Random pullback attractor of a non-autonomous local modified stochastic Swift-Hohenberg with multiplicative noise

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Abstract: In this paper, we study the existence of the random pullback attractor of a non-autonomous local modified stochastic Swift-Hohenberg equation with multiplicative noise in stratonovich sense. It is shown that a random pullback attractor exists in $D(\mathbb{R}^2)$ when its external force has exponential growth. Due to the stochastic term, the estimate are delicate, we overcome this difficulty by using the Ornstein-Uhlenbeck(O-U) transformation and its properties.

Keywords: Swift-Hohenberg equation; Random pullback attractor; Non-autonomous random dynamical system

1. Introduction

The Swift-Hohenberg(S-H) type equations arise in the study of convective hydrodynamical, plasma confinement in toroidal and viscous film flow, was introduced by authors in [1]. After that, Doelman and Standstede [2] proposed the following modified Swift-Hohenberg equation for a pattern formation system near the onset to instability

$$u_t + \Delta^3 u + 2\Delta u + au + b|\nabla u|^2 + u^3 = 0,$$

(1.1)

where $a$ and $b$ are arbitrary constants.

We can see from the above equation that there exist two operators $\Delta$ and $\Delta^2$, these two operators have some symmetry, for any $u \in H^1_0(D)$, the inner product $(\Delta u, u) = (\nabla u, \nabla u) = |\nabla u|^2$, for any $u \in H^2_0(D)$, $(\Delta^2 u, u) = (\Delta u, \Delta u) = |\Delta u|^2$, $\Delta$ is antisymmetry and $\Delta^2$ is symmetry. We will use the symmetry principle study S-H equation.

The dynamical properties of the S-H equation are important for the studies pattern formation system have been extensively investigated by many authors; see [3-8]. Polat [8] establish the existence of global attractor for the system (1.1), and then Song et al. [7] improved the result in $H^k$.

Recently for non-autonomous modified S-H equation:

$$du + (\Delta^2 u + 2\Delta u + au + b|\nabla u|^2 + u^3 - g(x,t))dt = lu \circ dW(t),$$

(1.2)

it has also attracted the interest of many authors. If $l = 0$, equation (1.2) becomes a non-autonomous modified S-H equaiton. Park [9] proved the existence of $\mathcal{D}$ -pullback attractor when the external force has exponential growth, Xu et al.[10] established the existence of uniform attractor when the external force $g(x,t)$ satisfies translation bounded, these results need the spatial variable in two dimensions.

When $l = 0$, equation (1.2) becomes a non-autonomous stochastic S-H equation, if $|b| < 1$ is a constant, Guo et al.[11] investigated the equation when $g(x,t) = 0$ and proved the existence of random attractor which need the spatial variable in one dimension. For $g(x,t) = 0$, to the best of our knowledge, the existence of random $\mathcal{D}$ -pullback attractor for equation (1.2) has not yet considered.
In this paper, we consider the following one dimensional non-autonomous local modified stochastic S-H equation with multiplicative noise:
\[
\begin{align*}
  du + (\Delta^2 u + 2u_{xx} + au + bu_x + u^3 - g(x,t))dt &= lu \circ dW(t), \text{ in } D \times [\tau, \infty), \\
  u &= u_{xx} = 0, x \in \partial D, \\
  u(x, \tau) &= u_\tau, x \in D
\end{align*}
\]  
(1.3)
(1.4)
(1.5)

Where \( D \) is a bounded open interval, \( \Delta u \) means \( u_{xx} \), and \( \Delta^2 u \) means \( u_{xxxx} \), \( l \) be a separable Banach space with Borel \( \mathcal{F} \) measurable, and \( \mathcal{F} \) is the identity on \( \mathcal{F} \). For the external force \( g(x) \in L^2_{\text{loc}}(R, L^2(D)) \), we assume that there exist \( M > 0 \) and \( \beta > 0 \) such that
\[
a - \beta - 5 < 0, \quad \| g(x, t) \|^2 \leq M e^{\gamma |t|}, \text{ for any } t \in \mathbb{R}, \quad 0 \leq \gamma < \frac{3\beta}{11}.
\]  
(1.6)

The assumption is same as \([8, 12]\), through simple calculation, for all \( t \in \mathbb{R} \), we have
\[
H(t) := \int_{-\infty}^t e^{\beta s} \| g(x, s) \|^2 ds < \infty, \quad \int_{-\infty}^t e^{-\frac{4\beta}{3} s} H^{\frac{11}{3}}(s) ds < \infty.
\]  
(1.7)

An outline of this paper is as follows: In section 2, we recall some basic concepts about random \( \mathcal{D} \)-pullback attractors. In Section 3, we prove that the stochastic dynamical system generated by (1.3) exists a random \( \mathcal{D} \)-pullback attractor in \( H^2_0(D) \).

