+2}^2 + \sum_{i=1}^{n} W_i^2, \\
\sum_{i=2}^{n+1} W_i^2 &= -W_1^2 + W_{n+1}^2 + \sum_{i=1}^{n} W_i^2, \\
\sum_{i=2}^{n+1} W_{i+1}W_i &= -W_2W_1 + W_{n+2}W_{n+1} + \sum_{i=1}^{n} W_{i+1}W_i,
\end{aligned}$$

We put them into the (2.1) we obtain

$$s^2\sum_{i=1}^{n}W_i^2 = (-W_1^2-W_2^2+W_{n+1}^2+W_{n+2}^2+\sum_{i=1}^{n}W_i^2)+r^2(-W_1^2+W_{n+1}^2+\sum_{i=1}^{n}W_i^2) -2r(-W_2W_1+W_{n+2}W_{n+1}+\sum_{i=1}^{n}W_{i+1}W_i). \tag{2.2}$$

Next we calculate $\sum_{i=1}^{n}W_{i+1}W_i$. Again, using the recurrence relation

$$W_{n+2}=rW_{n+1}+sW_n$$

i.e.

$$sW_n=W_{n+2}-rW_{n+1}$$

we obtain

$$\begin{aligned}
sW_{n+1}W_n &= W_{n+2}W_{n+1}-rW_{n+1}^2\\
sW_nW_{n-1} &= W_{n+1}W_n-rW_n^2\\
sW_{n-1}W_{n-2} &= W_nW_{n-1}-rW_{n-1}^2\\
&\vdots\\
sW_3W_2 &= W_4W_3-rW_3^2\\
sW_2W_1 &= W_3W_2-rW_2^2.
\end{aligned}$$

If we add the above equations by side by, we get

$$s\sum_{i=1}^{n}W_{i+1}W_i=\sum_{i=2}^{n+1}W_{i+1}W_i-r\sum_{i=2}^{n+1}W_i^2 \tag{2.3}$$

Note that

$$\begin{aligned}
\sum_{i=2}^{n+1}W_{i+1}W_i &= -W_2W_1+W_{n+2}W_{n+1}+\sum_{i=1}^{n}W_{i+1}W_i,\\
\sum_{i=2}^{n+1}W_i^2 &= -W_1^2+W_{n+1}^2+\sum_{i=1}^{n}W_i^2.
\end{aligned}$$

If we put them into the (2.3) then it follows that

$$s\sum_{i=1}^{n}W_{i+1}W_i=(-W_2W_1+W_{n+2}W_{n+1}+\sum_{i=1}^{n}W_{i+1}W_i)-r(-W_1^2+W_{n+1}^2+\sum_{i=1}^{n}W_i^2). \tag{2.4}$$

Then, using

$$W_2=(rW_1+sW_0)$$

and solving the system (2.2)-(2.4), the required results of (a) and (b) follow.

Taking $r=s=1$ in Theorem 2.1 (a) and (b), we obtain the following proposition.

Proposition 2.2. *If $r=s=1$ then for $n\geq 1$ we have the following formulas:*

(a): $\sum_{i=1}^{n} W_i^2 = \frac{1}{2}(-2W_{n+1}^2 + 2W_{n+2}W_{n+1} - 2W_1W_0)$.

(b): $\sum_{i=1}^{n} W_{i+1}W_i = \frac{1}{2}(W_{n+2}^2 + W_{n+1}^2 - W_{n+1}W_{n+2} - W_1^2 - W_0^2 - W_1W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 2.3. *For $n \geq 1$, Fibonacci numbers have the following properties:*

(a): $\sum_{i=1}^{n} F_i^2 = \frac{1}{2}(-2F_{n+1}^2 + 2F_{n+2}F_{n+1})$.

(b): $\sum_{i=1}^{n} F_{i+1}F_i = \frac{1}{2}(F_{n+2}^2 + F_{n+1}^2 - F_{n+1}F_{n+2} - 1)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 2.4. *For $n \geq 1$, Lucas numbers have the following properties:*

(a): $\sum_{i=1}^{n} L_i^2 = \frac{1}{2}(-2L_{n+1}^2 + 2L_{n+2}L_{n+1} - 4)$.

