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Conservation laws and travelling wave solutions for double dispersion equations in (1+1) and (2+1) dimensions

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Abstract

In this article, we investigate two types of double dispersion equations in two different dimensions. Double dispersion equation were derived to describe long nonlinear wave evolution in a thin hyperelastic rod. Conservation laws are obtained for these equations by the application of the multiplier method. Finally, travelling waves and line travelling waves are respectively considered for these two equations.

Key words: Conservation laws, Lie symmetries, Travelling wave solutions

1 Introduction

In this work, we study two equations of double dispersion in one and two dimensions. The double dispersion (DD) equation arises in several physical applications. For

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instance, it is used in analysing non-linear wave distribution in waveguide, interplay of waveguide and exterior medium and, therefore, likelihood of energy interchange through lateral coverings of waveguide.

The author of [1] concentrated on the theory, generation, simulation, and propagation of strain solitary waves in a non-linearly elastic, straight cylindrical rod under finite distortions. For this, the general theory of wave propagation in non-linearly elastic solids was introduced in which a new approach was developed to solve the corresponding DD equation with dissipative terms, that lead to new exact explicit solutions.

In physics of condensed matter, experiments dedicated to lucrative observation of solitary strain wave in solids is not mentioned. Undoubtedly, description of long wave propagation in solids and liquids is to a certain extent analogous, which provides good estimate of soliton existence in solids [2].

Just after a forceful strain wave inseminates in the non-linearly bounded and elastic solid, curving of wave front can escalate swiftly up to changeless deformation appearance. This sensation could be evened alongside the dispersion of wave inside the wave guide [3–5].

In [6], the authors have studied a multidimensional double dispersion equation

$$u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u + k\Delta u_t = \Delta f(u), \quad x \in \mathbb{R}^n, \quad t > 0, \tag{1.1}$$

with $n = 3$, $f(u) = |u|^l$ for $l > 1$ or $f(u) = u^{2m}$, $m = 1, 2, \cdots$.

Here we investigate equation (1.1) for $n = 1$ and $n = 2$.

The DD equation in $(1 + 1)$ dimensions is given by

$$u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} - f(u)_{xx} = 0, \tag{1.2}$$

which describes non-linear dispersive waves [1]. Here the function u is real and a, b > 0 are real numbers.

The DD equation in two dimensions is of the form:

$$u_{tt} - u_{xx} - u_{yy} - a(u_{ttxx} + u_{ttyy}) + b(u_{xxxx} + u_{yyyy}) + D_{xx}f(u) + D_{yy}f(u) = 0. \tag{1.3}$$

The Boussinesq equation reads

$$u_{tt} - u_{xx} + \alpha u_{xxxx} - \beta \left(u^2\right)_{xx} = 0, \tag{1.4}$$

and it arises in various physical applications. For example, it is used in the propagation of long waves in shallow water [7]. Researcher developed many generalizations of Boussinesq equation. One such genralization is the modified Boussinesq equation. In [8], generalization of (1.4) in the form

$$u_{tt} - u_{xx} + u_{xxxx} - \left(f(u)\right)_{xx} = g\left(x\right), \tag{1.5}$$

was studied and classical and nonclassical symmetries were investigated in [9].

Conservation laws have various utilizations in investigation of PDEs. For instance, the determination of conserved quantities and also the constants of motion. These can as well be applied to identify integrability and linearisation and moreover in verifying the correctness of numerical methods.

The celebrated theorem by Noether [10] can be used to derive conservation laws for variational problems. For any PDE, no matter whether it comes from variational problem or non-variational problem, conservation laws can be determined by a direct method [11–15].

Recently, double reduction of PDEs was performed by using the interrelation between symmetries and conservation laws [16–18]. Lately, non-linear $p-$power generalisations of the KP and Boussinesq equations were studied and line soliton solutions were constructed for $p > 0$ [19].

This paper is arranged as follows: In Section 2 and Section 3, conservations laws for the double dispersion equations (1.2) and (1.3) are obtained respectively. Finally, travelling waves $u = U(x - \nu t)$, $u = U(x + \mu y - \nu t)$ are respectively determined for DD equations (1.2) and (1.3), where μ determines direction and ν represents speed of travelling wave. The associated fourth-order non-linear ODEs for U are reduced to first-order variables separable equations by the application of conservation laws derived here for DD equations.

