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Random pullback attractor of a non-autonomous local modified stochastic Swift-Hohenberg with multiplicative noise

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Abstract: In this paper, we study the existence of the random $\mathcal{D}$-pullback attractor of a non-autonomous local modified stochastic Swift-Hohenberg equation with multiplicative noise in stratonovich sense. It is shown that a random $\mathcal{D}$-pullback attractor exists in $H_0^2(D)$ when its external force has exponential growth. Due to the stochastic term, the estimate are delicate, we overcome this difficulty by using the Ornstein-Uhlenbeck(O-U) transformation and its properties.

Keywords: Swift-Hohenberg equation; Random $\mathcal{D}$-pullback attractor; Non-autonomous random dynamical system

1. Introduction

The Swift-Hohenberg(S-H) type equations arise in the study of convective hydrodynamical, plasma confinement in toroidal and viscous film flow, was introduced by authors in [1]. After that, Doelman and Standstede [2] proposed the following modified Swift-Hohenberg equation for a pattern formation system near the onset to instability

$$u_t + \Delta^2 u + 2\Delta u + au + b|\nabla u|^2 + u^3 = 0, \tag{1.1}$$

where a and b are arbitrary constants.

We can see from the above equation that there exist two operators Δ and Δ^2, these two operators have some symmetry, for any $u \in H_0^1(D)$, the inner product $(\Delta u, u) = -(\nabla u, \nabla u) = \|\nabla u\|^2$, for any $u \in H_0^2(D)$, $(\Delta^2 u, u) = (\Delta u, \Delta u) = \|\Delta u\|^2$, Δ is antisymmetry and Δ^2 is symmetry. We will use the symmetry principle study S-H equation.

The dynamical properties of the S-H equation are important for the studies pattern formation system have been extensively investigated by many authors; see [3-8]. Polat [8] establish the existence of global attractor for the system (1.1), and then Song et al. [7] improved the result in H^k.

Recently for non-autonomous modified S-H equation:

$$du + (\Delta^2 u + 2\Delta u + au + b|\nabla u|^2 + u^3 - g(x,t))dt = lu \circ dW(t), \tag{1.2}$$

it has also attracted the interest of many authors. If $l = 0$, equation (1.2) becomes a non-autonomous modified S-H equaiton. Park [9] proved the existence of $\mathcal{D}$-pullback attractor when the external force has exponential growth, Xu et al.[10] established the existence of uniform attractor when the external force $g(x,t)$ satisfies translation bounded, these results need the spatial variable in two dimensions. When $l = 0$, equation (1.2) becomes a non-autonomous stochastic S-H equation, if $|b| << 1$ is a constant, Guo et al.[11] investigated the equation when $g(x,t) = 0$ and proved the existence of random attractor which need the spatial variable in one dimension. For $g(x,t) = 0$, to the best of our knowledge, the existence of random $\mathcal{D}$-pullback attractor for equation (1.2) has not yet considered.

In this paper, we consider the following one dimensional non-autonomous local modified stochastic S-H equation with multiplicative noise:

$$du+(\Delta^2 u+2u_{xx}+au+bu_x^2+u^3-g(x,t))dt = lu\circ dW(t),\ in\ D\times[\tau,\infty), \tag{1.3}$$

$$u=u_{xx}=0, x\in\partial D, \tag{1.4}$$

$$u(x,\tau)=u_\tau, x\in D \tag{1.5}$$

Where D is a bounded open interval, Δu means u_{xx}, and $\Delta^2 u$ means u_{xxxx}, $|b|<4$, a and l are arbitrary constants, $W(t)$ is a two-sided real-valued Wiener process on a probability space which will be specified later. For the external force $g(x)\in L^2_{loc}(R,L^2(D))$, we assume that there exist $M>0$ and $\beta>0$ such that

$$a-\beta-5<0,\ \ \|g(x,t)\|^2\le Me^{\gamma|t|},\ for\ any\ t\in\mathbb{R},\ \ 0\le\gamma<\tfrac{3\beta}{11}. \tag{1.6}$$

The assumption is same as [8, 12], through simple calculation, for all $t\in R$, we have

$$H(t):=\int_{-\infty}^t e^{\beta s}\|g(x,s)\|^2ds<\infty,\quad \int_{-\infty}^t e^{-\frac{8\beta}{3}s}H^{\frac{11}{3}}(s)ds<\infty. \tag{1.7}$$

An outline of this paper is as follows: In section 2, we recall some basic concepts about random $\mathcal{D}$-pullback attractors. In Section 3, we prove that the stochastic dynamical system generated by (1.3) exists a random $\mathcal{D}$-pullback attractor in $H_0^2(D)$.

2.2 Preliminaries

There are many research results on random attractors and related issues. The reader is referred to [13-19] for more details, we only list the definitions and abstract result

Let ($X,\|\cdot\|_X$) be a separable Banach space with Borel σ-algebra $\mathcal{B}(X)$ and $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. In this paper, the term $\mathbb{P}$-a.s.(the abbreviation for $\mathbb{P}$ almost surely) denotes that an event happens with probability one. In other words, the set of possible exception may be non-empty, but it has probability zero.

