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A Dissertation (1^{st} draft)

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On Bioquantum State Systems,

The Study of Algebraic Ecology and Macrocellular Biology

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SUMMARY

This dissertation is a rigorous study of ecology and macrocellular biology as a subfield of abstract algebra. We begin with the creation of an axiomatic paradigm, then move onto constructing a universal genetic code of biology. We use this to define increasingly complex algebraic structures (ecosystem, evolving populations, etc.). We prove a variety of theorems regarding to the members of the previous mathematical constructs, notably the following three:

1. There is one unique phenotypic representation of each organism. For example, if you subdivide any piece of genetic code into its phenotypic components, then two identical organisms have identical decomposed DNA.
2. There are a finite number of indivisible phenotypic traits.
3. The three sophioid-definitions are equivalent: (a) dynamical evolutionary enlargement of the medial temporal lobe and frontal lobe, (b) reliance upon intelligence, (c) the existence of an intellectually- or socially-hierarchical society.

Much has yet to be done on this work, but as a first draft, it stands as a jumping point for future exploits; I am working on an entirely revised second draft.

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1 Axiomatic Framing

1.1 Preamble

The study of this paper is algebraic biology. We differentiate this from computational biology, which aims to use statistical analysis or modeling to reduce biology to a mathematically comprehensible problem, instead defining it as a branch of abstract algebra that translates biology to a number system of sorts, specifically, one that lacks numbers. This algebra contains operations that take biological objects – ecosystems, individuals, etc. – as parameters.

This approach mirrors that of Euclid's influential *Elements* as it does of Claude Elwood Shannon's Ph.D. dissertation (in which he creates an algebra for genetics). In general, this paper attempts to redress the dearth of true biology-mathematical crossover. It seems that the only way to take biology out of its seat as a "soft science" is to reduce it, to remove variables and simplify the arbitrarily complex into mathematical models that supposedly indicate the simplicity of biology. However, my approach bucks this trend in that it creates a language with which biological concepts can be fully represented without compression or loss of information. The advantage to this is the ability to easily translate between standard biological language and mathematical logic. That is, after a result has been algebraically verified or discovered, the real-life application is immediately obvious. It suffices to say this interdisciplinary approach has unending potential.

That being said, this approach to algebraic biology, these axioms and definitions, are my own, and inherently flawed. An algebraic investigation into astrobiology might require different axioms. For this reason, I call my approach bioq, short for bioquantum state systems analysis, for reasons that will become apparent shortly.

Additionally, I intend to explore primarily macrocellular biology and ecology since my axioms are assumptions of what we already know of subcellular processes. In turn, we save ourselves from assuming that nature of astrobiological organisms is inherently cellular. With this said, we may proceed.

1.2 Axioms & Notation

The process of creating a subbranch of abstract algebra should not be unfamiliar to the reader. We create operations, create definitions with respect to these operations, and derive theorems and results from our definitions. The first operation is combination. Combination is the algebraic operation that mirrors geographic proximity. If a and b are combined, then we imagine a and b sitting close enough to each other that they affect each other on the biological scale. This is vague, but the vagueness in unimportant. When translating from biological language to a mathematical context, and *vice versa*, the wary algebraic biologist must apply and utilise combination – and any algebraic biology concept – at their own discretion. For now, it is enough to say that ab is the combination of a and b. We say that $ab = a$ if and only if b is the zero of this operation, which we notate as ∇.

Definition 1.1 *Combination of a and b is the conditionally*[a] *associative, conditionally commutative, unconditionally invertible operation ab such that $ab = a$ if and only if $b = \nabla$ where ∇ is the zero of this arbitrary operation. Combinatorial inverses are notated as $\bar{a}$ so that $a\bar{a} = \bar{a}a = \nabla$. Intuitively, combinatorial inverses should not exist as they violate the second law of thermodynamics (conservation of energy and matter). However, this reasoning only stands with the assumption that ∇ represents empty space when in fact it represents an inert matter. $a\bar{a} = \nabla$ says only that every substance a has a paired substance $\bar{a}$ that renders inert both a and $\bar{a}$. This inert matter constitutes ∇. Additionally, an pivotal operation without an inverse makes for an appalling mathematical scheme. Repetitive combination is marked as*

$$a^n = \underbrace{a \cdots a}_{n\ times}.$$

The proof of unique inverses (and identities) are classic problems in abstract algebra, and are likely covered in even the most cursory linear algebra class. Hence we will not cover it here. We borrow the standard notation that

$$x_0 x_1 \cdots x_n = \sum_{i=0} x_i$$

With

$$x_0 x_1 \cdots x_k \cdots = \sum_{i=0}^{\infty} x_i.$$

We have a final definition that will become invaluable in expanding the usefulness of combination. Consider $x^n = b$, then we define $b^{\frac{1}{n}} = a$. This will become incredibly valuable soon. We retain that $x^n x^m = x^{n+m}$ and $(x^n)^m = x^{nm}$. Therefore, $\overline{x^n} = \bar{x}^n = x^{-n}$.

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[a] To be discussed in the definition of specification.

As a basic exercise, we prove multiplictivity of combinatorial opposites: $\overline{kj} = \bar{k}\,\bar{j}$. First see that $\overline{kj}kj = \nabla$. Now remember that $\bar{k}k\bar{j}j = \nabla$. These are equal, so we have $\overline{kj}kj = \bar{k}k\bar{j}j$. Commutativity procures the result $\overline{kj}kj = \bar{k}\,\bar{j}kj$. If we multiply on both sides by $\overline{jk}$ we get $\overline{kj} = \bar{k}\,\bar{j}$, as desired.

Another quick verification is that $\bar{\bar{k}} = k$.

By definition, $\bar{k}k = \nabla$. If take the combinatorial opposite of both sides then we get $\overline{\bar{k}k} = \bar{\nabla}$. By the previous property, this is the same as $\bar{\bar{k}}\,\bar{k} = \nabla$. We multiply both sides by k to get $\bar{\bar{k}} = k$.

With this operation, we must declare the types of our parameters. That is, what does it mean to be a biological object? We first decree that there is an inherent difference between biological and non-biological objects: a piece of metal is not biological, and a cat is.

Definition 1.2 *λ is animated – a biological object – if it has distinct alive and dead states. We say that $\lambda\mathbf{.}$ represents the alive state of the biological object λ and $\lambda\mathbf{,}$ represents the dead state. Then we have $\lambda\mathbf{.} \neq \lambda\mathbf{,}$ where $\exists\lambda\mathbf{.}$ and $\lambda\mathbf{,}$ as our definition of animation. An animated object whose state is unknown is notated $\lambda_{\boldsymbol{x}}$. An inanimate object is usually notated without a subscript, or with the marker $\lambda_{\divideontimes}$.*

This leaves open space as to what constitutes these states.[1] Hence, we include the following:

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[1] The state-dependent definition of animation here first reminded me of quantum mechanics' bra-ket notation, hence the largely erroneous use of the word "quantum" in the name "bioquantum."

Definition 1.3 *Alive and dead states have the properties that:*

1. *Thermodynamic dependence:* $[\lambda_{\boldsymbol{x}}] = 0 \implies \lambda\boldsymbol{,}$
2. *Continuity:* $\lambda\boldsymbol{,} \not\mapsto \lambda\boldsymbol{.}$ *and* $\lambda\boldsymbol{.} \mapsto \lambda\boldsymbol{,}$
3. *Uniqueness:* $\exists!\ \lambda\boldsymbol{.}$ *and* $\lambda\boldsymbol{,}$

Where we assume the notation $[\lambda]$ *gives a quantitative assessment of the amount of energy in a system*

$$[\Lambda(t = v)] = \int_0^v \Lambda_I(T) - \Lambda_O(T)\ dT$$

given Λ_I *as energy influx per chronological unit* $\left(e.g.\ \frac{J}{s}\right)$*, and* Λ_O *as energy outflux per chronological unit* $\left(e.g.\ \frac{J}{s}\right)$*, both concepts inherited from thermodynamics. This is a hidden assumption of our system – that physical rules apply to our system, namely the second law of thermodynamics (conservation of energy and matter). In variant algebras, thermodynamic dependence might be omitted.*

Where we assume $\mapsto$ *to be a nonsymmetric equivalence relation on two objects* a*,* b *where* $a \mapsto a$ *always,* $a \mapsto b$ *is the same as* $b \mapsfrom a$*, and* $a \mapsto b$ *and* $b \mapsto c$ *means* $a \mapsto c$*.*

Where we realise that uniqueness is a fundamental property of earthly biology, but may not be apparent in other systems.

These three definitions are the crux of this algebra's divergence from more general algebraic biology (of which this is a subfield), as they make three key assumptions.

1. That biology obeys thermodynamic laws.
2. That biology obeys our theory of time.
3. That biology obeys our perceived reality that life and death are binary absolutes.

How much these laws affect the following mathematics indicates how important physics is with regards to biology.

Next, we make another fundamental definition: there exists an operation "specification" that is fundamentally separate from combination.

Definition 1.4 *Consider* $\lambda\boldsymbol{.}$*. Then* $\lambda\boldsymbol{.}[z] = \lambda\boldsymbol{.}G(z)$ *for some blackbox function* G *that has the properties* $G(z)G(w) = G(zw)$ *and by extension* $G(z)^n = G(z^n)$*. Specification is a function that we may understand as manipulating the genome. Additionally, we require that* $G(zw) = G(wz)$ *but not necessarily* $G(z)w = wG(z)$*, and not necessarily* $G(z)wx = G(z)xw$ *and not necessarily* $wxG(z) = xwG(z)$ *necessarily. That is to say, commutativity and associativity around* $G(z)$ *is preferential to* $G(z)$*. This prevents two individuals from swapping genetic information based on geographic proximity.* [a]

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[a] This is the aforementioned exception on commutativity and associativity of combination.

In order to understand how this function work, we shall investigate its properties.

We shall define a function $|\cdots|$ and animated object $\zeta\boldsymbol{.}$ such that $|\lambda\boldsymbol{.}|$ gives an object where $\zeta\boldsymbol{.}[|\lambda\boldsymbol{.}|] = \lambda\boldsymbol{.}$. Think of this $\zeta\boldsymbol{.}$ as an empty, unprogrammed cell (or astrobiological equivalent).

Now realise that

$$\zeta\boldsymbol{.}[|\zeta\boldsymbol{.}|] = \zeta\boldsymbol{.}$$

by the definition of $\zeta\boldsymbol{.}$ and $|\cdots|$, but also that

$$\zeta\boldsymbol{.}[|\zeta\boldsymbol{.}|] = \zeta\boldsymbol{.}G|\zeta\boldsymbol{.}|$$

by the definition of specification. These two facts together function to prove that $\zeta\boldsymbol{.} = \zeta\boldsymbol{.}G|\zeta\boldsymbol{.}|$, thus

$$\nabla = G|\zeta\boldsymbol{.}|$$

by definition of combination.