### 2.2 Preliminaries

There are many research results on random attractors and related issues. The reader is referred to \([13-19]\) for more details, we only list the definitions and abstract result

Let \((X, ||\cdot||_X)\) be a separable Banach space with Borel \( \sigma \)-algebra \( \mathcal{B}(X) \) and \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. In this paper, the term \( \mathbb{P}\text{-a.s.}(the abbreviation for \( \mathbb{P}\) almost surely) denotes that an event happens with probability one. In other words, the set of possible exception may be non-empty, but it has probability zero.

**Definition 2.1.** ([15,16,20,21]) \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{\omega \in \Omega})\) is called a metric dynamical systems if \( \theta: \mathbb{R} \times \Omega \to \Omega \) is \((\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})\)-measurable, and \( \theta_0 \) is the identity on \( \Omega \), \( \theta_s = \theta^s \theta_t \) for all \( t, s \in \mathbb{R} \) and \( \theta_0 \mathbb{P} = \mathbb{P} \) for all \( t \in \mathbb{R} \).

**Definition 2.2.** ([12,13,18]) A non-autonomous random dynamical system (NRDS) \((\varphi, \theta)\) on \( X \) over a metric dynamical system \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{\omega \in \Omega})\) is a mapping
\[
\varphi(t, \tau, \omega): X \to X, \quad (t, \tau, \omega, x) \to \varphi(t, \tau, \omega)x,
\]
which represents the dynamics in the state space \( X \) and satisfies the properties

(i) \( \varphi(t, \tau, \omega) \) is the identity on \( X \);

(ii) \( \varphi(t, \tau, \omega) = \varphi(t, s, \theta_{\tau-s} \omega) \varphi(s, \tau, \omega) \) for all \( \tau \leq s \leq t \);

(iii) \( \omega \to \varphi(t, \tau, \omega)x \) is \( \mathcal{F} \)-measurable for all \( t \geq \tau \) and \( x \in X \).

In the sequel, we use \( \mathcal{D} \) to denote a collection of some families of nonempty bounded subsets of \( X \):
\[
\mathcal{D}' \subseteq \mathcal{D}, D' = \{D(t, \omega) \in \mathcal{B}(X): t \in \mathbb{R}, \omega \in \Omega\}.
\]

**Definition 2.3.** ([12,13,18]) A set \( B' \subseteq \mathcal{D} \) is called a random \( \mathcal{D} \)-pullback bounded absorbing set for NRDS \((\varphi, \theta)\) if for any \( t \in \mathbb{R} \) and any \( D' \subseteq \mathcal{D} \), there exists \( \tau_0(t, D') \) such that
\[
\varphi(t, \tau, \theta_{\tau-\omega} \omega) D(\tau, \theta_{\tau-\omega} \omega) \subseteq B(t, \omega) \quad \text{for any } \tau \leq \tau_0.
\]

**Definition 2.4.** ([12,13,18]) A set \( \mathcal{A} = \{A(t, \omega): t \in \mathbb{R}, \omega \in \Omega\} \) is called a random \( \mathcal{D} \)-pullback attractor for \((\varphi, \theta)\) if the following hold:

(i) \( A(t, \omega) \) is a random compact set;

(ii) \( \mathcal{A} \) is invariant; that is, for \( \mathbb{P}\text{-a.s.} \) \( \omega \in \Omega \), and \( \tau \leq t \), \( \varphi(t, \tau, \omega) A(t, \omega) = A(t, \theta_{\tau-\omega} \omega) \).
(iii) $\mathcal{A}$ attracts all set in $\mathcal{D}$, that is, for all $B' \in \mathcal{D}$ and $\mathbb{P}$-a.s. $\omega \in \Omega$, 
$$
\lim_{t \to \pm \infty} d(\varphi(t, \tau, \vartheta_{\tau}, \omega)) B(\tau, \vartheta_{\tau}, \omega), A(t, \omega) = 0.
$$

Where $d$ is the Hausdorff semimetric given by $\text{dist}(B, A) = \sup_{b \in B} \inf_{a \in A} \| b - a \|_X$.

**Definition 2.5.** ([14, 17]) A NRDS $(\varphi, \Theta)$ on a Banach space $X$ is said to be pullback flattening if for every random bounded set $B' = \{B(t, \omega) : t \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, for any $\epsilon > 0$ and $\omega \in \Omega$ there exists a $T(B', \epsilon, \omega) < t$ and a finite dimensional subspace $X_\epsilon$ such that

(i) $P\left( \bigcup_{t \in \mathbb{R}} \varphi(t, \tau, \vartheta_{\tau}, \omega) B(\tau, \vartheta_{\tau}, \omega) \right)$ is bounded, and

(ii) $\left\| (I - P)\left( \bigcup_{t \in \mathbb{R}} \varphi(t, \tau, \vartheta_{\tau}, \omega) B(\tau, \vartheta_{\tau}, \omega) \right) \right\|_X < \epsilon$,

where $P : X \to X_\epsilon$ is a bounded projector.