(b): $\sum_{i=1}^{n} L_{i+1}L_i = \frac{1}{2}(L_{n+2}^2 + L_{n+1}^2 - L_{n+1}L_{n+2} - 7)$.

Taking $r = 2, s = 1$ in Theorem 2.1 (a) and (b), we obtain the following proposition.

Proposition 2.5. *If $r = 2, s = 1$ then for $n \geq 0$ we have the following formulas:*

(a): $\sum_{i=1}^{n} W_i^2 = \frac{1}{2}(-2W_{n+1}^2 + W_{n+2}W_{n+1} - W_1W_0)$.

(b): $\sum_{i=1}^{n} W_{i+1}W_i = \frac{1}{4}(W_{n+2}^2 + W_{n+1}^2 - 2W_{n+2}W_{n+1} - W_1^2 - W_0^2 - 2W_1W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 2.6. *For $n \geq 1$, Pell numbers have the following properties:*

(a): $\sum_{i=1}^{n} P_i^2 = \frac{1}{2}(-2P_{n+1}^2 + P_{n+2}P_{n+1}) = \frac{1}{2}P_nP_{n+1}$.

(b): $\sum_{i=1}^{n} P_{i+1}P_i = \frac{1}{4}(P_{n+2}^2 + P_{n+1}^2 - 2P_{n+2}P_{n+1} - 1)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 2.7. *For $n \geq 1$, Pell-Lucas numbers have the following properties:*

(a): $\sum_{i=1}^{n} Q_i^2 = \frac{1}{2}(-2Q_{n+1}^2 + Q_{n+2}Q_{n+1} - 4) = \frac{1}{2}(Q_nQ_{n+1} - 4)$.

(b): $\sum_{i=1}^{n} Q_{i+1}Q_i = \frac{1}{4}(Q_{n+2}^2 + Q_{n+1}^2 - 2Q_{n+2}Q_{n+1} - 16)$.

If $r = 1, s = 2$ then $(s+1)(r+s-1)(r-s+1) = 0$ so we can't use Theorem 2.1 directly. Therefore we need another method to find $\sum_{i=1}^{n} W_i^2$ and $\sum_{i=1}^{n} W_{i+1}W_i$ which is given in the following theorem.

Theorem 2.8. *If $r = 1, s = 2$ then for $n \geq 1$ we have the following formulas:*

(a): $\sum_{i=1}^{n} W_i^2 = \frac{1}{9}(W_{n+2}^2 - W_{n+1}^2 - 4(W_0 + W_1)W_0 + (2W_0 - W_1)^2 n)$.

(b): $\sum_{i=1}^{n} W_{i+1}W_i = \frac{1}{36}(5W_{n+2}^2 + 4W_{n+1}^2 + (-9W_1^2 - 20W_0^2 - 20W_1W_0) - 4(W_1 - 2W_0)^2 n)$.

Proof.

(a): The proof will be by induction on n. Before the proof, we recall some information on generalized Jacobsthal numbers. A generalized Jacobsthal sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$W_n = W_{n-1} + 2W_{n-2}; \quad W_0 = a, \ W_1 = b, \quad (n \geq 2) \tag{2.5}$$

with the initial values W_0, W_1 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{1}{2}W_{-(n-1)} + \frac{1}{2}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.5) holds for all integer n. The first few generalized Jacobsthal numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Jacobsthal numbers

n	W_n	W_{-n}
0	W_0	
1	W_1	$-\frac{1}{2}W_0 + \frac{1}{2}W_1$
2	$2W_0 + W_1$	$\frac{3}{4}W_0 - \frac{1}{4}W_1$
3	$2W_0 + 3W_1$	$-\frac{5}{8}W_0 + \frac{3}{8}W_1$
4	$6W_0 + 5W_1$	$\frac{11}{16}W_0 - \frac{5}{16}W_1$
5	$10W_0 + 11W_1$	$-\frac{21}{32}W_0 + \frac{11}{32}W_1$
6	$22W_0 + 21W_1$	$\frac{43}{64}W_0 - \frac{21}{64}W_1$

Binet formula of generalized Jacobsthal sequence can be calculated using its characteristic equation which is given as

$$t^2 - t - 2 = 0.$$

The roots of characteristic equation are

$$\alpha = 2, \quad \beta = -1$$

and the roots satisfy the following

$$\alpha + \beta = 1, \ \alpha\beta = -2, \ \alpha - \beta = 3.$$

Using these roots and the recurrence relation, Binet formula can be given as

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} = \frac{A \times 2^n - B(-1)^n}{3} \tag{2.6}$$

where $A = W_1 - W_0\beta = W_1 + W_0$ and $B = W_1 - W_0\alpha = W_1 - 2W_0$.