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2 Conservation laws for DD equation in (1+1) dimension

We obtain conservation laws for DD equation (1.2) by applying the general multiplier method [11–13, 21]. Conservation laws are also of basic importance in the study of evolution equations because they provide physical, conserved quantities for all solutions $u(x,t)$, and they can be used to check the accuracy of numerical solution methods [14, 22].

A local conservation law for equation (1.2) is a continuity equation

$$D_t T + D_x X = 0 \tag{2.6}$$

that holds for the whole set of solutions $u(x,t)$, where the conserved density T and the spatial flux X are functions of x, t, u, and derivatives of u [22]. Here D_t, D_x denote total derivatives with respect to t and x respectively. The pair of expressions (T, X) is called a conserved current.

Two local conservation laws are considered to be equivalent [21] if they differ by a trivial conservation law $T = D_x\Theta,\ X = -D_t\Theta$, where T and X are evaluated on the set of solutions of equation (1.2), and Θ is some function of x, t, u, and derivatives of u.

We begin by observing that equation (1.2) has a Cauchy-Kovaleskaya form. Consequently, the results in [12,13] show that all non-trivial conservation laws arise from multipliers. Specifically, when we move off of the set of solutions of equation (1.2), every non-trivial local conservation law (2.6) is equivalent to one that can be expressed in the characteristic form

$$D_t\tilde{T} + D_x\tilde{X} = [u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} - f(u)_{xx}]\,Q, \tag{2.7}$$

where $Q = Q(x, t, u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, u_{xxt}, u_{xtt})$ is a multiplier, and $(\tilde{T}, \tilde{X})$ differs from (T, X) by a trivial conserved current. On the set of solutions $u(x,t)$ of equation (1.2), the characteristic form (2.7) reduces to the conservation law (2.6).

In general, a function $Q(x, t, u, u_x, u_t, \ldots)$ is a multiplier if it is non-singular on the

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set of solutions $u(x,t)$ of equation (1.2), and if its product with equation (1.2) is a divergence expression with respect to t and x.

The determining equation to obtain all multipliers is

$$\frac{\delta}{\delta u}\Big[\{u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} - f(u)_{xx}\}\, Q\Big] = 0. \tag{2.8}$$

This equation must hold off of the set of solutions of equation (1.2). Once the multipliers are found, the corresponding non-trivial conservation laws are obtained either by using a homotopy formula [11–13] or by integrating the characteristic equation (2.7) [22].

In order to obtain local conservation laws of physical interest for nonlinear diffusion reaction equations, we typically focus on low-order multipliers [18, 21]. The general form of a low-order multiplier Q in terms of u and derivatives of u is given by variables which can be differentiated to obtain a leading derivative of the equation. We determine Q as the general form of a low-order multiplier for equation (1.2). The determining equation (2.8) splits with respect to the remaining variables.

This yields a linear determining system for Q, which can be solved by the same algorithmic method used to solve the determining equation for infinitesimal symmetries. By using Maple we solve this determining system subject to the classification conditions $f \neq \text{linear}, a \neq 0, b \neq 0$.

We obtain the following results:

Case 1: For $Q_1 = x$, we obtain the following corresponding conserved density and flux:

$$T_1 = bu_{xt} + xu_t, \tag{2.9}$$

$$X_1 = f'(u)xu_x - f(u) + (au_{xxx} - bu_{ttx} - cu_x)x - au_{xx} + u. \tag{2.10}$$

Case 2: For $Q_2 = t$, we obtain the corresponding conserved density and flux as

$$T_2 = tu_t - u, \tag{2.11}$$

$$X_2 = t(u_x f'(u) + au_{xxx} - bu_{ttx} - cu_x). \tag{2.12}$$

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Case 3: For $Q_3 = 1$, we obtain the following corresponding conserved density and flux:

$$T_3 = u_t, \tag{2.13}$$
$$X_3 = u_x(-c + f'(u)) + au_{xxx} - bu_{ttx}. \tag{2.14}$$