**Theorem 2.1.** ([14, 17]) Suppose that $(\varphi, \Theta)$ is a continuous NRDS on a uniformly convex Banach space $X$. If $(\varphi, \Theta)$ possesses a random $\mathcal{D}$-pullback bounded absorbing sets $B' = \{B(t, \omega) : t \in \mathbb{R}, \omega \in \Omega\}$ and $(\varphi, \Theta)$ is pullback flattening, then there exists a random $\mathcal{D}$-pullback attractor $\mathcal{A} = \{A(t, \omega) : t \in \mathbb{R}, \omega \in \Omega\}$.

### 3. Random pullback attractor for modified Swift-Hohenberg

In this section, we will use abstract theory in section 2 to obtain the random $\mathcal{D}$-pullback attractor for equation (1.3)-(1.5). First we introduce an Ornstein-Uhlenbeck process, 

$$
z(\Theta_{\omega}) := -\int_{-\infty}^{0} e^{t}(\Theta_{\omega})(\tau)d\tau, t \in \mathbb{R}.
$$

We known from [6], it is the solution of Langevin equation 

$$
dz + zdt = dW(t).
$$

$W(t)$ is a two-sided real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where 

$$
\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},
$$

$\mathcal{F}$ is the Borel algebra induced by the compact open topology of $\Omega$, and $\mathbb{P}$ is the corresponding Wiener measure on $\{\Omega, \mathcal{F}\}$. We identify $\omega(t)$ with $W(t)$, i.e., 

$$
W(t) = W(t, \omega) = \omega(t), t \in \mathbb{R}.
$$

Define the Wiener time shift by 

$$
\Theta_{t}\omega(s) = \omega(s + t) - \omega(t), \omega \in \Omega, t, s \in \mathbb{R}.
$$

Then $(\Omega, \mathcal{F}, \mathbb{P}, \Theta_{t})$ is an ergodic metric dynamical system.

From [15,16,20], it is known that the random variable $z(\omega)$ is tempered and there exists a $\Theta_{t}$-invariant set of full measure $\hat{\Omega} \subset \Omega$ such that for all $\omega \in \hat{\Omega}$:

$$
\lim_{t \to \pm \infty} \frac{|z(\Theta_{\omega})|}{|t|} = 0, \quad \lim_{t \to \pm \infty} \frac{1}{t} \int_{0}^{t} z(\Theta_{\omega}) d\omega = 0, \quad (3.1)
$$

and for any $\epsilon > 0$, there exists $\rho(\epsilon) > 0$, such that

$$
|z(\Theta_{\omega})| \leq \rho(\omega) + \epsilon |t|, \quad \left| \int_{0}^{t} z(\Theta_{\omega}) d\omega \right| \leq \rho(\omega) + \epsilon |t|. \quad (3.2)
$$

Let $v(s, \tau) = e^{-\epsilon z(\Theta_{\omega})} u(s), v = -le^{-\epsilon z(\Theta_{\omega})} u(s)dz + e^{-2\epsilon z(\Theta_{\omega})} du$. Using Langevin equation, combined with the original equation (1.3), we get

$$
\frac{dv}{ds} + \Delta^{2} v + 2v_{\nu} + (a - \epsilon z) v + e^{\epsilon z(\Theta_{\omega})} v^{2} = e^{2\epsilon z(\Theta_{\omega})} g(x, s), \text{in } D \times [\tau, \infty), \quad (3.3)
$$

$v = \Delta v = 0$ on $\partial D \times [\tau, \infty)$, \quad (3.4)

$v(x, \tau) = v_{\tau} = e^{-\epsilon z(\Theta_{\omega})} u_{\tau}(x) \text{ in } D. \quad (3.5)$


Equation (1.3)-(1.5) are equivalent to equation (3.3)-(3.5), by a standard method, it can be proved that the problem (3.3)-(3.5) is well posed in $H^2_0(D)$, that is, for every $\tau \in \mathbb{R}$ and $v_\tau \in H^2_0(D)$, there exits a unique solution $v \in C([\tau, \infty), H^2_0(D))$ (see e.g.[2,8,22]). Furthermore, the solution is continuous with respect to the initial condition $v_\tau$ in $H^2_0(D)$. To construct a non-autonomous random dynamical system $\{V(t, \tau, \omega)\}$ for problem (3.3)-(3.5), we define $V(t, \tau, \omega) : H^2_0(D) \rightarrow H^2_0(D)$ by $V(t, \tau, \omega)v_\tau$. Then the system $\{V(t, \tau, \omega)\}$ is a non-autonomous random dynamical system in $H^2_0(D)$.

We now apply abstract theory in Section 2 to obtain the random $\mathcal{D}$-pullback attractors for non-autonomous modified Swift-Hohenberg equation, by the equivalent, we only consider the random $\mathcal{D}$-pullback attractor of equation (3.3)-(3.5).

For convenience, the $L^p(D)$ norm of $u$ will be denoted by $\|u\|_{L^p}$, with the norm of Sobolev spaces $W^k_p(D)$ by $\|u\|_{W^k_p}$, we regard the space $H^2_0(D)$ endowed with the norm $\|u\|_{2,2} = \|\nabla u\|$, $c$ or $c(\omega)$ denote the arbitrary positive constants, which only depend on $\omega$ and may be different from line to line and even in the same line.

