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FIXED POINT RESULTS FOR A NEW CLASS OF MULTI-VALUED MAPPINGS UNDER $(\theta, \mathcal{R})$-CONTRACTIONS WITH AN APPLICATION

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ABSTRACT. In this article, we introduce a relatively new concept of multi-valued $(\theta, \mathcal{R})$-contractions and utilize the same to prove some fixed point results for a new class of multi-valued mappings in metric spaces endowed with an amorphous binary relation. Illustrative examples are also provided to exhibit the utility of our results proved herein. Finally, we utilize some of our results to investigate the existence and uniqueness of a positive solution for the integral equation of Volterra type.

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1. INTRODUCTION

The classical Banach contraction principle [6] continues to be the soul of metric fixed point theory which state that, every contraction mapping S defined on a complete metric space (M, ρ) has a unique fixed point. With a view to have wide range of applications, this principle has been improved, extended and generalized in many directions (e.g. [3, 7, 13, 16]) which contains several novel generalization. In the present context an effective generalization given by Jleli and Samet [13] is worth noting wherein authors introduce the idea of θ-contractions (or JS-contractions).

In 1969, Nadler [18] extended Banach contraction principle to multi-valued mappings and begun the study of fixed point theory of multi-valued contractions. For the sake of completeness, we recollect few basic notions and related results regarding multi-valued mappings.

Let (M, ρ) be a metric space and $CB(M)$ be the family of all nonempty closed and bounded subsets of M. Let $K(M)$ be the family of all nonempty compact subsets of M. Now, define $\mathcal{H} : CB(M) \times CB(M) \to \mathbb{R}$ by

$$\mathcal{H}(U, V) = max\Big\{\sup_{u\in U} D(u, V), \sup_{v\in V} D(v, U)\Big\}, \quad U, V \in CB(M).$$

Then $\mathcal{H}$ is a metric on $CB(M)$ known as Pompeiu-Hausdorff metric, where

$$D(u, V) := \inf\{\rho(u, v) : v \in V\}.$$

Let $\mathcal{P}(M)$ denotes the family of all nonempty subsets of M and $S : M \to P(M)$. An element $u \in M$ is said to be a fixed point of S if $u \in Su$ ($Fix(S)$ denotes the set of all such points).

Now, we are equipped to state Nadler's theorem as follows:

Theorem 1.1. *[18] Let (M, ρ) be a complete metric space and $S : M \to CB(M)$ a multi-valued contraction; i.e., there exists $\delta \in [0, 1)$ such that*

$$\mathcal{H}(Su, Sv) \leq \delta\rho(u, v), \quad \textit{for all } u, v \in M.$$

Then S has a fixed point.

Thereafter, vigorous studies were conducted to obtain a variety of generalizations, extensions, and applications of Theorem 1.1 (e.g. [1,8,14,17,19]). With a similar quest, Hançer et al. [9], extended the concept of θ-contractions to multi-valued mappings and prove two nice fixed point results. Furthermore, Baghani and Ramezani [5] introduced a new class of multi-valued mappings by utilizing the idea of arbitrary binary relations between two sets.

Continuing this direction of research, in this paper, we introduce a relatively new concept of multi-valued $(\theta, \mathcal{R})$-contractions and obtain some fixed point results for a new class of mappings proposed by Baghani and Ramezani [5]. Some illustrative examples are also furnished to exhibit the utility of our obtained results besides deducing some relation-theoretic existence and uniqueness results for single-valued mappings. Further, we show the applicability of our newly obtained results by investigating the existence and uniqueness of a solution for Volterra type integral equation under suitable conditions.

2. Preliminaries

We begin this section by describing some terminological and notational conventions that will be used through out the paper.

In what follows, we denote the sets of positive integers, nonnegative integers, rational numbers and real numbers by $\mathbb{N}, \mathbb{N}_0, \mathbb{Q}$ and $\mathbb{R}$ respectively.

Following [10, 13], let $\theta : (0, \infty) \to (1, \infty)$ be a function satisfying the following conditions:

(Θ_1) θ is nondecreasing;
(Θ_2) for each sequence $\{\beta_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(\beta_n) = 1 \iff \lim_{n\to\infty} \beta_n = 0^+$;
(Θ_3) there exist $\kappa \in (0, 1)$ and $\gamma \in (0, \infty]$ such that $\lim_{\beta\to 0^+} \frac{\theta(\beta)-1}{\beta^\kappa} = \gamma$;
(Θ_4) θ is continuous.

Also, We use the following notation:

- By $\Theta_{1,2,3,4}$, we denote all the functions θ satisfying $(\Theta_1) - (\Theta_4)$;
- By $\Theta_{1,2,3}$, we denote all the functions θ satisfying $(\Theta_1) - (\Theta_3)$;
- By $\Theta_{1,2,4}$, we denote all the functions θ satisfying $(\Theta_1), (\Theta_2)$ and (Θ_4);
- By $\Theta_{2,3}$, we denote all the functions θ satisfying (Θ_2) and (Θ_3);
- By $\Theta_{2,4}$, we denote all the functions θ satisfying (Θ_2) and (Θ_4);
- By Θ_2, we denote all the functions θ satisfying (Θ_2).

The following are some examples of such functions.

Example 2.1. [13] Define $\theta : (0, \infty) \to (1, \infty)$ by $\theta(\beta) = e^{\sqrt{\beta}}$, then $\theta \in \Theta_{1,2,3,4}$.

Example 2.2. [12] Define $\theta : (0, \infty) \to (1, \infty)$ by $\theta(\beta) = e^{\sqrt{\frac{\beta}{2}+sin\beta}}$, then $\theta \in \Theta_{2,3}$.

Example 2.3. [12] Define $\theta : (0, \infty) \to (1, \infty)$ by $\theta(\beta) = e^{\frac{\beta}{2}+sin\beta}$, then $\theta \in \Theta_{2,4}$.

Now, we add the following examples to this effect.

Example 2.4. Define $\theta : (0, \infty) \to (1, \infty)$ by

$$\theta(\beta) = \begin{cases} e^{\sqrt{\beta}}, & \beta \leq k, \\ e^{2(k+1)}, & \beta > k, \end{cases}$$

where k is any fixed real number greater than or equal to 1. Then $\theta \in \Theta_{1,2,3}$.