Definition 2.1.([15,16,20,21]) $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ *is called a metric dynamical systems if* $\theta:\mathbb{R}\times\Omega\to\Omega$ *is* $(\mathcal{B}(\mathbb{R})\otimes\mathcal{F},\mathcal{F})$*-measurable,and* θ_0 *is the identity on* Ω, $\theta_{s+t}=\theta_t\circ\theta_s$ *for all* $t,s\in\mathbb{R}$ *and* $\theta_t\mathbb{P}=\mathbb{P}$ *for all* $t\in\mathbb{R}$.

Definition 2.2.*([12,13,18]) A non-autonomous random dynamical system (NRDS)* (φ,θ) *on* X *over a metric dynamical system* $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ *is a mapping*

$$\varphi(t,\tau,\omega):X\to X,\qquad (t,\tau,\omega,x)\to\varphi(t,\tau,\omega)x,$$

which represents the dynamics in the state space X *and satisfies the properties*

(i) $\varphi(\tau,\tau,\omega)$ *is the identity on* X;

(ii) $\varphi(t,\tau,\omega)=\varphi(t,s,\theta_{s-\tau}\omega)\varphi(s,\tau,\omega)$ *for all* $\tau\le s\le t$;

(iii) $\omega\to\varphi(t,\tau,\omega)x$ *is* $\mathcal{F}$*-measurable for all* $t\ge\tau$ *and* $x\in X$.

In the sequel, we use $\mathcal{D}$ to denote a collection of some families of nonempty bounded subsets of X:

$$D'\in\mathcal{D}, D'=\{D(t,\omega)\in\mathcal{B}(X):t\in\mathbb{R},\omega\in\Omega\}.$$

Definition 2.3.([12,13,18]) *A set* $B'\in\mathcal{D}$ *is called a random* $\mathcal{D}$*-pullback bounded absorbing set for NRDS* (φ,θ) *if for any* $t\in\mathbb{R}$ *and any* $D'\in\mathcal{D}$, *there exists* $\tau_0(t,D')$ *such that*

$$\varphi(t,\tau,\theta_{\tau-t}\omega)D(\tau,\theta_{\tau-t}\omega)\subset B(t,\omega)\ \ for\ any\ \ \tau\le\tau_0.$$

Definition 2.4.*([12,13,18]) A set* $\mathcal{A}=\{A(t,\omega):t\in\mathbb{R},\omega\in\Omega\}$ *is called a random* $\mathcal{D}$*-pullback attractor for* $\{\varphi,\theta\}$ *if the following hold:*

(i) $A(t,\omega)$ *is a random compact set;*

(ii) $\mathcal{A}$ *is invariant; that is, for* $\mathbb{P}$*-a.s.* $\omega\in\Omega$, *and* $\tau\le t$, $\varphi(t,\tau,\omega)A(\tau,\omega)=A(t,\theta_{t-\tau}\omega)$;

(iii) $\mathcal{A}$ *attracts all set in* $\mathcal{D}$*; that is, for all* $B' \in \mathcal{D}$ *and* $\mathbb{P}$*-a.s.* $\omega \in \Omega$*,*

$$\lim_{\tau \to -\infty} d(\varphi(t,\tau,\theta_{\tau-t}\omega)B(\tau,\theta_{\tau-t}\omega), A(t,\omega)) = 0.$$

Where d is the Hausdorff semimetric given by $dist(B,\mathcal{A}) = \sup_{b\in B}\inf_{a\in\mathcal{A}} \| b-a\|_X$.

Definition 2.5.*([14, 17]) A NRDS* (φ,θ) *on a Banach space* X *is said to be pullback flattening if for every random bounded set* $B' = \{B(t,\omega): t\in\mathbb{R}, \omega\in\Omega\} \in \mathcal{D}$*, for any* $\varepsilon > 0$ *and* $\omega \in \Omega$ *there exists a* $T(B',\varepsilon,\omega) < t$ *and a finite dimensional subspace* X_ϵ *such that*

(i) $P(\bigcup_{\tau\le T_\varepsilon} \varphi(t,\tau,\theta_{\tau-t}\omega)B(\tau,\theta_{\tau-t}\omega))$ *is bounded, and*

(ii) $\| (I-P)(\bigcup_{\tau\le T_\varepsilon} \varphi(t,\tau,\theta_{\tau-t}\omega)B(\tau,\theta_{\tau-t}\omega)) \|_X < \varepsilon$*,*

where $P: X \to X_\varepsilon$ *is a bounded projector.*

Theorem 2.1.([14, 17]) *Suppose that* (φ,θ) *is a continuous NRDS on a uniformly convex Banach space* X*. If* (φ,θ) *possesses a random* $\mathcal{D}-$*pullback bounded absorbing sets* $B' = \{B(t,\omega): t\in\mathbb{R}, \omega\in\Omega\}$ *and* (φ,θ) *is pullback flattening, then there exists a random* $\mathcal{D}$*-pullback attractor* $\mathcal{A} = \{A(t,\omega): t\in\mathbb{R}, \omega\in\Omega\}$*.*

