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Article

Some Construction Methods of Aggregation Operators in Decision Making Problems: An Overview

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Abstract: Aggregating data is the main line of any discipline dealing with fusion of information from the knowledge-based systems to the decision-making. The purpose of aggregation methods is to convert a list of objects, all belonging to a given set, into a single representative object of the same set usually by an n-ary function, so-called aggregation operator. Since the useful aggregation functions for modeling real-life problems are limit, the basic problem is to construct a proper aggregation operator for each situation. During the last decades, a number of construction methods for aggregation functions have been developed to build new classes based on the well-known operators. This paper reviews some of these construction methods where they are based on transformation, composition and weighted rule.

Keywords: aggregation operators; composite aggregation operators; weighted aggregation operators; transformation; duality; group decision making

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1. Introduction

The importance of aggregating in fusion of information specially in decision making problems is to get an overview of data for taking the final action. However, there are diverse strategies to reach the aggregated value, aggregation functions are one of the most effective and simple methods in this area. Aggregation functions are a mathematical way to summarize an n-tuple of information into a single output by means of non-decreasing functions where the output should be remained in the same set as the input one. In literature, the non-decreasing function $A : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is usually considered as the standard definition of aggregation functions where the non-decreasing property of A shows increasing values of inputs increases the aggregated value. Moreover, the aggregation function A fulfills the boundary conditions $A(0,\cdots,0) = 0$ and $A(1,\cdots,1) = 1$ that guaranty the aggregating of minimal or maximal inputs are respectively minimal and maximal output.

The statistical tools and mathematical concepts, such as mean value, median, minimum, maximum, triangular norms [1] and fuzzy integrals [2,3] are widely used to aggregating data. However, the list of aggregation functions is not limited to these well-known operators. In fact, the class of aggregation functions is huge and choosing the right one for a given problem is the main task of decision makers. Since then, a survey on aggregation functions can give an insight to the experts to enhance their performance. Xu and Da [4] fulfilled the first overview of the aggregation operators in 2003 by reviewing the existing main aggregation operators. Authors in [5–7] focused on properties and classification of aggregation functions. Furthermore, Grabisch et al. [8] have made a review on

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averaging aggregation operators. Recently, Rosanisah Mohd and Lazim Abdullah [9] provided an overview of different types of aggregation functions used in decision making from year to year.

But, finding a proper aggregation operator for the situation with a complex consensus scenario and based on the existing formulas is not always straightforward. We usually need to develop the class of aggregation functions by constructing new ones based on the existing operators. There are a number of techniques to construct different classes of aggregation operators with different properties [6,10–17]. These methods start from given aggregation functions and continue by constructing new ones.

This paper aims to provide an overview of some key construction methods of aggregation functions that can give an insight into the future study on aggregation theory. We focus on consensus problems that may be solved by these construction techniques of aggregation operators and present a summary of some recent as well as older results. To this end, in Section 2 basic definitions and properties of aggregation functions are recalled. Sections 3, 4 and 5 are respectively devoted to review of construction methods for aggregation functions including transformation of aggregation operator, composite aggregations and weighted rule of aggregation operators. Finally, a summary is given in Conclusion Section.

2. Basic Definitions and Properties

In this section we recall some basic concepts about aggregation functions. Not that, throughout this paper, we use the following notations: $\mathbb{I} \subset \mathbb{R}$ is the closed unit interval $[0,1]$ and $\mathbb{I}^n = \{\mathbf{x}_{(n)} = (x_1, \cdots, x_n) : x_i \in \mathbb{I}, i = 1, \cdots, n\}$ denotes the set of all real n-dimensional vectors $\mathbf{x}_{(n)}$ whose components are in the interval $\mathbb{I}$. Moreover, we say $\mathbf{x}_{(n)} = (x_1, \cdots, x_n) \leq \mathbf{y}_{(n)} = (y_1, \cdots, y_n)$ if and only if $x_i \leq y_i$, for all i.