Example 2.5. Define $\theta : (0, \infty) \to (1, \infty)$ by $\theta(\beta) = e^{e^{-\frac{1}{\beta}}}$, then $\theta \in \Theta_{1,2,4}$.

For more examples one can see [11–13].

The notion of θ-contractions was introduced by Jleli and Samet [13] as follows:

Definition 2.1. [13] Let (M, ρ) be a metric space and $\theta \in \Theta_{1,2,3}$. Then $S : M \to M$ is called a θ-contraction mapping if $\exists \lambda \in (0, 1)$ such that

$$\rho(Su, Sv) > 0 \Rightarrow \theta(\rho(Su, Sv)) \leq [\theta(\rho(u, v))]^{\lambda}, \quad \text{for all } u, v \in M. \tag{2.1}$$

Considering this new concept, the authors in [13] proved the following result.

Theorem 2.1. *(Corollary 2.1 of [13]) On a complete metric space, every θ-contraction mapping has a unique fixed point.*

Imdad et al. [12] noticed that Theorem 2.1 can be proved without the assumption θ_1, from which they have introduced the notion of weak θ-contractions. Inspired by this, we deduce some relation-theoretic results (without assumption θ_1) for single-valued mappings.

On the other hand, the concept of multi-valued θ-contractions was introduced by Hançer et al. [9] as follows:

Definition 2.2. [9] Let (M, ρ) be a metric space and $S : M \to M$. Then S is said to be a multi-valued θ-contraction mapping if there exist $\lambda \in (0, 1)$ and $\theta \in \Theta_{1,2,3}$ such that

$$\mathcal{H}(Su, Sv) > 0 \Rightarrow \theta(\mathcal{H}(Su, Sv)) \leq [\theta(\rho(u, v))]^{\lambda}, \quad \text{for all } u, v \in M. \tag{2.2}$$

Utilizing the preceding definition, authors in [9] proved the following result.

Theorem 2.2. *[9] Let (M, ρ) be a complete metric space and $S : M \to K(M)$ a multi-valued θ-contraction for some $\theta \in \Theta_{1,2,3}$. Then S has a fixed point.*

Also, Hançer et al. [9] showed that one may replace $K(M)$ by $CB(M)$, by assuming the following additional condition on θ :
(θ'_4) $\theta(\inf B) = \inf \theta(B)$, $\forall B \subset (0, \infty)$ with $\inf B > 0$.
Notice that if θ satisfies (θ_1), then it satisfies $(\theta'_4) \iff \theta$ is right continuous.
Let $\Theta_{1,2,3,4'}$ be the class of all functions θ satisfying $(\theta_1), (\theta_2), (\theta_3)$ and (θ'_4).

Theorem 2.3. *[9] Let (M, ρ) be a complete metric space and $S : M \to CB(M)$ a multi-valued θ-contraction mapping for some $\theta \in \Theta_{1,2,3,4'}$. Then S has a fixed point.*

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3. Relation theoretic notions and auxiliary results

To make our paper self contained we provide the following definitions and notions. Let M be a nonempty set. A subset $\mathcal{R}$ of $M \times M$ is called a binary relation on M. Trivially, $\emptyset$ and $M \times M$ are binary relations on M known as the empty relation and the universal relation, respectively. A binary relation $\mathcal{R}$ on M is said to be transitive if $(u, v) \in \mathcal{R}$ and $(v, w) \in \mathcal{R}$ implies $(u, w) \in \mathcal{R}$, for any $u, v, w \in M$. Throughout this paper $\mathcal{R}$ stands for a nonempty binary relation. The inverse of $\mathcal{R}$ is denoted by $\mathcal{R}^{-1}$ and is defined by $\mathcal{R}^{-1} := \{(u, v) \in M \times M : (v, u) \in \mathcal{R}\}$ and $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$. The elements u and v of M are said to be $\mathcal{R}$-comparable if $(u, v) \in \mathcal{R}$ or $(v, u) \in \mathcal{R}$ and is denoted by $[u, v] \in \mathcal{R}$.

Proposition 3.1. *[3] For a binary relation $\mathcal{R}$ defined on a nonempty set M,*

$$(u, v) \in \mathcal{R}^s \Leftrightarrow [u, v] \in \mathcal{R}.$$

Definition 3.1. [3] Let $\mathcal{R}$ be a binary relation on a nonempty set M. A sequence $\{u_n\} \subseteq M$ is said to be $\mathcal{R}$-preserving if

$$(u_n, u_{n+1}) \in \mathcal{R}, \quad \forall n \in \mathbb{N}_0.$$

Definition 3.2. [2] Let (M, ρ) be a metric space and $\mathcal{R}$ a binary relation on M. Then M is said to be $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence converges to some point in M.

Remark 3.1. Every complete metric space is $\mathcal{R}$-complete, for arbitrary binary relation $\mathcal{R}$. On the other hand, under the universal relation the notion of $\mathcal{R}$-completeness coincides with the usual completeness.

Definition 3.3. [3] Let (M, ρ) be a metric space and $\mathcal{R}$ a binary relation on M. Then $\mathcal{R}$ is said to be ρ-self-closed if whenever $\mathcal{R}$-preserving sequence $\{u_n\}$ converges to u, then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $[u_{n_k}, u] \in \mathcal{R}$, $\forall k \in \mathbb{N}_0$.

Definition 3.4. [2] Let M be a nonempty set equipped with a binary relation $\mathcal{R}$. Then M is said to be locally transitive if for any (effective) $\mathcal{R}$-preserving sequence $\{u_n\} \subseteq M$ (with range $A := \{u_n : n \in \mathbb{N}_0\}$), the binary relation $\mathcal{R}|_A$ is transitive, where $\mathcal{R}|_A = \mathcal{R} \cap (A \times A)$.

Definition 3.5. [3] Let M be a nonempty set and $S : M \to M$. A binary relation $\mathcal{R}$ on M is called S-closed if for any $u, v \in M$,

$$(u, v) \in \mathcal{R} \Rightarrow (Su, Sv) \in \mathcal{R}.$$

Definition 3.6. [4] Let (M, ρ) be a metric space, $\mathcal{R}$ a binary relation on M, $S : M \to M$ and $u \in M$. We say that S is $\mathcal{R}$-continuous at u if for any $\mathcal{R}$-preserving sequence $\{u_n\} \subseteq M$ such that $u_n \xrightarrow{\rho} u$, we have $Su_n \xrightarrow{\mathcal{H}} Su$. Moreover, S is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at each point of M.