But we can prove that $G(\nabla) = \nabla$, as follows:

$$\nabla = \overline{G(z)}G(z) = \overline{G(z)}G(z\nabla) = \overline{G(z)}G(z)G(\nabla) = G(\nabla).$$

Thus $|\zeta\boldsymbol{.}| = \nabla$. Then it seems reasonable to create the notational convention $\nabla\boldsymbol{.} = \zeta\boldsymbol{.}$.[2]

We can prove as a corollary that $G(\bar{z}) = \overline{G(z)}$, as follows:

$$\overline{G(z)} = \overline{G(z)}\nabla = \overline{G(z)}G(\nabla) = \overline{G(z)}G(z\bar{z}) = \overline{G(z)}G(z)G(\bar{z}) = G(\bar{z}).$$

It should be intuitive that $\big|\lambda\boldsymbol{.}[\delta]\big| = \big|\lambda\boldsymbol{.}\big|\delta$ since $\lambda\boldsymbol{.} = \nabla\boldsymbol{.}\big[|\lambda\boldsymbol{.}|\big]$ so $\lambda\boldsymbol{.}[\delta] = \nabla\boldsymbol{.}\big[|\lambda\boldsymbol{.}|\delta\big]$ by the property of $G(z)$ that $G(zw) = G(z)G(w)$, so by definition of $\nabla\boldsymbol{.}$ *quod erat demonstrandum.*

Analysing $G(z)$ itself, we realise that $G(z) = \bar{g}f(z)g$ is a valid solution given some g for which the commutativity and associativity of combination does not apply, and $f(\nabla) = \nabla$ and $f(zw) = f(z)f(w)$. In fact, it is the only solution.[3]

2 Algebras and Factorisation

2.1 Mutations

The next pillar of this "bioquantum" algebraic biology will be the factorisation of states into basic genetic blocks. When we discuss genetics, we are less interested in alleles, codons, and base pairs, and are more interested in the resultant effects. Mapping genotypes with algebra is a very different pursuit, and it fits best to analyse only phenotypes. We can break these phenotypes into smaller units – in infinitesimal mutations we call "imus," written as μ. This subsection will investigate their uniqueness primarily, looking into various corollary definitions, primarily those with obvious analogies in number theory. In order to define them concisely, we write in a somewhat circular way. We first create a notation for them.

Notation 2.1 *For some multiset $x \in \tilde{\Omega}$ where we write $\tilde{\Omega}$ is the set of all multisets, where multisets are defined as sets that allow for repetition (i.e. $\{0,0\} \neq \{0\}$), we define the following to be true:*

$$\mu(x) = \sum_{i \in x} \mu_i.$$

If p can be expressed as $\mu(x)$ then p is perfect. Otherwise, p is imperfect. Notice that p cannot be both perfect and imperfect.

Definition 2.2 *We define divisibility[a] as follows: a/b if and only if $a = \mu(x)b$.*

[a] Note that a/b is said as "a is divisible by b".

Finally, one more definition before the big one:

Definition 2.3 *For static objects a and b and an integer n, the following notation is set:*

$$a \simeq^n b \iff a = b\mu(x) \text{ where } |x| = n.$$

By default, $n = 1$: if and only if $a \simeq b$ then $a \simeq^1 b$.

Perhaps it is worth noting that, while the above relation is not an equivalence relation, the relation $\sim$ where $a \sim b \iff \exists i$ such that $a \simeq^i b$.

With this background in mind, we may now properly define an imu.

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[2] Note the differences between $\boldsymbol{.}$ and . as they sit side-by-side in the text above.

[3] This, again, is fairly simple to prove based on the conditional commutativity and associativity of combination.

Definition 2.4 *An imu is defined as follows:*

$$\text{An } \mu \text{ satisfies}: \begin{cases} \mu_i \nmid \mu_j \ \forall i,j \\ \forall y, \exists (x_0, x_1) \ s.t. \ y = \mu(x_0)\overline{\mu(x_1)} \\ \forall i, \ \mu_i \simeq \nabla \end{cases}.$$

The above definition possibly looks suspicious, because it relies on the definition of divisibility, which relies on the definition of $\mu(x)$ which relies on the definition of an μ. However, the definition of divisibility is just a notational short hand, and $\mu(x)$ is a notation for the combination of imus. We can write the above definition as:

An μ is defined for all multisets x and all $j \notin x$ as follows:

$$\sum_{i \in x} \mu_i \neq \mu_j.$$

And $\forall y, \ \exists (x_0, x_1) \ s.t. \ y = \mu(x_0)\overline{\mu(x_1)}$.

This second definition is equivalent to the stated one, just without all of the useful notation. Before we move on, note that $\mu_i^n \neq \mu_j$ for all i and j, because the set of $\{i, \cdots, i\}$ with n instances of i is not any μ_j.

To prove that there actually exists a set of μ_i's, we construct them.

Axiom 2.5 *For the next result, we must assume the axiom of choice if and only if there are an infinite number of imus, since a step in a later proof requires it. It is important to stress that only if there are an infinite number of imus do we assume this axiom. Otherwise, we disregard the axiom of choice.*

Theorem 2.6 *There exists a set of μ_i's with the defined properties. That is, the three parts of the definition are not mutually exclusive and are entirely applicable.*

Proof Let Θ be the list of inanimate objects without the specification objects g and $\bar{g}$ or any scaling of these objects. Then see that Θ is a vector space over the rational numbers with operations defined as follows:

$$a + b = ab \text{ where } a \text{ and } b \text{ are inanimate, and}$$

$$c * a = a^c \text{ where } a \text{ is inanimate and } c \text{ is rational.}$$

Θ is a module (over the integers) because:

- Θ is closed over $+$, by the genesis axiom.
- By definition, $+$ is associative over Θ.
- By definition, $+$ is commutative over Θ.
- There is an identity ∇ for $+$.
- There is an inverse $\overline{\lambda}$ for any λ of $+$.
- There is an identity 1 of $*$.
- As can be easily verified, $*$ distributes over $+$ and *vice versa*.

As a result of this verification, we see that there must exist a basis: a linearly independent spanning set. This, in essence, is the definition of the μ's. Thus, there exists a basis, which we label as μ_i for some set of i's. However, an μ is a part of a linearly independent spanning set with one additional (trivial) parameter: $\mu_i \simeq \nabla$. By examining the definition of "$\simeq$", we see that this must be true.

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As mentioned initially, we create this object with the objective of creating an infinitesimal phenotype. We might desire from this biologically abstract definition that there is a single unique "factorization"[4] of some k. To get this point, we must prove various properties of imus and perfect objects.

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[4] A single, unique way of combining μs to create some genome.

For example, if a, b and c are perfect, then the following is true:

$$a/c \text{ or } b/c \implies ab/c.$$

Consider (without loss of generality) that a/c. So $a = pc$ where p is perfect. Now consider that $p' = pb$, so p' is also perfect (because b is perfect). Multiplying by b on both sides gives $ab = pbc = p'c$ which implies that ab/c.

It should be reasonably self-evident that $\mu(a)\mu(b) = \mu(a \cup b)$ because $\mu(x)$ is a function over multisets. With this in mind we can also show a intuitive algebraic result:

$$\nabla = \overline{\mu(a)}\mu(a) = \overline{\mu(a)}\mu(a \cup \nabla) = \overline{\mu(a)}\mu(a)\mu(\nabla) = \mu(\nabla).$$

The previous property gives us a lot of freedom when dealing with multisets. For example, we can now prove a property of the relation $\simeq^n$.

Property 2.7 *$\simeq$ is semitransitive in that $a \simeq^n b$ and $b \simeq^m c$ implies that $a \simeq^{n+m} c$.*

Proof First, there exist x's and y's such that $|x| = n$ and $|y| = m$. Then let $a = b\mu(x)$ and $b = c\mu(y)$. By substitution we have

$$a = c\mu(x)\mu(y) = c\mu(x \cup y).$$

And $|x \cup y| = |x| + |y| = n + m$ because x and y are multisets. Thus, by definition, $a \simeq^{n+m} c$.

. . .

Finally, we are ready to prove a theorem that gives imus their use: we can factor any object into them uniquely.

Theorem 2.8 ***Imu Invertability Theorem*** *It is the case that $\mu^{-1}(x)$ is a well-defined function over the set of perfect objects to the set of all multisets populated from $\mathbb{Z}$.*

Proof To verify this, we need to show that $\mu(x)$ is a bijection, and so we have to show that $\mu(x)$ is a surjection and an injection from the set of all repetitive sets to the set of all perfect objects.

Obviously, it is a surjection because the set of all perfect objects is the range of $\mu(x)$ (by definition).

To prove that it is an injection, we argue by contradiction.

Assume that $\mu(x) = \mu(y)$. Then $\mu_0 \cdots \mu_n = \mu'_0 \cdots \mu'_m$. We can also assume that no $\mu_i \neq \mu'_j$ for all i and j without loss of generality, because we can divide by any shared left-right terms to achieve this state. Then divide by μ_0. So,

$$\mu_1 \cdots \mu_n = \frac{\mu'_0 \cdots \mu'_m}{\mu_0}.$$

By the definition of an imu, the right side of this equation is imperfect and irreducible, but the left side is perfect. These two definitions are mutually exclusive, so we have a contradiction.$\rightarrow\leftarrow$

. . .

The imu invertability theorem inspires a definition of magnitude for inanimate objects.

Definition 2.9 *The magnitude of an inanimate object k is $|x| - |y|$ where $k = \mu(x)\overline{\mu(y)}$.*

Still remains the question of whether the operation $|\cdots|$ is a well-defined function. That is, can different representations of the same perfect object give different magnitudes? We will prove this is not the case.

For given x and y,

$$\frac{\mu(x)}{\mu(y)} = \frac{\mu(\hat{x})}{\mu(\hat{y})} \implies x \cap y = \hat{x} \cap \hat{y}.$$

To see that this is true, let $x = \{x_0, \cdots, x_n, s_0, \cdots, s_p\}$ and $y = \{y_0, \cdots, y_m, s_0, \cdots, s_p\}$ so that $x \cap y = \{s_0, \cdots, s_p\}$. Then allow $\hat{x} = \{x_0, \cdots, x_n\}$ and $\hat{y} = \{y_0, \cdots, y_m\}$. Thus

$$\left|\frac{\mu(x)}{\mu(y)}\right| = (n+p) - (m+p) = n - m = \left|\frac{\mu(\hat{x})}{\mu(\hat{y})}\right|.$$

This says that if we cancel the shared top-and-bottom terms, $|\cdots|$ outputs the same value. Hence, we can use this state as a stepping stone to see that two fractions with different shared top-and-bottom terms both equal the most minimal form (the form with no shared terms), hence must be equal. So $|\cdots|$ is properly defined as a function.