Definition 1.
- *An aggregation function of dimension $n \in \mathbb{N}$ is an n-ary function $A^{(n)} : [0,1]^n \to [0,1]$ satisfying:*

  A1. $A(x) = x$, *for* $n = 1$ *and any* $x \in [0,1]$;
  A2. $A^{(n)}(x_1, \cdots, x_n) \leq A^{(n)}(y_1, \cdots, y_n)$ *if* $(x_1, \cdots, x_n) \leq (y_1, \cdots, y_n)$;
  A3. $A^{(n)}(0, 0, \cdots, 0) = 0$ *and* $A^{(n)}(1, 1, \cdots, 1) = 1$.

- *An extended aggregation function is the function $A : \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$ whose restriction $A|_{\mathbb{I}^n} := A^{(n)}$ to $\mathbb{I}^n$ is the n-ary aggregation function $A^{(n)}$ for any $n \in \mathbb{N}$.*

Example 1. *The operators*

1. *median Med defined by* $Med(x_1, \cdots, x_n) = x_{\frac{n+1}{2}}$ *if n is odd and* $Med(x_1, \cdots, x_n) = \frac{1}{2}[x_{\frac{n}{2}} + x_{\frac{n}{2}+1}]$ *if n is even where* $x_1 \leq x_2 \leq \cdots \leq x_n$;
2. *arithmetic mean* $AM(\mathbf{x}_{(n)}) = \frac{1}{n} \sum_{i=1}^{n} x_i$;
3. *weighted arithmetic mean* $WAM(\mathbf{x}_{(n)}) = \sum_{i=1}^{n} w_i x_i$ *where* $w_i \in [0,1]$ *and* $\sum_{i=1}^{n} w_i = 1$;
4. *geometric mean* $GM(\mathbf{x}_{(n)}) = (\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$;
5. *harmonic mean* $HM(\mathbf{x}_{(n)}) = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$;
6. *minimum* $Min(\mathbf{x}_{(n)}) = \min_{i=1}^{n} x_i$ *and maximum* $Max(\mathbf{x}_{(n)}) = \max_{i=1}^{n} x_i$;
7. *product function* $\Pi(\mathbf{x}_{(n)}) = \prod_{i=1}^{n} x_i$;
8. *projection function to the kth coordinate* $P_k(\mathbf{x}_{(n)}) = x_k$

are several well-known examples of extended aggregation functions.

Remark 1. *Note that, we have the weakest aggregation operator A_w and the strongest aggregation operator A_s defined by*

$$A_w = \begin{cases} 1 & \text{if } x_1 = x_2 = \cdots = x_n = 1 \\ 0 & \text{else,} \end{cases} \qquad A_s = \begin{cases} 0 & \text{if } x_1 = x_2 = \cdots = x_n = 0 \\ 1 & \text{else,} \end{cases}$$

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where for any aggregation operator A:

$$A_w \leq A \leq A_s$$

Definition 2. *The aggregation function $A : [0,1]^n \to [0,1]$*

- *has a neutral element $e \in [0,1]$ if*

$$A(x_1, \cdots, x_{i-1}, e, x_{i+1}, \cdots, x_n) = A(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$$

- *has an annihilator element (absorbing element or zero element) $a \in [0,1]$ if*

$$A(x_1, \cdots, x_{i-1}, a, x_{i+1}, \cdots, x_n) = a$$

- *has no zero divisors if it has the zero element a, and*

$$A(x_1, \cdots, x_n) = a \implies \exists_{1 \leq s \leq n} s : x_s = a$$

Therefore, if an aggregation function has a neutral element e, it can be omitted from the initial list of objects without any influence on the final aggregated value. Thus, in a decision-making problem existence of the neutral element e means that the rest of arguments except e fulfill the aggregation process. If an aggregation function has the annihilator element a, then adding it to the list of arguments means that only the argument a fulfills the aggregation stage. In fact, the annihilator element a acts as a veto or qualifying score.