Remark 3.2. Every continuous mapping is $\mathcal{R}$-continuous, for any arbitrary binary relation $\mathcal{R}$. On the other hand, $\mathcal{R}$-continuity coincides with the usual continuity under the universal relation.

Definition 3.7. [15] For $u, v \in M$, a path of length n ($n \in \mathbb{N}$) in $\mathcal{R}$ from u to v is a finite sequence $\{u_0, u_1, u_2, ..., u_n\} \subseteq M$ such that $u_0 = u$, $u_n = v$ with $(u_i, u_{i+1}) \in \mathcal{R}$, for each $i \in \{0, 1, ..., n-1\}$.

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Definition 3.8. [4] A subset $S \subseteq M$ is called $\mathcal{R}$-connected if for each $u, v \in S$, there exists a path in $\mathcal{R}$ from u to v.

Now, we have some definitions which play a crucial role in the forthcoming sections.

Definition 3.9. [5] Let U, V be two nonempty subsets of a nonempty set M and $\mathcal{R}$ a binary relation on M. Define binary relations $\mathcal{R}_1$ and $\mathcal{R}_2$ between U and V as follows:
(i) $(U, V) \in \mathcal{R}_1$ if $(u, v) \in \mathcal{R}$, for all $u \in U$ and $v \in V$.
(ii) $(U, V) \in \mathcal{R}_2$ if for each $u \in U$, $\exists v \in V$ such that $(u, v) \in \mathcal{R}$.

Remark 3.3. Clearly, if $(U, V) \in \mathcal{R}_1$ then $(U, V) \in \mathcal{R}_2$ but the converse is not true in general.

Definition 3.10. [5] Let (M, ρ) be a metric space equipped with a binary relation $\mathcal{R}$ and $S : M \to CB(M)$. Then S is called
(i) monotone of type (I) if
$$u, v \in M,\ (u, v) \in \mathcal{R} \text{ implies that } (Su, Sv) \in \mathcal{R}_1;$$
(ii) monotone of type (II) if
$$u, v \in M,\ (u, v) \in \mathcal{R} \text{ implies that } (Su, Sv) \in \mathcal{R}_2.$$

Remark 3.4. If S is monotone of type (I) then by Remark 3.3 it is monotone of type (II), but the converse may not be true in general.

Definition 3.11. Let (M, ρ) be a metric space, $\mathcal{R}$ a binary relation on M, $S : M \to CB(M)$ and $u \in M$. We say that S is $\mathcal{R}_{\mathcal{H}}$-continuous at u if for any $\mathcal{R}$-preserving sequence $\{u_n\} \subseteq M$ such that $u_n \xrightarrow{\rho} u$, we have $Su_n \xrightarrow{\mathcal{H}} Su$ (as $n \to \infty$). Moreover, S is called $\mathcal{R}_{\mathcal{H}}$-continuous if it is $\mathcal{R}_{\mathcal{H}}$-continuous at each point of M.

4. Main Results

We begin this section by introducing the notion of multi-valued $(\theta, \mathcal{R})$-contractions as follows:

Definition 4.1. Let (M, ρ) be a metric space endowed with a binary relation $\mathcal{R}$ and $S : M \to CB(M)$. Given $\theta \in \Theta_{1,2,3}$ (or $\theta \in \Theta_{1,2,4}$), we say that S is multi-valued $(\theta, \mathcal{R})$-contraction mapping if there exists $\lambda \in (0, 1)$ such that
$$\theta(\mathcal{H}(Su, Sv)) \leq [\theta(\rho(u, v))]^{\lambda}, \quad \forall u, v \in M \text{ with } (u, v) \in \mathcal{R}^* \tag{4.1}$$
where $(u, v) \in \mathcal{R}^* := \{(u, v) \in \mathcal{R} : H(Su, Sv) > 0\}$.

Remark 4.1. Due to the symmetricity of the metrics ρ and $\mathcal{H}$ it is clear that, if equation (4.1) is satisfied for $(u, v) \in \mathcal{R}$, then it is also satisfied for $(v, u) \in \mathcal{R}$ and so for $[u, v] \in \mathcal{R}$.

Remark 4.2. Under the universal relation (in case $\theta \in \Theta_{1,2,3}$), Definition 4.1 coincides with Definition 2.2.

Now, we are in position to state and prove our first main result, which runs as follows.

Theorem 4.1. *Let (M, ρ) be a metric space endowed with a binary relation $\mathcal{R}$ and $S : M \to K(M)$. Suppose that the following conditions are fulfilled:*
(a) S is monotone of type (I);

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(b) there exists $u_0 \in M$ *such that* $u_0\mathcal{R}_2 Su_0$*;*
(c) S *is multi-valued* $(\theta, \mathcal{R})$*-contraction with* $\theta \in \Theta_{1,2,3}$*;*
(d) M *is* $\mathcal{R}$*-complete;*
(e) one of the following holds:
  (e′) S *is* $\mathcal{R}_{\mathcal{H}}$*-continuous, or*
  (e″) $\mathcal{R}$ *is* ρ*-self-closed.*

Then S *has a fixed point.*

Proof. In view of assumption (b), there exists $u_0 \in M$ such that $u_0\mathcal{R}_2 Su_0$. This implies that there exists $u_1 \in Su_0$ such that $u_0\mathcal{R}u_1$. As S is monotone of type (I), we have $Su_0\mathcal{R}_1 Su_1$. If $u_1 \in Su_1$, then u_1 is a fixed point of S and we are done. Assume that $u_1 \notin Su_1$, then $Su_0 \neq Su_1$, i.e. $\mathcal{H}(Su_0, Su_1) > 0$. Using the condition (c), we have

$$\theta(\mathcal{H}(Su_0, Su_1)) \leq [\theta(\rho(u_0, u_1))]^{\lambda}. \tag{4.2}$$