3. Random pullback attractor for modified Swift-Hohenberg

In this section, we will use abstract theory in section 2 to obtain the random $\mathcal{D}$-pullback attractor for equation (1.3)-(1.5). First we introduce an Ornstein-Uhlenbeck process,

$$z(\theta_t(\omega)) := -\int_{-\infty}^{0} e^{\tau}(\theta_t\omega)(\tau)d\tau, t\in\mathbb{R}.$$

We known from [6], it is the solution of Langevin equation

$$dz + zdt = dW(t).$$

$W(t)$ is a two-sided real-valued Wiener process on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, where

$$\Omega = \{\omega\in C(\mathbb{R},\mathbb{R}): \omega(0)=0\},$$

$\mathcal{F}$ is the Borel algebra induced by the compact open topology of Ω, and $\mathbb{P}$ is the corresponding Wiener measure on $\{\Omega,\mathcal{F}\}$. We identify $\omega(t)$ with $W(t)$, i.e.,

$$W(t) = W(t,\omega) = \omega(t),\ t\in\mathbb{R}.$$

Define the Wiener time shift by

$$\theta_t\omega(s) = \omega(s+t) - \omega(t), \omega\in\Omega, t,s\in\mathbb{R}.$$

Then $(\Omega,\mathcal{F},\mathbb{P},\theta_t)$ is an ergodic metric dynamical system.

From [15,16,20], it is known that the random variable $z(\omega)$ is tempered and there exists a θ_t-invariant set of full measure $\tilde{\Omega}\subset\Omega$ such that for all $\omega\in\tilde{\Omega}$:

$$\lim_{t\to\pm\infty}\frac{|z(\theta_t\omega)|}{|t|} = 0, \quad \lim_{t\to\pm\infty}\frac{1}{t}\int_0^t z(\theta_s\omega)ds = 0, \tag{3.1}$$

and for any $\varepsilon > 0$, there exists $\rho(\varepsilon) > 0$, such that

$$|z(\theta_t\omega)| \le \rho(\omega) + \varepsilon|t|, \quad |\int_0^t z(\theta_s\omega)ds| \le \rho(\omega) + \varepsilon|t|. \tag{3.2}$$

Let $v(s,\tau) = e^{-lz(\theta_{s-t}\omega)}u(s)$, then $dv = -le^{-lz(\theta_{s-t}\omega)}u(s)dz + e^{-lz(\theta_{s-t}\omega)}du$. Using Langevin equation, combined with the original equation (1.3), we get

$$\frac{dv}{ds} + \Delta^2 v + 2v_{xx} + (a-lz)v + be^{lz(\theta_{s-t}\omega)}|v_x|^2 + e^{2lz(\theta_{s-t}\omega)}v^3 = e^{-lz(\theta_{s-t}\omega)}g(x,s),\ in\ D\times[\tau,\infty), \tag{3.3}$$

$$v = \Delta v = 0\ on\ \partial D\times[\tau,\infty), \tag{3.4}$$

$$v(x,\tau) = v_\tau = e^{-lz(\theta_{s-t}\omega)}u_\tau(x)\ in\ D. \tag{3.5}$$

Equation (1.3)-(1.5) are equivalent to equation (3.3)-(3.5), by a standard method, it can be proved that the problem (3.3)-(3.5) is well posed in $H_0^2(D)$, that is, for every $\tau\in\mathbb{R}$ and $v_\tau\in H_0^2(D)$, there exits a unique solution $v\in C([\tau,\infty),H_0^2(D))$ (see e.g.[2,8,22]). Furthermore, the solution is continuous with respect to the initial condition v_τ in $H_0^2(D)$. To construct a non-autonomous random dynamical system $\{V(t,\tau,\omega)\}$ for problem (3.3)-(3.5), we define $V(t,\tau,\omega):H_0^2(D)\to H_0^2(D)$ by $V(t,\tau,\omega)v_\tau$. Then the system $\{V(t,\tau,\omega)\}$ is a non-autonomous random dynamical system in $H_0^2(D)$.

We now apply abstract theory in Section 2 to obtain the random $\mathcal{D}$-pullback attractors for non-autonomous modified Swift-Hohenberg equation, by the equivalent, we only consider the random $\mathcal{D}$-pullback attractor of equation (3.3)-(3.5).

For convenience, the $L^p(D)$ norm of u will be denoted by $\|\cdot\|_p$, $H=L^2(D)$ with a scalar product and the norm of Sobolev spaces $W_p^k(D)$ by $\|\cdot\|_{k,p}$, we regard the space $H_0^2(D)$ endowed with the norm $\|u\|_{2,2}=\|\Delta u\|$, c or $c(\omega)$ denote the arbitrary positive constants, which only depend on ω and may be different from line to line and even in the same line.

For our purpose that the following Gagliardo-Nirenberg inequality will be used.

