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Fractional Riccati equation and its applications to rough Heston model

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Abstract

Rough volatility models are popularized by (Gatheral, Jaisson, & Rosenbaum, 2018), where they have shown that the empirical volatility in the financial market is extremely consistent with rough volatility. Fractional Riccati equation as a part of computation for the characteristic function of rough Heston model is not known in explicit form as of now and therefore, we must rely on numerical methods to obtain a solution. In this paper, we give a short introduction to option pricing theory and an overview of the current advancements on the rough Heston model.

Keywords and Phrases: Fractional Riccati equation; Rough volatility models; Heston model

1 Introduction

The academic contribution by (Gatheral et al., 2018) enables us to price assets accurately by considering rough volatility in our framework. In particular, rough Heston model and its characteristic function has been proposed and derived in (El Euch & Rosenbaum, 2019) following the work of (El Euch, Fukasawa, & Rosenbaum, 2018) where it considers rough Heston model following the microscopic model - multivariate Hawkes processes. Coincidentally, it is found that the classical Riccati equation in the classical Heston model (Heston, 1993) has been replaced with fractional Riccati equation in the rough Heston model. Unlike the classical version of Riccati equation where the exact solution exists and the solution for the classical Heston can be obtained easily in an efficient manner, there is no explicit-form solution for the fractional Riccati equation as of now. This poses a huge disadvantage to use the rough Heston model for obtaining the call option price as we have to rely on

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numerical methods now. Unfortunately, it's also noted that to acquire an accurate solution, the general numerical approach requires high computational cost. In particular, the key factor contributing to the high computational costs originated from the lack of explicit solution for the fractional Riccati equation which is a part of the characteristic function of the rough Heston model. It can be seen as a huge sacrifice for the sake of incorporating rough volatility in its framework.

Besides the direct confrontation on the computation of rough Heston model using Fourier-type method, it is possible to use hybrid Brownian semistationary (BSS) method (Bennedsen, Lunde, & Pakkanen, 2017) as an efficient simulation method. In particular, the method involves the usage of appropriate power function to approximate when the kernel function near zero and Riemann-sum discretization elsewhere. It is found that the hybrid BSS scheme is able to reproduce the fine properties of rough Brownian semistationary process, but practical implementation remains hard. A variance reduction technique (McCrickerd & Pakkanen, 2018) took advantage of the hybrid BSS scheme for obtaining the solution of the rough Bergomi model. Furthermore, (Abi Jaber & El Euch, 2019) proposed a multifactor approximation method for the fractional Riccati equation. The specification of the method involves approximating fractional Riccati equation using a n-dimensional classical Riccati equation with large n (up to $n = 500$). A similar time-efficiency method is suggested in (Abi Jaber, 2019) with satisfactory result for the fits to the implied volatility smiles for short maturities with lesser parameters.

(El Euch, Gatheral, & Rosenbaum, 2018) managed to approximate the rough Heston model by using scaled "volatility of volatility" parameter in the classical Heston model. The method is somewhat extremely fast to calibrate (follows the time complexity of classical Heston model) with satisfactory accuracy. Besides that, (Callegaro, Grasselli, & Pages, 2018) used a hybrid method that involves Richardson-Romberg extrapolation method to approximate outside of convergence domain and a short-time series expansion. The authors noted that the hybrid method is both efficient and flexible.

Although it has only been recently discovered that solution to the fractional Riccati equation is essential for pricing call option, it has actually been studied extensively for general purposes by many researchers in the past. We first mention two of the most famous methods which are the Adomian decomposition method (Momani & Shawagfeh, 2006) and the variational iteration method (Odibat & Momani, 2006). Homotopy pertubation method (Odibat & Momani, 2008) and homotopy analysis method (Cang, Tan, Xu, & Liao, 2009) remain popular choices for approximating the analytical solutions. Other methods on approximating fractional Riccati equation include (Jafari, Tajadodi, & Matikolai, 2010; Raja, Khan, & Qureshi, 2010).

The usual algorithm dealing with fractional order differential equation is obtained by using the fractional Adams-Bashforth-Moulton method (Diethelm, Ford, & Freed, 2004). The error analysis is also provided in the paper. We refer the fractional Adams-Bashforth-Moulton method as fractional Adams method from now onwards. It is noted that (El Euch & Rosenbaum, 2019) has also demonstrated the use of fractional Adams method in its numerical

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application section for the rough Heston model. Although it is capable of providing extremely well solution with large time steps, the algorithm complexity to evaluate a call option is at $O(N_a n^2)$ where the N_a is the number of space steps used for Fourier-type method and n is the number of time steps for the fractional Adams method. The sole complexity of the algorithm makes computation of the call option to a certain accuracy not feasible to most practitioners. We will take advantage of this method by setting a high number of time steps to compare our numerical solution later in this paper.

In this paper, we will also focus heavily on the work of (Gatheral & Radoicic, 2019) where it uses multipoint Padé approximants on the asymptotical solutions ($t \to 0$ and $t \to \infty$) of the fractional Riccati equation and applying it to the rough Heston model. This is needed as the typical method (fractional Adams method) requires great computational effort, i.e. not feasible to most practitioners or perhaps even researchers. Before that, we wish to discuss some of the general aspects and history of Padé approximation method. Specifically, it was developed by Henri Padé in the 1890s with the constructing idea credited to Georg Frobenius where he investigated the usage of rational approximations of the power series. Coincidentally, Padé approximation turns out to be a great approximation to many application in physics, e.g. nuclear physics (Thompson, 1988), kinetic electron model (Lanti, Dominski, Brunner, McMillan, & Villard, 2016), quantum resonances (Rakityansky, Sofianos, & Elander, 2007) and many more (Baker, Baker Jr, Baker JR, Graves-Morris, & Baker, 1996). We follow closely on the literature from (Gatheral & Radoicic, 2019), we note that the theory of convergence of Padé approximants can be found in (Lubinsky, 2003). Also, as noted in (Baker et al., 1996), one of the theorems (Montessus de Ballore's theorem) states that if f is analytical in a ball centered at zero except for multiples of n, the sequence of $\{f^{(m,n)}\}_{m=1}^{\infty}$ for a fixed n, converges uniformly to f in compact subsets omitting the poles.