Definition 2.10 *For a number n (called a "set-element"), there exists a set-inverse $\overline{n}$. For ease of notation, the set which consists of the set-inverse of every element of S is written as S'. Additionally, if the set-element and set-inverse are in a set, then they cancel out. (I.e. $S \cup S' = \nabla$.) Finally, to keep the magnitude of a multiset well-defined, set-inverses count negatively towards the value of $|\cdots|$. There is no way to tell whether an an element of the set is a set-inverse or not. However, without loss of generality, we can choose this beforehand, and, if we are consistent, this does not cause any problems.*

Notation 2.11 *A set-inverse is interpreted by $\mu(\cdots)$ as an inverse-inanimate object. Put less succinctly:*

$$\mu(x) = \frac{\sum\limits_{i \in x \cap \tilde{\Omega}} \mu_i}{\sum\limits_{j \in x \cap \tilde{\Omega}'} \mu_j} = \sum_{k \in x} \mu_k^{\pm 1}.$$

Where the ± is dictated by whether or not k is a set-inverse of a number or not. This gives an easy way of expressing conditionals in a single function. Additionally, $\mu^+(x) = \sum\limits_{i \in x \cap \tilde{\Omega}} \mu_i$, the numerator of the previous expression, and $\mu^-(x) = \sum\limits_{j \in x \cap \tilde{\Omega}'} \mu_j$, the denominator. In short,

$$\mu(x) = \frac{\mu^+(x)}{\mu^-(x)}.$$

NOTE: *Any use of $\mu(\cdots)$ up to this point was actually a use of $\mu^+(\cdots)$.*

Theorem 2.12 ***Imu Invertability Theorem*** *It is the case that the imu invertability theorem still holds with the extended definition of $\mu(\cdots)$.*

Proof Consider that we have $\mu(x) = \mu(y)$. Then see that we can split up x so that all set-inverses are in one set and all set-elements are in another. So, $x = x_0 \cup x_1$ where $\mu(x_0) = \mu^+(x_0)$ and $\mu(x_1) = \overline{\mu^-(x_1)}$. Without loss of generality, let $x_0 \cap x_1' = \nabla$. In other words, there is no "shared" z term where $z \in x_0$, $\bar{z} \in x_1$. And likewise for y. For clarity's sake,

$$\mu(x) = \frac{\mu(x_0)}{\mu(x_1)}.$$

Then

$$\frac{\mu(x_0)}{\mu(x_1)} = \frac{\mu(y_0)}{\mu(y_1)},$$

So $\mu(x_0)\mu(y_1) = \mu(y_0)\mu(x_1)$. Using the additivity of $\mu(\cdots)$, this becomes $\mu(x_0 \cup y_1) = \mu(y_0 \cup x_1)$. We can take the inverse of this. Hence, $x_0 \cup y_1 = y_0 \cup x_1$.

Now consider that there is only one way to split up x and y into their set-inverse and set-element parts. In other words, for a set s, $\exists!(a, b)$ such that $\mu(a) = \mu^+(a)$, $\mu(b) = \overline{\mu^-(b)}$, and $s = a \cup b$.

Additionally, two sets are only equal if these partitions are equal, since no set-element is a set-inverse.

Remember that x_0 and y_0 contain only set-elements and x_1 and y_1 contain only set-inverses. In which case, $x_0 \cup y_1 = y_0 \cup x_1$ if and only if $x_0 = y_0$, since they are the set-element parts of each side of the equation, and $y_1 = x_1$, since they are the set-inverse parts of each side of the equation.

Now, take the union of x_0 and x_1:

$$x = x_0 \cup x_1 = y_0 \cup y_1 = y.$$

$$\cdots$$

2.2 Algebraic Structures

We may now begin to build some algebraic structures to investigate. Like all algebraic structures, bioq's structures have an operation and a set the operator acts on (a regular set, not a multiset).

Additionally, we define the magnitude of an algebraic structure is simply the size of its set: $\mathcal{P}$ with set Q, then $|\mathcal{P}| = |Q|$.

Finally, as a notation, we say if $\mathcal{P}$ be an algebraic structure with set Q, then $\rho \sim \mathcal{P}$ if and only if $\rho \in Q$.

Definition 2.13 *We will use the shorthand "population" to mean a set of elements. These elements may or may not actually be animated – but the intuition is that they are. Additionally, we include the notation $\tilde{P}$ the "carbon-copy" of a population, which satisfies:*

$$|\lambda\boldsymbol{.}|/\tilde{P},\ \forall \lambda\boldsymbol{.} \sim P.$$

Let us begin.

2.2.1 Pool Structures

We begin with the definition of a pool.

Definition 2.14 *A pool is a population with a mutation scheme M where the following are true:*

1. *P is closed over the entries of M (M produces a vector or tuple of creatures).*
2. *There exists a unique subset (made up of centres or centers, and called a base) of P such that any element of P can be expressed in terms of the mutation scheme and the elements of the basis. For example, $M(\lambda\boldsymbol{.}, M(M(\delta\boldsymbol{.}, \lambda\boldsymbol{.}), \delta\boldsymbol{.}))$ would be an expression in terms of the mutation scheme and the basis, if $\lambda\boldsymbol{.}$ and $\delta\boldsymbol{.}$ are in the basis.*

An example of a mutation scheme would be the following:

$$I(\lambda\boldsymbol{.}) = \lambda\boldsymbol{.}[\mu_0]$$

. Before we start our proofs, we must define some terms that help characterise pools:

Notation 2.15 *The dimension of a pool, notated as $\dim P$, is the magnitude of the base, i.e. the number of centres). Since there is only one base, this notation is properly defined.*

Definition 2.16 *The degree of a pool, notated as $\deg P$, is the size of the vector that is the input to the mutation scheme. That is, if $M(\lambda_{0}\boldsymbol{.}, \cdots, \lambda_{n}\boldsymbol{.})$ is well defined for all $\lambda_{i}\boldsymbol{.}$ in the aforementioned pool, then the degree of the pool is n.*

For simplicity, if M is a degree-1 mutation scheme, we will employ the notation:

$$M^n(\lambda\boldsymbol{.}) = \underbrace{M(\cdots M(}_{n \text{ times}}\lambda\boldsymbol{.})\cdots).$$

Definition 2.17 *The length of a pool, notated as $\operatorname{len} P$, is the size of the vector that is the output to the mutation scheme. That is, if $M(\delta\boldsymbol{.}) = (\lambda_{0}\boldsymbol{.}, \cdots, \lambda_{n}\boldsymbol{.})$ for all $\lambda_{i}\boldsymbol{.}$ in the aforementioned pool, then the length of the pool is n.*

To clarify a potential point of confusion, the length and degree of a pool do not function as they would in a linear algebra setting. Elements of a degree-n pool are not n-dimensional, they are 1-dimensional. We temporarily create a vector of length n as input values. Similarly, a pool of length n greater than 1 has a mutation scheme where the entries of the resultant n-dimensional vector are elements of the pool. Hence the power of the following proof.

The following lemma will be crucial for manipulating pools. It states that we can work with $\operatorname{len} P = 1$ pools instead of arbitrary pools. This gives us a lot of power over how we prove claims about pools.

Lemma 2.18 *Any theorem for is true for a pool P with $\operatorname{len} P = 1$ will work for a pool Q with $\operatorname{len} Q > 1$.*

Proof First, define a pool $\mathcal{P}$ with a mutation scheme $\mathcal{M}$. Then define $\mathcal{T}_1, \cdots, \mathcal{T}_{\operatorname{len} P}$ to be operators such that $\mathcal{T}_i(\lambda_{0\bullet}, \cdots, \lambda_{\deg P\bullet}) = \delta_{i\bullet}$ where $\mathcal{M}(\lambda_{0\bullet}, \cdots, \lambda_{n\bullet}) = (\delta_{0\bullet}, \cdots, \delta_{\operatorname{len} P\bullet})$. Then let $\mathcal{O}_1, \cdots, \mathcal{O}_{\operatorname{len} P}$ be pools such that each $\mathcal{O}_i$ shares its population with $\mathcal{P}$, and the mutation scheme for $\mathcal{O}_i$ is $\mathcal{T}_i$. See that all $\operatorname{len} \mathcal{O}_i = 1$ for all i.

Now consider that we prove some statement S about an length-1 pool. Then see that this statement applies entry-wise for any length-$n \geq 1$ pool, because we can decompose that mutation scheme to mutation schemes with image-length 1 based on the entries of elements of the image of that mutation scheme.

. . .

We have now simplified all pools to a length-1 case, which is very convenient, and much more intuitive.

Theorem 2.19 ***Degree**-1 **Pool Theorem*** *Let the mutation scheme $\mathcal{M}$ be closed over the set $\mathcal{P}$ which has a carbon-copy $\tilde{P}$. Additionally, let it be the case that $\deg \mathcal{P} = 1$*

Statement A: $\exists V$ such that $f(x) = V(\mathcal{M}^x(\delta\bullet))$ is invertible for all $\delta\bullet \in \mathcal{P}$ except the centres of $\mathcal{P}$.
Statement B: $\mathcal{M}$ is invertible.
Statement C: There does not exist a tuple $(\psi\bullet, n)$ such that $\mathcal{M}^n(\psi\bullet) = \psi\bullet$.
Statement D: $\mathcal{P}$ is a pool (with all previously mentioned components).

Then it is the case that only

$$A \wedge B \iff B \wedge C \iff D.$$

Proof This proof comprises six sub-proofs. These proofs will be the following:

1. $D \implies B$
2. $B \not\Rightarrow D$
3. $A \wedge B \implies D$
4. $A \not\Rightarrow D$
5. $D \implies A$
6. $A \iff C$

The D $\implies$ B Proof

Consider that $\mathcal{P}$ is a pool, and $\mathcal{M}$ is not invertible. Then that means that there exists at least one triplet $(\lambda\bullet, \delta\bullet, \epsilon\bullet)$ such that $\mathcal{M}(\lambda\bullet) = \mathcal{M}(\delta\bullet) = \epsilon\bullet$. Then, assuming that there exists an inverse for $\lambda\bullet$ and $\delta\bullet$, and all subsequent creatures created by taking some number of inverses of $\lambda\bullet$ and $\delta\bullet$, there will still be two potential centres. If $\mathcal{M}$ has more than one inverse on $\lambda\bullet$ or $\delta\bullet$ (and it must have at least one, or $\lambda\bullet$ or $\delta\bullet$ are the aforementioned centres), then the number of potential centres grows even greater. Thus, $\mathcal{P}$ is not a pool, by definition based on its absence or multiplicity of centres. $\rightarrow\leftarrow$

The B $\not\Rightarrow$ D Proof

Consider some mutation scheme $\mathcal{M}$ where there exists an integer x where $\mathcal{M}^x(\lambda\bullet) = \lambda\bullet$. An example of a scheme like this would be, for some multiset Q:

$$\mathcal{N}(\lambda\bullet) = \lambda\bullet \left[\begin{cases} k \text{ if } |\lambda\bullet| / k \\ \bar{k} \text{ if } |\lambda\bullet| \not| \, k \end{cases} \right],$$

Where $\mu(Q)$ is added to $\lambda\bullet$ if not already there, and is subtracted from $\lambda\bullet$ if present in $\lambda\bullet$'s genome. Notice that $\mathcal{N}^2 = \mathcal{I}$ where $\mathcal{I}$ is the identity mutation scheme. Then, obviously, $\mathcal{N}^{-1} = \mathcal{N}$, so $\mathcal{N}$ has an inverse. But, if we let $\zeta\bullet$ be a centre of the pool paired with $\mathcal{N}$. Similarly, let $\epsilon\bullet = \mathcal{N}(\zeta\bullet)$. Then

$\mathcal{N}(\epsilon.) = \zeta..$ The pool set contains only $\zeta.$ and $\epsilon.$, and both $\zeta.$ and $\epsilon.$ are centres for this pool. And we now have two options for centres, which violates the unique base clause of a pool's definition. Hence, $B \not\Longrightarrow C$.