Case 4: For $Q_4 = tx$, we obtain the corresponding conserved density and flux as

$$T_4 = (tu_{xt} - u_x)b + x(tu_t - u), \tag{2.15}$$
$$X_4 = t(f'(u)xu_x - f(u)) + (au_{xxx} - bu_{ttx} - u_x)x - au_{xx} + cu). \tag{2.16}$$

3 Conservation laws for DD equation in (2+1) dimension

Conservation laws are of basic importance because they provide physical, conserved quantities for all solutions, and they can be used to check the accuracy of numerical solution methods. A general discussion of conservation laws and their applications to differential equations can be found in Refs. [14, 22, 23]. A local conservation law for equation (1.3) is a continuity equation

$$D_tT + D_xX + D_yY = 0, \tag{3.17}$$

holding for all solutions $u(x, y, t)$ of Eq. (1.3), where the conserved density T and the spatial fluxes X, Y are functions of t, x, y, u, and x-y-derivatives of u. Every local conservation law can be expressed in an equivalent form by a divergence identity

$$\begin{aligned} D_tT + D_xX + D_yY = (&u_{tt} - u_{xx} - u_{yy} - a(u_{ttxx} + u_{ttyy}) \\ &+b(u_{xxxx} + u_{yyyy}) + D_{xx}f(u) + D_{yy}f(u))Q, \end{aligned}$$

which is called the characteristic equation. There is a one-to-one correspondence between non-trivial conservation laws (up to addition of a trivial conservation law) and non-zero multipliers.

All nontrivial conservation laws are characterized by a multiplier. The physically important conservation laws arise from multipliers of low order [23–25], namely $Q =$

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$Q(x,t,y,u,u_x,u_t,u_y,u_{xx},u_{tt},u_{yy},u_{tx},u_{ty},u_{xxx},u_{txx},u_{ttx})$. The determining equation for all such multipliers is given by

$$E_u\left((u_{tt}-u_{xx}-u_{yy}-a(u_{ttxx}+u_{ttyy})+b(u_{xxxx}+u_{yyyy})+D_{xx}f(u)+D_{yy}f(u))\,Q\right)=0,$$

where

$$E_u = u\partial_u - D_x\partial_{u_x} - D_y\partial_{u_y} - D_t\partial_{u_t} + D_x^2\partial_{u_{xx}} + D_xD_t\partial_{u_{xt}} + D_xD_y\partial_{u_{xy}} - \cdots$$

is the variational derivative (Euler operator) [14, 22, 23]. This operator has the property that it annihilates an expression identically if and only if the expressions is a space-time divergence. The determining equation will split with respect to all variables that do not appear in Q, which yields an overdetermined linear system of equations to solve for Q, $a \neq 0$ and $f(u) \neq$ linear:

$$Q_u = 0, \tag{3.18}$$

$$\begin{aligned}-a(Q_{xxtt}+Q_{yytt})+b(Q_{xxxx}+Q_{yyyy})+Q_{tt}\\ +Q_{xx}(f'(u)-1)+Q_{yy}(f'(u)-1)=0.\end{aligned} \tag{3.19}$$

This system is straightforward to solve by using Maple (rifsimp and pdsolve). We get the following classification results for Eq (1.3) with $a \neq 0$ and $f \neq$ linear and we find all multipliers Q and obtain all the low order conservation laws.

The corresponding non-trivial low-order conservation laws are given by the following cases:

Case 1: For $Q_1 = 1$

$$\begin{aligned}T_1 &= u_t,\\ X_1 &= bu_{xxx} - au_{ttx} + au_{tt} + (f'(u)-1)u_x,\\ Y_1 &= bu_{yyy}.\end{aligned} \tag{3.20}$$

Case 2: For $Q_2 = x$

$$\begin{aligned}T_2 &= xu_t,\\ X_2 &= xbu_{xxx} - bu_{xx} - xau_{ttx} + au_{tt} + (xf'(u)-x)u_x + u - f(u),\\ Y_2 &= xbu_{yyy} - xau_{tty} + (x(f'(u)-x)u_y.\end{aligned} \tag{3.21}$$