Furthermore, since we are dealing with *diagonal* Padé approximants for our fractional Riccati equation later on, it is only fair that we mention its convergence properties too. As noted in (Lubinsky, 2014), Baker-Gammel-Wills conjecture speculates that under some certain condition, the *diagonal* Padé approximant $\{f^{(n,n)}\}_{n=1}^{\infty}$ converges uniformly to f. Taken from (Lubinsky, 2003), a counterexample for the Baker-Gammel-Wills conjecture is provided. Consequently, Nutall-Pommerenke's theorem is provided where it states that there exists a subsequence of $\{f^{(n,n)}\}_{n=1}^{\infty}$ where it converges almost everywhere. Coupled with Monstessus de Ballor's theorem, we should be extremely careful when dealing with *diagonal* Padé approximant in our work. One of the functions we will mention later is the Mittag-Leffler function $E_\alpha(z)$. The uniform convergence of Mittag-Leffler function has been proven in (Starovoitov & Starovoitova, 2007) on the compact set $\{|z| \leq 1\}$. However, it is not neccessarily compatible with the asymptotic behavior of the Mittag-Leffler function for large t, although it performs better than their corresponding truncated Taylor series.

We now move to the discussion on the multipoint Padé approximation method. The method was originally derived in (Winitzki, 2003) as a global rational approximant. As mentioned previously, it aims to match asymptotical points of f using (m, n) Padé approximant. In addition, it has been successfully applied to both the Mittag-Leffler function (Atkinson & Osseiran, 2011) and generalized Mittag-Leffler function (Zeng & Chen, 2015). (Gatheral &

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Radoicic, 2019) demonstrated the use of global rational approximations for the fractional Riccati equation and found the excellent performance of the multipoint Padé approximation $(3, 3)$ especially when H is close to 0. However, the imaginary part of the solution deteriorates as $H \to 0.5$ or $\alpha \to 1$. This poses a challenge in terms of accuracy of the model and it would be erroneous to employ the same method to approximate the solution for the fractional Riccati equation when H is somewhat near 0.5.

1.1 Organization of the paper

We intend to organize our paper as following:

1. A short introduction to Riemann-Liouville fractional calculus and Mittag-Leffler function.
2. Fractional Adams-Bashforth-Moulton method and its error analysis.
3. Black Scholes equation, implied volatility, classical Heston, rough Heston model, their characteristic functions and their connection to call option pricing.
4. Small and long time expansion of solution for the fractional Riccati equation.
5. Padé, multipoint Padé approximation method and its application to the asymptotical solutions of the fractional Riccati equation.
6. Numerical experiments and performances.

2 Fractional Calculus and Mittag-Leffler function

2.1 Riemann-Liouville fractional integrals and fractional derivatives

Taken from (Dumitru, Kai, & Enrico, 2012) with some simplification to fit our area of study. We can define the Riemann-Liouville fractional integration for $\alpha \in (0, 1]$ as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)ds}{(t-s)^{1-\alpha}} \tag{2.1}$$

and Riemann-Liouville fractional derivatives as

$$D^\alpha f(t) = \frac{d}{dt}\left[I^{1-\alpha} f(t)\right] = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)ds}{(t-s)^\alpha} \tag{2.2}$$

Subsequently, we can determine the fractional derivative of $x^{k\alpha}$. With a little effort using change of variable and Beta function, we can easily obtain

$$D^\alpha x^{k\alpha} = \frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k-1)+1)} x^{\alpha(k-1)} \tag{2.3}$$

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2.2 Mittag-Leffler function

Following on to the discussion Mittag-Leffler function. Again from (Dumitru et al., 2012), we define the Mittag-Leffler function of one parameter $E_\alpha(z)$ for $\alpha \in (0,1]$ as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \tag{2.4}$$

An obvious deduction of $E_\alpha(Ax^\alpha)$ is as following

$$E_\alpha(Ax^\alpha) = \sum_{k=0}^{\infty} \frac{A^k x^{k\alpha}}{\Gamma(k\alpha+1)} \tag{2.5}$$

Furthermore, by the relationship of equation (2.3), we obtain

$$\begin{aligned} D^\alpha[E_\alpha(Ax^\alpha)-1] &= D^\alpha\left[\sum_{k=1}^{\infty} \frac{A^k x^{k\alpha}}{\Gamma(k\alpha+1)} + 1 - 1\right] \\ &= \sum_{k=1}^{\infty} \frac{A^k x^{\alpha(k-1)}}{\Gamma(\alpha(k-1)+1)} \\ &= A\sum_{k=1}^{\infty} \frac{A^{k-1} x^{\alpha(k-1)}}{\Gamma(\alpha(k-1)+1)} \\ &= AE_\alpha(Ax^\alpha) \end{aligned} \tag{2.6}$$

We now discuss the asymptotic expansion of Mittag-Leffler function from a review article of (Haubold, Mathai, & Saxena, 2011) originally taken from (Bateman, 1953). From equation (6.5) of (Haubold et al., 2011), we let $\alpha \in (0,1]$ and $\mu \in \mathbb{R}$ such that

$$\frac{\pi\alpha}{2} < \mu < \pi\alpha \tag{2.7}$$

Then for $N \in \mathbb{N},\ N \neq 1$, we have the asymptotic expansion as

$$E_\alpha(z) = -\sum_{k=1}^{N} \frac{1}{\Gamma(1-k\alpha)} \frac{1}{z^k} + \mathcal{O}\left(\frac{1}{|z|^{N+1}}\right), \quad |z| \to \infty, \quad \mu \leq |\arg(z)| \leq \pi \tag{2.8}$$

From Lemma B.2. and Corollary B.1. of (Gatheral & Radoicic, 2019), we have

Lemma 2.1 *For* $\alpha \in (0,1]$. *Let* $a = u + \mathrm{i}y$ *with* $\mathbb{R}_{\geq 0}$, $y \in \left[-1/(1-\rho^2), 0\right]$ *and let* $A = \sqrt{a(a+\mathrm{i}) - \rho^2 \mathrm{a}^2}$, *Then for any* $x \in \mathbb{R}_{>0}$

$$|\arg(-Ax^\alpha)| \in \left[\frac{3\pi}{4}, \pi\right] \tag{2.9}$$

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By setting $\mu = \frac{3}{4}\pi\alpha$ in equation (2.8), it follows from Lemma 2.1 that