The $\mathbf{A} \wedge \mathbf{B} \implies$ D Proof

Let there exist a V so that f is invertible in accordance with **B** being assumed. Then take some base $\zeta_0., \cdots, \zeta_n..$ This is not necessarily the only base – we must prove this is the case. Then we define

$$R(\lambda.) = \mathcal{M}^{-f^{-1}(V(\lambda.))}(\lambda.).$$

See that if and only if $R(\lambda.) = \lambda.$, then $\lambda.$ is a centre. In words, that is that if we cannot take any number of steps backwards and procure an earlier copy, then this $\lambda.$ must be the "smallest" element in the chain defined by $\mathcal{M}(\lambda.) = \delta. \implies \lambda. < \delta.$, thus must be a centre. By the same logic, $R(R(\lambda.)) = R(\lambda.)$. That is, all $R(\lambda.)$ are centres.

Let $\nu_0., \cdots, \nu_n.$ be $\nu_i. = R(\zeta_i.)$. Then either $\zeta_i. = \nu_i.$, in which case $\zeta_i. = \nu_i.$ is a centre, or $\nu_i.$ is. Then $\nu_0., \cdots, \nu_n.$ is a true base.

The D $\implies$ A Proof

Let V be defined in the following manner:

$$V(\lambda.) = \nu \text{ where } \exists i \text{ s.t. } \mathcal{M}^{\nu}(\zeta_i.) = \lambda..$$

In which case, we need to prove that V is well-defined. Since there is only one i for each $\lambda.$ such that $\mathcal{M}^{\nu}(\zeta_i.) = \lambda.$ by the definition of base uniqueness.

The A $\not\Longrightarrow$ D Proof

If $\mathcal{M}$ isn't invertible, then we could have the case where $\mathcal{M}^n(\zeta.) = \mathcal{M}^n(\epsilon.) = \lambda.$ but $\zeta. \neq \epsilon..$ Then, $f^{-1}(V(\lambda.))$ will be defined, but $\mathcal{P}$ will not be a pool.

Importantly, $f^{-1} \circ V$ says how many "steps" (applications of the inverse mutation scheme) backwards to go, and the invertability allows for the taking of those steps backward.

Consider the case where $\lambda. = M(\zeta_{0.}) = M(\zeta_{1.}) = M(\zeta_{2.},$ where $\zeta_i.$ form a "base". Hence, $f^{-1}(V(\lambda.)) = 1$, where V is the number of nodes between $\lambda.$ and a specific centre. However, M is obviously non-invertible, thus $\mathcal{P}$ is not a pool, based on the necessary uniqueness of its centres, which is violated by the existence of many $\zeta_i.$'s that are all performing the same "task" – that of allowing $\lambda.$ to exist.

The A $\iff$ C Proof

The only way that $V(\lambda.)$ given above is not a function is if there exists an element that either has at least two values. If it has two (or more) values, then, since we are dealing with a degree-1 pool, $\mathcal{M}^n(\psi.) = \psi..$ Then

$$x = V(\psi.) = V(\mathcal{M}^n(\psi.)) = x + n.$$

Similarly, if there exists a $\psi.$ and an n such that $\mathcal{M}^n(\psi.) = \psi.$, then there exist two values of V, and we see that A is no longer true.

Hence, $A \iff C$.

Since we have proven all of the six necessary parts of this proof, we can claim,

$$\mathcal{Q.E.D.}$$

. . .

This implies, in part, that mutation functions have no fixed points, and further that there is no n such that $\mathcal{M}^n$ has a fixed point.

The previous theorem works only for degree-1 pools. However, from this we can prove this incredible fact for degree-n pools with $n \geq 1$ too. To do this, we need a notation for the structure of a pool. We will use the following notation:

$$\mathcal{M}^{(\cdots(\lambda_{0.}, \cdots, \lambda_{n.}), \cdots)} = \mathcal{M}(\cdots \mathcal{M}(\lambda_{0.}, \cdots, \lambda_{n.}), \cdots)$$

In general, the structure of the mutation scheme is put as an exponent. So $\mathcal{M}S = \mathcal{M}^S$ when S is a tuple structure.

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Theorem 2.20 ***Pool Theorem*** *Let the mutation scheme $\mathcal{M}$ be closed over the set $\mathcal{P}$ which has a carbon-copy $\tilde{P}$.*

Statement A: $\exists V$ such that $f(x) = V(\mathcal{M}^x)$ is invertible.
Statement B: $\mathcal{M}$ is invertible.
Statement C: $\mathcal{P}$ is a pool (with all previously mentioned components).

Then it is the case that only

$$A \wedge B \iff C.$$

Proof This will be an inductive proof. Our base case has already been done: the degree-1 pool theorem proves that the claim is true for $\deg \mathcal{P} = 1$.

That leaves the inductive step: assume that the claim is true for $\deg \mathcal{P} = n$. Then construct a mutation scheme with degree $n + 1$. See that $\mathcal{M}(\lambda_{0.}, \cdots, \lambda_{n.}, \delta.)$ is equivalent to a degree-n mutation scheme if $\delta.$ is constant, so the claim is true by our inductive assumption. Then for each $\delta.$, the claim is true. When doing a proof, we prove that, for every combination values, the claim is true. Then, if for each $\delta.$ it is true, then we have proven the degree-$n+1$ case.

. . .

The proof of the previous is questionable – tenuous at best. Perhaps the statement may best be described as "justified conjecture."

We can now prove some trivial-sounding theorems that help us see how pools can accurately represent species reproduction.

Theorem 2.21 *If the degree-1 mutation scheme $\mathcal{M}$ is defined for all $\lambda. \sim \mathcal{P}$ a pool, then $|\mathcal{P}| \to \infty$.*

Proof Consider that a pool's structure is that of multiple parallel chains each with some initial value (a generator value) that we call a centre. Then for λ a value in some chain, there are three options: (a) that $\mathcal{M}(\lambda.)$ is undefined, thus $\lambda.$ is the terminal element of the chain (b) there is a $\delta.$ such that $\mathcal{M}^n(\lambda.) = \delta.$ with δ new in the chain, or (c) there is a $\delta.$ such that $\mathcal{M}^n(\lambda.) = \delta.$ where $\delta.$ is not new in the chain.

If we take (a) to be true, then we violate our original assumptions. If we take (c) to be true, then we clearly have a cycle structure. This violates our Pool Theorem. Hence, $\mathcal{P}$ is not a pool, and we have violated our original assumptions. If we take (b) to be true, then we ignore that case and choose $\mathcal{M}(\lambda.)$ and apply the same reasoning to it. Since (a) and (c) are invalid cases, we can repeat this process infinitely, thus $|\mathcal{P}| \to \infty$.

. . .

This feels reasonable intuitively, because it says that either a species has infinite individuals, or it has a terminal individual.

Next, inspired by direct sums of vector spaces we consider now sums of pools:

Definition 2.22 *Let the two pools $\mathcal{P}_1$ and $\mathcal{P}_2$ have respective mutation schemes $\mathcal{M}_1$ and $\mathcal{M}_2$. Then $\mathcal{P}_1 + \mathcal{P}_2 = \mathcal{O}$ implies that $\mathcal{O} = \mathcal{P}_1 \cup \mathcal{P}_2$ where $\mathcal{P}_1 \cap \mathcal{P}_2 = \nabla$ and the related mutation scheme of $\mathcal{O}$ is defined as $\mathcal{T}(\rho_i) = \mathcal{M}_i(\rho_i)$ where $\rho_i \sim \mathcal{P}_i$. We expand this definition to infinite sums.*

Notice the similarity here between the restriction operator of a subspace (from linear algebra), and the definition of the mutation scheme of $\mathcal{P}_1 + \mathcal{P}_2$.

However, the primary use of the sum of two pools is describing a collection of pools as one population. That is, even if $\mathcal{P}_1$ and $\mathcal{P}_2$ are geographically split and cannot inter-reproduce, the pool $\mathcal{P}_1 + \mathcal{P}_2$ is the pool that encompasses both of these pools. We will now prove a trivial but important theorem to do with the addition of pools.

Theorem 2.23 *If $\mathcal{O}$ is a pool with mutation scheme $\mathcal{T}$ such that $\dim \mathcal{O} = n$, then there exists a sum for each integer i from 1 to n such that $\mathcal{O} = \mathcal{P}_1 + \cdots + \mathcal{P}_i$ if $\mathcal{O}$ is a degree-1 pool. This is true for infinite sums too.*

Proof Let $o_1, \cdots, o_n$ be the base of $\mathcal{O}$. Then see that the sets $\mathcal{P}_i$ with i integers from 1 to n given by the mutation scheme $\mathcal{T}$ sum to $\mathcal{O}$ since only one creature can be the input of $\mathcal{T}$, thus the resulting chain structure makes is so that different chains do not interact with each other. Hence we have the largest sum (any more would cut into one of the chains because of the dimension of the pool, thus contradicting the definition of a sum). Now consider that we can group any number of subpools to create another sum. If we want i summands, then we choose $n - i + 1$ arbitrary smallest subpools and add them together. As can be verified, there are $i - 1$ smallest subpools left. Thus, in total, we have $(i-1)+1 = i$ subpools to add together.

A similar logic applies for infinite sums.

. . .

Now we may prove the following lemma...

Lemma 2.24 *For every population $\mathcal{P}$ there exists a mutation scheme $\mathcal{T}$ such that $(\mathcal{P}, \mathcal{T})$ is a pool.*

Proof Consider that we may define the following pools $\mathcal{O}_0, \cdots, \mathcal{O}_k, \cdots$ where $\lambda. \sim \mathcal{O}_k$ if and only if $\text{abs}(\mu^{-1}(|\lambda.|)) = k$ where we define $\text{abs}(x)$ as the smallest set length of a multiset, counting all elements as positive (that is, x and $\bar{x} \in X$ both contribute positively to the value abs X). For the case where k is infinity, we can assume that the definition of abs has a range of the one-point compactification of the reals, given that the number of is at most $\aleph_0$, which we can gloss as ∞ (this is shown by the proof that the μ's form a vector space over $\mathbb{Q}$).

