 doi:10.20944/preprints202002.0161.v1

New generalized the Hermite-Hadamard inequality and related integral inequalities involving Katugampola type fractional integrals

Ohud Almutairi[1] and Adem Kiliman[2]
Department of Mathematics, University of Hafr Al-Batin, Hafr Al-Batin 31991,Sudia Arabia.
Universiti Putra Malaysia, Serdang 43400 UPM, Selangor, Malaysia
[1]AhoudbAlmutairi@gmail.com and [2]akilic@upm.edu.my

February 12, 2020

Abstract.
In this paper, a new identity for the generalized fractional integral is defined, through which new integral inequality for functions whose first derivatives in absolute value are convex. The new generalized Hermite-Hadamard inequality for generalized convex function on fractal sets involving Katugampola type fractional integral is established. We derive trapezoid and mid-point type inequalities connected to these generalized Hermite-Hadamard inequality.
Keywords: Convex function; generalized convex function; HermiteHadamard inequality; Katugampola fractional integral.

1 Introduction

The emergence of convexity theory, in the field of mathematical analysis, has been considered as the remarkable development therein. The technique is robust in handling numerous problems, most of which exist in both pure and applied sciences. Due to the wide applications of convexity, a verity of new convex functions has been reported and studied in the literature. Therefore, the definition of a classical convex function is given below.

Definition 1. *A function* $\mathcal{G} : V \subseteq \mathbb{R} \to \mathbb{R}$ *is said to be convex if*

$$\mathcal{G}(\vartheta m + (1 - \vartheta n)) \leq \vartheta \mathcal{G}(m) + (1 - \vartheta)\mathcal{G}(n),$$

holds for all $\mathrm{m}, \mathrm{n} \in \mathbb{R}$ *and* $\vartheta \in [0, 1]$.

One of the most important applications of this function is the formulation of inequalities. Many new classes of inequalities related to the convex functions have been derived and applied to other field of studies (see [6]-[7]). The most interesting class of such inequalities, through which many problems in finance, engineering and science are investigated, is of Hermite-Hadamards type. Considering a convex function $\mathcal{G} : V \subseteq \mathbb{R} \to \mathbb{R}$, the Hermite-Hadamard inequality can be defined if and only if the following inequality is satisfied,

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$$\mathcal{G}\left(\frac{m+n}{2}\right) \leq \frac{1}{n-m}\int_{m}^{n}\mathcal{G}(x)dx \leq \frac{\mathcal{G}(m)+\mathcal{G}(n)}{2}. \tag{1}$$

This inequality is extended to include problems related to fractional calculus, a branch of calculus dealing with derivatives and integrals of non-integer order (see [9, 3, 20, 5, 16, 10, 15]). Nowadays, the real-life applications of fractional calculus exist in most areas of studies [1, 2]. To make the application of fractional calculus easier, Mathematicians defined its derivatives and integrals differently. One of the most widely used approaches is the Riemann-Liouville operator method. The detail of this method can found in the following references [4, 17]. The work of Sarikaya et al. [8] on the formulation of Hermite-Hadamard inequality, via Rieman-Liouville fractional integral, has fascinated many researchers to contribute to this field. Thus, the Sarikaya inequality is given as follows.

Theorem 1. *Let $\mathcal{G} : [m, n] \to \mathbb{R}$ be a positive function with $0 \leq m < n$ and $\mathcal{G} \in L[m, n]$. If $\mathcal{G}$ is a convex function on $[m, n]$, then the following inequalities hold*

$$\mathcal{G}\left(\frac{m+n}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(n-m)^{\lambda}}\left[J^{\lambda}_{m^+}\mathcal{G}(n) + J^{\lambda}_{n^-}\mathcal{G}(m)\right] \leq \frac{\mathcal{G}(m)+\mathcal{G}(n)}{2} \tag{2}$$

with $\lambda > 0$.

Using this approach, many new inequalities have been obtained and reported in the literature. For example, an important lemma was established through the Rieman-Liouville fractional calculus and reported in [22] as follows.

Theorem 2. *Suppse that $\mathcal{G} : [m, n] \to \mathbb{R}$ is a differentiable function on (m, n), where $m < n$. If $|\mathcal{G}'|$ is convex on $[m, n]$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{\Gamma(\lambda+1)}{2(n-m)^{\lambda}}\left[J^{\lambda}_{m^+}\mathcal{G}(n) + J^{\lambda}_{n^-}\mathcal{G}(n)\right] - \mathcal{G}\left(\frac{m+n}{2}\right)\right| \\ \leq &\frac{n-m}{4(\lambda+1)}\left(\lambda+3-\frac{1}{2^{\lambda-1}}\right)\left[|\mathcal{G}'(m)| + |\mathcal{G}'(n)|\right]. \end{aligned} \tag{3}$$

Other improvements, on Hermite-Hadamard type inequalities, include the introduction of generalized convex function on fractal sets [12]. Therefore, the definition of this concept is given below.

Definition 2. *Let $\mathcal{G} : V \subset \mathbb{R} \to \mathbb{R}^{\lambda}(0 < \lambda < 1)$. If the following inequality,*

$$\mathcal{G}(\vartheta m + (1-\vartheta)n) \leq \vartheta^{\lambda}\mathcal{G}(m) + (1-\vartheta)^{\lambda}\mathcal{G}(n), \tag{4}$$

holds for any $m, n \in V$ and $\vartheta \in [0, 1]$, then $\mathcal{G}$ is called a generalized convex on V.

The Rieman-Liouville fractional integral, along with the Hadamards fractional integral, is generalized through the recent work of Katugampola. These two integrals are given into a single form (see [13]).

Definition 3. *Let $[m, n] \subset \mathbb{R}$ be a finite interval. Then, the left-and right-sided Katugampola fractional integrals of order $\lambda > 0$ for $\mathcal{G} \in X^{p}_{c}(m, n)$ are defined by,*

$${}^{\rho}I^{\lambda}_{m^+}\mathcal{G}(x) = \frac{\rho^{1-\lambda}}{\Gamma(\alpha)}\int_{\mathrm{m}}^{x}\frac{\vartheta^{\rho-1}}{(x^{\rho}-\vartheta^{\rho})^{1-\lambda}}\mathcal{G}(\vartheta)d\vartheta \quad \text{and} \quad {}^{\rho}I^{\lambda}_{n^-}\mathcal{G}(x) = \frac{\rho^{1-\lambda}}{\Gamma(\lambda)}\int_{x}^{n}\frac{\vartheta^{\rho-1}}{(\vartheta^{\rho}-x^{\rho})^{1-\lambda}}\mathcal{G}(\vartheta)d\vartheta.$$

with $m < x < n$, $\rho > 0$. Given the space of complex-valued Lebesgue measurable function ω as $X_c^p(m,n)(c \in \mathbb{R}, 1 \le p \le \infty)$, we define the norm of the function on $[m,n]$ as follows

$$\|\mathcal{G}\|_{X_c^p} = \left(\int_m^n |\vartheta^c \mathcal{G}(\vartheta)|^p \frac{d\vartheta}{\vartheta} \right)^{1/p} < \infty,$$

whereby $1 \le p < \infty, c \in \mathbb{R}$. If $p = \infty$, we obtain

$$\|\mathcal{G}\|_{X_c^\infty} = \text{ess} \sup_{m \le \vartheta \le n} [\vartheta^c |\mathcal{G}(\vartheta)|] .$$

Other related works include the generalization of Hermite-Hadamard inequality for Katugampola fractional integrals [19], given in the following lemma, as well as the theorem that follows immediately.