Example 2. *The product function Π and the minimum function Min have the neutral element 1, while 0 is the neutral element of the maximum function Max. Obviously, 1 is the annihilator for Max and 0 is the annihilator element of Π, Min and GM where the latter has no neutral element. Operators A_w and A_s are without neutral element, while A_w has an annihilator $a = 0$ and A_s has an annihilator $a = 1$. The operators AM and P_k are examples of aggregation functions without neutral and annihilator elements.*

Example 3. *Operator $A_c : \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$ defined by*

$$A_c = (x_1, \cdots, x_n) = \max\left(0, \min\left(1, c + \sum_{i=1}^{n} (x_i - c)\right)\right)$$

where $c \in [0,1]$ is an aggregation function that has neutral element $e = c$ and without annihilator.

2.1. Classification of Aggregation Functions

There are four main classes of aggregation functions, including conjunctive, disjunctive, averaging and remaining aggregation functions, defined by their relationship to *Min* and *Max* functions proposed in [18] for the first time.

Definition 3. *The aggregation function $A : [0,1]^n \to [0,1]$ is called*

- *Conjunctive if for every $\mathbf{x}_{(n)} \in \mathbb{I}^n$, $A(\mathbf{x}_{(n)})$ is bounded by the minimum function:*

$$A(\mathbf{x}_{(n)}) \leq Min(\mathbf{x}_{(n)}) = \min(x_1, \cdots, x_n)$$

- *Disjunctive if for every $\mathbf{x}_{(n)} \in \mathbb{I}^n$, $A(\mathbf{x}_{(n)})$ is bounded by the maximum function:*

$$A(\mathbf{x}_{(n)}) \geq Max(\mathbf{x}_{(n)}) = \max(x_1, \cdots, x_n)$$

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- *Average whenever for all* $\mathbf{x}_{(n)} \in \mathbb{I}^n$ *we have:*

$$Min(\mathbf{x}_{(n)}) \leq A(\mathbf{x}_{(n)}) \leq Max(\mathbf{x}_{(n)})$$

Thus, if $\mathcal{A}$ is the collection of all aggregation functions and $\mathcal{C}$ shows the class of conjunctive functions, $\mathcal{D}$ is the class of disjunctive functions, the class $\mathcal{P}$ shows the pure averaging functions and $\mathcal{R} = \mathcal{A}/(\mathcal{C} \cup \mathcal{D} \cup \mathcal{P})$ is all remaining aggregation functions then $(\mathcal{C}, \mathcal{D}, \mathcal{P}, \mathcal{R})$ forms a partition on $\mathcal{A}$ [19].

Example 4. *Operators* Π, A_w *and Min are examples of conjunctive aggregation functions. While, the operators* $PS(x_1, x_2) = x_1 + x_2 - x_1 \cdot x_2$, A_s *and Max are disjunctive aggregation functions.*

Remark 2. *Note that if the conjunctive and disjunctive aggregation functions have neutral elements, they are, respectively,* $e = 1$ *and* $e = 0$.

However, the property of averaging is equivalent to the idempotency of aggregation function A where the aggregation function $A : [0,1]^n \to [0,1]$, for $n \in \mathbb{N}$, is called idempotent if $A(x, \cdots, x) = x$ for all $x \in [0,1]$. It is an immediate result from monotonic property of A. If an aggregation operator A has the neutral element e then A is idempotent at e. Similarly, if A has the annihilator a, then A is idempotent at a. So, 0 and 1 are idempotent elements for each aggregation operator, called trivial idempotent elements.

Example 5. *The arithmetic mean AM, the geometric mean GM, the operator Min, the operator Max and the projection operator* P_k *are average (idempotent) aggregation functions.* A_c *is not an average (idempotent) aggregation operator, but has the idempotent elements* $0, 1, c$ *where* $c \in [0,1]$.