For our purpose that the following Gagliardo-Nirenberg inequality will be used.

**Lemma 3.1. (Gagliardo-Nirenberg Inequality).** Let $D$ be an open, bounded domain of the Lipschitz class in $\mathbb{R}^n$. Assume that $1 \leq p \leq q \leq \infty, 1 \leq r \leq 0 < \theta \leq 1$ and let

$$k - \frac{n}{p} \leq \theta(m - \frac{n}{q}) + (1 - \theta) \frac{n}{r}.$$ 

Then the following inequality holds:

$$\|u\|_{k,p} \leq c(D) \|u\|^{\theta}_{r,m} \|u\|^{1-\theta}_{m,q}.$$ 

**Lemma 3.2.** For all $t \geq \tau$, the following inequality hold:

$$\|v(t, \tau, \theta_{\tau-\tau}, \omega)\|^2 \leq c(\omega)(e^{-\beta(l-t)} \|v_\tau\|^2 + 1 + e^{-\beta} H(t)),$$

and

$$\int_{\tau}^{t} e^{2\beta(l-t)+2} \|z(\theta_{\tau-\tau})\| dv \leq c(\omega)(e^{-\beta(l-t)} \|v_\tau\|^2 + 1 + e^{-\beta} H(t)).$$

**Proof.** Let $v(s)$ or $v$ denotes $v(s, \tau, \theta_{\tau-\tau}, \omega)$ be the solution of equation (3.3)-(3.5). Taking the inner product of equation (3.3) with $v$, we get

$$\frac{1}{2} \frac{d}{ds} \|v\|^2 + \|\Delta v\|^2 + 2(\Delta v, v) + (a - l\zeta)\|v\|^2 + be^{l(\theta_{\tau-\tau})}(v_\tau^2, v)\|v\|^4 + e^{2l(\theta_{\tau-\tau})}\|v\|^4 = e^{-l(\theta_{\tau-\tau})}(g(x, s), v).$$

Using Young inequality, we get

$$\|v(s)\| \leq \frac{1}{4} \|\Delta v\|^2 + 4\|v\|^2.$$ 

By integration by parts, we obtain

$$be^{l(\theta_{\tau-\tau})}(v_\tau^2, v) = be^{l(\theta_{\tau-\tau})} \int_D v_\tau^2 u dx = -be^{l(\theta_{\tau-\tau})} \int_D (v^2 v_{xx} + v v_{x}^2) dx,$$

thus

$$be^{l(\theta_{\tau-\tau})}(v_\tau^2, v) = -b e^{l(\theta_{\tau-\tau})} \int_D v_\tau^2 v_{xx} dx.$$ 

Applying the Hölder inequality and Young inequality, we get

$$be^{l(\theta_{\tau-\tau})} \|v_\tau^2\| \leq \frac{b}{2} e^{l(\theta_{\tau-\tau})} \int_D v_\tau^2 v_{xx} dx \leq \frac{b}{2} \|e^{l(\theta_{\tau-\tau})}\| v_{xx}^\| \|v\|_{4} \|v\|_{4} \leq \frac{b^2}{16\eta} \|v\|^2 + \frac{b^2}{16\eta} e^{2l(\theta_{\tau-\tau})}\|v\|_{4}^4.$$
and
\[ e^{-2c(\theta, \omega)} \| (g(x, s), v) \| \leq \| v \|^2 + \frac{1}{4} e^{-2c(\theta, \omega)} \| g(x, s) \|^2. \]

For convenience, we take \( \eta = \frac{1}{4} \), \( |b| < 2 \) (\( |b| < 4 \), the same conclusion hold), we obtain
\[ \frac{d}{ds} \| v \|^2 + \| \Delta v \|^2 + 2(a - l_z - 5) \| v \|^2 + 2(1 - \frac{b^2}{4}) e^{2c(\theta, \omega)} \| v \|^4 \leq \frac{1}{2} e^{-2c(\theta, \omega)} \| (g(x, s)) \|^2. \]

Taking \( \beta > 0 \) such that \( a - \beta - 5 < 0 \), we have
\[ \frac{d}{ds} \| v \|^2 + \| \Delta v \|^2 + 2(\beta - l_z) \| v \|^2 + 2(1 - \frac{b^2}{4}) e^{2c(\theta, \omega)} \| v \|^4 \leq -2(a - \beta - 5) \| v \|^2 + \frac{1}{2} e^{-2c(\theta, \omega)} \| (g(x, s)) \|^2. \]

By the Sobolev imbedding \( L^4(D) \subset L^2(D) \) and Young inequality, we get
\[ -2(a - \beta - 5) \| v \|^2 \leq c \| v \|^2 \leq 2(1 - \frac{b^2}{4}) e^{2c(\theta, \omega)} \| v \|^4 + ce^{-2c(\theta, \omega)}. \]

Thus we obtain
\[ \frac{d}{ds} \| v \|^2 + \| \Delta v \|^2 + 2(\beta - l_z) \| v \|^2 \leq ce^{-2c(\theta, \omega)} (1 + \| (g(x, s)) \|^2). \]