Also, we have

$$D(u_1, Su_1) \leq \mathcal{H}(Su_0, Su_1). \tag{4.3}$$

Making use of (θ_1), (4.2) and (4.3), we have

$$\theta(D(u_1, Su_1)) \leq \theta(\mathcal{H}(Su_0, Su_1)) \leq [\theta(\rho(u_0, u_1))]^{\lambda}. \tag{4.4}$$

As $u_1 \in Su_0$ and Su_1 is compact, there exists $u_2 \in Su_1$ with $(u_1, u_2) \in \mathcal{R}$ such that

$$D(u_1, Su_1) = \rho(u_1, u_2). \tag{4.5}$$

Now, from (4.4) and (4.5), we have

$$\theta(\rho(u_1, u_2)) \leq [\theta(\rho(u_1, u_0))]^{\lambda}.$$

Recursively, we obtain a sequence $\{u_n\}$ *in* M such that $u_{n+1} \in Su_n$ with $(u_n, u_{n+1}) \in \mathcal{R}$ (i.e $\{u_n\}$ is an $\mathcal{R}$-preserving sequence) and if $u_n \notin Su_n$ (for all $n \in \mathbb{N}$), then

$$\theta(\rho(u_n, u_{n+1})) \leq [\theta(\rho(u_n, u_{n-1}))]^{\lambda}, \quad \text{for all } n \in \mathbb{N}_0. \tag{4.6}$$

Otherwise, S has a fixed point. Denote $\alpha_n = \rho(u_n, u_{n+1})$, $\forall n \in \mathbb{N}_0$. Then $\alpha_n > 0$ $\forall n \in \mathbb{N}_0$. Now, in view of (4.6), we have $(\forall n \in \mathbb{N}_0)$

$$\theta(\alpha_n) \leq [\theta(\alpha_{n-1})]^{\lambda} \leq [\theta(\alpha_{n-1})]^{\lambda^2} \leq ... \leq [\theta(\alpha_0)]^{\lambda^n},$$

which yields that

$$1 < \theta(\alpha_n) \leq [\theta(\alpha_0)]^{\lambda^n}, \quad \forall n \in \mathbb{N}_0. \tag{4.7}$$

Taking $n \to \infty$ in (4.7), we obtain

$$\lim_{n\to\infty} \theta(\alpha_n) = 1,$$

which on using (θ_2) gives rise

$$\lim_{n\to\infty} \alpha_n = 0^+. \tag{4.8}$$

Using (θ_3), $\exists \kappa \in (0, 1)$ and $\gamma \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(\alpha_n) - 1}{(\alpha_n)^{\kappa}} = \gamma.$$

There are two cases depending on γ.
Case 1. when $\gamma < \infty$. Take $A = \frac{\gamma}{2}$, then by the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left|\frac{\theta(\alpha_n)-1}{(\alpha_n)^r}\right| \leq A, \quad \text{for all } n \geq n_0,$$

which implies that

$$\frac{\theta(\alpha_n)-1}{(\alpha_n)^r} \geq \gamma - A = A, \quad \text{for all } n \geq n_0,$$

yielding there by

$$n(\alpha_n)^r \leq nB[\theta(\alpha_n)-1], \ (where\ B = \frac{1}{A}) \ \text{ for all } n \geq n_0.$$

Case 2. when γ=∞. Let $A^* > 0$ be any positive real number. Then by the definition of limit, $\exists n_1 \in \mathbb{N}$ such that

$$\frac{\theta(\alpha_n)-1}{(\alpha_n)^r} \geq A^*, \quad \text{for all } n \geq n_1,$$

which yields

$$n(\alpha_n)^r \leq nB^*[\theta(\alpha_n)-1], \ (where\ B^* = \frac{1}{A^*}) \ \text{ for all } n \geq n_1.$$

Thus, in both the above cases, there exists $C > 0$ (real number) and a positive integer $n_2 \in \mathbb{N}$ (where $n_2 = max\{n_0, n_1\}$), such that

$$n(\alpha_n)^r \leq nC[\theta(\alpha_n)-1], \quad \text{for all } n \geq n_2.$$

Using (4.7), we have

$$n(\alpha_n)^r \leq nC[[\theta(\alpha_0)]^{\lambda^n} - 1].$$

Taking $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} n(\alpha_n)^r = 0.$$

Therefore, there exists $n_3 \in \mathbb{N}$ such that $n(\alpha_n)^r \leq 1$, for all $n \geq n_3$. Which amounts to say that

$$\alpha_n \leq \frac{1}{n^{\frac{1}{r}}}, \quad \text{for all } n \geq n_3.$$

Now, our aim is to show that $\{u_n\}$ is a Cauchy sequence, for this let $m, n \in \mathbb{N}$ with $m > n \geq n_2$, then we have

$$\begin{aligned} \rho(u_n, u_m) &\leq \rho(u_n, u_{n+1}) + \rho(u_{n+1}, u_{n+2}) + ... + \rho(u_{m-1}, u_m) \\ &= \sum_{j=n}^{m-1} \alpha_j \leq \sum_{j=n}^{\infty} \alpha_j \leq \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{r}}}. \end{aligned}$$

As $\sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{r}}} < \infty$, we get

$$\lim_{n,m\to\infty} \rho(u_n, u_m) = 0.$$

Thus, the sequence $\{u_n\}$ is $\mathcal{R}$-preserving Cauchy sequence in (M, ρ). By the condition (d), M is $\mathcal{R}$-complete, then there exists $u^* \in M$ such that $\lim_{n\to\infty} u_n = u^*$. Now, in view of the condition (e), we have two alternative cases. Firstly, if (e') holds, then

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due to $\mathcal{R}_{\mathcal{H}}$-continuity of S, we must have $\mathcal{H}(Su_n, Su^*) \to 0$ as $n \to \infty$. Now, as $u_{n+1} \in Su_n, \forall n \in \mathbb{N}_0$, we get

$$0 \leq D(u_{n+1}, Su^*) \leq \mathcal{H}(Su_n, Su^*), \forall n \in \mathbb{N}_0$$

which implies that

$$0 \leq \lim_{n\to\infty} D(u_{n+1}, Su^*) \leq \lim_{n\to\infty} \mathcal{H}(Su_n, Su^*) = 0.$$

That is, $\lim_{n\to\infty} D(u_{n+1}, Su^*) = 0$. From which we obtain $u_{n+1} \in \overline{Su^*}$ (as $n \to \infty$). Since Su^* is closed and $u_{n+1} \to u^*$(as $n \to \infty$) then $u^* \in Su^*$. Hence, S has a fixed point.