Remark 3. *In multi-criteria decision making problem, idempotency (averaging) means if all criteria (or decision makers) are satisfied at the same degree like* x *then the global score should be* x.

2.1.1. Triangular Norms and Conorms

Triangular norms and conorms, or t-norms and t-conorms in brief, are well-known examples of conjunctive and disjunctive aggregation operators with the neutral elements $e = 1$ and $e = 0$, respectively, that are associative and commutative.

Definition 4. *([1]) A binary operation* $T : [0,1]^2 \to [0,1]$ *is called a triangular norm or t-norm if it is symmetric, associative, non-decreasing function and satisfies the boundary condition* $T(x, 1) = x, \forall x \in [0,1]$. *A binary operation* $S : [0,1]^2 \to [0,1]$ *is called a triangular conorm or t-conorm if it is symmetric, associative, non-decreasing function and satisfies the boundary condition* $S(x, 0) = x, \forall x \in [0,1]$.

Example 6. $T_D(x_1, x_2) = x_1$ *or* x_2 *if* $x_2 = 1$ *or* $x_1 = 1$ *and otherwise is zero,* $T_M(x_1, x_2) = \min(x_1, x_2)$, $T_P(x_1, x_2) = x_1 x_2$ *and* $T_L(x_1, x_2) = \max(x_1 + x_2 - 1, 0)$ *are some examples for t-norms where*

$$T_D \leq T_L \leq T_P \leq T_M$$

On the other hand, operators $S_D(x_1, x_2) = x_1$ *or* x_2 *if* $x_2 = 0$ *or* $x_1 = 0$ *and otherwise is one,* $S_M(x_1, x_2) = \max(x_1, x_2)$, $S_P(x_1, x_2) = x_1 + x_2 - x_1 x_2$ *and* $S_L(x_1, x_2) = \min(x_1 + x_2, 1)$ *are t-conorms such that*

$$S_M \leq S_P \leq S_L \leq S_D$$

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2.2. Properties of Aggregation Functions

The algebraic and analytic properties of an arbitrary n-ary real functions, such as continuity and associativity can be naturally defined for the aggregation function $A : [0,1]^n \to [0,1]$.

Definition 5. *The aggregation function* $A : [0,1]^n \to [0,1]$*, for* $n \in \mathbb{N}$*, is called*