Lemma 1. *Let $\mathcal{G} : [m^\rho, n^\rho] \to \mathbb{R}$ be a differentiable mapping on (m^ρ, n^ρ), with $0 \le m < n$. If the fractional integrals exist, we obtain the following inequality,*

$$\frac{\mathcal{G}(m^\rho) + \mathcal{G}(n^\rho)}{2} - \frac{\lambda \rho^\lambda \Gamma(\lambda+1)}{2(n^\rho - m^\rho)^\lambda} \left[{}^\rho I_{m^+}^\lambda \mathcal{G}(n^\rho) + {}^\rho I_{n^-}^\lambda \mathcal{G}(m^\rho) \right] = \frac{n^\rho - m^\rho}{2} \int_0^1 \left[(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda} \right] \vartheta^{\rho-1} \times \mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho) n^\rho\right) d\vartheta. \quad (5)$$

Theorem 3. *Let $\lambda > 0$ and $\rho > 0$. Let $\mathcal{G} : [m^\rho, n^\rho] \to \mathbb{R}$ be a non-negative function with $0 \le m < n$ and $\mathcal{G} \in X_c^p(m^\rho, n^\rho)$. If $\mathcal{G}$ is also a convex function on $[m,n]$, then we have*

$$\mathcal{G}\left(\frac{m^\rho + n^\rho}{2}\right) \le \frac{\rho^\alpha \Gamma(\lambda+1)}{2(n^\rho - m^\rho)^\lambda} \left[{}^\rho I_{m^+}^\lambda \mathcal{G}(n^\rho) + {}^\rho I_{n^-}^\lambda \mathcal{G}(m^\rho) \right] \le \frac{\mathcal{G}(m^\rho) + \mathcal{G}(n^\rho)}{2}, \quad (6)$$

whereby the fractional integrals are given for the function $\mathcal{G}(x^\rho)$ and evaluated at m and n, respectively.

Katugampola fractional integrals have many applications in the fields of science and technology, some of which can be found in the following references [21], [25]. Therefore, many generalizations of different inequalities are studied via these fractional integrals. For example, Kermausuor [24] and Mumcu et al, [14] generalized Ostrowski-type and Hermite-Hadamard type inequalities for harmonically convex functions, respectively. Therefore, the aim of this paper is to generalize the Hermite-Hadamard inequality for generalized convex functions on fractal sets via Katugampola fractional integrals. This can be the generalization of the work of Chen and Katugampola [19], who proposed the inequality stated in Theorem 3. Another objective of this study is to define a new identity for generalized fractional integrals, through which generalized Hermite-Hadamard type inequalities for convex function are derived. The trapezoid and mid-point type inequalities are also proposed for the generalized convex function involving Katugampola fractional integrals, which would generalize the Riemann-Liouville and the Hadamard integrals into a single form.

 doi:10.20944/preprints202002.0161.v1

2 New generalized fractional integrals identity and new integral inequality for Katugampola fractional integrals

In order to improve the identity established in [22] for generalized fractional integrals, the following lemma can be used to prove our results.

Lemma 2. *Let $\mathcal{G} : [m^\rho, n^\rho] \to \mathbb{R}$ be a differentiable mapping on (m^ρ, n^ρ), where $m < n$. The following equality holds, if the fractional integrals exist,*

$$\begin{aligned}
&\frac{\lambda\rho^\alpha\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^\lambda}[{}^\rho J^\lambda_{m^+}\mathcal{G}(n^\rho) + {}^\rho J^\lambda_{n^-}\mathcal{G}(m^\rho)] - \mathcal{G}\left(\frac{m^\rho+n^\rho}{2}\right)\\
=&\frac{n^\rho-m^\rho}{2}\left[\int_0^1 M\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta - \int_0^1\left[(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda}\right]\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta\right],
\end{aligned} \tag{7}$$

where

$$M = \begin{cases} \vartheta^{\rho-1}, & 0 \le \vartheta < \frac{1}{\rho\sqrt{2}} \\ -\vartheta^{\rho-1}, & \frac{1}{\rho\sqrt{2}} \le \vartheta < 1. \end{cases}$$

Proof. It suffices to note that

$$\begin{aligned}
I =&\left[\int_0^{\frac{1}{\rho\sqrt{2}}}\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta\right] + \left[-\int_{\frac{1}{\rho\sqrt{2}}}^1\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta\right]\\
&+\left[-\int_0^1\left[(1-\vartheta^\rho)^\lambda\right]\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta\right] + \left[\int_0^1\left[\vartheta^{\rho\lambda}\right]\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta\right]\\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{8}$$

Integrating by parts, we get I_1 and I_2 as follows,

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{\rho\sqrt{2}}}\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)dt = \frac{1}{m^\rho-n^\rho}\mathcal{G}(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)\Big|_0^{\frac{1}{\rho\sqrt{2}}}\\
&= \frac{1}{\rho(n^\rho-m^\rho)}\left[-\mathcal{G}\left(\frac{m^\rho+n^\rho}{2}\right) + \mathcal{G}(n^\rho)\right],
\end{aligned} \tag{9}$$

$$\begin{aligned}
I_2 &= -\int_{\frac{1}{\rho\sqrt{2}}}^1\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta = \frac{-1}{m^\rho-n^\rho}\mathcal{G}(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)\Big|_{\frac{1}{\rho\sqrt{2}}}^1\\
&= \frac{1}{n^\rho-m^\rho}\left[\mathcal{G}(m^\rho) - \mathcal{G}\left(\frac{m^\rho+n^\rho}{2}\right)\right].
\end{aligned} \tag{10}$$

Set $x^\rho = \vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho$ for calculating I_3 and I_4,