Secondly, assume that the condition (e'') holds. Then by Definition 3.3, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $[u_{n_k}, u] \in \mathcal{R}$, $\forall k \in \mathbb{N}_0$. Also, from (θ_1) and (4.1), we have

$$\mathcal{H}(Su, Sv) < \rho(u, v), \quad \forall u, v \in M \text{ with } (u, v) \in \mathcal{R}^*$$

Now, using the condition (c), we obtain

$$D(u_{n_k+1}, Su^*) \leq \mathcal{H}(Su_{n_k}, Su^*) \leq \rho(u_{n_k}, u^*), \ \forall k \in \mathbb{N}_0.$$

Taking limit as $n \to \infty$, we have $D(u^*, Su^*) = 0$, which implies that $u^* \in \overline{Su^*} = Su^*$ (as Su^* is closed). Thus, u^* is a fixed point of S. This finishes the proof. □

Remark 4.3. The following question naturally arises: Can we replace $K(M)$ by $CB(M)$ in Theorem 4.1? The answer to this question is No. The following example substantiates the answer.

Example 4.1. Let $M = [0, 2]$ and define a metric ρ on M by (for all $u, v \in M$)

$$\rho(u, v) = \begin{cases} 0, & u = v \\ \mu + |u - v|, & u \neq v, \end{cases}$$

where μ be any fixed real number such that $\mu \geq 1$. Define a binary relation $\mathcal{R}$ on M as follows:

$$\mathcal{R} := \{(u, v) \in \mathcal{R} \Leftrightarrow \{u, v\} \cap \mathbb{Q} \text{ is singeltion, for all } \ u, v \in M\}.$$

Then M is $\mathcal{R}$-complete and $\mathcal{R}$ is d-self closed. Also, (M, ρ) is bounded metric space. All subsets of M are closed as τ_ρ generates discrete topology. Define a mapping $S : M \to CB(M)$ by

$$Su = \begin{cases} \mathbb{Q}_M, & u \in M \setminus \mathbb{Q}_M, \\ M \setminus \mathbb{Q}_M, & u \in \mathbb{Q}_M, \end{cases}$$

where $\mathbb{Q}_M = \mathbb{Q} \cap M$. Then S is not compact valued. Now, define $\theta : (0, \infty) \to (1, \infty)$ by

$$\theta(\beta) = \begin{cases} e^{\sqrt{\beta}}, & \beta \leq \mu, \\ e^{2(\mu+1)}, & \beta > \mu. \end{cases}$$

Clearly $\theta \in \Theta_{1,2,3}$ and does not satisfy Θ_4. Next, we will show that

$$\theta(\mathcal{H}(Su, Sv)) \leq [\theta(\rho(u, v))]^{1/2}, \forall u, v \in M \ with \ (u, v) \in \mathcal{R}^*$$

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. Observe that (for all $(u,v)\in\mathcal{R}$)

$$\begin{aligned}&\mathcal{H}(Su,Sv)=\mu \text{ and } \rho(u,v)=\mu+|u-v|>\mu\\ \Rightarrow\ &\theta(\mathcal{H}(Su,Sv))=e^{\sqrt{\mu}} \text{ and } [\theta(\rho(u,v))]^{1/2}=e^{(\mu+1)}\\ \Rightarrow\ &\theta(\mathcal{H}(Su,Sv))\le[\theta(\rho(u,v))]^{1/2}.\end{aligned}$$

Therefore, S is a multi-valued $(\theta,\mathcal{R})$-contraction with $\theta\in\Theta_{1,2,3}$. Hence, all the conditions of Theorem 4.1 are satisfied but still S has no fixed point.

Next, we present the following result employing the relatively larger class $CB(M)$ instead of $K(M)$.

Theorem 4.2. *Let (M,ρ) be a complete metric space endowed with a locally transitive binary relation $\mathcal{R}$ and $S:M\to CB(M)$. Suppose that the following conditions are fulfilled:*
(a) S is monotone of type (I);
(b) there exists $u_0\in M$ such that $(\{u_0\},Su_0)\in\mathcal{R}_2$;
(c) S is multi-valued $(\theta,\mathcal{R})$-contraction with $\theta\in\Theta_{1,2,4}$;
(d) M is $\mathcal{R}$-complete;
(e) one of the following holds:
(e') either S is $\mathcal{R}_{\mathcal{H}}$-continuous, or
(e'') $\mathcal{R}$ is ρ-self-closed.
Then S has a fixed point.

Proof. In view of assumption (b), there exists $u_0\in M$ such that $(\{u_0\},Su_0)\in\mathcal{R}_2$. This implies that there exists $u_1\in Su_0$ such that $(u_0,u_1)\in\mathcal{R}$. As S is monotone of type (I), we have $(Su_0,Su_1)\in\mathcal{R}_1$. Now, if $u_1\in Su_1$, then u_1 is a fixed point of S and the proof is completed. Assume that $u_1\notin Su_1$, then $Su_0\ne Su_1, i.e., \mathcal{H}(Su_0,Su_1)>0$. Now, making use of the condition (c), we have

$$\theta(\mathcal{H}(Su_0,Su_1))\le[\theta(\rho(u_0,u_1))]^{\lambda}. \tag{4.9}$$

Also, we have

$$D(u_1,Su_1)\le\mathcal{H}(Su_0,Su_1).$$

Using (θ_1) and (4.9), we obtain

$$\theta(D(u_1,Su_1))\le\theta(\mathcal{H}(Su_0,Su_1))\le[\theta(\rho(u_0,u_1))]^{\lambda}. \tag{4.10}$$