We now prove (a) by induction on n. If $n=1$ wee that the sum formula reduces to the relation

$$W_1^2=\frac{1}{9}(W_3^2-W_2^2+W_1^2-8W_1W_0). \tag{2.7}$$

Since

$$\begin{aligned}W_2 &= 2W_0+W_1,\\ W_3 &= 2W_0+3W_1,\end{aligned}$$

(2.7) is true. Assume that the relation in (a) is true for $n=m$, i.e.,

$$\sum_{i=1}^{m}W_i^2=\frac{1}{9}(W_{m+2}^2-W_{m+1}^2-4\left(W_0+W_1\right)W_0+\left(2W_0-W_1\right)^2m).$$

Then we get

$$\begin{aligned}\sum_{i=1}^{m+1}W_i^2 &= W_{m+1}^2+\sum_{i=1}^{m}W_i^2\\ &= W_{m+1}^2+\frac{1}{9}(W_{m+2}^2-W_{m+1}^2-4\left(W_0+W_1\right)W_0+\left(2W_0-W_1\right)^2m)\\ &= \frac{1}{9}(W_{m+2}^2+8W_{m+1}^2-4\left(W_0+W_1\right)W_0+\left(2W_0-W_1\right)^2m)\\ &= \frac{1}{9}(W_{m+2}^2+8W_{m+1}^2-\left(2W_0-W_1\right)^2-4\left(W_0+W_1\right)W_0+\left(2W_0-W_1\right)^2(m+1))\\ &= \frac{1}{9}(W_{m+3}^2-W_{m+2}^2-4\left(W_0+W_1\right)W_0+\left(2W_0-W_1\right)^2(m+1))\\ &= \frac{1}{9}(W_{(m+1)+2}^2-W_{(m+1)+1}^2-4\left(W_0+W_1\right)W_0+\left(2W_0-W_1\right)^2(m+1))\end{aligned}$$

where

$$W_{m+2}^2+8W_{m+1}^2-\left(2W_0-W_1\right)^2=W_{m+3}^2-W_{m+2}^2. \tag{2.8}$$

(2.8) can be proved by using Binet formula of W_n. Hence, the relation in (a) holds also for $n=m+1$.

(b): We now prove (b) by induction on n. If $n=1$ wee that the sum formula reduces to the relation

$$W_2W_1=\frac{1}{36}(5W_3^2+4W_2^2-13W_1^2-36W_0^2-4W_1W_0). \tag{2.9}$$

Since

$$\begin{aligned}W_2 &= 2W_0+W_1,\\ W_3 &= 2W_0+3W_1,\end{aligned}$$

(2.9) is true. Assume that the relation in (b) is true for $n=m$, i.e.,

$$\sum_{i=1}^{m}W_{i+1}W_i=\frac{1}{36}(5W_{m+2}^2+4W_{m+1}^2+(-9W_1^2-20W_0^2-20W_1W_0)-4\left(W_1-2W_0\right)^2m).$$

Then we get

$$\begin{aligned}
\sum_{i=1}^{m+1} W_{i+1}W_i &= W_{m+2}W_{m+1} + \sum_{i=1}^{m} W_{i+1}W_i \\
&= \frac{1}{36}(5W_{m+2}^2 + 4W_{m+1}^2 + 36W_{m+2}W_{m+1} + (-9W_1^2 - 20W_0^2 - 20W_1W_0) - 4(W_1 - 2W_0)^2 m) \\
&= \frac{1}{36}(5W_{m+2}^2 + 4W_{m+1}^2 + 36W_{m+2}W_{m+1} + 4(W_1 - 2W_0)^2 + (-9W_1^2 - 20W_0^2 - 20W_1W_0) \\
&\quad -4(W_1 - 2W_0)^2 (m+1)) \\
&= \frac{1}{36}(5W_{m+3}^2 + 4W_{m+2}^2 + (-9W_1^2 - 20W_0^2 - 20W_1W_0) - 4(W_1 - 2W_0)^2 (m+1)) \\
&= \frac{1}{36}(5W_{(m+1)+2}^2 + 4W_{(m+1)+1}^2 + (-9W_1^2 - 20W_0^2 - 20W_1W_0) - 4(W_1 - 2W_0)^2 (m+1))
\end{aligned}$$

where

$$5W_{m+2}^2 + 4W_{m+1}^2 + 36W_{m+2}W_{m+1} + 4(W_1 - 2W_0)^2 = 5W_{m+3}^2 + 4W_{m+2}^2. \tag{2.10}$$