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$$
\begin{aligned}
I_3 &= -\int_0^1 (1-\vartheta^\rho)^\lambda \mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)d\vartheta \\
&= -\frac{(1-\vartheta^\rho)^\lambda}{m^\rho - n^\rho}\mathcal{G}(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n)\Big|_0^1 - \frac{\lambda}{m^\rho - n^\rho}\int_0^1 (1-\vartheta^\rho)^{\lambda-1}\mathcal{G}(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n)d\vartheta \\
&= -\frac{\mathcal{G}(n^\rho)}{n^\rho - m^\rho} + \frac{\lambda}{n^\rho - m^\rho}\int_{n^\rho}^{m^\rho}\left(\frac{x^\rho - m^\rho}{n^\rho - m^\rho}\right)^{\lambda-1}\mathcal{G}(x^\rho)\frac{x^\rho}{m^\rho - n^\rho}dx \\
&= -\frac{\mathcal{G}(n^\rho)}{\rho(n^\rho - m^\rho)} + \frac{\lambda\rho^{\lambda-1}\Gamma(\lambda+1)}{(n^\rho - m^\rho)^{\lambda+1}}{}^\rho I_{n^-}^\lambda \mathcal{G}(m^\rho),
\end{aligned} \tag{11}
$$

$$
I_4 = -\int_0^1 \vartheta^{\rho\alpha}\cdot\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)d\vartheta = -\frac{\mathcal{G}(m^\rho)}{\rho(n^\rho - m^\rho)} + \frac{\lambda\rho^{\lambda-1}\Gamma(\lambda+1)}{(n^\rho - m^\rho)^{\lambda+1}}{}^\rho I_{m^+}^\lambda \mathcal{G}(n^\rho). \tag{12}
$$

Now substituting inequalities (9), (10), (11) and (12) into (8) compeletes the proof. □

Remark 1. *If $\rho = 1$, then the identity (7) in Lemma 2 reduces to identity (3) in Lemma 2.1 [22].*

Using Lemma 2, the following result for differentiable function is obtained.

Theorem 4. *Let $\mathcal{G} : [m^\rho, n^\rho] \to \mathbb{R}$ be a differentiable mapping on (m^ρ, n^ρ) with $0 \le m < n$. If $|\mathcal{G}'|$ is convex on $[m^\rho, n^\rho]$, then the following inequality holds:*

$$
\left|\frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho - m^\rho)^\lambda}\left[{}^\rho I_{m^+}^\lambda \mathcal{G}(n^\rho) + {}^\rho I_{n^-}^\alpha \mathcal{G}(m^\rho)\right] - \mathcal{G}\left(\frac{m^\rho + n^\rho}{2}\right)\right| \le \frac{n^\rho - m^\rho}{4\rho(\lambda+1)}\left[3+\lambda-\frac{1}{2^{\lambda-1}}\right]\left[|\mathcal{G}'(m^\rho)| + |\mathcal{G}'(n^\rho)|\right]. \tag{13}
$$

Proof. Usining Lemma 2 and the convexity of $|\mathcal{G}'|$, we get

$$
\begin{aligned}
&\left|\frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho - m^\rho)^\lambda}\left[{}^\rho I_{m^+}^\lambda \mathcal{G}(n^\rho) + {}^\rho I_{n^-}^\lambda \mathcal{G}(m^\rho)\right] - \mathcal{G}\left(\frac{m^\rho + n^\rho}{2}\right)\right| \\
&\le \frac{n^\rho - m^\rho}{2}\Bigg[\int_0^{\frac{1}{\rho\sqrt{2}}} \vartheta^{\rho-1}|\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)|d\vartheta + \int_{\frac{1}{\rho\sqrt{2}}}^1 t^{\rho-1}|\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)|d\vartheta \\
&+ \int_0^1 |(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda}|\vartheta^{\rho-1}|\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)|d\vartheta\Bigg] \\
&\le \frac{n^\rho - m^\rho}{2}\Bigg[\int_0^{\frac{1}{\rho\sqrt{2}}} \vartheta^{\rho-1}[\vartheta^\rho|\mathcal{G}'(m^\rho)| + (1-\vartheta^\rho)|\mathcal{G}'(n^\rho)|]d\vartheta + \int_{\frac{1}{\rho\sqrt{2}}}^1 \vartheta^{\rho-1}[\vartheta^\rho|\mathcal{G}'(m^\rho)| + (1-\vartheta^\rho)|\mathcal{G}'(n^\rho)|]d\vartheta \\
&+ \int_0^1 |(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda}|\vartheta^{\rho-1}|\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)|d\vartheta\Bigg].
\end{aligned} \tag{14}
$$

$$\left|\frac{\lambda\rho^{\lambda}\Gamma(\lambda+1)}{2(n^{\rho}-m^{\rho})^{\lambda}}\left[{}^{\rho}I^{\lambda}_{m+}\mathcal{G}\left(n^{\rho}\right)+{}^{\rho}I^{\lambda}_{n-}\mathcal{G}\left(m^{\rho}\right)\right]-\mathcal{G}\left(\frac{m^{\rho}+n^{\rho}}{2}\right)\right|=\frac{n^{\rho}-m^{\rho}}{2}[I_1+I_2+I_3], \tag{15}$$

whereby I_1, I_2 and I_3 are the first, second and third integrals in inequality (14).
When calculating I_1 and I_2, we get the following

$$I_1=\int_0^{\frac{1}{\rho\sqrt{2}}}\vartheta^{\rho-1}|\mathcal{G}'(\vartheta^{\rho}m^{\rho}+(1-\vartheta^{\rho})n^{\rho})|d\vartheta=\frac{1}{\rho 8}\left|\mathcal{G}'(m^{\rho})\right|+\frac{3}{\rho 8}\left|\mathcal{G}'(n^{\rho})\right|. \tag{16}$$

$$I_2=\int_{\frac{1}{\rho\sqrt{2}}}^{1}\vartheta^{\rho-1}|\mathcal{G}'(\vartheta^{\rho}m^{\rho}+(1-\vartheta^{\rho})n^{\rho})|=\frac{3}{\rho 8}\left|\mathcal{G}'(m^{\rho})\right|+\frac{1}{\rho 8}\left|\mathcal{G}'(n^{\rho})\right|. \tag{17}$$

A similar line of argument for the proof of Theorem 2.5 in [19] can be used to calculate I_3,

$$I_3=\int_0^1\left[(1-\vartheta^{\rho})^{\lambda}-\vartheta^{\rho\lambda}\right]\vartheta^{\rho-1}\mathcal{G}'(\vartheta^{\rho}m^{\rho}+(1-\vartheta)^{\rho}b^{\rho})dt=\frac{1}{\rho(\lambda+1)}\left(1-\frac{1}{2^{\lambda}}\right)\left[\left|\mathcal{G}'\left(m^{\rho}\right)\right|+\left|\mathcal{G}'\left(n^{\rho}\right)\right|\right]. \tag{18}$$

Submiting inqualities (16), (17) and (18) in (15), we get (13). This completes the proof. □

Remark 2. i. *Choosing* $\rho=1$ *in* Theorem 4 will reduces inequality (13) to inequality (3) of Theorem 2.

ii. Choosing $\rho=1$ and $\alpha=1$ reduces inequality (13) to inequality (16) in [22], which is given as follows

$$\left|\frac{1}{n-m}\int_m^n\mathcal{G}(x)dx-\mathcal{G}\left(\frac{m+n}{2}\right)\right|\leq\frac{3(n-m)}{8}\left(|\mathcal{G}'(m)|+|\mathcal{G}'(n)|\right).$$

3 Generalized Hermite-Hadamard inequality and related integral inequalities for Katugampola fractional integral on fractal set

The following theorem generalizes the result obtained by [19] of the Hermite-Hadamard inequality invoving the Katugampola fractional integrals for generalized convex function on fractal sets.