Due to (θ_4), we have

$$\theta(D(u_1,Su_1))=\inf_{v\in Su_1}\theta(\rho(u_1,v)).$$

This together with (4.10) give rise

$$\inf_{v\in Su_1}\theta(\rho(u_1,v))\le[\theta(\rho(u_0,u_1))]^{\lambda}<[\theta(\rho(u_0,u_1))]^{\lambda_1}, \tag{4.11}$$

where $\lambda_1\in(c,1)$. From (4.11), there exists $u_2\in Su_1$ with $(u_1,u_2)\in\mathcal{R}$ such that

$$\theta(\rho(u_1,u_2))\le[\theta(\rho(u_0,u_1))]^{\lambda_1}.$$

Again if $u_2\in Su_2$, then we are done. Otherwise, by the same way we can find $u_3\in Su_2$ with $(u_2,u_3)\in\mathcal{R}$ such that

$$\theta(\rho(u_2,u_3))\le[\theta(\rho(u_1,u_2))]^{\lambda_1}.$$

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Continuing this process, we construct a sequence $\{u_n\}$ in M such that $u_{n+1} \in Su_n$ with $(u_n, u_{n+1}) \in \mathcal{R}$ and if $u_n \notin Su_n$, then

$$\theta(\rho(u_n, u_{n+1})) \leq [\theta(\rho(u_{n-1}, u_n))]^{\lambda_1}, \quad \text{for all } n \in \mathbb{N}. \tag{4.12}$$

Otherwise, u_n is a fixed point of S. Denote $\alpha_n = \rho(u_n, u_{n+1})$, for all $n \in \mathbb{N}_0$. Then $\alpha_n > 0$, for all $n \in \mathbb{N}_0$. Now, in view of (4.12), we have

$$\theta(\alpha_n) \leq [\theta(\alpha_{n-1})]^{\lambda_1} \leq [\theta(\alpha_{n-1})]^{\lambda_1^2} \leq ... \leq [\theta(\alpha_0)]^{\lambda_1^n},$$

which implise that

$$1 < \theta(\alpha_n) \leq [\theta(\alpha_0)]^{\lambda_1^n}, \quad \text{for all } n \in \mathbb{N}_0. \tag{4.13}$$

Letting $n \to \infty$ in (4.13), we obtain

$$\lim_{n\to\infty} \theta(\alpha_n) = 1. \tag{4.14}$$

This together with (θ_2), give rises $\lim_{n\to\infty} \alpha_n = 0^+$, that is

$$\lim_{n\to\infty} \rho(u_n, u_{n+1}) = 0. \tag{4.15}$$

Now, we show that $\{u_n\}$ is a Cauchy sequence. Let on contrary $\{u_n\}$ is not Cauchy, then there exists an $\epsilon > 0$ and two subsequences $\{u_{n(k)}\}$ and $\{u_{m(k)}\}$ of $\{u_n\}$ such that

$$k \leq n(k) < m(k), \quad \rho(u_{m(k)-1}, u_{n(k)}) < \epsilon \leq \rho(u_{m(k)}, u_{n(k)}) \quad \forall\ k \geq 0, \tag{4.16}$$

and

$$\lim_{k\to\infty} \rho(u_{m(k)}, u_{n(k)}) = \epsilon. \tag{4.17}$$

Now, observe that

$$\begin{aligned} \epsilon &\leq \rho(u_{m(k)}, u_{n(k)}) \\ &\leq \rho(u_{m(k)}, u_{(m(k)-1)}) + \rho(u_{m(k)-1} + u_{n(k)-1}) + \rho(u_{n(k)-1}, u_{n(k)}) \\ &\leq \rho(u_{m(k)}, u_{(m(k)-1)}) + \rho(u_{m(k)-1} + u_{n(k)}) + 2\rho(u_{n(k)-1}, u_{n(k)}). \end{aligned}$$

Making use of (4.15),(4.16),(4.17) and letting $k \to \infty$, we have

$$\lim_{k\to\infty} \rho(u_{m(k)-1}, u_{n(k)-1}) = \epsilon. \tag{4.18}$$

which implies that there exists $n_0 \in \mathbb{N}_0$ such that $\rho(u_{m(k)}, u_{n(k)}) > 0$ for all $k \geq n_0$ (due to (4.17)). Since $\mathcal{R}$ is locally transitive, we have $(u_{n(k)-1}, u_{m(k)-1}) \in \mathcal{R}$ (as $n(k) - 1 < m(k) - 1$). Using the condition (c), we have (for all $k \geq n_0$)

$$\theta(\rho(u_{n(k)}, u_{m(k)})) \leq \theta(\mathcal{H}(Su_{n(k)-1}, Su_{m(k)-1})) \leq [\theta(\rho(u_{n(k)-1}, u_{m(k)-1}))]^{\lambda}. \tag{4.19}$$

Letting $k \to \infty$ in (4.19) and making use of $\Theta4$, (4.17) and (4.18), we obtain $\theta(\epsilon) \leq \theta(\epsilon)^{\lambda}$, which is a contradiction. Thus, $\{u_n\}$ is an $\mathcal{R}$-preserving Cauchy sequence. The rest of the proof follows same lines as in the proof of Theorem 2.1. □

Now, we give the following example which exhibits the utility of our results.

Example 4.2. Let $M = (0, \infty)$ equipped with the usual metric ρ. Define a sequence $\{\sigma_n\}$ in M by

$$\sigma_n = 1 + 2 + ... + n = \frac{n(n+1)}{2}, \quad \text{for all } n \in \mathbb{N}.$$

Now, consider a binary relation $\mathcal{R}$ on M as follows:

$$\mathcal{R} := \{(\sigma_1, \sigma_1), (\sigma_i, \sigma_j) : for\ 1 \leq i < j,\ where\ i, j \in \mathbb{N}\}.$$

Then, it is obvious that $\mathcal{R}$ is locally transitive and d-self-closed. Also M is $\mathcal{R}$-complete. Now, define a mapping $S : M \to CB(M)$ by

$$Su = \begin{cases} \{u\}, & if\ 0 < u \le \sigma_1; \\ \{\sigma_1\}, & if\ \sigma_1 \le u \le \sigma_2; \\ \{\sigma_1, \sigma_i + \left(\frac{\sigma_{i+1}-\sigma_i}{\sigma_{i+2}-\sigma_{i+1}}\right)(u-\sigma_{i+1})\}, & if\ \sigma_{i+1} \le u \le \sigma_{i+2}, \ \ i = 1, 2... \ . \end{cases}$$

Clearly, S is a monotone mapping of type (I) and $\{\sigma_1\}\mathcal{R}_2 S\sigma_1$. Now, observe that

$$\sigma_i \mathcal{R} \sigma_j, S\sigma_i \neq S\sigma_j \Leftrightarrow \ (i \ge 1, j > 3).$$