Corollary 2.1 *For* $\alpha \in (0,1]$. *Let* $a = u + \mathrm{i}y$ *with* $\mathbb{R}_{\geq 0}$, $y \in \left[-1/(1-\rho^2), 0\right]$ *and let* $A = \sqrt{a(a+\mathrm{i}) - \rho^2 \mathrm{a}^2}$*, Then for any* $x \in \mathbb{R}_{>0}$ *and* $N \in \mathbb{N}$

$$E(-Ax^\alpha) = \sum_{k=1}^{N} \frac{(-1)^{k-1}}{A^k x^{k\alpha}\Gamma(1-k\alpha)} + \mathcal{O}\left(\frac{1}{|Ax^\alpha|^{N+1}}\right) \tag{2.10}$$

3 Fractional Adams-Bashforth-Moulton method and its error analysis

We discuss the general method of obtaining a solution for the fractional ordinary differential equations. In particular, we refer to (Dumitru et al., 2012). The original work on the error analysis belongs to (Diethelm et al., 2004). Furthermore, the full algorithmic description with pseudocode can be found in (Diethelm, Ford, Freed, & Luchko, 2005). It is also noted that this method is also known as one of broader method called *predictor corrector method* or PECE (predict-evaluate-correct evaluate). This method relies on product trapezoidal method and product rectangle method in the quadrature theory. The fractional Adams method is shown as following

$$D^\alpha h(a,x) = F(a, h(a,x)) \tag{3.1}$$

$$h_{k,0} = \sum_{j=0}^{\lceil\alpha\rceil-1} \frac{x_k^j}{j!} h_0^{(j)} + \Delta^\alpha \sum_{j=0}^{k-1} b_{jk} F(a, h(a, x_j)), \tag{3.2}$$

$$h_k = \sum_{j=0}^{\lceil\alpha\rceil-1} \frac{x_k^j}{j!} h_0^{(j)} + \Delta^\alpha \sum_{j=0}^{k-1} a_{jk} F(a, h(a, x_j)) + \Delta^\alpha a_{kk} F(a, h_{k.0}) \tag{3.3}$$

$$h(a,0) = 0 \tag{3.4}$$

with $a_{j,k}$ and $b_{j,k}$ as

$$a_{j,k} = \frac{1}{\Gamma(2+\alpha)} \times \begin{cases} ((k-1)^{1+\alpha} - (k-\alpha-1)k^\alpha) & \text{if } j = 0, \\ ((k-j+1)^{1+\alpha} + (k-j-1)^{1+\alpha} \\ \qquad - 2(k-j)^{1+\alpha}) & \text{if } 1 \leq j \leq k-1, \\ 1 & \text{if } j = k. \end{cases} \tag{3.5}$$

$$b_{j,k} = \frac{1}{\Gamma(1+\alpha)}((k-j)^\alpha - (k-1-j)^\alpha) \quad \text{if } 1 \leq j \leq k \tag{3.6}$$

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In particular, we can let $n = k$, pick T and Δ such that $n = T/\Delta$. It is noted that the complexity of this algorithm is at $O(n^2)$. In the later section, we will discuss why employing this algorithm for the rough Heston model has higher computational complexity than $O(n^2)$.

We follow similarly from (El Euch & Rosenbaum, 2019) and note that (Li & Tao, 2009) provides convergence proofs of fractional Adams method. In particular, for $x > 0$ and $a \in \mathbb{R}$

$$\max_{x_j \in [0,x]} |h_j - h(x_j)| = O(\Delta) \tag{3.7}$$

and

$$\max_{x_j \in [\varepsilon,x]} |h_j - h(x_j)| = O(\Delta^{2-\alpha}) \tag{3.8}$$

for any $\varepsilon > 0$.

4 Option Pricing Models, Implied Volatility, Characteristic Functions of the Option Pricing models

We discuss some of the financial related models and their aspects in this section. We intend to organize them as following

1. Introduce the famous Black-Scholes pricing model and discuss about implied volatility as well as its importance in the financial market.
2. Focus on one of the most favourable financial model amongst the practitioners - classical Heston model and the newly developed rough Heston model.
3. Display its characteristic functions and its connection to call option pricing formula using the inversion of characteristic function.

4.1 Black-Scholes model and Implied Volatility

The Nobel Prizes of economics were awarded to the Myron S. Scholes and Robert C. Merton in 1997 for their extreme marvellous derivation of the now known as Black-Scholes equations. Sadly, one of the authors Fisher Black passed away before receiving the prizes. In this subsection, we briefly discuss about the option pricing equation (Black & Scholes, 1973) from the Black-Scholes equation derived under some conditions and the implied volatility. In particular, Black and Scholes derived the Black-Scholes equation using two different methods, i.e. constructing a hedged portfolio and the capital asset pricing. Both of which led to the same equation. We now provide the closed-form solution for the European call option:

$$\text{BS}(S, K, \sigma, r, T) = SN(d_1) - Ke^{-rT}N(d_2) \tag{4.1}$$

$$d_1 = \frac{\ln\frac{S}{K} + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

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where S is the stock price, K is the strike price, σ is the volatility of the stock return, r is the interest rate, T is the time to maturity and $N(\cdot)$ is the cumulative normal density function. The short and concise equation (4.1) plays an important in the financial industry; literally everyone who trades stock and options would have known or heard about it. However, some of the assumptions that led to equation (4.1) are unrealistic. One of them is the assumption of constant volatility. Nevertheless, we will discuss why equation (4.1) is still popular among the practitioners.