Lemma 3.1. (*Gagliardo-Nirenberg Inequality*). *Let* D *be an open, bounded domain of the Lipschitz class in* $\mathbb{R}^n$. *Assume that* $1\le p\le\infty, 1\le q\le\infty, 1\le r, 0<\theta\le 1$ *and let*

$$k-\frac{n}{p}\le\theta(m-\frac{n}{q})+(1-\theta)\frac{n}{r}.$$

Then the following inequality holds:

$$\|u\|_{k,p}\le c(D)\|u\|_r^{1-\theta}\|u\|_{m,q}^{\theta}$$

Lemma 3.2. *For all* $t\ge\tau$, *the following inequality hold:*

$$\|v(t,\tau,\theta_{\tau-t}\omega)\|^2\le c(\omega)(e^{-\beta(t-\tau)}\|v_\tau\|^2+1+e^{-\beta t}H(t)), \tag{3.6}$$

and

$$\int_\tau^t e^{2\beta(s-t)+2l\int_s^t z(\theta_{r-t}\omega)dr}\|v_{xx}\|^2ds\le c(\omega)(e^{-\beta(t-\tau)}\|v_\tau\|^2+1+e^{-\beta t}H(t)). \tag{3.7}$$

Proof. Let $v(s)$ or v denotes $v(s,\tau,\theta_{s-t}\omega)$ be the solution of equation (3.3)-(3.5). Taking the inner product of equation (3.3) with v, we get

$$\frac{1}{2}\frac{d}{ds}\|v\|^2+\|\Delta v\|^2+2(\Delta v,v)+(a-lz)\|v\|^2+be^{lz(\theta_{s-t}\omega)}(v_x^2,v)\|v\|_4^4$$
$$+e^{2lz(\theta_{s-t}\omega)}\|v\|_4^4=e^{-lz(\theta_{s-t}\omega)}(g(x,s),v).$$

Using Young inequality, we get

$$|(2\Delta v,v)|\le\frac{1}{4}\|\Delta v\|^2+4\|v\|^2.$$

By integration by parts, we obtain

$$be^{lz(\theta_{s-t}\omega)}(v_x^2,v)=be^{lz(\theta_{s-t}\omega)}\int_D v_x^2vdx=-be^{lz(\theta_{s-t}\omega)}\int_D(v^2v_{xx}+vv_x^2)dx,$$

thus

$$be^{lz(\theta_{s-t}\omega)}(v_x^2,v)=-\frac{b}{2}e^{lz(\theta_{s-t}\omega)}\int_D v^2v_{xx}dx.$$

Applying the Hölder inequality and Young inequality, we get

$$be^{lz(\theta_{s-t}\omega)}|(v_x^2,v)|=|\frac{b}{2}e^{lz(\theta_{s-t}\omega)}\int_D v^2v_{xx}dx|\le|\frac{b}{2}|e^{lz(\theta_{s-t}\omega)}\|v_{xx}\|\ \|v\|_4^2\le\eta\|v_{xx}\|^2+\frac{b^2}{16\eta}e^{2lz(\theta_{s-t}\omega)}\|v\|_4^4,$$

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and

$$e^{-lz(\theta_{s-t}\omega)}|(g(x,s),v)|\le\|v\|^2+\frac{1}{4}e^{-2lz(\theta_{s-t}\omega)}\|(g(x,s)\|^2.$$

For convenience, we take $\eta=\frac{1}{4}$, $|b|<2$ ($|b|<4$, the same conclusion hold), we obtain

$$\frac{d}{ds}\|v\|^2+\|\Delta v\|^2+2(a-lz-5)\|v\|^2+2(1-\frac{b^2}{4})e^{2lz(\theta_{s-t}\omega)}\|v\|_4^4\le\frac{1}{2}e^{-2lz(\theta_{s-t}\omega)}\|(g(x,s)\|^2.$$

Taking $\beta>0$ such that $a-\beta-5<0$, we have

$$\frac{d}{ds}\|v\|^2+\|\Delta v\|^2+2(\beta-lz)\|v\|^2+2(1-\frac{b^2}{4})e^{2lz(\theta_{s-t}\omega)}\|v\|_4^4$$
$$\le-2(a-\beta-5)\|v\|^2+\frac{1}{2}e^{-2lz(\theta_{s-t}\omega)}\|(g(x,s)\|^2.$$

By the Sobolev imbedding $L^4(D)\subset L^2(D)$ and Young inequality, we get

$$-2(a-\beta-5)\|v\|^2\le c\|v\|_4^2\le 2(1-\frac{b^2}{4})e^{2lz(\theta_{s-t}\omega)}\|v\|_4^4+ce^{-2lz(\theta_{s-t}\omega)}.$$

Thus we obtain

$$\frac{d}{ds}\|v\|^2+\|\Delta v\|^2+2(\beta-lz)\|v\|^2\le ce^{-2lz(\theta_{s-t}\omega)}(1+\|(g(x,s)\|^2).$$