Let the function $\theta : (0, \infty) \to (1, \infty)$ be defined by

$$\theta(\beta) := e^{\sqrt{\beta e^{\beta}}}, \quad \forall\ \beta > 0.$$

Then $\theta \in \Theta_{1,2,3,4}$. Now, we show that S satisfies (4.1), that is

$$\mathcal{H}(S\sigma_i, S\sigma_j) \neq 0 \Rightarrow e^{\sqrt{\mathcal{H}(S\sigma_i,S\sigma_j)e^{\mathcal{H}(S\sigma_i,S\sigma_j)}}} \le e^{\lambda\sqrt{\rho(\sigma_i,\sigma_j)e^{\rho(\sigma_i,\sigma_j)}}}, \quad \text{for some } \lambda \in (0,1),$$

or

$$\mathcal{H}(S\sigma_i, S\sigma_j) \neq 0 \Rightarrow \frac{\mathcal{H}(S\sigma_i, S\sigma_j)e^{\mathcal{H}(S\sigma_i,S\sigma_j)-\rho(\sigma_i,\sigma_j)}}{\rho(\sigma_i, \sigma_j)} \le \lambda^2, \quad \text{for some } \lambda \in (0,1). \tag{4.20}$$

Now, consider two cases as follows:

Case 1. when $i = 1$ *or* 2, and $j > 3$. In this case, we get

$$\begin{aligned} \frac{\mathcal{H}(S\sigma_1, S\sigma_j)e^{\mathcal{H}(S\sigma_1,S\sigma_j)-\rho(\sigma_1,\sigma_j)}}{\rho(\sigma_1, \sigma_j)} &= \frac{j^2-j-2}{j^2+j-2}e^{-j} \\ &\le e^{-1}. \end{aligned} \tag{4.21}$$

Case 2. when $j > i > 2$, we have

$$\begin{aligned} \frac{\mathcal{H}(S\sigma_i, S\sigma_j)e^{\mathcal{H}(S\sigma_i,S\sigma_j)-\rho(\sigma_i,\sigma_j)}}{\rho(\sigma_i, \sigma_j)} &= \frac{j+i-1}{j+i+1}e^{i-j} \\ &\le e^{-1}. \end{aligned} \tag{4.22}$$

Therefore, the inequality (4.20) is satisfied with $\lambda = e^{-1/2}$. Hence, all the requirement of Theorem 4.1 (also Theorem 4.2) are fulfilled ($Fix(S) = (0, \sigma_1]$).

Remark 4.4. Observe that the results due to Hançer et al. [9] are not usable in the context of Example 4.2 as S does not satisfy equation (2.2) on $(0, \sigma_1]$ and also the underlying space is incomplete.

By putting $Su = \{Su\}$ (for all $u \in M$), every single valued map can be treated as a multi-valued map. Therefore using Theorems 4.1 and 4.2, we deduce two fixed point results for single valued mappings as follows:

Theorem 4.3. *Let (M, ρ) be a metric space endowed with a binary relation $\mathcal{R}$ and $S : M \to M$. Suppose the following conditions are fulfilled:*

(a) $\mathcal{R}$ is S-closed;
(b) $\exists u_0 \in M$ such that $(u_0, Su_0) \in \mathcal{R}$;
(c) S is $(\theta, \mathcal{R})$-contraction with $\theta \in \Theta_{2,3}$;
(d) M is $\mathcal{R}$-complete;
(e) one of the following holds:

(e') *S is continuous, or*
(e'') *$\mathcal{R}$ is ρ-self-closed.*

Then S has a fixed point.

Theorem 4.4. *Let (M, ρ) be a complete metric space endowed with a locally transitive binary relation $\mathcal{R}$ and $S : M \to M$. Suppose the following conditions are fulfilled:*

(a) $\mathcal{R}$ is S-closed;
(b) $\exists u_0 \in M$ such that $(u_0, Su_0) \in \mathcal{R}$;
(c) S is $(\theta, \mathcal{R})$-contraction with $\theta \in \Theta_{2,4}$;
(d) M is $\mathcal{R}$-complete;
(e) one of the following holds:
(e') *S is continuous, or*
(e'') *$\mathcal{R}$ is ρ-self-closed.*

Then S has a fixed point.

Remark 4.5. The monotonicity assumption on θ (namely θ_1) can be removed in the context of single-valued mappings and hence it is omitted in Theorems 4.3 and 4.4.

Next, we obtain a corresponding uniqueness result in this sequel as follows.

Theorem 4.5. *Besides the assumptions of Theorem 4.3 (or Theorem 4.4), if $Fix(S)$ is $\mathcal{R}^s$-connected, then the fixed point of S is unique.*

Proof. On contrary, let us suppose that $u, v \in Fix(S)$ such that $u \neq v$. Then we construct a path of some finite length n from u to v in $\mathcal{R}^s$ say $\{u = u_0, u_1, u_2, ..., u_n = v\} \subseteq Fix(S)$ (where $u_i \neq u_{i+1}$ for each i, $(0 \leq i \leq n-1)$, otherwise $u = v$, a contradiction) with $[u_i, u_{i+1}] \in \mathcal{R}$ for each i, $(0 \leq i \leq n-1)$. As $u_i \in Fix(S)$, then $Su_i = u_i$, for each $i \in \{0, 1, 2, ..., n\}$. By using the fact that S is $(\theta, \mathcal{R})$-contraction, we have (for all i, $(0 \leq i \leq n-1)$)

$$\theta(\rho(u_i, u_{i+1})) = \theta(\rho(Su_i, Su_{i+1})) \leq [\theta(\rho(u_i, u_{i+1}))]^{\lambda}, \ \textit{where}\ \lambda \in (0, 1),$$

a contradiction. This concludes the proof. □

Remark 4.6. If we take $\theta(\beta) = e^{\sqrt{\beta}}$ $(\theta \in \Theta_{2,3})$, then Theorem 4.5 is a sharpened version of the main result due to Alam and Imdad [3].