1. *Additive if* $A(\mathbf{x}_{(n)} + \mathbf{y}_{(n)}) = A(\mathbf{x}_{(n)}) + A(\mathbf{y}_{(n)})$ *for all* $\mathbf{x}_{(n)}, \mathbf{y}_{(n)} \in \mathbb{I}^n$ *such that* $\mathbf{x}_{(n)} + \mathbf{y}_{(n)} \in \mathbb{I}^n$*;*
2. *Idempotent if* $A(x, \cdots, x) = x$ *for all* $x \in [0,1]$*;*
3. *Symmetric if* $A(x_1, \cdots, x_n) = A(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$ *for all* $\mathbf{x}_{(n)} = (x_1, \cdots, x_n) \in \mathbb{I}^n$ *where* σ *is any permutation of* $\{1, \cdots, n\}$*;*
4. *Bisymmetric if for all* $x_{ij} \in [0,1]$ *where* $i, j \in \{1, \cdots, n\}$ *we have:* $A(A(x_{11}, \cdots, x_{1n}), \cdots, A(x_{n1}, \cdots, x_{nn})) = A(A(x_{11}, \cdots, x_{n1}), \cdots, A(x_{1n}, \cdots, x_{nn}))$*;*
5. *Strongly bisymmetric if* $A(\mathbf{x}_{(n)}) = \mathbf{x}_{(n)}$ *for all* $\mathbf{x}_{(n)} \in \mathbb{I}^n$ *and* $A(A(x_{11}, \cdots, x_{1n}), \cdots, A(x_{m1}, \cdots, x_{mn})) = A(A(x_{11}, \cdots, x_{m1}), \cdots, A(x_{1n}, \cdots, x_{mn}))$ *for any* $m, n \in \mathbb{N}$*;*
6. *Associative if for all* $(x_1, x_2, x_3) \in \mathbb{I}^3$ *we have:* $A(A(x_1, x_2), x_3) = A(x_1, A(x_2, x_3))$*;*
7. *Continuous if for any* $\mathbf{x}_{(n)} \in \mathbb{I}^n$ *and* $(x_{i_j})_{j \in \mathbb{N}} \in \mathbb{I}^{\mathbb{N}}$ *where* $i \in \{1, \cdots, n\}$*, if* $\lim_{j \to \infty} x_{i_j} = x_i$ *then* $\lim_{j \to \infty} A(x_{1_j}, \cdots, x_{n_j}) = A(x_1, \cdots, x_n)$ *or equivalently* $\forall \epsilon > 0, \exists \delta > 0 :$ *if* $\mid x_i - y_i \mid < \delta$ *where* $i \in \{1, \cdots, n\}$ *then* $\mid A(x_1, \cdots, x_n) - A(y_1, \cdots, y_n) \mid < \epsilon$*;*
8. *c-Lipschitz with respect to the norm* $\|.\| : \mathbb{R} \to [0, +\infty)$*, if for some constant* $c \in (0, +\infty)$ *we have:* $\mid A(\mathbf{x}_{(n)}) - A(\mathbf{y}_{(n)}) \mid \leq c \|\mathbf{x}_{(n)} - \mathbf{y}_{(n)}\|$ *for all* $\mathbf{x}_{(n)}, \mathbf{y}_{(n)} \in \mathbb{I}^n$*.*

Example 7. *It is evident that, the arithmetic mean AM is an example of an additive function. Operator* Π *is not idempotent. Operators* $AM, \prod, Min, Max, P_k$ *and* A_c *are continuous aggregation operators which all also fulfill the 1-Lipschitz property. Operator AM is the only aggregation function that is* $\frac{1}{n}$*-Lipschitz for all* $n \in \mathbb{N}$*:*

$$1 = \mid M(1, \cdots, 1) - M(0, \cdots, 0) \mid = \frac{1}{n} \sum_{i=1}^{n} \mid 1 - 0 \mid$$

while GM is an example of a continuous aggregation operator which is not Lipschitz. A_w *and* A_s *are examples of non-continuous operators.*

Remark 4. *Note that, the symmetric property of an aggregation function reflects the same importance of single criteria in a multi-criteria decision making problem, i.e., knowledge of the order of input score is irrelevant. Moreover, the bisymmetry property of an aggregation operator allows us to obtain an overall score of each candidate according to n judges from different decision makers based on m criteria by any two following ways. We can first aggregate the numerical scores of each candidate over all criteria given by each decision maker and then aggregate these values. Or we may first aggregate the scores of the candidate on the basis of each criteria and then merge them over all decision makers.*

Example 8. $AM, GM, \prod, Med, Min, Max, OWA$ *and* OWG *are examples of symmetric aggregation functions. While* P_k *and WAM are non-symmetric aggregation functions.*

Remark 5. *The associativity property of an aggregation function allows us to start with aggregation procedure before knowing all inputs to be aggregated. Then additional input can be simply aggregated with the output of the previous aggregating step.*

Example 9. *Aggregation operators* $A_w, A_s, Min, Max, \prod$ *and* P_k *are examples of associative functions. Operators* A_c *and GM are non-associative aggregation operators.*

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2.3. Construction Methods of Aggregation Functions

In aggregation operators theory, construction methods are one of the important issues that should be addressed. There exist a large number of aggregation operators, some simple and straightforward and some very complex, with different properties. However, finding a proper one for a specific situation may be difficult. It is sometimes happened that the traditional operators are not suitable and experts try to develop new aggregation functions from the existing ones. The generic problem that should be solved is:

Problem I. Constructing an optimal aggregation function $A \in \mathcal{A}$, possibly with some additional properties, to find the best output.