Theorem 5. *Suppose that* $\mathcal{G}:[m^{\rho},n^{\rho}]\subset\mathbb{R}_+\to\mathbb{R}^{\lambda}$ *is a positive function with* $0\leq m<n$ *and* $\mathcal{G}\in X^p_c\left(m^{\rho},n^{\rho}\right)$ *for* $\lambda>0$ *and* $\rho>0$. *If* $\mathcal{G}$ *is a generalized convex function on* $[m^{\rho},n^{\rho}]$, *then we obtain*

$$\begin{aligned}\mathcal{G}\left(\frac{m^{\rho}+n^{\rho}}{2}\right)&\leq\frac{\rho^{\lambda}\Gamma(\lambda+1)}{2^{\lambda}\left(n^{\rho}-m^{\rho}\right)^{\lambda}}\left[{}^{\rho}I^{\alpha}_{m+}\mathcal{G}\left(n^{\rho}\right)+{}^{\rho}I^{\lambda}_{n-}\mathcal{G}\left(m^{\rho}\right)\right]\\&\leq\frac{\mathcal{G}\left(m^{\rho}\right)+\mathcal{G}\left(n^{\rho}\right)}{2^{\lambda}}.\end{aligned} \tag{19}$$

Proof. Suppose that $x,y\in[m,n],a>0$, defined by $x^{\rho}=\vartheta^{\rho}m^{\rho}+\left(1-\vartheta^{\rho}\right)n^{\rho}$ and $y^{\rho}=\vartheta^{\rho}n^{\rho}+\left(1-\vartheta^{\rho}\right)m^{\rho}$, where $\vartheta\in[0,1]$. Since $\mathcal{G}$ is generalized convex function, we have

$$2^{\lambda}\mathcal{G}\left(\frac{m^{\rho}+n^{\rho}}{2}\right)\leq\mathcal{G}\left(\vartheta^{\rho}m^{\rho}+\left(1-\vartheta^{\rho}\right)n^{\rho}\right)+\mathcal{G}\left(\vartheta^{\rho}n^{\rho}+\left(1-\vartheta^{\rho}\right)m^{\rho}\right). \tag{20}$$

Multiplying both sides of the inequality (20) by $\vartheta^{\alpha\rho-1}$, for $\lambda > 0$ and then integrating over $[0,1]$ with respect to ϑ, we obtain the following

$$\begin{aligned}\frac{2^\lambda}{\lambda\rho}\mathcal{G}\left(\frac{m^\rho+n^\rho}{2}\right) &\leq \int_0^1 \vartheta^{\lambda\rho-1}\mathcal{G}\left(\vartheta^\rho m^\rho+(1-\vartheta^\rho)\,n^\rho\right)d\vartheta + \int_0^1 \vartheta^{\lambda\rho-1}\mathcal{G}\left(\vartheta^\rho n^\rho+(1-\vartheta^\rho)\,m^\rho\right)d\vartheta \\ &= \int_n^m \left(\frac{n^\rho-x^\rho}{n^\rho-m^\rho}\right)^{\lambda-1}\mathcal{G}\left(x^\rho\right)\frac{x^{\rho-1}}{m^\rho-n^\rho}dx \\ &\quad + \int_m^n \left(\frac{y^\rho-m^\rho}{n^\rho-m^\rho}\right)^{\lambda-1}\mathcal{G}\left(y^\rho\right)\frac{y^{\rho-1}}{n^\rho-m^\rho}dy \\ &= \frac{\rho^{\lambda-1}\Gamma(\alpha)}{(n^\rho-m^\rho)^\lambda}\left[{}^\rho I^\lambda_{m+}\mathcal{G}\left(n^\rho\right)+{}^\rho I^\lambda_{n-}\mathcal{G}\left(m^\rho\right)\right].\end{aligned} \tag{21}$$

This establishes the first inequality. When proving the second inequality (19), we first observed generalized convex functions $\mathcal{G}$, which is given as

$$\mathcal{G}\left(\vartheta^\rho m^\rho+(1-\vartheta^\rho)\,n^\rho\right) \leq \left(\vartheta^\rho\right)^\lambda\mathcal{G}\left(m^\rho\right)+\left(1-\vartheta^\rho\right)^\lambda\mathcal{G}\left(n^\rho\right), \tag{22}$$

and

$$\mathcal{G}\left(\vartheta^\rho n^\rho+(1-\vartheta^\rho)\,m^\rho\right) \leq \left(\vartheta^\rho\right)^\lambda\mathcal{G}\left(n^\rho\right)+\left(1-\vartheta^\rho\right)^\lambda\mathcal{G}\left(m^\rho\right). \tag{23}$$

Summing the above inequalities, we have

$$\mathcal{G}\left(\vartheta^\rho m^\rho+(1-\vartheta^\rho)\,n^\rho\right)+\mathcal{G}\left(\vartheta^\rho n^\rho+(1-\vartheta^\rho)\,m^\rho\right) \leq \mathcal{G}\left(m^\rho\right)+\mathcal{G}\left(n^\rho\right). \tag{24}$$

Multiplying both sides of inequality (24) by $\vartheta^{\alpha\rho-1}$, for $\alpha > 0$ and integrating the result over $[0,1]$ with respect to ϑ, we obtain

$$\frac{\rho^{\lambda-1}\Gamma(\lambda)}{(n^\rho-m^\rho)^\lambda}\left[{}^\rho I^\lambda_{m+}\mathcal{G}\left(n^\rho\right)+{}^\rho I^\lambda_{n-}\mathcal{G}\left(m^\rho\right)\right] \leq \frac{\mathcal{G}\left(m^\rho\right)+\mathcal{G}\left(n^\rho\right)}{\lambda\rho}. \tag{25}$$

This completes the proof. □

Remark 3. *Taking $\lambda = 1$ in inequality (19) of Theorem 5 reduces the result to inequality (6) of Theorem 3.*

Now, we derive the mid-point type inequalities via generalized convex functions on fractal set for Katugampola fractional integral. Therefore, the definition of generalized beta function is given as follows

$$\beta_\rho(m,n) = \int_0^1 \rho\left(1-x^\rho\right)^{n-1}\left(x^\rho\right)^{m-1}x^{\rho-1}dx.$$

Note that, as $\rho \to 1, \beta_\rho(m,n) \to \beta(m,n)$.