Multiply this by \( e^{2\beta t - 2(\beta t + 2) \int_0^t z(\theta, \omega) \, dt} \) and integrating from \( \tau \) to \( t \), we have
\[ \| v(t) \|^2 + \int_\tau^t e^{2\beta(t - s) + 2(\beta \int_0^s z(\theta, \omega) \, ds)} \| \Delta v \|^2 \, ds \]
\[ \leq e^{-2\beta(t - \tau) + 2(\beta \int_\tau^t z(\theta, \omega) \, ds)} \| v(\tau) \|^2 + \int_\tau^t e^{-2\beta(t - s) + 2(\beta \int_s^t z(\theta, \omega) \, ds)} (1 + \| g(x, s) \|^2) \, ds \]
\[ \leq e^{-2\beta(t - \tau) + 2(\beta \int_\tau^t z(\theta, \omega) \, ds)} \| v(\tau) \|^2 + \int_\tau^t e^{-\beta(t - s) + 2(\beta \int_s^t z(\theta, \omega) \, ds)} (1 + \| g(x, s) \|^2) \, ds \]

From (3.2), we get
\[ 2 \int_{t-\tau}^0 z(\theta, \omega) \, dt \leq \rho(\omega) + \beta(t - \tau), \quad 2 \int_{t-\tau}^0 z(\theta, \omega) - 2l_z(\theta, \omega) \leq \rho(\omega) + \beta(t - s). \]

Then we have
\[ \| v(t) \|^2 + \int_\tau^t e^{2\beta(t - s) + 2(\beta \int_s^t z(\theta, \omega) \, ds)} \| \Delta v \|^2 \, ds \]
\[ \leq c(\omega) e^{-\beta(t - \tau)} \| v(\tau) \|^2 + \int_\tau^t e^{-\beta(t - s)} (1 + \| g(x, s) \|^2) \, ds \]
\[ \leq c(\omega) e^{-\beta(t - \tau)} \| v(\tau) \|^2 + 1 + e^{-\beta(t - \tau)} \| g(x, s) \|^2 \, ds \]

Thus we get the desired results. \( \square \)

**Lemma 3.3.** For all \( t \geq \tau \), the following inequality hold:
\[ \| \Delta v(t, \tau, \theta, \omega) \|^2 \leq c(\omega) \left[ (1 + \frac{2}{t - \tau}) e^{-\beta(t - \tau)} \| v(\tau) \|^2 + e^{-\frac{11}{3} \beta(t - \tau)} \| v \|_1^{22} + e^{-\frac{8}{3} \beta(t - \tau)} H(\tau) + e^{-\beta(t - \tau)} \int_{\tau}^{t} e^{\frac{5}{2} \beta(s) H(s) \, ds} \right]. \]

**Proof.** Taking inner product of equation (3.3) with \( \Delta^2 v \), we have
\[ \frac{1}{2} \frac{d}{ds} \| \Delta v \|^2 + \| \Delta^2 v \|^2 + 2(\Delta v, \Delta^2 v) + (a - l_z) \| \Delta v \|^2 \]
\[ + b e^{c(\theta, \omega)} (v_1, \Delta^2 v) + e^{2c(\theta, \omega)} (v_3, \Delta^2 v) = e^{-\xi(\theta, \omega)} (g(x, s), \Delta^2 v). \]
By the Hölder inequality, Young inequality and Gagliardo-Nirenberg inequality, we get
\[ -2(\Delta \nu, \Delta^2 \nu) \leq \frac{1}{8} \| \Delta^2 \nu \|^2 + 8 \| \Delta \nu \|^2, \]
\[ |b| e^{4z(t, \nu)} \left| \left( x^2, \Delta^2 \nu \right) \right| \leq b \left| e^{4z(t, \nu)} \right| \| \nu \|^2 \| \Delta^2 \nu \| \leq ce^{5z(t, \nu)} \| \nu \|^2 \Delta^2 \nu \| \leq \frac{1}{8} \| \Delta^2 \nu \|^2 + ce^{5z(t, \nu)} \| \nu \|^2 \Delta^2 \nu \|, \]
\[ e^{2z(t, \nu)} \left| \left( x^3, \Delta^2 \nu \right) \right| \leq e^{2z(t, \nu)} \| \nu \|^2 \| \Delta^2 \nu \| \leq ce^{2z(t, \nu)} \| \nu \|^2 \| \Delta^2 \nu \| \leq ce^{2z(t, \nu)} \| \nu \|^2 \| \Delta^2 \nu \| \]
\[ = ce^{2z(t, \nu)} \| \Delta^2 \nu \|^2 \| \nu \|^2 \| \Delta^2 \nu \|^2 \leq \frac{1}{8} \| \Delta^2 \nu \|^2 + ce^{5z(t, \nu)} \| \nu \|^2 \Delta^2 \nu \|. \]

Putting all these inequalities together, we deduce
\[ \frac{d}{ds} \| \Delta \nu \|^2 + 2(\beta - I \nu) \| \Delta \nu \|^2 \leq 2(8 + \beta - \alpha) \| \Delta \nu \|^2 + c(e^{16z(t, \nu)} \| \nu \|^3 + e^{2z(t, \nu)} \| g(x, s) \|^3). \]