(2.10) can be proved by using Binet formula of W_n. Hence, the relation in (b) holds also for $n = m+1$.

From the last theorem we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 2.9. *For $n \geq 1$, Jacobsthal numbers have the following property:*

- **(a):** $\sum_{i=1}^{n} J_i^2 = \frac{1}{9}(J_{n+2}^2 - J_{n+1}^2 + n)$.
- **(b):** $\sum_{i=1}^{n} J_{i+1}J_i = \frac{1}{36}(5J_{n+2}^2 + 4J_{n+1}^2 - 9 - 4n)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 2.10. *For $n \geq 1$, Jacobsthal-Lucas numbers have the following property:*

- **(a):** $\sum_{i=1}^{n} j_i^2 = \frac{1}{9}(j_{n+2}^2 - j_{n+1}^2 - 24 + 9n)$.
- **(b):** $\sum_{i=1}^{n} j_{i+1}j_i = \frac{1}{36}(5j_{n+2}^2 + 4j_{n+1}^2 - 129 - 36n)$.

3. Summing Formulas of Generalized Fibonacci Numbers with Negative Subscripts

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 3.1. *For $n \geq 1$ we have the following formulas: If $(s+1)(r+s-1)(r-s+1) \neq 0$ then*

(a):

$$\sum_{i=1}^{n} W_{-i}^2 = \frac{\begin{array}{c}(s-1)W_{-n+1}^2 + (r^2 + r^2 s + s - 1)W_{-n}^2 - 2rsW_{-n+1}W_{-n} + 2rsW_1W_0 + (1-s)W_1^2 \\ +(1 - s - r^2 - r^2 s)W_0^2\end{array}}{(s+1)(r+s-1)(r-s+1)}$$

(b):

$$\sum_{i=1}^{n} W_{-i+1}W_{-i} = \frac{-rW_{-n+1}^2 - rs^2W_{-n}^2 + (r^2+s^2-1)W_{-n+1}W_{-n} + (1-r^2-s^2)W_1W_0 + rW_1^2 + rs^2W_0^2}{(s+1)(r+s-1)(r-s+1)}$$

Proof. Using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n} \Rightarrow W_{-n} = -\frac{r}{s}W_{-n+1} + \frac{1}{s}W_{-n+2}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

we obtain

$$\begin{aligned}
s^2W_{-n}^2 &= W_{-n+2}^2 + r^2W_{-n+1}^2 - 2rW_{-n+2}W_{-n+1} \\
s^2W_{-n+1}^2 &= W_{-n+3}^2 + r^2W_{-n+2}^2 - 2rW_{-n+3}W_{-n+2} \\
s^2W_{-n+2}^2 &= W_{-n+4}^2 + r^2W_{-n+3}^2 - 2rW_{-n+4}W_{-n+3} \\
s^2W_{-n+3}^2 &= W_{-n+5}^2 + r^2W_{-n+4}^2 - 2rW_{-n+5}W_{-n+4} \\
&\vdots \\
s^2W_{-3}^2 &= W_{-1}^2 + r^2W_{-2}^2 - 2rW_{-1}W_{-2} \\
s^2W_{-2}^2 &= W_0^2 + r^2W_{-1}^2 - 2rW_0W_{-1} \\
s^2W_{-1}^2 &= W_1^2 + r^2W_0^2 - 2rW_1W_0.
\end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned}
s^2\sum_{i=1}^{n} W_{-i}^2 &= (W_1^2 + W_0^2 - W_{-n+1}^2 - W_{-n}^2 + \sum_{i=1}^{n} W_{-i}^2) + r^2(W_0^2 - W_{-n}^2 + \sum_{i=1}^{n} W_{-i}^2) \\
&\quad -2r(W_1W_0 - W_{-n+1}W_{-n} + \sum_{i=1}^{n} W_{-i+1}W_{-i})
\end{aligned} \tag{3.1}$$