Theorem 6. *Suppose that $\alpha > 0$ and $\rho > 0$. Let $f : [a^\rho, b^\rho] \subset \mathbb{R}_+ \to \mathbb{R}^\alpha$ be a differentiable function on (a^ρ, b^ρ), and $f' \in L^1[a,b]$ with $0 \leq a < b$. If $|f'|^q$ is generalized convex on $[a^\rho, b^\rho]$, we obtain*

$$\left|\frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^\lambda}\left[{}^\rho J^\lambda_{m+}\mathcal{G}(n^\rho)+{}^\rho J^\lambda_{n-}\mathcal{G}(m^\rho)\right]-\mathcal{G}\left(\frac{m^\rho+n^\rho}{2}\right)\right| \leq \frac{n^\rho-m^\rho}{2}\left[\frac{\beta_\rho(\lambda+1,\lambda+1)}{\rho}\right]\left[|\mathcal{G}'(m^\rho)|+|\mathcal{G}'(n^\rho)|\right]. \tag{26}$$

 doi:10.20944/preprints202002.0161.v1

Proof. From Lemma 2, we have

$$\begin{aligned}
&\left|\frac{\lambda\rho^{\lambda}\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^{\lambda}}[^\rho J^{\lambda}_{m+}\mathcal{G}(n^\rho)+^\rho J^{\lambda}_{n-}\mathcal{G}(m^\rho)]-\mathcal{G}\left(\frac{m^\rho+n^\rho}{2}\right)\right|\\
\leq\ &\frac{n^\rho-m^\rho}{2}\left|\int_0^{\frac{1}{\rho\sqrt{2}}}\vartheta^{\rho-1}[\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)]d\vartheta+\int_{\frac{1}{\rho\sqrt{2}}}^{1}\vartheta^{\rho-1}[\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)]d\vartheta\right.\\
&\left.-\int_0^1\left[(1-\vartheta^\rho)^{\lambda}-\vartheta^{\rho\lambda}\right]\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta\right|\\
\leq\ &\frac{n^\rho-m^\rho}{2}\left|\int_0^{\frac{1}{\rho\sqrt{2}}}\vartheta^{\rho-1}[\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)]d\vartheta+\int_{\frac{1}{\rho\sqrt{2}}}^{1}t^{\rho-1}[\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)]d\vartheta\right|\\
+\ &\left|\int_0^1[(1-\vartheta^\rho)^{\lambda}-\vartheta^{\rho\lambda}]\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta\right|. \qquad (27)
\end{aligned}$$

Using the fact that the function $|\mathcal{G}'|$ is generalized convex on $[m^\rho,n^\rho]$, we obtain the following

$$\begin{aligned}
\left|\int_0^{\frac{1}{\rho\sqrt{2}}}\vartheta^{\rho-1}[\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)]d\vartheta\right| &\leq\int_0^{\frac{1}{\rho\sqrt{2}}}\vartheta^{\rho-1}[\vartheta^{\rho\lambda}|\mathcal{G}'(m^\rho)|+(1-\vartheta^\rho)^{\lambda}|\mathcal{G}'(n^\rho)|]dt\\
&\leq|\mathcal{G}'(m^\rho)|\int_0^{\frac{1}{\rho\sqrt{2}}}\vartheta^{\rho\lambda+\rho-1}d\vartheta+|\mathcal{G}'(n^\rho)|\int_0^{\frac{1}{\rho\sqrt{2}}}t^{\rho-1}(1-t^\rho)^{\lambda}d\vartheta\\
&=|\mathcal{G}'(m^\rho)|\left[\frac{1}{2^{\lambda+1}\rho(\lambda+1)}\right]+|\mathcal{G}'(n^\rho)|\left[\frac{2^{\lambda+1}-1}{2^{\lambda+1}\rho(\lambda+1)}\right]. \qquad (28)
\end{aligned}$$

In the same way, we have

$$\left|\int_{\frac{1}{\rho\sqrt{2}}}^{1}\vartheta^{\rho-1}[\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)]d\vartheta\right|\leq|\mathcal{G}'(m^\rho)|\left[\frac{2^{\lambda+1}-1}{2^{\lambda+1}\rho(\lambda+1)}\right]+|\mathcal{G}'(n^\rho)|\left[\frac{1}{2^{\lambda+1}\rho(\lambda+1)}\right] \qquad (29)$$

and

$$\begin{aligned}
\left|\int_0^1[(1-\vartheta^\rho)^{\lambda}-\vartheta^{\rho\lambda}]\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta\right| &\leq\int_0^1\left[(1-\vartheta^\rho)^{\lambda}+(\vartheta^\rho)^{\lambda}\right]\vartheta^{\rho-1}\left|\mathcal{G}'\left(\vartheta^\rho m^\rho+(1-\vartheta^\rho)\,n^\rho\right)\right|d\vartheta\\
&\leq\left|\mathcal{G}'\left(m^\rho\right)\right|\int_0^1[\vartheta^{\rho-1}(\vartheta^\rho)^{\lambda}(1-\vartheta^\rho)^{\lambda}+t\vartheta^{\rho-1}(\vartheta^\rho)^{\lambda}(\vartheta^\rho)^{\lambda}]d\vartheta\\
&+|\mathcal{G}'(n^\rho)|\int_0^1[\vartheta^{\rho-1}(1-\vartheta^\rho)^{\lambda}(1-\vartheta^\rho)^{\alpha}+\vartheta^{\rho-1}(\vartheta^\rho)^{\alpha}(1-\vartheta^\rho)^{\lambda}]d\vartheta\\
&\leq\left|\mathcal{G}'\left(m^\rho\right)\right|\left[\frac{\beta_\rho(\alpha+1,\alpha+1)}{\rho}+\frac{1}{\rho(\lambda+1)}\right]\\
&+|\mathcal{G}'(n^\rho)|\left[\frac{\beta_\rho(\lambda+1,\lambda+1)}{\rho}+\frac{1}{\rho(\lambda+1)}\right]. \qquad (30)
\end{aligned}$$

Substituting the inequalities (28), (29) and (30) in (27), we deduce the inequality (26). □

Corollary 1. *In Theorem 6, if we take $\alpha=\frac{2}{3}$ in inequality (26), we have*

$$\left|\frac{\alpha\rho^{\lambda}\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^{\lambda}}[^\rho J^{\alpha}_{m+}\mathcal{G}(n^\rho)+^\rho J^{\lambda}_{n-}\mathcal{G}(m^\rho)]-\mathcal{G}\left(\frac{m^\rho+n^\rho}{2}\right)\right|\leq\frac{(n^\rho-m^\rho)\beta_\rho(\frac{5}{3},\frac{5}{3})}{\rho}\left[\frac{|\mathcal{G}'(m^\rho)|+|\mathcal{G}'(n^\rho)|}{2}\right].$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 February 2020 doi:10.20944/preprints202002.0161.v1

The trapezoid-type inequalities via generalized convex function on fractal sets for Katugampola fractional integrals can be derived using Lemma 1.