Then clearly $\mathcal{P} = \mathcal{O}_0 + \cdots + \mathcal{O}_k + \cdots$. We can now define $\mathcal{M}_k$ for each $\mathcal{O}_k$. Once we do this, $\mathcal{T}$ will be defined with respect to the $\mathcal{M}_k$, as dictated in our pool-sum procedure. So we will be done

Define the total ordering $<'$:

$$a <' \bar{a} <' b <' \bar{b} \iff a < b.$$

We quickly make the notation that $\min_i x$ is the i^{th} smallest element of x. Then we define $\mathcal{M}_i$ such that:

$$\mathcal{M}_i(\lambda.) = \delta. \iff \min_i \mu^{-1}|\lambda.| <' \min_i \mu^{-1}|\delta.| \text{ and } \min_j \mu^{-1}|\lambda.| = \min_j \mu^{-1}|\delta.|$$

For all $j < i$. Since each $\mathcal{M}_k$ is undeniably bijective, and there does not exist a cycle structure (based on our clear total ordering $<'$), each $\mathcal{M}_i$ is a mutation scheme, and $\mathcal{T}$ is a mutation scheme.

Thus we are done.

. . .

Lemma 2.25 *A degree-*1 *pool is a poset.*

Proof Consider a partial ordering of $\mathcal{P}$ where each chain is disjoint from every other chain. Then let the smallest elements of each chain be a centre of a basis. Then let the mutation scheme $\mathcal{M}$ be of degree 1 and defined as follows: for each chain P_i, let $m_{0i.}$ be the minimal element, and let $m_{ji.}$ be greater than $m_{ki.}$ if and only if $j > k$. Then $\mathcal{M}(m_{ji.}) = m_{(j+1)i.}$. Obviously, there is one unique base—the set of minimal elements of $\mathcal{P}$. And, since each chain is disjoint with every other chain, each element can only be mapped as $\mathcal{M}^n(m_{0i.})$ for one tuple (n, i).

Similarly, we can define a partial ordering of $\mathcal{P}$ based on $\mathcal{M}$.

Finally, see that the relation R on $\mathcal{P}$ defined as $R(\lambda., \delta.)$ if and only if there exists a positive n such that $\mathcal{M}^n(\lambda.) = \delta..$ If $\mathcal{M}^n(\lambda.) = \lambda.$ then $n = 0$ by the pool theorem, so we have reflexivity. Now see that $\mathcal{M}^n(\lambda.) = \delta.$ and $\mathcal{M}^m(\delta.) = \lambda.$ implies that $\mathcal{M}^{n+m}(\lambda.) = \lambda.$, so $m + n = 0$ and $m = -n$. But since m and n are both positive by our assumption, $m = n = 0$. Thus $\mathcal{M}^0(\lambda.) = \delta.$ and so $\lambda. = \delta.$ because $\mathcal{M}^0$ is the identity.

. . .

This new thought process yields some exciting results, as we can begin to pull from the vast repository of knowledge created by past mathematicians, such as Zorn's Lemma, as we shall now see.

Previously, we tentatively assumed the axiom of choice, if and only if there are an infinite number of imus. We would like to prove that there are a finite number of imus by disproving Zorn's lemma, which is equivalent to the axiom of choice. That is, if we can prove that Zorn's lemma is not true with our present axioms, then we must have a finite number of imus. This is tenuous logic – perhaps better described as justification. We shall proceed nonetheless.

Lemma 2.26 *We cannot prove Zorn's lemma. Consider a partially ordered set. Zorn's lemma states that if each chain of this set has an upper bound, then the set has at least one maximal element.*

Proof If we were to prove this, then we would use our most basic algebraic structure: the degree-1 pool which is, as we have discussed, a poset. Additionally, we have proven that every degree-1 pool either has a maximal element, or has no upper bound. Finally, we have shown that any *discrete* – that is, non-dense – poset can be represented by a degree-1 pool.

However, the fundamental nature of imus is that they are discrete. The very reason that we can define a mutation scheme on any population is because we can deconstruct its genome into discrete factors (the imus) and compare these imus. Hence, any pool is non-dense, and we can never prove Zorn's lemma without creating a dense algebraic structure. To do this in our current situation would be to create a definition of some truly infinitesimal, indivisible unit. We would then prove that these objects exist. However, this would be to *de facto* assume the axiom of choice, because we attempt to prove their existence under the parameter that we need an infinite number of them to construct anything (hence their infinitesimal construction).

So the only way to prove Zorn's lemma is to create a new set of objects, and in doing so assume the axiom of choice, which is equivalent to Zorn's lemma. Therefore, we have not yet already assumed the axiom of choice.

...

Theorem 2.27 *There are a finite number of imus.*

Proof By the fact that we have not assumed the axiom of choice, there must be a finite number of imus.

...

This is a logical and acceptable claim: to say anything else would be to imply that an infinite amount of data can be encoded in finite space, which violates our understanding of the physical universe our real ecology conforms to. In a sense, this result should make more plausible the claim that algebraic biology deserves the name "biology."

For a future analysis of mutation schemes, we will define some terminology and notation.

Notation 2.28 *The function on a mutation scheme* $\mathcal{T}$ *of a degree-*1 *pool* $\mathcal{O}$ *has numbers* $\Delta\mathcal{T}_{\lambda.} = \mu^{-1}|\lambda.\overline{|\mu^{-1}|\mathcal{T}(\lambda.)|}$. *The number* $\Delta\mathcal{T}$ *is the mean of the "cline set"* $\{\text{abs}\,\Delta\mathcal{T}_{\lambda.}|\forall\lambda. \in \mathcal{O}\}$.

The previous notation and terminology seems useless at first, but will become useful in a future chapter, and in future analysis.

2.2.2 Convolutional Pools

The convolutional pool structure is an extended pool where the mutation scheme changes. Notice that all of the previous properties of a pool still apply with only slight modification. So we can still think of a convolutional pool as a poset.

In order to be concise, we retain all of the notation (such as len, deg, and dim) from before. First, we define a convolutional pool.

In order to create the definition for a convolutional pool, we define an operation on mutation schemes:

Definition 2.29 *A convolutional pool V is a population with elements that are tuples of the form $(\lambda.,\mathcal{T}_{\lambda.})$. Additionally, each convolutional pool has a hypermutation scheme $\mathcal{J}$ that takes an input of n degree-n mutation schemes and returns a degree-n mutation scheme.*
For simplicity, $\lambda. \sim V$ if and only if $(\lambda.,\mathcal{T}_{\lambda.})$ is an element of the population.
We use the notation T^S to denote $T_1(\cdots T_m(\lambda.,\cdots,\delta.),\cdots)$ when S is a structure (in the example, $S=(\cdots(\lambda.,\cdots,\delta.),\cdots)$).

1. *V is closed over the entries of T_i (all T_i's produce vectors or tuples of creatures that are part of the population).*
2. *There exists a unique subset (made up of centres or centers, and called a base) of V such that any element of V can be expressed in terms of the mutation scheme and the elements of the basis. For example, $T_2(\lambda.,T_1(T_0(\delta.,\lambda.),\delta.))$ would be an expression in terms of the mutation scheme and the basis, if $\lambda.$ and $\delta.$ are in the basis.*
3. *n creatures $\alpha_{1.},\cdots,\alpha_{n.}$ procreate in the following way: let A_i be the mutation scheme for α_i. Then the procreation of these is the function*
$$\mathcal{J}(A_1,\cdots,A_n)$$
And the creature
$$\left(\mathcal{J}(A_1,\cdots,A_n)\right)(\alpha_{1.},\cdots,\alpha_{n.}).$$

In this way, we demoted the mutation scheme to being an element of the set, and made $\mathcal{J}$ a hyper-parameter of the convolutional pool.

Theorem 2.30 ***Convolutional Pool Theorem*** *Let the mutation schemes $\mathcal{M}_0,\cdots,\mathcal{M}_i,\cdots$ be closed over the set $\mathcal{V}$ which has a carbon-copy $\tilde{V}$.*

Statement A: $\exists V$ such that $f(x)=V(\mathcal{T}^x)$ is invertible.
Statement B: All $\mathcal{T}_i$ are invertible when $\mathcal{T}_i$ are mutation schemes of elements of the base. Additionally, $\mathcal{J}$ is invertible.
Statement C: All mutation schemes $\mathcal{M}_i$ are invertible and $\mathcal{J}$ is invertible.
Statement D: $\mathcal{P}$ is a convolutional pool (with all previously mentioned components).

Then it is the case that only
$$A\wedge B \iff A\wedge C \iff D.$$

Proof The same logic applies as in the pool theorem for the equivalence of $A\wedge C$ and D.

See that $\mathcal{J}$ and the set of all $\mathcal{T}_i$ form a pool with centres that relate to the centres of $\mathcal{V}$, if and only if there exists a V function with the properties given in statement A (because V acts on the tuple of creature and mutation scheme). So we have $A\wedge B \iff D$.

Now remember that $B \iff C$ because for a composition function to be invertible, it requires that both the input functions and the composition function be invertible. And the converse of this is true: if all functions involved are invertible, then the result is invertible.

$\cdots$

Similarly to the pool theorem, this is one of the most important theorems in the book, because it allows us to manipulate the elements of a convolutional pool in a way that we could not before.

Next, we create another useful theorem that builds on the last one.

Theorem 2.31 *Every theorem that applies to a pool applies to a convolutional pool, and vice versa.*

Proof Consider that we have a naming convention where $T_{S_1,\cdots,S_m}$ is the mutation scheme that acts on all $S_i=(\lambda_{1i.},\cdots,\lambda_{ni.})$. Then define a mutation scheme over this convolutional pool $\mathcal{M}$ where:

$$\mathcal{M}v = T_{S_1,\cdots,S_m}v \text{ if } v\in\{S_1,\cdots,S_m\}$$

Then there exists a mutation scheme. And since each T_i is invertible (by the convolutional pool theorem) $\mathcal{M}$ is invertible, thus is a mutation scheme of a pool. The carbon-copy of a convolutional pool is the same as a carbon-copy of a pool, so the we finally see that for every pool there is an equivalent convolutional pool and vice versa.