Multiply this by $e^{2\beta s-2l\int_\tau^s z(\theta_{r-t}\omega)dr}$ and integrating from τ to t, we have

$$\|v(t)\|^2+\int_\tau^t e^{2\beta(s-t)+2l\int_s^t z(\theta_{r-t}\omega)dr}\|\Delta v\|^2ds$$
$$\le e^{-2\beta(t-\tau)+2l\int_\tau^t z(\theta_{r-t}\omega)dr}\|v_\tau\|^2+\int_\tau^t e^{-2\beta(t-s)+2l\int_s^t z(\theta_{r-t}\omega)-2lz(\theta_{s-t}\omega)}(1+\|g(x,s)\|^2)ds$$
$$\le e^{-2\beta(t-\tau)+2l\int_{\tau-t}^0 z(\theta_r\omega)dr}\|v_\tau\|^2+\int_\tau^t e^{-2\beta(t-s)+2l\int_{s-t}^0 z(\theta_r\omega)-2lz(\theta_{s-t}\omega)}(1+\|g(x,s)\|^2)ds$$

From (3.2), we get

$$2l\int_{\tau-t}^0 z(\theta_r\omega)dr\le\rho(\omega)+\beta(t-\tau),\ \ 2l\int_{s-t}^0 z(\theta_r\omega)-2lz(\theta_{s-t}\omega)\le\rho(\omega)+\beta(t-s).$$

Then we have

$$\|v(t)\|^2+\int_\tau^t e^{2\beta(s-t)+2l\int_s^t z(\theta_{r-t}\omega)dr}\|\Delta v\|^2ds$$
$$\le c(\omega)(e^{-\beta(t-\tau)}\|v_\tau\|^2+\int_\tau^t e^{-\beta(t-s)}(1+\|g(x,s)\|^2)ds)$$
$$\le c(\omega)(e^{-\beta(t-\tau)}\|v_\tau\|^2+1+e^{-\beta t}\int_\tau^t e^{\beta s}\|g(x,s)\|^2ds).$$

Thus we get the desired results. □

Lemma 3.3. *For all* $t\ge\tau$, *the following inequality hold:*

$$\|\Delta v(t,\tau,\theta_{-t}\omega)\|^2\le c(\omega)[(1+\frac{1}{t-\tau})e^{-\beta(t-\tau)}\|v_\tau\|^2+e^{-\frac{11}{3}\beta(t-\tau)}\|v_\tau\|^{\frac{22}{3}}$$
$$+e^{-\beta t}(H(t)+\int_{-\infty}^t e^{-\frac{8}{3}\beta s}H^{\frac{11}{3}}(s)ds)] \quad (3.8)$$

Proof. Taking inner product of equation (3.3) with $\Delta^2 v$, we have

$$\frac{1}{2}\frac{d}{ds}\|\Delta v\|^2+\|\Delta^2 v\|^2+2(\Delta v,\Delta^2 v)+(a-lz)\|\Delta v\|^2$$
$$+be^{lz(\theta_{s-t}\omega)}(v_x^2,\Delta^2 v)+e^{2lz(\theta_{s-t}\omega)}(v^3,\Delta^2 v)=e^{-lz(\theta_{s-t}\omega)}(g(x,s),\Delta^2 v). \quad (3.9)$$

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By the Hölder inequality, Young inequality and Gagliardo-Nirenberg inequality, we get

$$-2(\Delta v,\Delta^2 v)\le \frac{1}{8}\|\Delta^2 v\|^2+8\|\Delta v\|^2,$$

$$|b|e^{lz(\theta_{s-t}\omega)}|(v_x^2,\Delta^2 v)|\le |b|e^{lz(\theta_{s-t}\omega)}\|v_x\|_4^2\|\Delta^2 v\|\le ce^{lz(\theta_{s-t}\omega)}\|v\|^{\frac{11}{8}}\|\Delta^2 v\|^{\frac{13}{8}}$$

$$\le \frac{1}{8}\|\Delta^2 v\|^2+ce^{\frac{16}{3}lz(\theta_{s-t}\omega)}\|v\|^{\frac{22}{3}},$$

$$e^{2lz(\theta_{s-t}\omega)}|(v^3,\Delta^2 v)|\le e^{2lz(\theta_{s-t}\omega)}\|v\|_6^3\|\Delta^2 v\|\le ce^{2lz(\theta_{s-t}\omega)}\|\Delta^2 v\|\ \ \|v\|^{\frac{11}{4}}\|\Delta^2 v\|^{\frac{1}{4}}$$

$$=ce^{2lz(\theta_{s-t}\omega)}\|\Delta^2 v\|^{\frac{5}{4}}\|v\|^{\frac{11}{4}}\le \frac{1}{8}\|\Delta^2 v\|^2+ce^{\frac{16}{3}lz(\theta_{s-t}\omega)}\|v\|^{\frac{22}{3}}$$

$$e^{-lz(\theta_{s-t}\omega)}(g(x,s),\Delta^2 v)\le \frac{1}{8}\|\Delta^2 v\|^2+2e^{-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2.$$

Putting all these inequalities together, we deduce

$$\frac{d}{ds}\|\Delta v\|^2+2(\beta-lz)\|\Delta v\|^2\le 2(8+\beta-a)\|\Delta v\|^2+c(e^{\frac{16}{3}lz(\theta_{s-t}\omega)}\|v\|^{\frac{22}{3}}+e^{-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2).$$