Multiplying this by \((s - \tau)e^{2\beta(t-s)\int z(t, \nu)dr}\) and integrating it over \((\tau, t)\), we get
\[ (t - \tau)e^{2\beta(t-s)\int z(t, \nu)dr} \| \Delta \nu(t) \|^2 \leq c\int_{\tau}^{t} e^{2\beta(t-s)\int z(t, \nu)dr} \| \Delta \nu \|^2 ds \]
\[ + \int_{\tau}^{t} (s - \tau)e^{2\beta(t-s)\int z(t, \nu)dr} \| \Delta \nu \|^2 + e^{\beta(t-\tau)\int z(t, \nu)dr} \| g(x, s) \|^2 ds. \]

Then we have
\[ \| \Delta \nu(t) \|^2 \leq c(1 + \frac{1}{t - \tau}) \int_{\tau}^{t} e^{2\beta(t-s)\int z(t, \nu)dr} \| \Delta \nu \|^2 ds \]
\[ + \int_{\tau}^{t} e^{2\beta(t-s)\int z(t, \nu)dr} \| \Delta \nu \|^2 + e^{\beta(t-\tau)\int z(t, \nu)dr} \| g(x, s) \|^2 ds. \]

By (3.2), (3.6) and the inequality \((a + b)^r \leq c(a^r + b^r)\) \((a, b > 0, r \geq 1)\), we get
\[ \int_{\tau}^{t} e^{2\beta(t-s)\int z(t, \nu)dr} \| \Delta \nu \|^2 \leq c(\omega) \int_{\tau}^{t} e^{\beta(t-\tau)\int z(t, \nu)dr} \| \Delta \nu \|^2 ds \]
\[ \leq c(\omega)e^{\beta(t-\tau)\int z(t, \nu)dr} \| \nu \|^2 + e^{\beta(t-\tau)\int z(t, \nu)dr} H(s)^{\frac{11}{3}} ds \]
\[ \leq c(\omega)e^{\beta(t-\tau)\int z(t, \nu)dr} \| \nu \|^2 + e^{\beta(t-\tau)\int z(t, \nu)dr} H(s)^{\frac{11}{3}} ds \]
\[ \leq c(\omega)(e^{\frac{11}{3} \beta(t-\tau)} \| \nu \|^2 + e^{\frac{11}{3} \beta(t-\tau)} H(s)^{\frac{11}{3}} ds). \]

Thus we have
\[ \| \Delta \nu(t, \tau) \|^2 \leq c(\omega)(1 + \frac{1}{t - \tau}) e^{\beta(t-\tau)\int z(t, \nu)dr} \| \nu \|^2 + e^{\beta(t-\tau)\int z(t, \nu)dr} H(s)^{\frac{11}{3}} ds \]
\[ + e^{\beta(t-\tau)\int z(t, \nu)dr} \| \nu \|^2 + e^{\beta(t-\tau)\int z(t, \nu)dr} H(s)^{\frac{11}{3}} ds]. \]

We complete the proof of Lemma 3.3. □

Let \( \mathcal{R} \) be the set of all function \( r : \mathbb{R} \rightarrow (0, +\infty) \) such that \( \lim_{t \to +\infty} e^r(t) = 0 \) and denote by \( \mathcal{D} \)
the class of all families \( \tilde{D} = \{ D(t) : t \in \mathbb{R} \} \) such that \( D(t) \subset \overline{B}(r(t)) \) for some \( r(t) \in \mathcal{R} \), \( \overline{B}(r(t)) \)
denote the closed ball in \( H^s_{\tilde{\nu}}(D) \) with radius \( r(t) \). Let
\[ r_1^2(t) = 2c(\omega)[1 + e^{-\beta t} (H(t) + \int_{-\infty}^t e^{-\frac{\beta s}{3}} H^3(s)ds)] \]  

(3.10)

By lemma 3.3 for any \( \hat{D} \in \mathcal{D} \) and \( t \in \mathbb{R} \), there exists \( \tau_0(\hat{D}, t, \omega) < t \) such that  
\[ \| \Delta v(t, \tau, \theta \omega) \| \leq r_1(t), \text{ for any } \tau < \tau_0. \]

Since \( 0 \leq \gamma < \frac{3\beta}{11} \), simple calculation imply that \( r_1(t) \in \mathcal{R} \), which say that the \( \overline{B}(r_1(t)) \) be a family of random \( \mathcal{D} \)-pullback bounded absorbing sets in \( H_0^2(D) \) and \( \{ \overline{B}(r_1(t)) \} \in \mathcal{D} \).

**Theorem 3.1.** The non-autonomous random dynamical system to problem (1.1)-(1.3) possesses a unique random \( \mathcal{D} \)-pullback attractor in \( H_0^2(D) \).