Alternatively, consider that we construct a pool $\mathcal{O}$ with mutation scheme $\mathcal{T}$ for some convolutional pool $\mathcal{P}$ with mutation schemes $\mathcal{M}_{\lambda.}$. Of course, the population of $\mathcal{P}$ will work for $\mathcal{O}$. Then all we need to do is choose a convenient $\mathcal{T}$. So be it: let $\mathcal{T}(\lambda.) = \mathcal{M}_{\lambda.)(\lambda.)}$. We now must demonstrate that this is a valid mutation scheme that creates a pool. To do this, we create a new convolutional pool with mutation schemes all $\mathcal{T}$ and hypermutation scheme $\mathcal{J}$ the identity function. This convolutional pool is the same as our first convolutional pool in practice, so $\mathcal{T}$ over this shared population must satisfy the requirements of the Convolutional Pool Theorem, which are the same as the requirements of the Pool Theorem. Hence, $\mathcal{T}$ over this population also forms a pool. This says that we can construct a pool for any convolutional pool that functions in the exact same way. They are equivalent. And, of course, a pool is merely a convolutional pool with an identity hypermutation scheme.

...

This is a pivotal proof in this chapter, because it says that pools are alternative framings of the same algebraic structure – we can use pools to prove complex theorems that we want to see are true in convolutional pools.

2.2.3 Dynamic Pools

We may now build on our definition of convolutional pools, which built on pools, to create a convolutional pool in which the landscape changes, and there are many species. This is the closest we have come to defining an ecosystem.

Definition 2.32 *A landscape is a set of conditionals and a mutation scheme. If L is a landscape with mutation scheme $\mathfrak{L}$, then $L_{t=k}$ is the i^{th} conditional with respect to this mutation scheme. It can be thought of as the landscape at time $t = k$. Thus we have a base element $L_{t=0}$ and from there we define $L_{t=k+1} = \mathfrak{L}(L_{t=k})$. Notice that this definition is similar to the definition of a dimension-*1 *pool, except that we allow cycle structures. We add a note here that the energy in ℓ a discrete part of L (discrete part meaning $L_{t=t_k}/\ell$) is defined by a landscape energy function $E(\ell)$.*

A landscape is a description of how an environment changes. Additionally, levels of compounds are put in the elements of the landscape. These compounds could include calcium, carbonic acid, non-biological food, or less tangible elements, *e.g.* free space.

Definition 2.33 *A dynamic pool is a pair of the following form: $\Phi = (\{V_0, \cdots, V_n\}, L)$ where L is a landscape and V_i are pools. The term for the set of V_i's is the pod, and is written as* pod V*. A dynamic pool has the following properties:*

1. *At any given time $t = t_k$, certain elements of each pool in the pod are alive, with the constraint that $\mathcal{M}(\xi_x)$ cannot be alive at time $t = t_i < t_j$ where t_j is the smallest time at which ξ_x is alive.*
2. *For any given time period $t \in [t_0, t_f]$ and a landscape energy function E:*

$$\int_{t_0}^{t_f} \sum_{\xi.\sim\Phi} \left[\frac{d}{d\tau} [\xi.(t = \tau)] \right] d\tau \leq E(L_{t=t_f}) - E(L_{t=t_0}).$$

   The left-hand side of this inequality can be simplified to: $\sum_{\xi.\sim\Phi} [\xi.(t = t_f)] - [\xi.(t = t_0)]$.

3. *Individuals in each pool are capable of causing each others' deaths.*

Additionally, we may say that a dynamic convolutional pool is a dynamic pool where members of the pod are convolutional pools.

Since we have proven the congruence between these two structures, we will prove all of the following theorems on non-convolutional pools with thought in mind that the same theorems apply to dynamic convolutional pools.

Terminology 2.34 *A singleton-pod (convolutional) pool is dynamic (convolutional) pool that has* $\operatorname{pod}\Phi = 1$.

In order to analyse how dynamic pools work, we must create some definitions, notation, and terminology. With this, we may begin to understand how dynamic pools and dynamic convolutional pools work.[5]

First, we define the gene pool:

Definition 2.35 *The "gene pool" is the population* Γ *is defined for a (convolutional) pool* V *with*[a] *mutation schemes* Ξ_j *as follows:*

$$\Gamma = \{\xi_\bullet \mid \exists\{\eta_{i\bullet}\}, j \text{ s.t. } (\eta_{0\bullet}, \cdots, \eta_{n\bullet}, \xi_\bullet, \eta_{n+1\bullet}, \cdots, \eta_{n+m\bullet}) \in domain\ \Xi_i \mid \forall i\}$$

[a] We use the notation Γ_V if the pool V is not obvious from context.

Our first theorem is very useful and explains our ostensibly arbitrary use of the word "pool" in the definition of a gene pool.

Theorem 2.36 *If* $\mathcal{O}$ *is a dynamic pool with mutation scheme* $\mathcal{T}$ *then* $\Gamma_{\mathcal{O}}$ *is a pool over* $\hat{\mathcal{T}}$, *a restricted* $\mathcal{T}$.

Proof Consider a pool $\mathcal{O}$ with mutation scheme $\mathcal{T}$. Then let us construct a mutation scheme over $\Gamma_{\mathcal{O}}$ in the following fashion: $\hat{\mathcal{T}}(\lambda_{0.}, \cdots, \lambda_{n.}) = \delta_\bullet$ if and only if $\delta_\bullet \sim \Gamma_{\mathcal{O}}$. See that the $\lambda_{i.}$'s are all in $\Gamma_{\mathcal{O}}$ because they create $\delta_\bullet$. However, $\delta_\bullet$ may not be in $\Gamma_{\mathcal{O}}$, so $\hat{\mathcal{T}}(\lambda_{0.}, \cdots, \lambda_{n.})$ may be left undefined. This means that the gene pool of the gene pool may be smaller still, and eventually, the gene pool of the $\cdots$ of the gene pool may be the empty set.

. . .

Now we shall define extinction, so that we may properly discuss interspecies interactions.

Definition 2.37 *A (convolutional) pool* $\mathcal{O}$ *is extinct if and only if* $|\mathcal{O}| = 0$. *The notation for this is* $\mathcal{O} \mapsto \emptyset$.

The next proof is that an empty gene pool results in extinction of the species. While this seems obvious, we have not proven it. It is critical to our understanding of evolution and extinction, thus is a pivotal theorem in this sub-section.

. . .

Definition 2.38 *If the definition of a quasi-x is* $A \implies B$, *then anti-x is* $B \implies A$ *and x or full-x is* $A \iff B$. *In other words, anti-x is the converse of quasi-x. x is the combination of quasi-x and anti-x.*

Terminology 2.39 *We create the terminology "mutation" which means* $\mu(x)$ *for some* x. *There is nothing new about mutations – they are simply the objects we have dealt with since the beginning. However, whenever we call an object a mutation, the indication is that we aim to specify it.*

Now, we look at invariance and whether or not a series of mutation schemes is invariant over a mutation. For this terminology definition, we show how quasi-, anti- and full-invariance are all related. All following (terminology) definitions will do this implicitly.

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[5] Convolutional pools are equivalent to pools, so we do not discuss them explicitly here.

Definition 2.40 *We now create a definition of quasi-invariance. Let $\mathcal{T}_1, \cdots, \mathcal{T}_n$ be mutation schemes of a convolutional pool. Let $\mathcal{T}_i(\xi_{1.}, \cdots, \xi_{n.}) = \eta.$. Then the following definitions stand:*

- *$\mathcal{T}_i$ is quasi-invariant over a mutation γ if and only if there is an i such that $|\xi_{i.}|/\gamma^n$ implies $|\eta.|/\gamma^n$.*
- *$\mathcal{T}_i$ is anti-invariant over a mutation γ if and only if $|\eta.|/\gamma$ implies there exists an i such that $|\xi_{i.}|/\gamma$.*
- *$\mathcal{T}_i$ is invariant over a mutation γ if and only if it is quasi- and anti-invariant over γ.*

This definition allows us to understand what will stay constant in a species. Note though that if $\mathcal{T}$ is invariant over γ in the pool $\mathcal{O}$ then $\tilde{O}$ is not necessarily divisible by γ, because a centre does not necessarily contain γ in its genome. However, when γ does appear in the genome then it will stay for all successive generations.

Definition 2.41 *Consider the mutation γ. Now let $\mathcal{T}_i$ stand for all mutation schemes of a (convolutional) pool. Let the following definitions hold:*

- *γ is quasi-necessary if $\xi. \sim \Gamma$ implies $|\xi.|/\gamma$.*
- *γ is quasi-malignant if $|\xi.|/\gamma$ implies $\xi. \sim \Gamma$.*
- *γ is quasi-specific to ℓ if $\xi. \sim \Gamma$ implies that $|\xi.|/\gamma$ at times t with $i \leq t \leq j$ only when $L_{t=k} \approx \ell$ for all $k \in [i, j]$.*
- *γ is quasi-general if it is not specific for any ℓ.*

Now we can properly talk about mutations, we can begin to analyse them. The following theorem shows some simple facts about mutations.

Theorem 2.42 ***Mutation Indicators*** *The following statements are all true for a degree-1 dynamic pools $\mathcal{O}$ with mutation scheme Ξ:*

1. *$\tilde{\Gamma}/\gamma$ is equivalent to γ is quasi-necessary.*
2. *If γ is always quasi-necessary then it is quasi-general (the same is true with anti- and full-necessity and -generality)*
3. *Ξ is invariant over γ if γ is quasi-malignant.*
4. *Given that only $\xi. \sim \mathcal{O}$, if Ξ is quasi-invariant over the quasi-specific mutation γ on $\ell andthetimes[i,j]then L_{t=k} \not\approx \ell \implies \mathcal{O} \mapsto \varnothing$ where $k >> j$.*
5. *Ξ not quasi-invariant over a quasi-necessary mutation γ implies that $|\Gamma| < |\mathcal{O}|$.*
6. *If and only if γ is (quasi-)necessary to the pool $\mathcal{O}$ with mutation scheme Ξ, the pool with population $\Gamma_{\mathcal{O}}$ and mutation scheme $\hat{\Xi}$ is (quasi-)invariant over γ. [By (quasi-), we mean that if γ is quasi-necessary then Ξ is quasi-invariant over γ, if it is anti-necessary, then it is anti-invariant, and if it is necessary then it is invariant.]*
7. *If and only if Ξ is quasi-invariant over γ then $\Delta\Xi_{\xi.} \nmid \overline{\gamma}$ given that $|\xi.|/\gamma$.*

Proof The proof for these lemmas will be given as an ordered list.

1. $\tilde{\Gamma}/\gamma$ implies that $|\xi.|/\gamma$ if $\xi. \sim \Gamma$, which is the definition of quasi-necessity.
2. If γ is always quasi-necessary then there does not exist a landscape for which it is not quasi-necessary, which is the definition of quasi-generality.
3. If γ is quasi-malignant then $|\xi.|/\gamma \implies \xi. \not\sim \Gamma$, so Ξ is not not invariant over γ. In a useless way, this is vacuously true.

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4. The contrapositive of this statement is the definition of a quasi-specific mutation. This lemma says that quasi-invariance over quasi-specific mutation is dangerous.