Multiplying this by $(s-\tau)e^{2\beta s-2l\int_\tau^s z(\theta_{r-t}\omega)dr}$ and integrating it over (τ,t), we get

$$(t-\tau)e^{2\beta t-2l\int_\tau^t z(\theta_{r-t}\omega)dr}\|\Delta v(t)\|^2\le c[\int_\tau^t e^{2\beta s-2l\int_\tau^s z(\theta_{r-t}\omega)dr}\|\Delta v\|^2 ds$$

$$+\int_\tau^t(s-\tau)e^{2\beta s-2l\int_\tau^s z(\theta_{r-t}\omega)dr}(\|\Delta v\|^2+e^{\frac{16}{3}lz(\theta_{s-t}\omega)}\|v\|^{\frac{22}{3}}+e^{-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2)ds].$$

Then we have

$$\|\Delta v(t)\|^2\le c[(1+\frac{1}{t-\tau})\int_\tau^t e^{-2\beta(t-s)+2l\int_s^t z(\theta_{r-t}\omega)dr}\|\Delta v\|^2 ds$$

$$+\int_\tau^t e^{-2\beta(t-s)+2l\int_s^t z(\theta_{r-t}\omega)dr+\frac{16}{3}lz(\theta_{s-t}\omega)}\|v\|^{\frac{22}{3}}ds+\int_\tau^t e^{-2\beta(t-s)+2l\int_s^t z(\theta_{r-t}\omega)dr-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2].$$

By (3.2), (3.6) and the inequality $(a+b)^r\le c(a^r+b^r)\,(a,b>0,r\ge 1)$, we get

$$\int_\tau^t e^{-2\beta(t-s)+2l\int_s^t z(\theta_{r-t}\omega)dr+\frac{16}{3}lz(\theta_{s-t}\omega)}\|v\|^{\frac{22}{3}}ds\le c(\omega)\int_\tau^t e^{-\beta(t-s)}\|v\|^{\frac{22}{3}}ds$$

$$\le c(\omega)e^{-\beta t}\int_\tau^t e^{\beta s}(e^{-\beta(s-\tau)}\|v_\tau\|^2+1+e^{-\beta s}H(s))^{\frac{11}{3}}ds$$

$$\le c(\omega)e^{-\beta t}\int_\tau^t e^{\beta s}(e^{-\frac{11}{3}\beta(s-\tau)}\|v_\tau\|^{\frac{22}{3}}+1+e^{-\frac{11}{3}\beta s}H^{\frac{11}{3}}(s))ds$$

$$\le c(\omega)(e^{-\frac{11}{3}\beta(t-\tau)}\|v_\tau\|^{\frac{22}{3}}+1+e^{-\beta t}\int_\tau^t e^{-\frac{8}{3}\beta s}H^{\frac{11}{3}}(s)ds).$$

Thus we have

$$\|\Delta v(t,\tau)\|^2\le c(\omega)[(1+\frac{1}{t-\tau})e^{-\beta(t-s)}\|v_\tau\|^2+e^{-\frac{11}{3}\beta(t-\tau)}\|v_\tau\|^{\frac{22}{3}}+1$$

$$+e^{-\beta t}(H(t)+\int_{-\infty}^t e^{-\frac{8}{3}\beta s}H^{\frac{11}{3}}(s))ds)].$$

We complete the proof of Lemma 3.3. □

Let $\mathcal{R}$ be the set of all function $r:\mathbb{R}\to(0,+\infty)$ such that $\lim_{t\to-\infty}e^{\beta t}r^2(t)=0$ and denote by $\mathcal{D}$ the class of all families $\hat{D}=\{D(t):t\in\mathbb{R}\}$ such that $D(t)\subset\bar{B}(r(t))$ for some $r(t)\in\mathcal{R}$, $\bar{B}(r(t))$ denote the closed ball in $H_0^2(D)$ with radius $r(t)$. Let

$$r_1^2(t) = 2c(\omega)[1 + e^{-\beta t}(H(t) + \int_{-\infty}^{t} e^{-\frac{8}{3}\beta s} H^{\frac{11}{3}}(s)ds)] \tag{3.10}$$

By lemma 3.3 for any $\hat{D} \in \mathcal{D}$ and $t \in \mathbb{R}$, there exists $\tau_0(\hat{D}, t, \omega) < t$ such that

$$\| \Delta v(t, \tau, \theta_{\tau-t}\omega) \| \le r_1(t), \ for\ any\ \tau < \tau_0.$$

Since $0 \le \gamma < \frac{3\beta}{11}$, simple calculation imply that $r_1(t) \in \mathcal{R}$, which say that the $\bar{B}(r_1(t))$ be a family of random $\mathcal{D}$-pullback bounded absorbing sets in $H_0^2(D)$ and $\{\bar{B}(r_1(t))\} \in \mathcal{D}$.

Theorem 3.1. *The non-autonomous random dynamical system to problem (1.1)-(1.3) possesses a unique random* $\mathcal{D}$*-pullback attractor in* $H_0^2(D)$.