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Case 3: For $Q_3 = y$

$$\begin{aligned} &T_3 = yu_t,\\ &X_3 = ybu_{xxx} - yau_{ttx} + (yf'(u) - y)u_x,\\ &Y_3 = ybu_{yyy} - bu_{yy} - yau_{tty} + au_{tt} + (yf'(u) - y)u_y + u - f(u). \end{aligned} \tag{3.22}$$

Case 4: For $Q_4 = t$

$$\begin{aligned} &T_4 = tu_t - u,\\ &X_4 = tbu_{xxx} - atu_{ttx} + (tf'(u) - t)u_x,\\ &Y_4 = tbu_{yyy} - atu_{tty} + (tf'(u) - t)u_y. \end{aligned} \tag{3.23}$$

Case 5: For $Q_5 = xy$

$$\begin{aligned} &T_5 = xyu_t,\\ &X_5 = xybu_{xxx} - ybu_{xx} - xyau_{ttx} + yau_{tt} + (xy(f'(u) - xy)u_x + yu - yf(u),\\ &Y_5 = xybu_{yyy} - xyau_{tty} - xbu_{yy} + xau_{tt} + (xyf'(u) - xy)u_y + xu - xf(u). \end{aligned} \tag{3.24}$$

Case 6: For $Q_6 = tx$

$$\begin{aligned} &T_6 = txu_t - xu,\\ &X_6 = txbu_{xxx} - tbu_{xx} - txau_{ttx} + tau_{tt} + (tx(f'(u) - tx)u_x + tu - tf(u),\\ &Y_6 = txbu_{yyy} - txau_{tty} + (tx(f'(u) - tx)u_y. \end{aligned} \tag{3.25}$$

Case 7: For $Q_7 = ty$

$$\begin{aligned} &T_7 = tyu_t - yu,\\ &X_7 = tybu_{xxx} - tyau_{ttx} + (ty(f'(u) - ty)u_x,\\ &Y_7 = txbu_{yyy} - tyau_{tty} + tbu_{yy} + tau_{tt} + (ty(f'(u) - ty)u_y + tu - tf(u). \end{aligned} \tag{3.26}$$

Case 8: For $Q_8 = txy$

$$\begin{aligned} &T_8 = txyu_t - xyu,\\ &X_8 = txybu_{xxx} - tybu_{xx} - txyau_{ttx} + tyau_{tt} + (txy(f'(u) - txy)u_x + ytu - ytf(u),\\ &Y8 = txybu_{yyy} - txyau_{tty} - txbu_{yy} + txau_{tt} + (txy(f'(u) - txy)u_y + txu - txf(u). \end{aligned} \tag{3.27}$$

4 Travelling waves for equations (1.2) and (1.3)

In this section, we consider line travelling waves for the DD equations (1.2) and (1.3) in two and one dimensions respectively. Conservation laws that are symmetry

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invariant have some important applications. It is well known that when a differential equation admits a Noether symmetry, a conservation law is associated with this symmetry, and furthermore that a double reduction can be achieved as a result of this association [14, 27]. Moreover, any symmetry-invariant conservation law will reduce to a first integral for the ODE obtained by symmetry reduction of the given PDE when symmetry-invariant solutions u are sought. In [24,26,27] the relationship between symmetries and conservation laws has been used to find a double reduction of partial differential equations with two independent variables. This provides a direct reduction of order of the ODE.

In [24,25], a further connection between symmetries and conservation laws is explored by focusing on conservation laws that are invariant or, more generally, homogeneous under the action of a given set of symmetries. This yields an explicit condition for invariance and, more generally, homogeneity, which are formulated in terms of multipliers. Some applications to finding symmetry-invariant conservation laws and finding symmetry-invariant solutions of PDEs were outlined.