There are several construction methods to create new aggregation functions. In the following sections, we discuss some of them.

3. Transformation of Aggregation Functions

The idea of transformation of functions can be used to construct new aggregation functions that inherit the algebraic and topological properties of the original aggregation functions.

Proposition 1. *Let $A : \mathbb{I}^n \to \mathbb{I}$ be an n-ary aggregation function and $\phi : \mathbb{J} \to \mathbb{I}$ be a monotone bijection where $\mathbb{I}, \mathbb{J}$ are real intervals. Then $A_\phi : \mathbb{J}^n \to \mathbb{J}$ given by*

$$A_\phi(x_1, \cdots, x_n) = \phi^{-1}(A(\phi(x_1), \cdots, \phi(x_n))) \tag{1}$$

is an n-ary aggregation function on $\mathbb{J}^n$.

However, the analytical properties of A such as Lipschitz property, additivity or linearity are not be inherited by A_ϕ.

Observe that the transformation formula can be successfully applied to change the scale from example $\mathbb{I}^n$ into the $\mathbb{J}^n$ by means of a monotone bijection $\phi : \mathbb{J} \to \mathbb{I}$ where $\mathbb{J}$ is any real interval.

If A has the neutral element e, then by applying transformation formula on scales $[0, e]$ and $[e, 1]$ we can get two new aggregation operators based on A and with the neutral elements 1 and 0, respectively. For example, let $A : \bigcup_{n\in\mathbb{N}}[0, 1]^n \to [0, 1]$ be an aggregation function defined by

$$A(x_1, \cdots, x_n) = \frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n x_i + \prod_{i=1}^n (1 - x_i)}$$

with $\frac{0}{0} = 0$, that is an associative and symmetric aggregation function with neutral element $e = 0.5$. Then by assumption $\phi_0 : [0, 1] \to [0, 0.5]$ where $\phi_0(x) = \frac{1}{2}x$ we can define $A_{[0]} : \bigcup_{n\in\mathbb{N}}[0, 1]^n \to [0, 1]$ by

$$A_{[0]}(x_1, \cdots, x_n) = \frac{2\prod_{i=1}^n x_i}{\prod_{i=1}^n x_i + \prod_{i=1}^n (2 - x_i)}$$

that is an associative and symmetric aggregation function with neutral element $e = 1$. Moreover, if we put $\phi_1 : [0, 1] \to [0.5, 1]$ where $\phi_1(x) = \frac{x+1}{2}$ we can define $A_{[1]} : \bigcup_{n\in\mathbb{N}}[0, 1]^n \to [0, 1]$ by

$$A_{[1]}(x_1, \cdots, x_n) = \frac{2\prod_{i=1}^n (x_i + 1)}{\prod_{i=1}^n (x_i + 1) + \prod_{i=1}^n (1 - x_i)} - 1$$

that is an associative and symmetric aggregation function with neutral element $e = 0$. Similarly, if A has the annihilator a then by applying transformation formula on scales $[0, a]$ and $[a, 1]$ we can get two aggregation operators with annihilators 1 and 0, respectively.

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3.1. Duality of Aggregation Functions

One of the most applied transformations is the duality transformation where $\phi(x) = 1 - x$. Applying this transformation to any aggregation operator A, the *dual aggregation function* A^d of A is obtained. Indeed, duality is one of the simplest methods to develop an aggregation function on the basis of a given one.

For any (extended) aggregation function A, the mapping $A^d : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ defined by

$$A^d(x_1, \cdots, x_n) = 1 - A(1 - x_1, \cdots, 1 - x_n) \tag{2}$$

is called the dual aggregation function of A.