Proof. We need only prove that the dynamical system (3.3)-(3.5) satisfies the pullback flattening condition. Since A^{-1} is a continuous compact operator in $H_0^2(D)$, by the classical spectral theorem, there exists a sequence $\{\lambda_j\}_{j=1}^{\infty}$ satisfing

$$0 < \lambda_1 \le \lambda_2 \le \cdots \lambda_j \le \cdots, \lambda_j \to +\infty, as\ j \to +\infty,$$

and a family of elements $\{e_j\}_{j=1}^{\infty}$ of $H_0^2(D)$ which are orthonormal in H such that

$$Ae_j = \lambda_j e_j, \forall j \in \mathbb{N}^+.$$

Let $H_m = span\{e_1, e_2, \cdots, e_m\}$ in H and $P_m : H \to H_m$ be an orthogonal projector. For any $v \in H$ we write

$$v = P_m v + (I - P_m)v = v_1 + v_2.$$

Taking inner product of （3.3） with $\Delta^2 v_2$ in H, we get

$$\frac{1}{2}\frac{d}{ds}\| \Delta v_2\|^2 + \| \Delta^2 v_2\|^2 + 2(\Delta v, \Delta^2 v_2) + (a - lz)\|\Delta v_2\|^2$$
$$+be^{lz(\theta_{s-t}\omega)}(|\nabla v|^2, \Delta^2 v_2) + e^{2lz(\theta_{s-t}\omega)}(v^3, \Delta^2 v_2) = e^{-lz(\theta_{s-t}\omega)}(g(x,s), \Delta^2 v_2).$$

By the Hölder inequality, Young inequality and Gagliardo-Nirenberg inequality, we get

$$-2(\Delta v, \Delta^2 v_2) \le \frac{1}{8}\| \Delta^2 v_2\|^2 + 8\|\Delta v\|^2,$$

$$-be^{lz(\theta_{s-t}\omega)}(v_x^2, \Delta^2 v_2) \le \frac{1}{8}\| \Delta^2 v_2\|^2 + 2b^2 e^{2lz(\theta_{s-t}\omega)}\| v_x\|_4^4$$
$$\le \frac{1}{8}\| \Delta^2 v_2\|^2 + ce^{2lz(\theta_{s-t}\omega)}\| v\|^{4(1-\theta)}\| \Delta v\|^{4\theta}$$
$$= \frac{1}{8}\| \Delta^2 v_2\|^2 + ce^{2lz(\theta_{s-t}\omega)}\| v\|^3\| \Delta v\| \qquad (\theta = \frac{1}{4})$$
$$\le \frac{1}{8}\| \Delta^2 v_2\|^2 + \frac{1}{2}\| \Delta v\|^2 + ce^{4lz(\theta_{s-t}\omega)}\| v\|^6,$$

$$-e^{2lz(\theta_{s-t}\omega)}(v^3,\Delta^2 v_2)\le \frac{1}{8}\|\Delta^2 v_2\|^2+2e^{4lz(\theta_{s-t}\omega)}\|v\|_6^3$$
$$\le \frac{1}{8}\|\Delta^2 v_2\|^2+ce^{4lz(\theta_{s-t}\omega)}\|v\|^{3(1-\theta)}\|\Delta v\|^{3\theta}$$
$$=\frac{1}{8}\|\Delta^2 v_2\|^2+ce^{4lz(\theta_{s-t}\omega)}\|v\|^2\|\Delta v\| \quad (\theta=\frac{1}{3})$$
$$\le \frac{1}{8}\|\Delta^2 v_2\|^2+\frac{1}{2}\|\Delta v\|^2+ce^{8lz(\theta_{s-t}\omega)}\|v\|^4,$$
$$e^{-lz(\theta_{s-t}\omega)}(g(x,s),\Delta^2 v_2)\le \frac{1}{8}\|\Delta^2 v_2\|^2+2e^{-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2.$$

Putting all these inequalities together, we have

$$\frac{d}{ds}\|\Delta v_2\|^2+\|\Delta v_2^2\|^2+2(a-lz)\|\Delta v_2\|^2$$
$$\le 18\|\Delta v\|^2+c(e^{4lz(\theta_{s-t}\omega)}\|v\|^6+e^{8lz(\theta_{s-t}\omega)}\|v\|^4+e^{-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2).$$

$\lambda_n\|\Delta v_2\|^2\le\|\Delta^2 v_2\|^2$, which imply that

$$\frac{d}{ds}\|\Delta v_2\|^2+(\lambda_n-2lz)\|\Delta v_2\|^2$$
$$\le c(\|\Delta v\|^2+e^{4lz(\theta_{s-t}\omega)}\|v\|^6+e^{8lz(\theta_{s-t}\omega)}\|v\|^4+e^{-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2).$$

Multiply this by $(s-\tau)e^{\lambda_n s-2l\int_\tau^s z(\theta_{r-t}\omega)dr}$ and integrating from τ to t, we obtain

$$(t-\tau)e^{\lambda_n t-2l\int_\tau^t z(\theta_{r-t}\omega)dr}\|\Delta v_2\|^2\le c[\int_\tau^t(1+s-\tau)e^{\lambda_n s-2l\int_\tau^s z(\theta_{r-t}\omega)dr}\|\Delta v\|^2 ds$$
$$+\int_\tau^t(s-\tau)e^{\lambda_n s-2l\int_\tau^s z(\theta_{r-t}\omega)dr}(e^{4lz(\theta_{s-t}\omega)}\|v\|^6+e^{8lz(\theta_{s-t}\omega)}\|v\|^4+e^{-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2)ds]$$