Although no practitioners would use equation (4.1) to price their options in this era, many of them took interest on its implied volatility σ_{imp}. The implied volatility has become the standard tool to evaluate the volatility surface of a particular stock. Volatility surface is a combination of volatility skew/smile and volatility term structure, normally seen as a three-dimensional graph for comparison purposes. Let the F_m as the market price of the model or price of other option model, then

$$F_m = \text{BS}(S, K, \sigma_{imp}, r, T) \tag{4.2}$$

$$\sigma_{imp} = \text{BS}^{-1}(F_m) \tag{4.3}$$

While there is no explicit closed form solution for obtaining equation (4.3), the monotonic relationship of the Black-Scholes pricing model and the volatility σ makes the implied volatility extremely easy to be obtained. The monotonicity of σ can be seen from one of the greeks of Black-Scholes pricing model - *vega* or $\dfrac{\partial C}{\partial \sigma}$ where C is the call option price of (4.1). In particular, bisection method is one of the easy and fast way to obtain the implied volatility σ_{imp}. What practitioners normally do with the implied volatility σ_{imp} is that they compare the figures across different strike price K and maturity T in between the volatility implied by the financial market prices and the model prices. As a result, it is easy to observe whether the model is capable of fitting the empirical data and reconstruct the surfaces of the market prices.

4.2 Classical Heston model and rough Heston model

We now turn to the discussion of Heston models, both classical and rough version. Suppose from (Heston, 1993), we let some security or stock price S and its instantaneous variance $V = \sigma^2$ such that S and V follows the stochastic process:

$$\frac{dS_t}{S_t} = \sqrt{V_t}\{\rho\ dB_t + \sqrt{1-\rho^2}\ dB_t^{\perp}\} \tag{4.4}$$

$$dV_t = \lambda(\theta - V_t)dt + \lambda\nu\sqrt{V_t}dB_t \tag{4.5}$$

where $\lambda, \theta, \nu, \rho$ are the parameters of the model along with B_t and $B_t^{\perp}$ as the independent Brownian motions. From (El Euch & Rosenbaum, 2019), we discuss why the classical Heston model is popular among the practitioners.

1. The model reproduce several stylized facts of low frequency stock data, e.g. the leverage

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effect, time-varying volatility and fat tails.

2. It generates similar shapes and dynamics for the implied volatility surface.
3. Efficient computation for the classical Heston model using the explicit formula for the characteristic function of the asset log-price (we will discuss it later).

Subsequently, we discuss about the the rough volatility model shown in (El Euch & Rosenbaum, 2019), the one-dimensional asset price S is in the form of

$$\frac{dS_t}{S_t} = \sqrt{V_t}\,\{\rho\, dB_t + \sqrt{1-\rho^2}\, dB_t^{\perp}\} \tag{4.6}$$

with

$$V_u = V_t + \frac{\lambda}{\Gamma(H+\frac{1}{2})}\int_t^u \frac{\theta^t(s)-V_s}{(u-s)^{\frac{1}{2}-H}}ds + \frac{\lambda\nu}{\Gamma(H+\frac{1}{2})}\int_t^u \frac{\sqrt{V_s}}{(u-\frac{1}{2})^{\frac{1}{2}-H}}dB_s \quad u \geq t \tag{4.7}$$

where $\lambda \geq 0$, $\rho \in [-1,1]$ is the correlation between spot and volatility movement, $\Gamma(\cdot)$ is the Gamma function, ν is the volatility of the volatility and $\theta^t(s)$ is the $\mathcal{F}_t$-measurable mean reversion level that makes the model time consistent (El Euch, Rosenbaum, et al., 2018). However, as shown in (El Euch, Rosenbaum, et al., 2018), equation (4.7) can be rewritten in the form of

$$V_u = \xi_t(u) + \frac{\nu}{\Gamma(H+\frac{1}{2})}\int_t^u \frac{\sqrt{V_s}}{(u-\frac{1}{2})^{\frac{1}{2}-H}}dB_s \quad u \geq t \tag{4.8}$$

where $(\xi_t(u))_{u\geq t}$ is the forward variance curve observed at time t. This is because $\lambda\theta^t(s)$ can be inferred from the forward variance curve, therefore λ is set to 0 and rewritten as the forward variance form of (4.8). Furthermore, from (Gatheral, 2011) and (El Euch, Gatheral, & Rosenbaum, 2018), it is possible to obtain the forward variance curve by differentiating the variance swap curve. In particular, by assuming continuous sample paths, the well-known fair value variance swaps can be obtained from an infinite log-strip of out-of-money options. Note that, the absence of λ in the second term of equation (4.8) is due to a different way of expressing the notation between different authors, we will be sure to specify the appropriate form of the characteristic function in the next section accordingly.

4.3 Characteristic functions and their connection to call option pricing

In this subsection, we discuss both of the characteristic functions for classical Heston and rough Heston model. Subsequently, we show their connection to call option pricing. Although, the characteristic function was derived in (Heston, 1993), we refer to (El Euch & Rosenbaum, 2019) for an overall review for the classical Heston model. We let the characteristic function of the log-price $X_t = \log(S_t/S_0)$ as $\phi_t(a,T)$, then it is given by

$$\phi_t(a,T) = \mathbb{E}[e^{\mathrm{iaX_t}}] = \exp(g(a,t) + V_0 h(a,t)) \tag{4.9}$$

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and

$$g(a,t) = \theta\lambda \int_0^t h(a,s)ds \tag{4.10}$$

where h is the unique solution to the following Riccati equation:

$$\partial_t h(a,t) = -\frac{1}{2}a(a+i) + \lambda(\mathrm{i}a\rho\nu - 1)h(a,t) + \frac{(\theta\nu)^2}{2}h^2(a,t), \quad h(a,0) = 0 \tag{4.11}$$

As stated before, equation (4.11) has an explicit closed-form solution which can be easily obtained. From page 18 of (Gatheral, 2011), the solution is given as

$$h(a,t) = r_- \frac{1-e^{-dt}}{1-ge^{-dt}} \tag{4.12}$$

with

$$d = \sqrt{\beta^2 - 4\Omega\kappa}, \quad g = \frac{r_-}{r_+}, \qquad r_\pm = \frac{-\beta \pm \sqrt{\beta^2 - 4\Omega\kappa}}{2\gamma} \tag{4.13}$$

$$\kappa = \frac{(\theta\nu)^2}{2}, \qquad \beta = \lambda(\mathrm{i}a\rho\nu - 1), \quad \Omega = -\frac{1}{2}a(a+\mathrm{i}) \tag{4.14}$$

Eventually, the closed-form solution from equations (4.12), (4.13) and (4.14) enable us to price the call option price with little effort. The exact pricing formula using the inverse Fourier method will be discussed after the introduction on characteristic function for rough Heston model.