4.1 The (1+1)-DD travelling waves

We will derive travelling waves for (1.2), it will be convenient to use the coordinate form of travelling wave variable expressed as $z = x - \lambda t$. A travelling wave solution is of the form

$$u(x,t) = U(z), \quad z = x - \lambda t \tag{4.28}$$

and substituting this expression into (1.2) we get the reduced ODE

$$(a - b\lambda^2)U'''' + (-\lambda^3 + f'(U) - 1)U'' + f''(U)U'^2 = 0. \tag{4.29}$$

We observe that a travelling wave solution is invariant under the two-parameter group of translations $t \to t + \epsilon_1$, $x \to x + \epsilon_1\lambda$, with $\epsilon_1, \epsilon_2 \in \mathbb{R}$. Suppose a conservation law $D_tT + D_xX = 0$ does not contain the variables t, x explicitly. Then the conservation law gives rise to a reduction of order of the travelling wave ODE by the reductions

$$D_t\big|_{u=U(z)} = -\lambda\frac{d}{dz}, \quad D_x\big|_{u=U(z)} = \frac{d}{dz}, \tag{4.30}$$

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yielding

$$\frac{d}{dz}\left((X-\lambda T)\big|_{u=U(z)}\right)=0. \tag{4.31}$$

The first integral is thus given by

$$X-\lambda T=C. \tag{4.32}$$

The corresponding symmetry-invariant conservation law will be a first integral of (4.29) as $D_t(T_3)+D_x(X_3)=0$ does not contain the variables t,x explicitly. Then the conservation law gives rise to the following reduction of order of the travelling wave ODE:

$$(a-b\lambda^2)U'''+(-\lambda^3+f'(U)-1)U'=C_1. \tag{4.33}$$

Moreover, although conservations laws $D_t(T_1)+D_x(X_1)=0$ and $D_t(T_2)+D_x(X_2)=0$ contain explicitly the variables x and t however the linear combination

$$D_t(T_1-\lambda T_2)+D_x(X_1-\lambda X_2)=0$$

is invariant under the generator $\partial_x-\lambda\partial_t$ yielding a second first integral

$$(-b\lambda^2+a)U'''+(b\lambda^2-a)U''+(-\lambda^3+f'(U)-1)zU'+(\lambda^3+1)U-f'(U)=C_2. \tag{4.34}$$

From (4.33) and (4.34) we get the second order ODE

$$U''-\frac{\lambda^3U-C_2-f(U)+U}{-b\lambda^2+a}=0. \tag{4.35}$$

4.2 The (2+1)-DD line travelling waves

We will derive the explicit line travelling waves for (1.3), for this it will be considered to use the coordinate form of travelling wave variable $z=-\mu y-\nu x+t$ for this derivation. A line travelling wave in two dimensions is given by

$$u(x,y,t)=U(z),\quad z=-\mu y-\nu x+t, \tag{4.36}$$

where the parameters ν and μ determine the direction and the speed of the wave. Specifically, the speed is $c=1/\sqrt{\nu^2+\mu^2}>0$, and the direction is given by the unit

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vector $\hat{k} = (c\nu, c\mu)$ with respect to (x, y) coordinates. Substitution of expression (4.36) into Eq. (1.3) yields fourth-order nonlinear differential equation for $U(\zeta)$:

$$\begin{aligned}&\left(b\mu^4 - a\mu^2 + \nu^2\left(b\nu^2 - a\right)\right)U'''' + \left(\left(\mu^2+\nu^2\right)f'(U) + \left(-\mu^2 - \nu^2 + 1\right)\right)U'' \\ &+ \left(\mu^2+\nu^2\right)f''(U)U'^2 = 0.\end{aligned} \tag{4.37}$$

We observe that a travelling wave solution is invariant under the three-parameter group of translations $t \to t - \epsilon_1$, $x \to x + \epsilon_2\nu$, $y \to y + \epsilon_3\mu$, with $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}$.

Suppose a conservation law $D_tT + D_xX + +D_yY$ does not contain the variables t, x, y explicitly. Then the conservation law gives rise to a reduction of order of the travelling wave ODE by the reductions

$$D_t\big|_{u=U(z)} = \frac{d}{dz}, \quad D_x\big|_{u=U(z)} = -\nu\frac{d}{dz}, \qquad D_y\big|_{u=U(z)} = -\mu\frac{d}{dz}, \tag{4.38}$$

yielding

$$\frac{d}{dz}\left((\nu X + \mu Y - T)\big|_{u=U(z)}\right) = 0. \tag{4.39}$$

The first integral is thus given by

$$\nu X + \mu Y - T = C. \tag{4.40}$$

The corresponding symmetry-invariant conservation law will be a first integral of (4.37) as $D_t(T_1) + D_x(X_1) + D_y(Y_1) = 0$ does not contain the variables t, x, y explicitly. Then the conservation law gives rise to the following reduction of order of the travelling wave ODE:

$$(b\mu^4 + b\nu^4 - a\mu^2 - a\nu^2)U''' + (1 - \mu^2 - \nu^2)U' + f'(U)(\mu^2+\nu^2) = C_1. \tag{4.41}$$