**Proof.** We need only prove that the dynamical system (3.3)-(3.5) satisfies the pullback flattening condition. Since \( A^{-1} \) is a continuous compact operator in \( H_0^2(D) \), by the classical spectral theorem, there exists a sequence \( (\lambda_j)_{j=1}^{\infty} \) satisfying  
\[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_j \to +\infty, \text{ as } j \to +\infty, \]
and a family of elements \( \{ e_j \}_{j=1}^{\infty} \) of \( H_0^2(D) \) which are orthonormal in \( H \) such that  
\[ Ae_j = \lambda_j e_j, \forall j \in \mathbb{N}^+. \]

Let \( H_m = \text{span}\{e_1, e_2, \cdots, e_m\} \) in \( H \) and \( P_m : H \to H_m \) be an orthogonal projector. For any \( v \in H \) we write  
\[ v = P_m v + (I - P_m)v = v_1 + v_2. \]

Taking inner product of (3.3) with \( \Delta^2 v_2 \) in \( H \), we get
\[
\frac{1}{2} \frac{d}{ds} \| \Delta v_2 \|^2 + \| \Delta^2 v_2 \|^2 + 2(\Delta v_1, \Delta^2 v_2) + (a - lz) \| \Delta v_2 \|^2
\]
\[ + be^{\varepsilon(v, \omega)}(\| \nabla v_1 \|^2, \| \Delta^2 v_2 \|) + e^{2\varepsilon(v, \omega)}(v_1^2, \Delta^2 v_2) = e^{-\varepsilon(v, \omega)}(g(x, s), \Delta^2 v_2). \]

By the Hölder inequality, Young inequality and Gagliardo-Nirenberg inequality, we get
\[
-2(\Delta v_1, \Delta^2 v_2) \leq \frac{1}{8} \| \Delta^2 v_1 \|^2 + 8 \| \Delta v_2 \|,
\]
\[
- be^{\varepsilon(v, \omega)}(v_1^2, \Delta^2 v_2) \leq \frac{1}{8} \| \Delta^2 v_1 \|^2 + 2b^2 e^{2\varepsilon(v, \omega)} \| v_1 \|_4^2
\]
\[
\leq \frac{1}{8} \| \Delta^2 v_1 \|^2 + ce^{2\varepsilon(v, \omega)} \| v \|^{11-\theta} \| \Delta v \|^{4\theta}
\]
\[
= \frac{1}{8} \| \Delta^2 v_1 \|^2 + ce^{2\varepsilon(v, \omega)} \| v \|^{11} \| \Delta v \| \quad (\theta = \frac{1}{4})
\]
\[
\leq \frac{1}{8} \| \Delta^2 v_1 \|^2 + \frac{1}{2} \| \Delta v \|^2 + ce^{4\varepsilon(v, \omega)} \| v \|^{6},
\]
\[-e^{2z(\theta_n, o)}(v^3, \Delta^2v_2) \leq \frac{1}{8} \| \Delta^2v_2 \|^2 + 2e^{4z(\theta_n, o)} \| \| v \|^3 \]

\[ \leq \frac{1}{8} \| \Delta^2v_2 \|^2 + ce^{4z(\theta_n, o)} \| v \|^{4} \Delta v \|^{4} \]

\[ = \frac{1}{8} \| \Delta^2v_2 \|^2 + ce^{4z(\theta_n, o)} \| v \|^{2} \| \Delta v \| \quad (\theta = \frac{1}{3}) \]

\[ \leq \frac{1}{8} \| \Delta^2v_2 \|^2 + \frac{1}{2} \| \Delta v \|^2 + ce^{8z(\theta_n, o)} \| v \|^{4}, \]

\[ e^{-2z(\theta_n, o)}(g(x, s), \Delta^2v_2) \leq \frac{1}{8} \| \Delta^2v_2 \|^2 + 2e^{-2z(\theta_n, o)} \| g(x, s) \|^2. \]

Putting all these inequalities together, we have

\[ \frac{d}{ds} \| \Delta v \|^2 + \| \Delta v \|^2 + 2(a - lz) \| \Delta v \|^2 \]

\[ \leq 18 \| \Delta v \|^{2} + c(e^{4z(\theta_n, o)} \| v \|^{6} + e^{8z(\theta_n, o)} \| v \|^{4} + e^{-2z(\theta_n, o)} \| g(x, s) \|^{2}). \]

\[ \lambda_n \| \Delta v \|^{2} \leq \| \Delta v \|^2 , \text{ which implies that} \]

\[ \frac{d}{ds} \| \Delta v \|^2 + (\lambda_n - 2lz) \| \Delta v \|^2 \]

\[ \leq c(\| \Delta v \|^{2} + e^{4z(\theta_n, o)} \| v \|^{6} + e^{8z(\theta_n, o)} \| v \|^{4} + e^{-2z(\theta_n, o)} \| g(x, s) \|^{2}). \]