Surprisingly, the rough case of Heston model (we are referring to equation (4.7) first) that corresponds to the version of equation (4.11) differs with little modification to its Riccati form. The only difference is that the classical Riccati equation is replaced with a fractional Riccati equation. In a precise form, the characteristic function of the rough Heston model is as following

$$\phi_t(a,T) = \mathbb{E}[e^{\mathrm{i}aX_t}] = \exp(g_1(a,t) + V_0 g_2(a,t)) \tag{4.15}$$

and

$$g_1(a,t) = \theta\lambda \int_0^t h(a,s)ds, \quad g_2(a,t) = I^{1-\alpha}h(a,t) \tag{4.16}$$

where $\alpha = H + \frac{1}{2}$ and h is the solution for the fractional Riccati equation with the following form:

$$D^\alpha h(a,t) = \frac{1}{2}(-a^2 - \mathrm{i}a) + \lambda(\mathrm{i}a\rho\nu - 1)h(a,t) + \frac{(\lambda\nu)^2}{2}h^2(a,t), \quad I^{1-\alpha}h(a,0) = 0, \tag{4.17}$$

with the fractional derivative D^α and integral $I^{1-\alpha}$ discussed in section (2.1). Note that, when $\alpha = 1$, equation (4.17) reverts back to equation (4.11) and the rough Heston model coincides with the classical Heston model. Unfortunately, there is no known explicit closed-form solution for the fractional Riccati equation, i.e. no closed form solution when $\alpha < 1$.

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Therefore, the only available choice is to rely on numerical methods. In addition, we wish to discuss another version of rough Heston model based on the work of (El Euch, Rosenbaum, et al., 2018) which is based upon equation (4.8). We let $x^\alpha = \nu t^\alpha$ as a dimensionless quantity and from (Alos, Gatheral, & Radoičić, 2020), the characteristic function of equation (4.8) can be denoted as following

$$\phi_t(a,T) = \mathbb{E}[e^{\mathrm{i}aX_t}] = \exp\{\mathrm{i}aX_0 + \int_0^t D^\alpha h(a,T-u)\xi_t(u)du\} \tag{4.18}$$

where $h(a,\cdot)$ as the unique continuous solution of the fractional Riccati equation with the form of

$$D^\alpha h(a,x) = -\frac{1}{2}a(a+\mathrm{i}) + \mathrm{i}\rho a h(a,x) + \frac{1}{2}h^2(a,x), \quad I^{1-\alpha}h(a,0) = 0 \tag{4.19}$$

where all the parameters have the same definition as from equation (4.8). Note that, we are using $X_t = \log(S_t)$ in equations (4.18) and (4.19) as compared to $X_t = \log(S_t/S_0)$ in equations (4.9) and (4.15). From now onwards, we will focus on this second version of characteristic function for rough Heston model, i.e. equations (4.18) and (4.19).

We have now discussed the characteristic function of both the classical Heston model and rough Heston model. Now, we turn to the inversion of the characteristic function which leads to the option price, the key aspect of this work. From (Carr & Madan, 1999), (A. Lewis, 2000) and (A. L. Lewis, 2001), the call option price can be denoted in the form of

$$C(S,K,T) = Se^{-qT} - \frac{1}{\pi}\sqrt{SK}e^{-(r+q)T/2}\int_0^\infty \mathrm{Re}[e^{\mathrm{i}uk}\phi_T(u-\mathrm{i}/2)]\frac{du}{u^2+\frac{1}{4}} \tag{4.20}$$

where S is the current stock price, q is the dividend yield, T is the time to maturity, K is the strike price of the option, r is the interest rate or borrowing rate and $k = \log(\frac{K}{S})$. Now, using equation (4.20), it is possible to evaluate the call option price under the classical Heston model immediately since the explicit closed-form solution for the classical Riccati equation is available. As for the case of fractional Riccati equation in equation (4.19), the typical method to solve the equation by using the fractional Adams method as shown in section 3. The fractional Adams method has been widely employed in (El Euch & Rosenbaum, 2019; Gatheral & Radoicic, 2019; El Euch, Gatheral, & Rosenbaum, 2018).

5 Small and long time expansion of solution for the fractional Riccati equation

The short and long time expansion of the solution $h(a,x)$ with $x^\alpha = \nu t^\alpha$ are the key components for our efficient approximation method (Padé approximation method) for the solution of fractional Riccati equation which we will discuss in section 6. Therefore, we take this opportunity to discuss on the expansions in this section.

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5.1 Small time expansion on solution of fractional Riccati equation

We first discuss on the work of (Alos et al., 2020), the short-time $x \to 0$ expansion of $h(a,x)$. In particular, $h(a,x)$ can be represented as

$$h(a,x) = \sum_{j=0}^{\infty} \frac{\Gamma(1+j\alpha)}{\Gamma(1+(j+1)\alpha)} \beta_j(a) x^{(j+1)\alpha} \tag{5.1}$$

with

$$\beta_0(a) = -\frac{1}{2}a(a+\mathrm{i}) \tag{5.2}$$

$$\beta_k(a) = \frac{1}{2}\sum_{i,j=0}^{k-2} \mathbb{1}_{i+j=k-2}\beta_i(a)\beta_j(a)\frac{\Gamma(1+i\alpha)}{\Gamma(1+(i+1)\alpha)}\frac{\Gamma(1+j\alpha)}{\Gamma(1+(j+1)\alpha)} + \mathrm{i}\rho a \frac{\Gamma(1+(k-1)\alpha)}{\Gamma(1+k\alpha)}\beta_{k-1}(a) \tag{5.3}$$

Although equation (5.3) appears as a complicated set of summation, we note that only first few terms will be used in the approximation method which we will discuss in the next section. Earlier terms of equation (5.3) are fairly quite manageable and easy to obtain.