Moreover, although conservations laws $D_t(T_2) + D_x(X_2) + D_y(Y_2) = 0$, $D_t(T_3) + D_x(X_3) + D_y(Y_3) = 0$ and $D_t(T_4) + D_x(X_4) + D_y(Y_4) = 0$ contain explicitly the variables x, y and t however the linear combination

$$D_t(T_2 - \lambda T_3 - \gamma T_4) + D_x(X_2 - \lambda X_3 - \gamma X_4) + D_y(Y_2 - \lambda Y_3 - \gamma Y_4) = 0$$

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is invariant under the generator $\lambda\partial_x + \gamma\partial_y - \partial_t$ yielding a second first integral

$$\begin{aligned}&(b(\mu^4+\nu^4)z - a(\mu^2-\nu^2)z)U''' + (a(\mu^2+\nu^2) - b(\mu^4+\nu^4))U'' \\ &+(\mu^2+\nu^2-1)U + z((f'(U)-1)(\mu^2+\nu^2)+1)f'(U) - (\mu^2+\nu^2)f(U) = C_2.\end{aligned} \tag{4.42}$$

From (4.41) and (4.42) we get the second order ODE

$$U'' - \frac{(\mu^2+\nu^2)f\,(U) - (\mu^2+\nu^2)U - C_2\,z + C_1 + U}{a(\mu^2+\nu^2) - b(\mu^4+\nu^4)} = 0. \tag{4.43}$$

Setting $C_2 = 0$ we get a separable ODE that can be intagrated once yielding the following first order ODE:

$$\begin{aligned}&U'^2 + \frac{(\mu^2+\nu^2-1)\,U^2}{a(\mu^2+\nu^2) - b(\mu^4+\nu^4)} - \frac{2\,C_1\,U}{a(\mu^2+\nu^2) - b(\mu^4+\nu^4)} \\ &\quad - \frac{2\,(\mu^2+\nu^2)\int f\,(U)\;\mathrm{d}U}{a(\mu^2+\nu^2) - b(\mu^4+\nu^4)} = C_3.\end{aligned} \tag{4.44}$$

We are interested in soliton solutions, so we set $f(U) = U^2$ and we get the following solution:

$$U(z) = -\frac{3}{4}tanh^2(z) + \frac{3}{4}$$

with $z = -\mu y - \nu x + t$. In figure 1, we show this solution for equation (4.43) for $C_2 = 0$, $\mu = 1.$ $\nu = 1$, $a = -\frac{1}{4}$ and $b = -\frac{1}{8}$.

5 Conclusions

In this work, we have investigated the (1+1)-dimensional and (2+1)-dimensional double dispersion equations (1.2) and (1.3). Our results, first determine all low-order conservation laws by using the multiplier method for equations (1.2) and (1.3). Second, travelling waves have been obtained for these equations. In the case of translation symmetries, we have showed how conservation laws that explicity

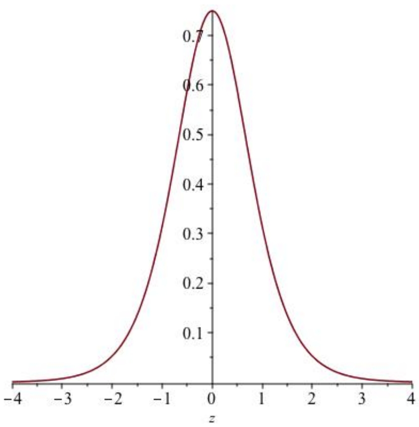

Figure 1: Solution for the ODE (4.43)

contain the independent variables can nevertheless be used under certain conditions to obtain a double redaction. We have obtained some exacts solutions, in particular we have found a line soliton solution for equation (1.3).

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