5. Anti-invariance is the property of a mutation scheme that says that a mutation cannot enter a system. That is, if the output of a mutation scheme has a mutation in their genome, then the input that generated that output also had that mutation in their genome. If a quasi-necessary mutation is not in the genome of any creature in that pool, then that mutation cannot enter the pool.

6. Since Ξ is not invariant over γ, there exists a $\lambda.$ such that $|\Xi(\lambda.)| \nmid \gamma$ and $|\lambda.|/\gamma$. But γ is quasi-necessary, so $\Xi(\lambda.) \nsim \Gamma$. Since $\Gamma \subseteq \mathcal{O}$ and there exists an element of $\mathcal{O}$ not in Γ, $|\Gamma| < |\mathcal{O}|$.

7. See that (quasi-)invariance over a mutation is given by a mutation's continued existence in a pool, and (quasi-)necessity is given by a mutation's continued existence in the gene pool (in the set of procreating creatures). Then the combination of these statements proves the claim.

8. If Ξ is quasi-invariant over γ then if γ^n is in $\xi.$'s genome then it will stay in $\xi.$'s offspring's genome. Hence, no mutation scheme can subtract a γ^i from the input's genome. Thus, there cannot be fewer γ's in the output's genome than in the input's genome. We have,

$$|\xi.| = x\gamma^i$$

With $x \nmid \gamma$ and $i \geq 1$ and

$$|\mathcal{T}(\xi.)| = y\gamma^j$$

With $y \nmid \gamma$ and $j \geq 1$. So $j \geq i$. By definition,

$$\Delta\Xi_{\xi.} = \frac{y\gamma^j}{x\gamma^i} = c\gamma^{j-i}$$

With $c \nmid \gamma$. Since $j - i \geq 0$, $c\gamma^{j-i} = c\gamma^k$ with $k \geq 0$. Thus $k \nless 0$ therefore $\Delta\Xi_{\xi.} \nmid \overline{\gamma}$.

. . .

The previous lemmas are mainly practical. That is, when trying to apply bioq to observed biology, the mutation indicators will often be referenced (hence, they have a name). Now we will deal with an important theorem that has useful applications in applied biology. The proof is poor and the claim is vague, so is only really used as an application, or to prove other primarily-applicable theorems.

Theorem 2.43 *If γ is quasi-necessary then probably each $\Xi_{\lambda.}$ elements of dynamic pool Ξ will be quasi-invariant over γ (given $\lambda. \sim \Gamma_\Xi$).*

Proof Elements of a (convolutional) pool that have mutation schemes that are invariant over some quasi-necessary mutation γ probably have more procreating offspring (by an extension of mutation indicator 7) because eventually, there must be a terminal node or (if there is no terminal node)

. Call these elements $\rho_{i.}$ As a result of this, the ratio of $\rho_{i.}$ offspring to non-$\rho_{i.}$ offspring is probably fairly large, because γ is quasi-necessary. As a result of this, a majority of the convolutional pool (and possibly eventually all of the convolutional pool) has mutation schemes that are quasi-invariant over γ.

. . .

The above proof is an example of when it is easier to prove a statement with a convolutional pool instead of a pool. This theorem suggests that there is a danger in specific mutations, because there is a large probability that a pool's mutation scheme will be quasi-invariant over a specific mutation. This dangerous, as shown by mutation indicator 4.

We now turn to a more specific case of dynamic pools in a hunt for the definition of "intelligence."

2.3 The Sophioid Theorem[6]

Now we use what we have learnt from dynamic pools and we create a new group of algebraic structures: societies. With these we can begin to study sophioid creatures and understand intelligence.

We must begin with the definition of a "society."

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[6] This chapter is inspired by a previous zoological paper of mine called "On Sophioid Theory" (abridged version in the appendix).

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Definition 2.44 *A society is a singleton-pod dynamic pool* S *and an operation* D *(called the gene-value) on that dynamic pool where any element* $s.$*'s survival is determined by the value of* $D(s.)$ *and other elements of the society's gene-values. For convenience,* $D(\xi.) \in \mathbb{R}$ *for any* $\xi. \in S$*.*

This definition does not yield many interesting theorems because it is not specific. However, it creates a space that can be filled with more applicable definitions, such as the following one:

Definition 2.45 *A hierarchy is a society where the gene-value function* H *follows the requirement that there is some value* j *where the* j *largest values of* $\{H(\lambda_{i.})\}$ *correlate with the* j *creatures that procreate. That is,*

$$\Gamma = \{\lambda_{i.} | H(\lambda_{i.}) > H(\delta_{k.}) \text{ iff } \delta_{k.} \nsim \Gamma\}$$

With $|\Gamma| = j$*.*

This gives us some more theorems to prove about mutations and their impact on the gene pool.

Theorem 2.46 *If* $H(\lambda.[\mu(x)]) > H(\lambda.)$ *for all* $\lambda.$ *then* $\mu(x)$ *is mostly quasi-necessary.*

Proof By definition, γ is quasi-necessary if $\xi. \sim \Gamma$ implies $|\xi.|/\gamma$. See that, since there are only j slots to fill in the Γ pool and $\xi.[\mu(x)]$'s will be valued more highly than $\xi.$'s without $\mu(x)$, $\mu(x)$ is mostly quasi-necessary. But notice that if there are enough $\lambda.$'s where $|\lambda.|/\mu(x)$, γ will be completely quasi-necessary.

...

The next theorem says that in a hierarchy, if creatures of one generation compete with creatures of another generation to procreate and there exists a mutation that increases their probability to procreate, then the levels of that mutation will increase over time.

Theorem 2.47 *For the next few proofs, let* $\lceil \xi. \rceil_k$ *be the greatest* i *such that* $|\xi.|/k^i$*. Then if* $H(\xi.[k^i]) > H(\xi.)$ *where* $i \geq 0$ *and different generations complete to procreate then* $\lceil \mathcal{O}(\lambda_{0.}. \cdots, \lambda_{n.}) \rceil_k > \lambda_{i.}$ *for any* i *from* 0 *to* n*.*

See briefly that the notation $\lceil \xi. \rceil_{\mu(x)}$ is actually the number of repetitions of some subset x in $\mu^{-1}|\xi.|$.

Proof The proof for this is fairly intuitive. Consider that the lowest procreating value of H is ℓ. That is:

$$\ell = \min\{H(\lambda.) | \lambda. \sim \Gamma\}.$$

Then consider that it is most likely (by the definition of a hierarchy) that, in order to procreate, any other creature $\xi.$ must have a value $H(\xi.) > \ell$. If $\xi. = \mathcal{O}(\lambda_{0.}, \cdots, \lambda_{n.})$ then we see that the statement is true.

...

Now that we understand a hierarchy, we can explore a sophioid hierarchy. However, this does not presently mean anything, because pinning down intelligence – as opposed to consciousness – is a difficult task. Thus, we define a sophioid function: $\int \xi.$ that has a codomain of the reals.

Definition 2.48 *A sophioid function has the properties that:*

1. $\int \xi. \in \mathbb{R}$ *for any* $\xi.$*.*
2. *There exists a mutation* n *(called the neural mutation) such that* $\int \xi - \lceil \xi. \rceil_n$ *is a monotonic function of* $\xi.$*.*
3. *The neural mutation is quasi-general.*

These definitions seem arbitrary, but there is a reasoning behind each one:

1. This is useful to have because it means that we can totally order any set of $\xi_i.$'s using the function $\int$.

2. Intuitively, this means that there is some genetic and physiological component to "intelligence," which is what we are trying to define. This should explain why the neural mutation is named as it is. Additionally, this aspect of the definition assumes that the physical augmentation of the "intelligence centres" will, on the whole, increase intelligence.

3. This says that sophioid intelligence is always critical to a species's survival.

Notably, this definition allows many non-intelligence attributes to be treated as intelligence. This is a case where a careful algebraic biologist will have to choose when sophioid theory truly applies. Indeed, sophioid theory most likely applies to traits other than intelligence, and so the theorems still stand up to interrogation in these circumstances. This will be an interesting result in its own right. If this still causes the reader difficulty, consider "true intelligence" to be a single case of sophioid-intelligence, which is an arbitrary name for a larger group of traits.

In the following section, we will be working with sophioid hierarchies – hierarchies with a sophioid function. For utter clarity of terminology, we say that:

Terminology 2.49 *A sophioid society is a society and a sophioid function. In general, the word "sophioid" can be adjoined to the beginning of any type of society's name (such as a hierarchy) to procure that same society with a sophioid function.*

Now we may begin to prove the theorem. First, we must create the "definitions" – the statements pertinent to the claim.

For simplicity, we will use the plash-structure instead of the dynamic pool. We can do this by a previous theorem that says that any theorem that applies to a convolutional pool applies to a pool, and vice versa. Let S be a sophioid society a gene-value function H and mutation scheme $\mathcal{T}$.

The first of these is Def 1: the dynamical enlargement of a cortex in the brain. The bioq representation of this is $\lceil \mathcal{T}(\xi_\bullet) \rceil_n > \lceil \xi_\bullet \rceil_n$ where $\xi_\bullet \sim \Gamma$ given the neural mutation n of S's sophioid function.

The second definition is Def 2: the reliance on sophioid intelligence. Bioqically, this is that n is quasi-necessary.

Finally, we have Def 3: the existence of an intellectual hiearchy.

By "intellectual hierarchy", we mean a sophioid hierarchy where $(H - \int)$ is one consistent sign (+, −, or 0) throughout S. Notice that this function exists because H and $\int$ both give real values and both take inputs from S.

To be clear, a species is sophioid-intelligent if it satisfies one of the definitions.

We are now set to take on the sophioid theorem.

Theorem 2.50 ***Sophioid Theorem*** *Def 1, Def 2, and Def 3 are roughly equivalent given that different generations compete to procreate. That is:*

$$\lceil \mathcal{T}(\xi_\bullet) \rceil_n > \lceil \xi_\bullet \rceil_n \text{ with } \xi_\bullet \sim \Gamma \iff n \text{ is quasi-necessary} \iff H - \int \text{ is monotonic.}$$

Proof Let I stand for Def 1, J stand for Def 2, and K stand for Def 3. We aim to prove that $I \iff J \iff K$. To do this, we will prove the following:

$$K \iff I \wedge J \text{ and } I \iff J$$

First, assume that K is true. Then by the definition of a sophioid function, the "magnitude" of the neural mutation in a genome is positively correlated with the value of the sophioid function, and the society in which they exist is an intellectual hierarchy, theorem 2.43 says that $\lceil \xi_{i\bullet} \rceil_n$ will get larger for successive generations.

We continue to assume that K holds. Then see that we have previously proven that n is mostly quasi-necessary, as desired. If there are enough sophioid intelligent creatures in a given society, then we can remove the word "roughly" from the theorem claim.