Next we calculate $\sum_{i=1}^{n} W_{-i+1}W_{-i}$. Again using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n} \Rightarrow W_{-n} = -\frac{r}{s}W_{-n+1} + \frac{1}{s}W_{-n+2}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

we obtain

$$
\begin{aligned}
sW_{-n+1}W_{-n} &= W_{-n+2}W_{-n+1} - rW_{-n+1}^2 \\
sW_{-n+2}W_{-n+1} &= W_{-n+3}W_{-n+2} - rW_{-n+2}^2 \\
sW_{-n+3}W_{-n+2} &= W_{-n+4}W_{-n+3} - rW_{-n+3}^2 \\
sW_{-n+4}W_{-n+3} &= W_{-n+5}W_{-n+4} - rW_{-n+4}^2 \\
&\vdots \\
sW_{-2}W_{-3} &= W_{-1}W_{-2} - rW_{-2}^2 \\
sW_{-1}W_{-2} &= W_0W_{-1} - rW_{-1}^2 \\
sW_0W_{-1} &= W_1W_0 - rW_0^2.
\end{aligned}
$$

If we add the above equations by side by, we get

$$
s\sum_{i=1}^{n} W_{-i+1}W_{-i} = (W_1W_0 - W_{-n+1}W_{-n} + \sum_{i=1}^{n} W_{-i+1}W_{-i}) - r(W_0^2 - W_{-n}^2 + \sum_{i=1}^{n} W_{-i}^2) \tag{3.2}
$$

Then, solving the system (3.1)-(3.2), the required results of (a) and (b) follow.

Taking $r = s = 1$ in Theorem 3.1 (a) and (b), we obtain the following proposition.

PROPOSITION 3.2. *If $r = s = 1$ then for $n \geq 1$ we have the following formulas:*

(a): $\sum_{i=1}^{n} W_{-i}^2 = \frac{1}{2}(2W_{-n}^2 - 2W_{-n+1}W_{-n} + 2W_1W_0 - 2W_0^2)$.

(b): $\sum_{i=1}^{n} W_{-i+1}W_{-i} = \frac{1}{2}(-W_{-n+1}^2 - W_{-n}^2 + W_{-n+1}W_{-n} - W_1W_0 + W_1^2 + W_0^2)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

COROLLARY 3.3. *For $n \geq 1$, Fibonacci numbers have the following properties.*

(a): $\sum_{i=1}^{n} F_{-i}^2 = \frac{1}{2}(2F_{-n}^2 - 2F_{-n+1}F_{-n})$.

(b): $\sum_{i=1}^{n} F_{-i+1}F_{-i} = \frac{1}{2}(-F_{-n+1}^2 - F_{-n}^2 + F_{-n+1}F_{-n} + 1)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

COROLLARY 3.4. *For $n \geq 1$, Lucas numbers have the following properties.*

(a): $\sum_{i=1}^{n} L_{-i}^2 = \frac{1}{2}(2L_{-n}^2 - 2L_{-n+1}L_{-n} - 4)$.

(b): $\sum_{i=1}^{n} L_{-i+1}L_{-i} = \frac{1}{2}(-L_{-n+1}^2 - L_{-n}^2 + L_{-n+1}L_{-n} + 3)$.

Taking $r = 2, s = 1$ in Theorem 3.1 (a) and (b), we obtain the following proposition.

PROPOSITION 3.5. *If $r = 2, s = 1$ then for $n \geq 1$ we have the following formulas:*

(a): $\sum_{i=1}^{n} W_{-i}^2 = \frac{1}{2}(2W_{-n}^2 - W_{-n+1}W_{-n} - 2W_0^2 + W_1W_0)$.