Theorem 7. *Suppose that* $\lambda > 0$ *and* $\rho > 0$. *Let* $\mathcal{G} : [m^\rho, n^\rho] \subset \mathbb{R}_+ \to \mathbb{R}^\alpha$ *be a differentiable function on* (m^ρ, n^ρ), *and* $\mathcal{G}' \in L^1[m,n]$ *with* $0 \le m < n$. *If* $|\mathcal{G}'|^q$ *is generalized convex on* $[m^\rho, n^\rho]$ *for* $q \ge 1$, *we obtain*

$$\left|\frac{\mathcal{G}(m^\rho)+\mathcal{G}(b^\rho)}{2} - \frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^\lambda}\left[{}^\rho I^\lambda_{m+}\mathcal{G}(n^\rho) + {}^\rho I^\alpha_{n-}\mathcal{G}(m^\rho)\right]\right| \le \frac{n^\rho-m^\rho}{2}\left(\frac{1}{\rho(\lambda+1)}\right)^{1-\frac{1}{q}}\left[\frac{\beta_\rho(\lambda+1,\lambda+1)}{\rho}+\frac{1}{\rho(\lambda+1)}\right]^{\frac{1}{q}} \times \left(|\mathcal{G}'(m^\rho)|^q + |\mathcal{G}'(n^\rho)|^q\right)^{\frac{1}{q}}. \quad (31)$$

Proof. From Lemma 1, we have

$$\left|\frac{\mathcal{G}(m^\rho)+\mathcal{G}(n^\rho)}{2} - \frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^\alpha}\left[{}^\rho I^\alpha_{m+}\mathcal{G}(n^\rho) + {}^\rho I^\lambda_{n-}\mathcal{G}(m^\rho)\right]\right| \le \frac{n^\rho-m^\rho}{2}\left|\int_0^1\left[(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda}\right]\vartheta^{\rho-1}\mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho\right)d\vartheta\right|. \quad (32)$$

In the first case, suppose that $q = 1$. Since the function $|\mathcal{G}'|$ is generalized convex on $[m^\rho, n^\rho]$, we have

$$\mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho\right) \le (\vartheta^\rho)^\lambda|\mathcal{G}'(m^\rho)| + (1-\vartheta^\rho)^\lambda|\mathcal{G}'(n^\rho)|.$$

Therefore,

$$\begin{aligned}
&\left|\int_0^1\left[(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda}\right]\vartheta^{\rho-1}\mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho\right)d\vartheta\right| \\
&\le \int_0^1\left[(1-\vartheta^\rho)^\lambda + \vartheta^{\rho\lambda}\right]\vartheta^{\rho-1}[(\vartheta^\rho)^\alpha\mathcal{G}'|(m^\rho)| + (1-\vartheta^\rho)^\lambda|\mathcal{G}'(n^\rho)|]d\vartheta \\
&\le |\mathcal{G}'(m^\rho)|\int_0^1[\vartheta^{\rho-1}(\vartheta^\rho)^\lambda(1-\vartheta^\rho)^\lambda + \vartheta^{\rho-1}(\vartheta^\rho)^\lambda(\vartheta^\rho)^\lambda]d\vartheta \\
&+|\mathcal{G}'(n^\rho)|\int_0^1[\vartheta^{\rho-1}(1-\vartheta^\rho)^\lambda(1-\vartheta^\rho)^\lambda + \vartheta^{\rho-1}(\vartheta^\rho)^\lambda(1-\vartheta^\rho)^\lambda]d\vartheta \\
&\le |\mathcal{G}'(m^\rho)|\left[\frac{\beta_\rho(\lambda+1,\lambda+1)}{\rho}+\frac{1}{\rho(\lambda+1)}\right] \\
&\quad +|\mathcal{G}'(n^\rho)|\left[\frac{\beta_\rho(\lambda+1,\lambda+1)}{\rho}+\frac{1}{\rho(\lambda+1)}\right]. \quad (33)
\end{aligned}$$

Hence, the inequalities (32) and (33) complete the proof.

The second case can be evaluated when $q > 1$. Using the Hölder's inequality and generalized convexity of $|\mathcal{G}'|$, for $p = \frac{q}{q-1}$, we obtain

$$\begin{aligned}
\left|\int_0^1 \left[(1-\vartheta^\rho)^\alpha - \vartheta^{\rho\lambda}\right]\vartheta^{\rho-1}\mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho\right)d\vartheta\right| &\le \left(\int_0^1 \left[(1-\vartheta^\rho)^\lambda + (\vartheta^\rho)^\lambda\right]\vartheta^{\rho-1}\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \left[(1-\vartheta^\rho)^\lambda + (\vartheta^\rho)^\lambda\right]\vartheta^{\rho-1}\left|\mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho\right)\right|^q d\vartheta\right)^{\frac{1}{q}} \\
&\le \left(\int_0^1 \left[(1-\vartheta^\rho)^\lambda + (\vartheta^\rho)^\lambda\right]\vartheta^{\rho-1}\right)^{1-\frac{1}{q}} \\
&\times\Bigg(\left|\mathcal{G}'(m^\rho)\right|^q \int_0^1 [\vartheta^{\rho-1}(\vartheta^\rho)^\lambda(1-\vartheta^\rho)^\lambda + \vartheta^{\rho-1}(\vartheta^\rho)^\lambda(\vartheta^\rho)^\lambda]d\vartheta \\
&+|\mathcal{G}'(n^\rho)|^q \int_0^1 [\vartheta^{\rho-1}(1-\vartheta^\rho)^\lambda(1-\vartheta^\rho)^\lambda + \vartheta^{\rho-1}(\vartheta^\rho)^\lambda(1-\vartheta^\rho)^\lambda]d\vartheta\Bigg)^{\frac{1}{q}} \\
&\le \left(\frac{1}{\rho(\lambda+1)}\right)^{1-\frac{1}{q}} \\
&\times\Bigg(\left|\mathcal{G}'(m^\rho)\right|^q\left[\frac{\beta_\rho(\lambda+1,\lambda+1)}{\rho} + \frac{1}{\rho(\lambda+1)}\right] \\
&\quad +|\mathcal{G}'(n^\rho)|^q\left[\frac{\beta_\rho(\lambda+1,\lambda+1)}{\rho} + \frac{1}{\rho(\lambda+1)}\right]\Bigg)^{\frac{1}{q}}. \quad (34)
\end{aligned}$$

The inequalities (32) and (34) complete the proof. □

Other special casses related to Theorem 7 are stated in the following corollary.

Corollary 2. *Consider inequality (31) of the Theorem 7,*

1. *If $\alpha = \frac{1}{3}$ and $\rho = 1$, we have the trapezoid inequality:*

$$\begin{aligned}
\left|\frac{\mathcal{G}(m)+\mathcal{G}(n)}{2} - \frac{\lambda\Gamma(\lambda+1)}{2(n-m)^\lambda}\left[I^\lambda_{m+}\mathcal{G}(n) + I^\lambda_{n-}\mathcal{G}(m)\right]\right| &\le \frac{n-m}{2}\left(\frac{3}{4}\right)^{1-\frac{1}{q}}\left[\beta\left(\frac{4}{3},\frac{4}{3}\right)+\frac{3}{4}\right]^{\frac{1}{q}} \\
&\times\left(|\mathcal{G}'(m)|^q + |\mathcal{G}'(n)|^q\right)^{\frac{1}{q}}.
\end{aligned} \quad (35)$$