5. Application to Integral Equation

In this section, we show the applicability of our newly obtained by proving the existence and uniqueness of a positive solution for the integral equation of Volterra type as follows:

$$u(t) = \int_0^t g(t, r, u(r))dr + \beta(t), \quad \forall t \in I = [0, 1], \tag{5.1}$$

where $g : I \times I \times \mathbb{R} \to \mathbb{R}$ be an integrable function and $\beta : I \to [1, \infty)$ is a given function.

Consider $M = \{u \in C(I, \mathbb{R}) : u(t) > 0, \ for\ all\ t \in I\}$, where $C(I, \mathbb{R})$ is the space of all continuous functions $u : I \to \mathbb{R}$ equipped with the Bielecki's norm

$$\|u\| = \sup_{t \in I}\ e^{-t}|u(t)|.$$

Define a metric ρ on M by $\rho(u,v) = \|u-v\|$, for all $u,v \in M$. Then (M,ρ) is a metric space which is not complete.

Now, we are equipped to state and prove our result of the section which runs as follows:

Theorem 5.1. *Assume that the following conditions are satisfied:*

(a_1) $g(t_1, r_1, u) \geq 0$, *for all* $u \geq 0$ *and* $t_1, r_1 \in I$,

(a_2) g *is non-decreasing in the third variable and there exists* $h > 0$ *such that*

$$|g(t,r,u) - g(t,r,v)| \leq \frac{|u(t)-v(t)|}{h\|u-v\|+1},$$

$$\forall\ t, r \in I \text{ and } u, v \geq 0 \text{ with } uv \geq (u \vee v), \text{ where } u \vee v = u \text{ or } v.$$

Then the integral equation (5.1) *has a positive solution.*

Proof. Let us define a binary relation $\mathcal{R}$ on M as follows:

$$\mathcal{R} := \{u\mathcal{R}v \Leftrightarrow u(t)v(t) \geq (u(t) \vee v(t)), \quad \text{for all } t \in I\}.$$

Since $C(I,\mathcal{R})$ is a Banach space with Bielecki's norm then for any $\mathcal{R}$-preserving Cauchy sequence $\{u_n\}$ in M, it converges to some point $u \in C(I,\mathbb{R})$. Now, fix $t \in I$, then by the definition of $\mathcal{R}$, we have

$$u_n(t)u_{n+1}(t) \geq (u_n(t) \vee u_{n+1}(t)), \text{ for all } n \in \mathbb{N}.$$

As $u_n(t) > 0$, $\forall\ n \in \mathbb{N}$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k}(t) \geq 1$, $\forall k \in \mathbb{N}$. This subsequence $\{u_{n_k}(t)\}$ of real numbers converges to $u(t)$ which gives rise $u(t) \geq 1$. As $t \in I$ is arbitrary, we have $u \geq 1$ and consequently $u \in M$. Therefore, (M,ρ) is $\mathcal{R}$-complete. In a similar fashion, one may prove that $\mathcal{R}$ is ρ-self-closed. Consider a mapping $S : M \to M$ defined by

$$Su(t) = \int_0^t g(t,r,u(r))dr + \beta(t), \quad u \in C(I,\mathbb{R}).$$

Clearly, the solutions of (5.1) are nothing but fixed points of S.
Now, for all $u,v \in M$ with $(u,v) \in \mathcal{R}$ and $t \in I$, we have

$$\begin{aligned} &Su(t) = \int_0^t g(t,r,u(r))dr + \beta(t) \geq 1 \\ &\Rightarrow Su(t)Sv(t) \geq Su(t) \end{aligned}$$

so that by the definition of $\mathcal{R}$, we have $(Su,Sv) \in \mathcal{R}$, *i.e.*, $\mathcal{R}$ is S-closed. By the definition of $\mathcal{R}$, it is clear that $\mathcal{R}$ is also locally transitive. Furthermore, for any $u \in M$, $(u,Su) \in \mathcal{R}$.

Next, for all $u, v \in M$ with $(u, v) \in \mathcal{R}$ and $t \in I$, consider

$$\begin{aligned} |Su(t) - Sv(t)| &= \left| \int_0^t (g(t, r, u(r)) - g(t, r, v(r)))dr \right| \\ &\leq \int_0^t |(g(t, r, u(r)) - g(t, r, v(r)))|\, dr \\ &\leq \int_0^t \frac{1}{h\|u - v\| + 1}(|u - v|e^{-t})e^t dr \\ &\leq \frac{1}{h\|u - v\| + 1} \int_0^t \|u - v\|e^t dr \\ &\leq \frac{\|u - v\|}{h\|u - v\| + 1} e^t. \end{aligned}$$

Thus, we obtain

$$|Su(t) - Sv(t)|e^{-t} \leq \frac{\|u - v\|}{h\|u - v\| + 1}, \ \forall t \in I.$$

Taking supremum over both the sides, we have

$$\|Su - Sv\| \leq \frac{\|u - v\|}{h\|u - v\| + 1},$$

or

$$\frac{-1}{\|Su - Sv\|} \leq \frac{-1}{\|u - v\|} - h,$$

or

$$\frac{-1}{\rho(Su, Sv)} \leq \frac{-1}{\rho(u, v)} - h.$$

Now, define $\theta : (0, \infty) \to (1, \infty)$ by $\theta(\beta) = e^{e^{-\frac{1}{\beta}}}$, then $\theta \in \Theta_{1,2,4}$. Also S satisfies (4.1) with this θ (and $\lambda = e^{-h}, h > 0$). Therefore, all the requirement of Theorem 4.4 are fulfilled. Consequently, S has a fixed point. □

Next, we obtain a corresponding uniqueness result of Theorem 5.1 as follows.

Theorem 5.2. *Besides the assumptions of Theorem 5.1, if $Fix(S) \subseteq \{u \in M : u(t) \geq 1, \ \forall t \in I\}$, then the solution of the integral equation* (5.1) *is unique.*

Proof. Due to Theorem 5.1, the set $Fix(S)$ is nonempty. Now, if $Fix(S) \subseteq \{u \in M : u(t) \geq 1, \ \forall t \in I\}$, then by the definition of $\mathcal{R}$, we have $Fix(S)$ is $\mathcal{R}^s$-connected. Hence, Theorem 4.5, ensures that $Fix(S)$ is singleton set. Thus, the solution of the integral equation (5.1) is unique. This establishes our result.

□

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