(b): $\sum_{i=1}^{n} W_{-i+1}W_{-i} = \frac{1}{4}(-W_{-n+1}^2 - W_{-n}^2 + 2W_{-n+1}W_{-n} + W_1^2 + W_0^2 - 2W_1W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

COROLLARY 3.6. *For $n \geq 1$, Pell numbers have the following properties.*

(a): $\sum_{i=1}^{n} P_{-i}^2 = \frac{1}{2}(2P_{-n}^2 - P_{-n+1}P_{-n})$.
(b): $\sum_{i=1}^{n} P_{-i+1}P_{-i} = \frac{1}{4}(-P_{-n+1}^2 - P_{-n}^2 + 2P_{-n+1}P_{-n} + 1)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

COROLLARY 3.7. *For $n \geq 1$, Pell-Lucas numbers have the following properties.*

(a): $\sum_{i=1}^{n} Q_{-i}^2 = \frac{1}{2}(2Q_{-n}^2 - Q_{-n+1}Q_{-n} - 4)$.
(b): $\sum_{i=1}^{n} Q_{-i+1}Q_{-i} = \frac{1}{4}(-Q_{-n+1}^2 - Q_{-n}^2 + 2Q_{-n+1}Q_{-n})$.

If $r = 1, s = 2$ then $(s+1)(r+s-1)(r-s+1) = 0$ so we can't use Theorem 3.1 directly. Therefore we need another method to find $\sum_{i=1}^{n} W_{-i}^2$ and $\sum_{i=1}^{n} W_{-i+1}W_{-i}$ which is given in the following theorem.

THEOREM 3.8. *If $r = 1, s = 2$ then for $n \geq 1$ we have the following formulas:*

(a): $\sum_{i=1}^{n} W_{-i}^2 = \frac{1}{9}(-W_{-n+1}^2 + W_{-n}^2 + (W_1^2 - W_0^2) + (W_1 - 2W_0)^2 n)$.
(b): $\sum_{i=1}^{n} W_{-i+1}W_{-i} = \frac{1}{27}(-2W_{-n+1}^2 + 4W_{-n}^2 - 7W_{-n+1}W_{-n} + (W_1 + 4W_0)(2W_1 - W_0) - 3(W_1 - 2W_0)^2 n)$.

Proof. (a) and (b) can be proved by mathematical induction.

(a): The proof will be by induction on n. If $n = 1$ wee that the sum formula reduces to the relation

$$W_{-1}^2 = \frac{1}{9}(2W_0^2 - 4W_0W_1 + 2W_1^2 + W_{-1}^2) \tag{3.3}$$

.

Since

$$W_{-1} = (-\frac{1}{2}W_0 + \frac{1}{2}W_1)$$

(3.3) is true. Assume that the relation in (a) is true for $n = m$, i.e.

$$\sum_{i=1}^{m} W_{-i}^2 = \frac{1}{9}(-W_{-m+1}^2 + W_{-m}^2 + (W_1^2 - W_0^2) + (W_1 - 2W_0)^2 m).$$

Then we get

$$\begin{aligned}
\sum_{i=1}^{m+1} W_{-i}^2 &= W_{-(m+1)}^2 + \sum_{i=1}^{m} W_{-i}^2 \\
&= W_{-m-1}^2 + \frac{1}{9}(-W_{-m+1}^2 + W_{-m}^2 + (W_1^2 - W_0^2) + (W_1 - 2W_0)^2 m) \\
&= \frac{1}{9}(-W_{-m+1}^2 + W_{-m}^2 + 9W_{-m-1}^2 + (W_1^2 - W_0^2) + (W_1 - 2W_0)^2 m) \\
&= \frac{1}{9}(-W_{-m+1}^2 + W_{-m}^2 + 9W_{-m-1}^2 - (W_1 - 2W_0)^2 + (W_1^2 - W_0^2) + (W_1 - 2W_0)^2 (m+1)) \\
&= \frac{1}{9}(-W_{-m}^2 + W_{-m-1}^2 + (W_1^2 - W_0^2) + (W_1 - 2W_0)^2 (m+1)) \\
&= \frac{1}{9}(-W_{-(m+1)+1}^2 + W_{-(m+1)}^2 + (W_1^2 - W_0^2) + (W_1 - 2W_0)^2 (m+1))
\end{aligned}$$

where

$$-W_{-m+1}^2 + W_{-m}^2 + 9W_{-m-1}^2 - (W_1 - 2W_0)^2 = -W_{-m}^2 + W_{-m-1}^2. \tag{3.4}$$

(3.4) can be proved by using Binet formula of W_n. Hence, the relation in (a) holds also for $n = m+1$.