2. *For $\alpha = \frac{3}{5}$, we have*

$$\begin{aligned}
\left|\frac{\mathcal{G}(m^\rho)+\mathcal{G}(n^\rho)}{2} - \frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^\lambda}\left[{}^\rho I^\lambda_{m+}\mathcal{G}(n^\rho) +{}^\rho I^\lambda_{n-}\mathcal{G}(m^\rho)\right]\right| &\le \frac{n^\rho-m^\rho}{2}\left(\frac{5}{8\rho}\right)^{1-\frac{1}{q}}\left[\frac{\beta_\rho(\frac{8}{5},\frac{8}{5})}{\rho}+\frac{5}{8\rho}\right]^{\frac{1}{q}} \\
&\times\left(|\mathcal{G}'(m^\rho)|^q + |\mathcal{G}'(n^\rho)|^q\right)^{\frac{1}{q}}.
\end{aligned} \quad (36)$$

Theorem 8. *Let $\lambda > 0$ and $\rho > 0$. Let $\mathcal{G} : [m^\rho, n^\rho] \subset \mathbb{R}_+ \to \mathbb{R}^\alpha$ be a differentiable function on (m^ρ, n^ρ), and $\mathcal{G}' \in L^1[m,n]$ with $0 \le a < b$. If $|\mathcal{G}'|^q$is generalized convex on $[m^\rho, n^\rho]$ for $q \ge 1$, we obtain*

$$\begin{aligned}
\left|\frac{\mathcal{G}(m^\rho)+\mathcal{G}(n^\rho)}{2} - \frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^\alpha}\left[{}^\rho I^\lambda_{m+}\mathcal{G}(n^\rho) +{}^\rho I^\lambda_{n-}\mathcal{G}(m^\rho)\right]\right| &\le \frac{n^\rho-m^\rho}{2}\left(\frac{1}{p(\rho-1)+1}\right)^{\frac{1}{p}} \\
&\times\left[\frac{\beta_\rho(\lambda+1,\lambda+1)}{\rho}+\frac{1}{\rho(\lambda+1)}\right]^{\frac{1}{q}}\left(|\mathcal{G}'(m^\rho)|^q + |\mathcal{G}'(n^\rho)|^q\right)^{\frac{1}{q}}. \quad (37)
\end{aligned}$$

Proof. From Lemma 1, we have

$$\left|\frac{\mathcal{G}(m^\rho)+\mathcal{G}(n^\rho)}{2}-\frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^\lambda}\left[{}^\rho I^\lambda_{m+}\mathcal{G}(n^\rho)+{}^\rho I^\lambda_{n-}\mathcal{G}(m^\rho)\right]\right| \le \frac{n^\rho-m^\rho}{2}$$
$$\left|\int_0^1\left[(1-\vartheta^\rho)^\lambda-\vartheta^{\rho\alpha}\right]\vartheta^{\rho-1}\mathcal{G}'\left(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho\right)d\vartheta\right|. \quad (38)$$

Using the Hölder's inequality and generalized convexity of $|f'|$, we obtain

$$\left|\int_0^1\left[(1-\vartheta^\rho)^\lambda-\vartheta^{\rho\lambda}\right]\vartheta^{\rho-1}\mathcal{G}'\left(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho\right)d\vartheta\right| \le \left(\int_0^1(\vartheta^{\rho-1})^p\,d\vartheta\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left[(1-\vartheta^\rho)^\lambda+(\vartheta^\rho)^\lambda\right]\vartheta^{\rho-1}\left|\mathcal{G}'\left(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho\right)\right|^q d\vartheta\right)^{\frac{1}{q}}$$
$$\le\left(\frac{1}{p(\rho-1)+1}\right)^{\frac{1}{p}}$$
$$\times\left(|\mathcal{G}'(m^\rho)|^q\int_0^1[\vartheta^{\rho-1}(\vartheta^\rho)^\lambda(1-\vartheta^\rho)^\lambda+\vartheta^{\rho-1}(\vartheta^\rho)^\lambda(\vartheta^\rho)^\lambda]d\vartheta\right.$$
$$\left.+|\mathcal{G}'(n^\rho)|^q\int_0^1[\vartheta^{\rho-1}(1-\vartheta^\rho)^\lambda(1-\vartheta^\rho)^\lambda+\vartheta^{\rho-1}(\vartheta^\rho)^\lambda(1-\vartheta^\rho)^\lambda]d\vartheta\right)^{\frac{1}{q}}$$
$$\le\left(\frac{1}{p(\rho-1)+1}\right)^{\frac{1}{p}}$$
$$\times\left(|\mathcal{G}'(m^\rho)|^q\left[\frac{\beta_\rho(\lambda+1,\lambda+1)}{\rho}+\frac{1}{\rho(\lambda+1)}\right]\right.$$
$$\left.+|\mathcal{G}'(n^\rho)|^q\left[\frac{\beta_\rho(\lambda+1,\lambda+1)}{\rho}+\frac{1}{\rho(\lambda+1)}\right]\right)^{\frac{1}{q}}.$$

□

Corollary 3. *Considering inequality (37) of the Theorem 8, we have the following trapezoid inequality*

1. *For* $\alpha=\frac{1}{2}$, *we get*

$$\left|\frac{\mathcal{G}(m^\rho)+\mathcal{G}(n^\rho)}{2}-\frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(n^\rho-m^\rho)^\alpha}\left[{}^\rho I^\alpha_{m+}\mathcal{G}(n^\rho)+{}^\rho I^\alpha_{n-}\mathcal{G}(m^\rho)\right]\right| \le \frac{n^\rho-m^\rho}{2}\left(\frac{1}{p(\rho-1)+1}\right)^{\frac{1}{p}}$$
$$\times\left[\frac{\beta_\rho(\frac{3}{2},\frac{3}{2})}{\rho}+\frac{2}{3\rho}\right]^{\frac{1}{q}}\left(|\mathcal{G}'(m^\rho)|^q+|\mathcal{G}'(n^\rho)|^q\right)^{\frac{1}{q}}. \quad (39)$$

2. *If* $\alpha=\frac{4}{9}$, *we have*

$$\left|\frac{\mathcal{G}(m^\rho)+\mathcal{G}(n^\rho)}{2}-\frac{\lambda\rho^\lambda\Gamma(\alpha+1)}{2(n^\rho-m^\rho)^\lambda}\left[{}^\rho I^\lambda_{m+}\mathcal{G}(n^\rho)+{}^\rho I^\lambda_{n-}\mathcal{G}(m^\rho)\right]\right| \le \frac{n^\rho-m^\rho}{2}\left(\frac{1}{p(\rho-1)+1}\right)^{\frac{1}{p}}$$
$$\times\left[\frac{\beta_\rho(\frac{13}{9},\frac{13}{9})}{\rho}+\frac{9}{13\rho}\right]^{\frac{1}{q}}\left(|\mathcal{G}'(m^\rho)|^q+|\mathcal{G}'(n^\rho)|^q\right)^{\frac{1}{q}}. \quad (40)$$