5.2 Large time expansion on solution of fractional Riccati equation

Next, we obtain the following work from (Gatheral & Radoicic, 2019). Under the expectation for $\lim_{x\to\infty} h(a,x) = r_-$ from classical Riccati solution and linearization of the fractional Riccati equation. The authors managed to derive out the large time expansion for the solution of the fractional Riccati equation. We begin by noting that equation (4.19) can be rewritten as

$$D^\alpha h(a,x) = \frac{1}{2}(h(a,x)-r_-)(h(a,x)-r_+) \tag{5.4}$$

with

$$A = \sqrt{a(a+\mathrm{i}) - \rho^2 a^2}; \qquad r_\pm = -\mathrm{i}\rho a \pm A \tag{5.5}$$

In linear form, the fractional Riccati equation is approximated as following

$$\begin{aligned} D^\alpha h(a,x) &= \frac{1}{2}(h(a,x)-r_-)(h(a,x)-r_+) \\ &\approx -\frac{1}{2}(r_+ - r_-)(h(a,x)-r_-) \\ &= -A(h(a,x)-r_-) \end{aligned} \tag{5.6}$$

The solution to equation (5.6) is of the form

$$h_\infty(a,x) = r_-[1 - E_\alpha(-Ax^\alpha)] \tag{5.7}$$

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where $E_\alpha(\cdot)$ is the Mittag-Leffler function denoted in section (2.2). Equation (5.7) is apparent from the derivation of equation (2.6). We take a proposition from (Gatheral & Radoicic, 2019) as:

Proposition 5.1 *For $x \in \mathbb{R}_{\geq 0}$ and $a \in \mathcal{A} = \{z \in \mathbb{C} : Re(z) \geq 0, -\frac{1}{1-\rho^2} \leq Im(z) \leq 0\}$. Let $h_\infty(a,x) = r_-[1 - E_\alpha(-Ax^\alpha)]$ as $x \to \infty$, then the $h_\infty(a,x)$ solves the equation (5.4) up to error term of $\mathcal{O}(|Ax^\alpha|^{-2})$.*

Proof. See proposition 3.1 in (Gatheral & Radoicic, 2019) with the help of corollary 2.1.

From equation (2.8), the authors of (Gatheral & Radoicic, 2019) were able to derive the following result:

$$h(a,x) = r_- \sum_{k=0}^{\infty} \gamma_k \frac{x^{-k\alpha}}{A^k \Gamma(1-k\alpha)} \tag{5.8}$$

as $x \to \infty$ along with $\gamma_0, \gamma_1, \gamma_2$ in the forms of

$$\gamma_1 = -\gamma_0 = -1 \tag{5.9}$$

$$\gamma_2 = 1 + \frac{r_-}{2A} \frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)^2} \tag{5.10}$$

and the general recursion formula as

$$\gamma_k = -\gamma_{k-1} + \frac{r_-}{2A} \sum_{i,j=1}^{\infty} \mathbb{1}_{i+j=k} \gamma_i \gamma_j \frac{\Gamma(1-k\alpha)}{\Gamma(1-\mathrm{i}\alpha)\Gamma(1-j\alpha)} \tag{5.11}$$

Equation (5.8) is our large time expansion for the fractional Riccati equation. We have now the necessary tools to employ the approximation method on fractional Riccati equation.

6 Multipoint Padé approximation method for fractional Riccati equation

As we have mentioned before, the main issue for the rough Heston model is at its high computational cost. Coupled with fractional Adams method, in order to compute for the option price, a computation complexity of $O(N_a n^2)$ is needed, where N_a is the number of space steps for equation (4.20) and n arises from the use of fractional Adams method noted in section 3. One of the important advancements for rough Heston model belongs to the use of multipoint Padé method for obtaining the fractional Riccati equation's solution at $O(1)$ complexity. This is a huge leap in computational aspect as from (Gatheral & Radoicic, 2019) has shown virtually that $n \approx 20,000$ would result to a satisfactory solution indifferent from the $n \approx 50,000$ case. By employing the multipoint Padé method, the computational complexity of the evaluation for option price using (4.20) reduces to only $O(N_a)$ same as the case of the classical Heston model.

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We first introduce the work of (Atkinson & Osseiran, 2011) which concerns with the global rational approximation on $h(a, x)$:

$$h^{(m,n)}(a, x) = \frac{\sum_{i=1}^{m} p_m y^m}{\sum_{j=0}^{n} q_n y^n} \tag{6.1}$$

where we make the substitution $y = x^{\alpha}$ on both equation (5.1) and (5.8). Based on the argument for (Gatheral & Radoicic, 2019), we note that only diagonal Padé approximation is admissible. This is due to when $x \to \infty$, Padé approximation should approach as following

$$h^{(m,n)}(a, x) \sim \frac{p_m}{q_n} y^{m-n} \to r_- \tag{6.2}$$

where $0 < |r_-| < \infty$ only if $m = n$. Notice that if $m \neq n$ and $x \to \infty$, $h^{m,n}(a, x)$ will either not admit constant other than zero or does not exist (involves ∞ or $-\infty$). Furthermore, equation (6.2) corresponds to an assumption we made at section 5.2. The possibilities of choosing $m = n$ equals to some constant is endless. However, the authors (Gatheral & Radoicic, 2019) have noted that $m = n = 2$ and $m = n = 4$ do not perform well for the fractional Riccati equation case. Fortunately, the case $m = n = 3$ produces excellent quality of solution for the fractional Riccati equation, at least for the $H \to 0$ solution. Therefore, we will work out the case for multipoint Padé approximation method for $m = n = 3$ in this section. We follow closely from the presentation of (Gatheral & Radoicic, 2019) on this section.

Suppose that from equation (5.1), we obtain the small time expansion in the form of

$$h_s(y) = b_1 y + b_2 y^2 + b_3 y^3 + \mathcal{O}(y^4) \tag{6.3}$$

and from equation (5.8), we have the series expansion for small time on the solution of fractional Riccati equation as

$$h_\ell(y) = w_0 + \frac{w_1}{y} + \frac{w_2}{y^2} + \mathcal{O}\left(\frac{1}{y^3}\right) \tag{6.4}$$

The Padé $(3, 3)$ approximation can be written as

$$h^{(3,3)}(y) = \frac{p_1 y + p_2 y^2 + p_3 y^3}{1 + q_1 y + q_2 y^2 + q_3 y^3} \tag{6.5}$$

with q_0 set to 1. Matching the coefficients to equations (5.1) and (5.8) using Padé approximation