Thus we get

$$\|\Delta v_2\|^2\le c[(1+\frac{1}{t-\tau})\int_\tau^t e^{\lambda_n(s-t)+2l\int_s^t z(\theta_{r-t}\omega)dr}\|\Delta v\|^2 ds+\int_\tau^t e^{\lambda_n(s-t)+2l\int_s^t z(\theta_{r-t}\omega)dr+4lz(\theta_{s-t}\omega)}\|v\|^6 ds$$
$$+\int_\tau^t e^{\lambda_n(s-t)+2l\int_s^t z(\theta_{r-t}\omega)dr+8lz(\theta_{s-t}\omega)}\|v\|^4 ds+\int_\tau^t e^{\lambda_n(s-t)+2l\int_s^t z(\theta_{r-t}\omega)dr-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2 ds$$
$$\le c(\omega)[(1+\frac{1}{t-\tau})\int_\tau^t e^{(\lambda_n-\beta)(s-t)}\|\Delta v\|^2 ds+\int_\tau^t e^{(\lambda_n-\beta)(s-t)}\|v\|^6 ds$$
$$+\int_\tau^t e^{(\lambda_n-\beta)(s-t)}\|v\|^4 ds+\int_\tau^t e^{(\lambda_n-\beta)(s-t)}\|g(x,s)\|^2 ds]$$
$$=c(\omega)(I_1+I_2+I_3+I_4)).$$

By simple calculation, we find that there exists $N\in\mathbb{N}$, $\forall n>N$, $\lambda_n-\beta>\beta$, and

$$I_1\le(1+\frac{1}{t-\tau})\int_\tau^t e^{\beta(s-t)}\|\Delta v\|^2 ds<\infty,\ e^{\lambda_n(s-t)}\|\Delta v(s)\|^2\to 0\ as\ n\to\infty,$$

According to Lebesgue dominated convergent theorem, we obtain

$$I_1\to 0\ as\ n\to\infty.$$

Using (3.6), we get

$$I_2\le c\int_\tau^t e^{(\lambda_n-\beta)(s-t)}(e^{-\beta(s-\tau)}\|v_\tau\|^2+1+e^{-\beta s}H(s))^3 ds$$
$$\le c\int_\tau^t e^{(\lambda_n-\beta)(s-t)}(e^{-3\beta(s-\tau)}\|v_\tau\|^6+1+e^{-3\beta s}H^3(s))ds$$
$$\le c[e^{-\beta t}\frac{e^{-3\beta(t-\tau)}}{\lambda_n-4\beta}\|v_\tau\|^6+\frac{1}{\lambda_n-\beta}+\frac{e^{-4\beta t}}{\lambda_n-4\beta}H^3(t)]\to 0\quad as\quad n\to\infty,$$

$$
\begin{aligned}
I_3 &\le c\int_\tau^t e^{(\lambda_n-\beta)(s-t)}(e^{-\beta(s-\tau)}\| v_\tau\|^2+1+e^{-\beta s}H(s))^2 ds \\
&\le c\int_\tau^t e^{(\lambda_n-\beta)(s-t)}(e^{-2\beta(s-\tau)}\| v_\tau\|^4+1+e^{-2\beta s}H^2(s))ds \\
&\le c[e^{-\beta t}\frac{e^{-2\beta(t-\tau)}}{\lambda_n-3\beta}\| v_\tau\|^4+\frac{1}{\lambda_n-\beta}+\frac{e^{-3\beta t}}{\lambda_n-3\beta}H^2(t)]\to 0, as\ n\to\infty,
\end{aligned}
$$

and

$$I_4 \le e^{-\beta t}\int_\tau^t e^{\beta s}\| g(x,s)\|^2 ds,\ e^{(\lambda_n-\beta)(s-t)}\| g(x,s)\|^2 \to 0\ as\ n\to\infty,$$

Again using Lebesgue dominated convergent theorem, we get

$$e^{(\lambda_n-\beta)(s-t)}\| g(x,s)\|^2 ds \to 0\ \ as\ \ n\to\infty.$$

In summary, we obtain that the terms on the right hand of inequality (3.13) tend to 0 as $n\to\infty$, which say that $\| v_2(t,\tau,\theta_{\tau-t}\omega)\|\to 0$, i.e., the random dynamical system (3.3)-(3.5) satisfies pullback flattening.

□ 5.

4.Conclusions

This paper extends the existence of pullback attractor of non-autonomous modified S-H equation to the case of non-autonomous stochastic modified S-H equation with multiplicative noise. In the concrete experiment, the random term in the equation is more consistent with the actual problem. For S-H equation with multiplicative noise, the external force has exponential growth, we have proved that the equation exists a random $\mathcal{D}$-pullback attractor in one dimension. In the future work, we will continue to investigate whether the same results can be obtained when the spatial dimension is two-dimensional or n-dimensional.

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