Theorem 9. *Let* $\lambda>0$ *and* $\rho>0$. *Let* $\mathcal{G}:[m^\rho,n^\rho]\subset\mathbb{R}_+\to\mathbb{R}^\lambda$ *be a differentiable function on* (m^ρ,n^ρ), *and* $\mathcal{G}'\in L^1[m,n]$ *with* $0\le m<n$. *If* $|\mathcal{G}'|^q$*is generalized convex on* $[m^\rho,n^\rho]$ *for* $q\ge1$, *we obtain*

$$\left|\frac{\mathcal{G}(m^\rho)+\mathcal{G}(n^\rho)}{2}-\frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^\lambda}\left[{}^\rho I^\lambda_{m+}\mathcal{G}(n^\rho)+{}^\rho I^\lambda_{n-}\mathcal{G}(m^\rho)\right]\right| \le \frac{n^\rho-m^\rho}{2}\left(\frac{1}{\rho}\right)^{1-\frac{1}{q}}$$
$$\times\left[\frac{\beta_\rho(\lambda+1,\lambda+1)}{\rho}+\frac{1}{\rho(\lambda+1)}\right]^{\frac{1}{q}}\left(|\mathcal{G}'(m^\rho)|^q+|\mathcal{G}'(n^\rho)|^q\right)^{\frac{1}{q}}. \quad (41)$$

Proof. Using the fact $|\mathcal{G}'|^q$, a generalized convex on $[m^\rho, n^\rho]$ with $q \geq 1$, we obtain

$$\mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho) n^\rho\right) \leq (\vartheta^\rho)^\lambda \mathcal{G}'(m^\rho) + (1-\vartheta^\rho)^\lambda \mathcal{G}'(n^\rho). \tag{42}$$

Applying inequality (42), together with the power mean inequality, on (38), we have

$$\begin{aligned}
\left|\int_0^1 \left[(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda}\right]\vartheta^{\rho-1}\mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho) n^\rho\right) d\vartheta\right| &\leq \left(\int_0^1 \vartheta^{\rho-1} d\vartheta\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \left[(1-\vartheta^\rho)^\lambda + (\vartheta^\rho)^\lambda\right]\vartheta^{\rho-1}\left|\mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho) n^\rho\right)\right|^q d\vartheta\right)^{\frac{1}{q}} \\
&\leq \left(\frac{1}{\rho}\right)^{1-\frac{1}{q}}\left(\left|\mathcal{G}'(m^\rho)\right|^q \int_0^1 [\vartheta^{\rho-1}(\vartheta^\rho)^\lambda(1-\vartheta^\rho)^\lambda + \vartheta^{\rho-1}(\vartheta^\rho)^\lambda(\vartheta^\rho)^\lambda] d\vartheta\right. \\
&\left.+|\mathcal{G}'(n^\rho)|^q \int_0^1 [\vartheta^{\rho-1}(1-\vartheta^\rho)^\lambda(1-\vartheta^\rho)^\lambda + \vartheta^{\rho-1}(\vartheta^\rho)^\lambda(1-\vartheta^\rho)^\lambda] d\vartheta\right)^{\frac{1}{q}} \\
&\leq \left(\frac{1}{\rho}\right)^{1-\frac{1}{q}}\left(\left|\mathcal{G}'(m^\rho)\right|^q\left[\frac{\beta_\rho(\lambda+1,\lambda+1)}{\rho} + \frac{1}{\rho(\lambda+1)}\right]\right. \\
&\left.+|\mathcal{G}'(n^\rho)|^q\left[\frac{\beta_\rho(\lambda+1,\lambda+1)}{\rho} + \frac{1}{\rho(\lambda+1)}\right]\right)^{\frac{1}{q}}.
\end{aligned}$$

□

Corollary 4. *Considering inequality (41) of the Theorem 9, for $\lambda = \frac{3}{7}$, we get*

$$\begin{aligned}
\left|\frac{\mathcal{G}(m^\rho) + \mathcal{G}(n^\rho)}{2} - \frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho - m^\rho)^\lambda}\left[{}^\rho I_{m+}^\alpha \mathcal{G}(n^\rho) + {}^\rho I_{n-}^\alpha \mathcal{G}(m^\rho)\right]\right| &\leq \frac{n^\rho - m^\rho}{2}\left(\frac{1}{\rho}\right)^{1-\frac{1}{q}} \\
&\times\left[\frac{\beta_\rho(\frac{10}{3},\frac{10}{3})}{\rho} + \frac{3}{10\rho}\right]^{\frac{1}{q}}\left(\left|\mathcal{G}'(m^\rho)\right|^q + |\mathcal{G}'(n^\rho)|^q\right)^{\frac{1}{q}}.
\end{aligned} \tag{43}$$

Corollary 5. *From Theorems 7, 8 and 9 for $q > 1$, we obtain the following:*

$$\left|\frac{\mathcal{G}(m^\rho) + \mathcal{G}(n^\rho)}{2} - \frac{\alpha\rho^\lambda\Gamma(\alpha+1)}{2(n^\rho - m^\rho)^\lambda}\left[{}^\rho I_{m+}^\lambda \mathcal{G}(n^\rho) + {}^\rho I_{n-}^\lambda \mathcal{G}(m^\rho)\right]\right| \leq \min\{S_1, S_2, S_3\}\frac{n^\rho - m^\rho}{2},$$

where,

$$S_1 = \left(\frac{1}{\rho(\lambda+1)}\right)^{1-\frac{1}{q}}\left[\frac{\beta_\rho(\alpha+1,\lambda+1)}{\rho} + \frac{1}{\rho(\lambda+1)}\right]^{\frac{1}{q}}\left(\left|\mathcal{G}'(m^\rho)\right|^q + |\mathcal{G}'(n^\rho)|^q\right)^{\frac{1}{q}},$$

$$S_2 = \left(\frac{1}{p(\rho-1)+1}\right)^{\frac{1}{p}}\left[\frac{\beta_\rho(\lambda+1,\lambda+1)}{\rho} + \frac{1}{\rho(\lambda+1)}\right]^{\frac{1}{q}}\left(\left|\mathcal{G}'(m^\rho)\right|^q + |\mathcal{G}'(n^\rho)|^q\right)^{\frac{1}{q}},$$

and

$$S_3 = \left(\frac{1}{\rho}\right)^{1-\frac{1}{q}}\left[\frac{\beta_\rho(\lambda+1,\lambda+1)}{\rho} + \frac{1}{\rho(\lambda+1)}\right]^{\frac{1}{q}}\left(\left|\mathcal{G}'(m^\rho)\right|^q + |\mathcal{G}'(n^\rho)|^q\right)^{\frac{1}{q}}.$$

 doi:10.20944/preprints202002.0161.v1

4 Conclusion

In this paper, we defined a new identity for the generalized fractional integrals. Connected to this, the new integral inequality for a differentiable convex function is derived. We obtained the generalization of Theorem 2 introduced by Chen and Katugampola. Also, the trapezoid and mid-point type inequalities are studied, along with generalized Hermite-Hadamard inequality, for Katugampola fractional integrals.

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