(b): We now prove (b) by induction on n. If $n = 1$ wee see that the sum formula reduces to the relation

$$W_0 W_{-1} = \frac{1}{27}(-W_1^2 - 18W_0^2 + 4W_{-1}^2 + 19W_0 W_1 - 7W_0 W_{-1}). \tag{3.5}$$

Since

$$W_{-1} = (-\frac{1}{2}W_0 + \frac{1}{2}W_1),$$

(3.5) is true. Assume that the relation in (b) is true for $n = m$ i.e.,

$$\sum_{i=1}^{m} W_{-i+1} W_{-i} = \frac{1}{27}(-2W_{-m+1}^2 + 4W_{-m}^2 - 7W_{-m+1} W_{-m} + (W_1 + 4W_0)(2W_1 - W_0) - 3(W_1 - 2W_0)^2 m).$$

Then we get

$$\begin{aligned}
\sum_{i=1}^{m+1} W_{-i+1}W_{-i} &= W_{-(m+1)+1}W_{-(m+1)} + \sum_{i=1}^{m} W_{-i+1}W_{-i} \\
&= W_{-m}W_{-m-1} + \frac{1}{27}(-2W_{-m+1}^2 + 4W_{-m}^2 - 7W_{-m+1}W_{-m} \\
&\quad + (W_1 + 4W_0)(2W_1 - W_0) - 3(W_1 - 2W_0)^2 m) \\
&= \frac{1}{27}(-2W_{-m+1}^2 + 4W_{-m}^2 - 7W_{-m+1}W_{-m} + 27W_{-m}W_{-m-1} \\
&\quad + (W_1 + 4W_0)(2W_1 - W_0) - 3(W_1 - 2W_0)^2 m) \\
&= \frac{1}{27}(-2W_{-m+1}^2 + 4W_{-m}^2 - 7W_{-m+1}W_{-m} + 27W_{-m}W_{-m-1} + 3(W_1 - 2W_0)^2 \\
&\quad + (W_1 + 4W_0)(2W_1 - W_0) - 3(W_1 - 2W_0)^2 (m+1)) \\
&= \frac{1}{27}(-2W_{-m}^2 + 4W_{-m-1}^2 - 7W_{-m}W_{-m-1} + (W_1 + 4W_0)(2W_1 - W_0) \\
&\quad -3(W_1 - 2W_0)^2 (m+1)) \\
&= \frac{1}{27}(-2W_{-(m+1)+1}^2 + 4W_{-(m+1)}^2 - 7W_{-(m+1)+1}W_{-(m+1)} + (W_1 + 4W_0)(2W_1 - W_0) \\
&\quad -3(W_1 - 2W_0)^2 (m+1))
\end{aligned}$$

where

(3.6)

$$-2W_{-m+1}^2 + 4W_{-m}^2 - 7W_{-m+1}W_{-m} + 27W_{-m}W_{-m-1} + 3(W_1 - 2W_0)^2 = -2W_{-m}^2 + 4W_{-m-1}^2 - 7W_{-m}W_{-m-1}.$$

(3.6) can be proved by using Binet formula of W_n. Hence, the relation in (b) holds also for $n = m+1$.

From the last theorem, we have the following corollary which gives sum formula of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 3.9. *For $n \geq 1$, Jacobsthal numbers have the following property:*

(a): $\sum_{i=1}^{n} J_{-i}^2 = \frac{1}{9}(-J_{-n+1}^2 + J_{-n}^2 + 1 + n)$.

(b): $\sum_{i=1}^{n} J_{-i+1}J_{-i} = \frac{1}{27}(-2J_{-n+1}^2 + 4J_{-n}^2 - 7J_{-n+1}J_{-n} + 2 - 3n)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 3.10. *For $n \geq 1$, Jacobsthal-Lucas numbers have the following property:*

(a): $\sum_{i=1}^{n} j_{-i}^2 = \frac{1}{9}(-j_{-n+1}^2 + j_{-n}^2 - 3 + 9n)$.

(b): $\sum_{i=1}^{n} j_{-i+1}j_{-i} = \frac{1}{27}(-2j_{-n+1}^2 + 4j_{-n}^2 - 7j_{-n+1}j_{-n} - 27n)$.

4. Conclusion

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Fibonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

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