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method, we have the system of equations as

$$\begin{aligned} p_1 &= b_1 \\ p_2 - p_1 q_1 &= b_2 \\ p_1 q_1^2 - p_1 q_2 - p_2 q_1 + p_3 &= b_3 \\ \frac{p_3}{q_3} &= w_0 \\ \frac{p_2 q_3 - p_3 q_2}{q_3^2} &= w_1 \\ \frac{p_1 q_3^2 - p_2 q_2 q_3 - p_3 q_1 q_3 + p_3 q_2^2}{q_3^3} &= w_2 \end{aligned} \tag{6.6}$$

Accordingly, we obtain the constants $p_1, p_2, p_3, q_1, q_2, q_3$ as

$$\begin{aligned} p_1 &= b_1 \\ p_2 &= \frac{b_1^3 w_1 + b_1^2 w_0^2 + b_1 b_2 w_0 w_1 - b_1 b_3 w_0 w_2 + b_1 b_3 w_1^2 + b_2^2 w_0 w_2 - b_2^2 w_1^2 + b_2 w_0^3}{b_1^2 w_2 + 2b_1 w_0 w_1 + b_2 w_0 w_1 + b_2 w_0 w_2 - b_2 w_1^2 + w_0^3} \\ p_3 &= \frac{w_0(b_1^3 + 2b_1 b_2 w_0 + b_1 b_3 w_1 - b_2^2 w_1 + b_3 w_0^2)}{b_1^2 + 2b_1 w_0 w_1 + b_2 w_0 w_2 - b_2 w_1^2 + w_0^3} = w_0 q_3 \\ q_1 &= \frac{b_1^2 w_1 - b_1 b_2 w_2 + b_1 w_0^2 - b_2 w_0 w_1 - b_3 w_0 w_2 + b_3 w_1^2}{b_1^2 w_2 + 2b_1 w_0 w_1 + b_2 w_0 w_2 - b_2 w_1^2 + w_0^3} \\ q_2 &= \frac{b_1^2 w_0 - b_1 b_2 w_1 - b_1 b_3 w_2 + b_2^2 w_2 + b_2 w_0^2 - b_3 w_0 w_1}{b_1^2 w_2 + 2b_1 w_0 w_1 + b_2 w_0 w_2 - b_2 w_1^2 + w_0^3} \\ q_3 &= \frac{b_1^3 + 2b_1 b_2 w_0 + b_1 b_3 w_1 - b_2^2 w_1 + b_3 w_0^2}{b_1^2 w_2 + 2b_1 w_0 w_1 + b_2 w_0 w_2 - b_2 w_1^2 + w_0^3} \end{aligned}$$

Note that, the solutions $p_1, p_2, p_3, q_1, q_2, q_3$ can be easily obtained using `solve` function in MATLAB in less than one second. Now that we have the required multipoint Padé method, it is time we test it against the general approach (fractional Adams method).

7 Numerical experiment and performances

In the previous, we have demonstrated the fractional Adams method and the multipoint Padé approximation method. Therefore, it is time that we conduct some numerical experiment on the fractional Riccati equation. In particular, we have decided to compare the result of the fractional Riccati equation using fractional Adams method and the multipoint Padé approximation on a different scale of the Hurst parameter H.

The parameters used in our numerical studies are as following unless specifically stated otherwise:

$$\nu = 0.4; \quad \rho = -0.65; \quad N = 20,000 \tag{7.1}$$

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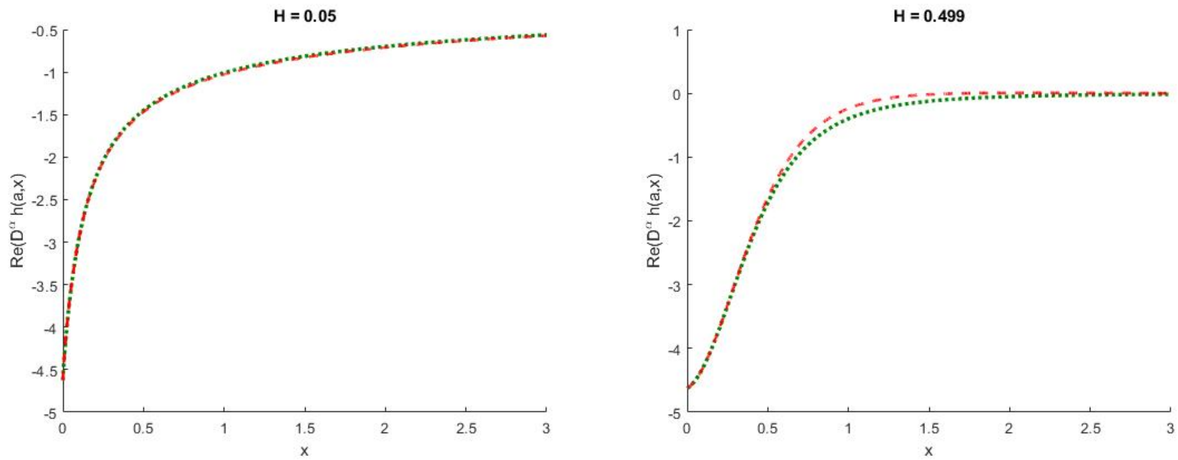

Figure 7.1: $\mathrm{Re}[D^\alpha h(3-\frac{1}{2}\mathrm{i},x)]$ for $H=0.05$ and $H=0.499$. The long red dashed line is produced using fractional Adams method with 20,000 time steps, whereas the short green dashed line is produced using the multipoint Padé method

where N belongs to the time steps of fractional Adams method. Although we have given the ν value, but note that if we use x in $x^\alpha=\nu t^\alpha$ as a substitute of ν and t for the fractional Adams method and multipoint Padé method, we will not directly be affected by the value ν is computation for the plotting of $D^\alpha h(a,x)$ and x.

For perspective, we first display the $\mathrm{Re}(D^\alpha h(a,x))$ for $H=0.05$ and $H=0.499$ in figure 7.1. Notice that, the solution for the real part of the $D^\alpha h(a,x)$ performs extremely well for small H and moderate for $H\to 0.5$. The following observation is taken from (Gatheral & Radoicic, 2019). In particular, the deterioration effect is deeply enlarged when we observe for the imaginary part of $D^\alpha h(a,x)$ as seen in figure 7.2. This in return would provide an erroneous result for the option price if not fixed. In general, the performance of multipoint Padé approximation method shouldn't be too surprised as we have matched on only two points which are $x\to 0$ and $x\to\infty$. Therefore, we will make some recommendation on the end of the summary for this particular issue.