Multiply this by \((s - \tau)e^{4z(\theta_n, o)\tau} + 2(\lambda_n - 2lz) \| \Delta v \|^2\) and integrating from \(\tau\) to \(t\), we obtain

\[ (t - \tau)e^{4z(\theta_n, o)\tau} \| \Delta v \|^2 \leq c\int_{\tau}^{t}(1 + s - \tau)e^{4z(\theta_n, o)\tau} \| \Delta v \|^2 ds \]

\[ + \int_{\tau}^{t}(s - \tau)e^{4z(\theta_n, o)\tau} (e^{4z(\theta_n, o)\tau} \| \| v \|^{6} + e^{8z(\theta_n, o)} \| \| v \|^{4} + e^{-2z(\theta_n, o)} \| g(x, s) \|^{2})ds \]

Thus we get

\[ \| \Delta v \|^2 \leq c(1 + \frac{1}{t - \tau}) \int_{\tau}^{t}e^{4z(\theta_n, o)\tau} \| \Delta v \|^2 ds + \int_{\tau}^{t}e^{4z(\theta_n, o)\tau} (e^{4z(\theta_n, o)\tau} \| \| v \|^{6} + e^{8z(\theta_n, o)} \| \| v \|^{4} + e^{-2z(\theta_n, o)} \| g(x, s) \|^{2})ds \]

\[ \leq c(\omega)(1 + \frac{1}{t - \tau}) \int_{\tau}^{t}e^{(\lambda_n - \beta)(s - \tau)} \| \Delta v \|^2 ds + \int_{\tau}^{t}e^{(\lambda_n - \beta)(s - \tau)} \| g(x, s) \|^{2} ds \]

\[ + \int_{\tau}^{t}e^{(\lambda_n - \beta)(s - \tau)} \| v \|^{6} + \int_{\tau}^{t}e^{(\lambda_n - \beta)(s - \tau)} \| g(x, s) \|^{2} ds \]

By simple calculation, we find that there exists \(N \in \mathbb{N}, \forall n > N, \lambda_n - \beta > 0, \text{ and} \)

\[ I_1 \leq (1 + \frac{1}{t - \tau}) \int_{\tau}^{t}e^{(\lambda_n - \beta)(s - \tau)} \| \Delta v \|^2 ds < \infty, \quad e^{(\lambda_n - \beta)(s - \tau)} \| \Delta v(s) \|^2 \rightarrow 0 \text{ as } n \rightarrow \infty, \]

According to Lebesgue dominated convergent theorem, we obtain

\[ I_1 \rightarrow 0 \text{ as } n \rightarrow \infty. \]

Using (3.6), we get

\[ I_2 \leq c \int_{\tau}^{t}e^{(\lambda_n - \beta)(s - \tau)} (e^{-\beta(s - \tau)} \| v \|^{2} + 1 + e^{-\beta t} H(s))^{3} ds \]

\[ \leq c \int_{\tau}^{t}e^{(\lambda_n - \beta)(s - \tau)} (e^{-3\beta(s - \tau)} \| v \|^{6} + 1 + e^{-3\beta t} H^{3}(s))ds \]

\[ \leq c(e^{-\beta t} \| v \|^{6} + \frac{1}{\lambda_n - \beta} + e^{-\beta t} \lambda_n - 4\beta \rightarrow 0 \text{ as } n \rightarrow \infty. \]
\[
I_3 \leq e \int_{T}^T e^{(\lambda - \beta) (s-t)} \left( e^{-\beta (s-t)} \right) v_s^2 \, ds + 1 + e^{-\beta s} H(s)^2 \, ds \\
\leq e \int_{T}^T e^{(\lambda - \beta) (s-t)} \left( e^{-2\beta (s-t)} \right) v_s^4 \, ds + 1 + e^{-2\beta s} H^2(s) \, ds \\
\leq e^{[e^{-\beta} e^{-2\beta (t-t)} \left( \frac{1}{\lambda_n - 3\beta} + \frac{1}{\lambda_n - 3\beta} \right) H^2(t)]} \to 0, \text{ as } n \to \infty,
\]
and
\[
I_4 \leq e^{-\beta} \int_{T}^T e^{\beta s} \left( g(x,s) \right) \left[ e^{(\lambda - \beta) (s-t)} \right] g(x,s) \, ds \to 0 \text{ as } n \to \infty,
\]
Again using Lebesgue dominated convergent theorem, we get
\[
e^{[e^{(\lambda - \beta) (s-t)} \left( g(x,s) \right) ^2]} \to 0 \text{ as } n \to \infty,
\]
In summary, we obtain that the terms on the right hand of inequality (3.13) tend to 0 as \( n \to \infty \), which say that \( \left\| v_{3}(t, \tau, \theta_{\tau}, \omega) \right\| \to 0 \), i.e., the random dynamical system (3.3)-(3.5) satisfies pullback flattening.

4. Conclusions

This paper extends the existence of pullback attractor of non-autonomous modified S-H equation to the case of non-autonomous stochastic modified S-H equation with multiplicative noise. In the concrete experiment, the random term in the equation is more consistent with the actual problem. For S-H equation with multiplicative noise, the external force has exponential growth, we have proved that the equation exists a random pullback attractor in one dimension. In the future work, we will continue to investigate whether the same results can be obtained when the spatial dimension is two-dimensional or n-dimensional.

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