Next, we assume $I \wedge J$ and aim to prove K. To do this, we will show that $H - \int$ is one consistent sign by constructing a function $\int$ such that this is true and proving that it is a sophioid function. First, let there exist a real-valued function $\int$ such that $H - \int$ is one consistent value. Then we can limit it more so that $\int \xi_\bullet - \lceil \xi_\bullet \rceil_n$ is one consistent sign. We now only need to prove that n is quasi-general. Since n is quasi-necessary at any time, it is quasi-general, by our previous theorems.

Finally, we must prove that I and J are roughly equivalent. First we prove that roughly $J \implies I$. See that if n^i is quasi-necessary with $i = \lceil \xi . \rceil_n$, then $\lceil \mathcal{T}(\xi .) \rceil_n \geq \lceil \xi . \rceil_n$, by definition. Going the other way, if $\lceil \mathcal{T}(\xi .) \rceil_n > \lceil \xi . \rceil_n$ then $\lceil \mathcal{T}(\xi .) \rceil_n \geq \lceil \xi . \rceil_n$, so n^i is necessary with the same reasoning as before, so n is necessary.

...

Thus we conclude the sub-section on the sophioid theorem.

3 A Short Conclusion

To briefly conclude this document, I will now enumerate the various future directions bioq could take:

- Discussion of bacterial conjugation and lateral gene transfer as a unique case where $G(x)$ allows for complete associativity.
- Study of parasitism, mutualism, and commensualism.
- Investigation into disease and infection as a unique case of the previous.
- Creation of generalised theorems that discuss interpod (interspecies) interactions.
- Deconstruction of pools into singleton dynamic pools to treat different populations of one species separately.
- Prove more concretely the less rigorously justified theorems.
- Going deeper into non-degree-1 (convolutional) pools.

These are my present goals for trying to make algebraic biological language more versatile and useful. At this point, I feel that there is more than enough raw material (definitions, axioms, and structures) to generate unique insight that mathematics can deliver aside from that which the “logical intuition” of data analysis-driven biology can.

Finally, intentionally omitted from this document are examples of applications. While there are many examples I have on my mind, I feel that to put them here would be to dilute the hypothetical nature of this document and limit it. Rather, I would prefer to release such ideas separately from this, and leave it untainted as a central, core, forever-unfinished text.

4 Appendix

4.1 On Sophioid Theory[7] (as public in 2015, abridged)

4.1.1 Abstract

The definition of intelligence and the existence of non-human intelligence are two open-ended questions that have failed to be answered. Many answers have failed to find every intelligent species, whereas others rely on the mental capacities of an animal, specifically in ways that require written or verbal language. Because of this, many intelligent species (such as chelonians) were categorised as intuitive rather than decision-making animals. To fill this hole in neuroscience, three main definitions and three corollary ones (the sophioid indicators) have been created. Sophioid-intelligence (the specific intelligence described in this paper (see glossary) is (Def. 1) An evolutionary, dynamical enlargement of the medial temporal lobe and frontal lobe, specifically the hippocampus and prefrontal cortex; (Def 1.2) The exhibition of curiosity or non-traumatic memory recall, or repeating actions that have no biological benefit (nor negative consequence); (Def 1.3) The comprehension of perception in relation to reality; (Def. 2) The reliance upon sophioid-intelligence to survive or be chosen as a mate; (Def 2.2) The reliance upon catching sophioid-prey; and (Def. 3) The existence or use of an intellectual or sociological hierarchy. Where main indicators are (Def. #) and corollary ones are (Def. #.#). And, obviously, where a sophioid is an animal with sophioid-intelligence. There are many other nuances of terminology: there are classes of sophioid – lower, lower-medial, medial, higher-medial, and higher. These are primarily subjective, and so have little to no scientific value in research papers, but are helpful indicators when writing notes on an animal's behaviour, or a species' behaviour. Additionally, in a research paper, classes can be used to show an estimate of a species' intelligence. Other terminology includes sophioidism – the act of being a sophioid – and sophioid-prey. Obviously, sophioid-prey are animals that are preyed upon be other animals and are sophioids. Every sophioid-prey's predator, if they have a brain, are sophioids to some degree. Animal neuroscience is an interesting and developing field of work in which new sophioids are being discovered every day. This paper aims to fully explain sophioid theory so that species can be fully recognised before it is too late for them.

4.1.2 A Brief Introduction

The existence of animal intelligence is a long debate that has held the world's greatest neurobiologists and philosophers. Intuitively, humans cannot be the only "intelligent" species, so there must be other intelligent animals. The problem is the proof, though. In fact, the very definition of intelligence has been a topic of debate for just as long; how can we know that an animal is intelligent if we don't know what intelligence is? Humans value ourselves as sentient beings, that is, intelligent. With this, definitions have been rigorously proven and rigorously disproven, and still the scientific community has no functioning definition of intelligence. There is one point at which most agree, however. If an animal exhibits advanced neurocognitive behaviours, it is intelligent. But which 'neurocognitive behaviours' are eligible to be considered 'advanced', and at what level? This paper aims to show that a certain kind of intelligence (sophioid-intelligence) has created intelligence hierarchies – an advanced form of society – and should classically be considered "smart". Animals, including humans, with this intelligence are called sophioids, from the Greek: $\sigma o \phi \iota \alpha$, or wisdom.

4.1.3 Sophioid Definition

Many species exhibit behaviours that could be interpreted as 'intelligence', but the debate about whether anything non-human is intelligent still stands, as do many definitions. To avoid this problem of false negative definitions, the type of intelligence referenced in this paper will be named differently from every other kind of intelligence: sophioid-intelligence. Obviously, any creature that exhibits sophioid-intelligence is a sophioid. There are three main definitions of sophioids, and each one implies the others. These will be explored in the first portion of this paper, as well as the dynamical evolution of a Def. 2 sophioid.

The family Trochilidae is the most advanced neurocognition of all avian families known. This is because of one of the processes that creates sophioids. They show all of the definitions of a sophioid.

Hummingbirds memorise the location of every flower in their territory, as well as the calculation when each one will be ready to drink from again. This requires sophioid-intelligence, in both the medial

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temporal lobe and the frontal lobe, specifically, the hippocampus and prefrontal cortex. The females choose the males based on the size of their domain and their plumage. This shows that the females choose the most intelligent, or at least, the most intelligent male available. This renders four possibilities.

Assuming the female in question was not a sophioid individual, one of the offspring hummingbirds would either be (a) a non-sophioid male, (b) a sophioid male, (c) a non-sophioid female, or (d) a sophioid female. In case (a), the male would not be chosen and his DNA would not be passed down, or, if it did, the set of cases would repeat. In cases (b) and (c) the set of cases would repeat, but, in case (d), the second set rendered would be (a) a sophioid male, or 9b) a sophioid female. Both of these repeat either the original set of possibilities or the secondary one. Because of this system, the whole species becomes more intelligent as the number of sophioid females increases the secondary system's use.

Simply, the result is sophioid-intelligence. This creates one of the definitions of sophioid-intelligence, and sophioids in general. The following 'definitions' are more technically indicators, and so should not be held to definition standards, hence their name: sophioid indicators:

> (Def. 1) An evolutionary, dynamical enlargement of the medial temporal lobe and frontal lobe, specifically the hippocampus and prefrontal cortex.

Essentially, proof of time-dependent, evolutionary growth of the specific sections of the brain. A corollary definition is:

> (Def 1.2) The exhibition of curiosity or non-traumatic memory recall, or repeating actions that have no biological benefit

. An example of a "repeating action that has no biological benefit" is spinning and jumping. Spinner dolphins spin and jump for no apparent reason, showing, at least a form of, enjoyment. A second following definition is Def 1.3:

> (Def 1.3) The comprehension of perception in relation to reality.

The Trochilidae family also clearly shows a second of the three definitions:

> (Def. 2) The reliance upon sophioid-intelligence to survive or be chosen as a mate.

There exists a corollary definition for Def. 2:

> (Def 2.2) The reliance upon catching sophioid-prey.

Given that sophioid-prey is prey that is sophioid-intelligent. If an animal such as a jellyfish, for example, relies of catching sophioids, then it is obviously null, as a jellyfish has no brain.

A third definition remains:

> (Def. 3) The existence or use of an intellectual or sociological hierarchy.

The intellectual hierarchy that the third indicator references is the system of male selection from the species' point of view as opposed to the male's. Effectively, for Trochilidae, the alpha male is the most intelligent, and then the beta, etcetera.

Types of Sophioids There are three main classes of sophioid. These are primarily subjective as no accurate system can completely measure intelligence. This system of types is called the 'class' of the sophioid. The first class is a lower sophioid. A lower sophioid is an animal that has limited hierarchical and social structure, yet still has a basic understanding of its surroundings.

The next type is a higher sophioid. This category encompasses cephalopods, simians, mice – anything with 'advanced' neurocognition. A sure sign of this is metacognition, though species without it may still be higher sophioids.

Medial sophioids are the bridge between the higher and lower sophioids. Anything that does not belong in the higher or lower classes is a form of medial sophioid. Another reason to use the term 'medial sophioid' is if a collection of animals is a sophioid as a whole. For example, if a family consists of higher and lower sophioids, then the collection is said to be medially sophioid. If a collection of sophioids is skewed towards one more than the other (one higher sophioids and five lower sophioids) then the mode sophioid will be chosen. If the levels of higher sophioid to lower sophioid are approximately equal, then the group will be deemed a medial sophioid. And, obviously, for any single species, a combination class can be used as a class.

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Another way of expressing a sophioid's differences is by including the definition it was discovered to be a sophioid as. In general, this is the most obvious definition, thus giving more information than the simple tag 'sophioid'. A cetacean would be a Def. 3 Sophioid (also called an S3 species), while a hummingbird would be a Def. 2 Sophioid. This system is not infallible, as the signs of a Def. 2 Sophioid may not be recognised until dissection, and so will be a Def. 1 Sophioid instead.

Every sophioid shows every definition, but some stand out more than others. This produces some interesting results.

4.1.4 Scientific Usage of the Three Sophioid Definitions

Every sophioid must show all three definitions of a sophioid, at least by some degree, but can be identified by only one. This means that additional information can be obtained by observation of an animal alone. If a proof is submitted that one animal fits one of the definitions, then it must fit all of them. This means that an insight into a species' brain, society or life can be obtained with minimal intrusion and the absence of death (unless physical corroboration is required; if such a did arise where this paper is treated as false, the definitions can be used to direct the experiments towards an accurate conclusion more efficiently than the estimation method).

4.1.5 Conclusion

Many sophioids are unknown to man, but, equipped with this knowledge, we can discover them and learn from them. And, with sophioid theory, they can be discovered and studied with minimal, if no, interference. This is an important aspect of the theory, as interaction with an intelligent animal can be disastrous. Sophioid theory can now be used to protect injured animals from trusting humans or associating themselves with humans. In general, sophioid theory can be used to help both sophioids and scientists alike.