8 Concluding Remarks

In previous sections, we have provided with literature review for the application of fractional Riccati equation in rough Heston model. Firstly, we discussed on the introductory level of the fractional calculus and Mittag-Leffler function. They are subsequently used in describing the dynamics of rough Heston model and represented as solution of fractional Riccati equation. Later, the fractional Adams-Bashforth-Moulton method and its error analysis are provided. Fractional Adams method still remain classical and frequently used method of solving any fractional ordinary differential equation. Unfortunately, fractional Adams method typically

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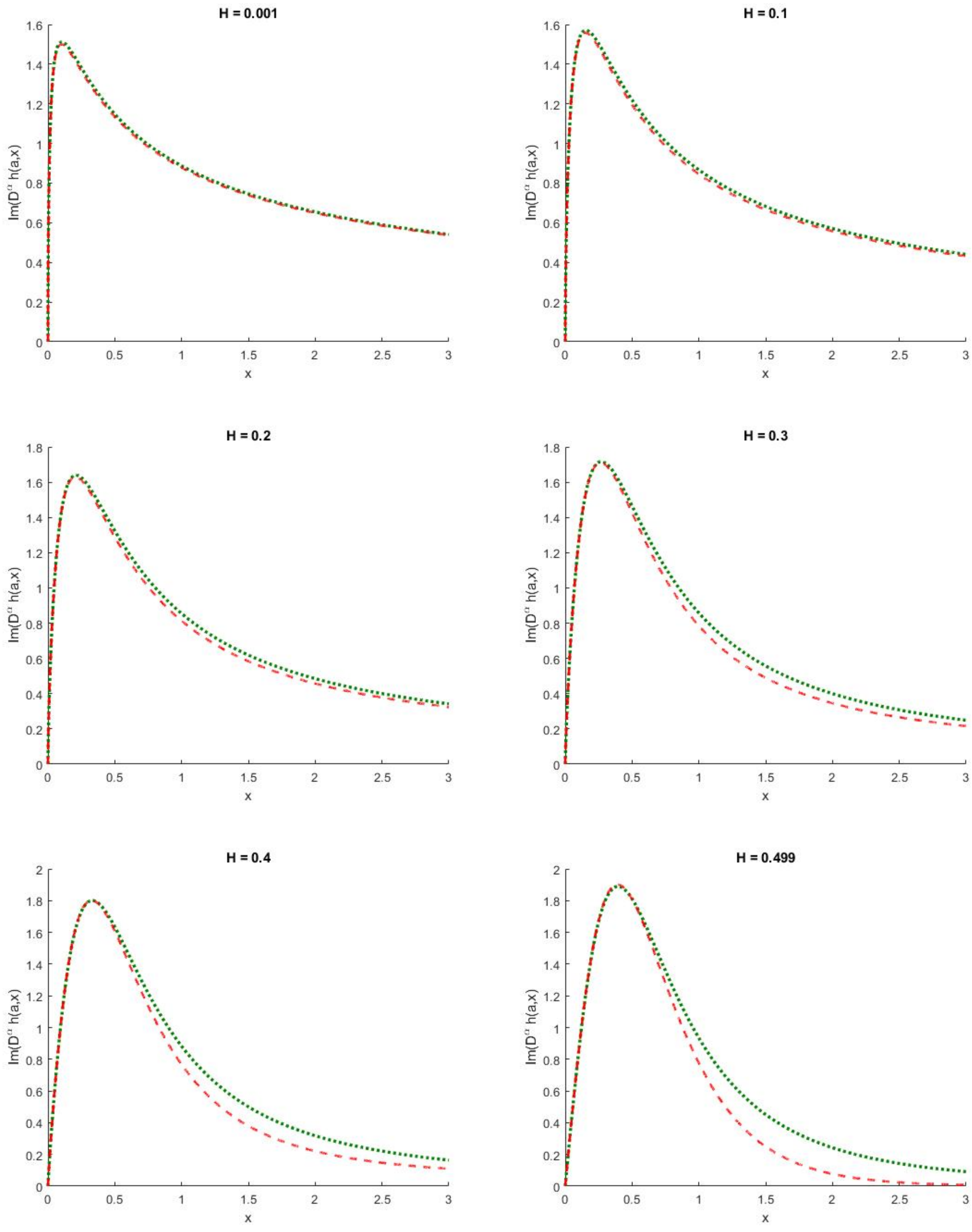

Figure 7.2: $\mathrm{Im}[D^\alpha h(3-\frac{1}{2}\mathrm{i},x)]$ for $H = 0.001, 0.1, 0.2, 0.3, 0.4, 0.499$. The long red dashed line is produced using fractional Adams method with 20,000 time steps, whereas the short green dashed line is produced using the multipoint Padé method

requires large number of time steps which in return requires large computational cost. But before that, we have introduced some recent advancements in the option pricing theory. Classical Black-Scholes option pricing models, classical Heston model and the newly rough Heston models have been discussed subsequently. In addition, we gave a slight touch on the implied volatility topic in case readers are keen on testing the models on real data. Not forgetting, the characteristic functions and their inversions to the option prices were given. Since the multipoint Padé approximation method requires some expansion on multiple points for matching purposes, we introduce some literature on the small and large time expansion of the solution of the fractional Riccati equation which is being used in the characteristic function of rough Heston model. We later then discuss the required multipoint Padé method $(3, 3)$ as suggested by some authors in a great effort of using the the small and large time expansion to obtain for the solution on fractional Riccati equation. Finally, we test the multipoint Padé method $(3, 3)$ on a setting and compare it with fractional Adams method. What we have found coincides with the previous authors such that multipoint Padé method $(3, 3)$ performed extremely well with $H \to 0$ but not $H \to 0.5$. Above all, we wish to make a minor recommendation on this issue - since the solution for the classical Riccati equation or fractional Riccati equation with $H = 0.5$ exists, in our opinion, we believe it is possible for an attempt of using a hybrid model between the existing multipoint Padé method and the exact solution for the classical Riccati equation. An attempt to match it at $H \to 0$ and $H \to 0.5$ would result in a more robust setting and applicable to altering volatility behaviour.

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Competing Interests.

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