Clearly, the minimum and the maximum functions are dual of each other. The operators A_w and A_s are also dual of each other. Moreover, if the product function $\Pi(\mathbf{x}_{(n)}) = \prod_{i=1}^n x_i$ is given then its dual i.e., the probabilistic sum $PS(\mathbf{x}_{(n)}) = 1 - \prod_{i=1}^n (1 - x_i)$ is also an aggregation function.

Remark 6. *The aggregation operators A and A^d have the same analytical and algebraic properties, e.g. if A is continuous or symmetric then A^d is also continuous and symmetric.*

Since the operators *Min, Max* are dual of each other, we clearly have the following result.

Proposition 2. *Let $A : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ be an aggregation operator and A^d be its dual.*

- *If A is average (or alternatively idempotent), then A^d is also average (idempotent).*
- *If A is conjunctive (disjunctive), then A^d is disjunctive (conjunctive).*

Remark 7. *If A has the neutral element e, then $1 - e$ is the neutral element of A^d. Moreover, if a is the annihilator of A, then A^d has the annihilator element $1 - a$.*

The aggregation operator A is called *self-dual* (or *symmetric sum*) if and only if $A^d = A$ or equivalently for any $\mathbf{x}_{(n)} \in \mathbb{I}^n$: $A(\mathbf{1} - \mathbf{x}_{(n)}) = 1 - A(\mathbf{x}_{(n)})$. The arithmetic mean *AM* is a well-known example for self-dual aggregation functions, since $AM^d(x_1, \cdots, x_n) = 1 - \frac{n - \sum_{i=1}^n x_i}{n} = \frac{\sum_{i=1}^n x_i}{n}$. The median operator *Med* and the weighted arithmetic mean *WAM* are also self-dual aggregation functions.

However, in general case, the most aggregation functions are not self-dual. But, there is a technique (c.f. [11], Propositions 6 and 8) to make a self-dual function by using the average of aggregation function A and its dual, i.e. A^d.

Proposition 3. *([11]) An aggregation operator $A : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is self-dual if and only if there exists an aggregation operator B such that $A = \hat{B}$ where*

$$\hat{B}(x_1, \cdots, x_n) = \frac{B(x_1, \cdots, x_n) + B^d(x_1, \cdots, x_n)}{2} \tag{3}$$

and it is called the core of aggregation operator B.

According to Proposition 3, by having an aggregation function B we can generate a self-dual aggregation function A. For example, the arithmetic mean of *Min* and *Max*, i.e.

$$A(\mathbf{x}_{(n)}) = \frac{\min(\mathbf{x}_{(n)}) + \max(\mathbf{x}_{(n)})}{2}$$

or the arithmetic mean of product function and probabilistic sum, i.e.

$$A(\mathbf{x}_{(n)}) = \frac{1 + \prod_{i=1}^n x_i - \prod_{i=1}^n (1 - x_i)}{2}$$

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where the latter is mean value for $n = 2$, are two new aggregation operators that are self-dual functions.

Remark 8. *Triangular norms and their dual, i.e, the class of triangular conorms are the most important and useful examples for the concept of duality in aggregation theory. Furthermore, since t-norms and t-conorms are dual to each other, by condition $T(x_1, x_2) = 1 - S(1 - x_1, 1 - x_2)$, it will be sufficient to deal with t-norms only and accordingly, the properties of t-conorms are obtained.*

3.2. Quasi-Arithmetic Means

One of the most important classes of aggregation functions generated by the transformation formula is the *quasi-arithmetic means* as the transformation of AM [12]. If the arithmetic mean AM defined by $AM(x_1, \cdots, x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$ is given and $f : [0,1] \to \mathbb{R}$ is a continuous strictly monotone function, then by (1) we can get the aggregation function $AM_f : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ defined by

$$AM_f(x_1, \cdots, x_n) = f^{-1}\Big(AM(f(x_1), \cdots, f(x_n))\Big) = f^{-1}\Big(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\Big) \tag{4}$$

that is called quasi–arithmetic mean. The function f defined in (4) is called a *generator of AM_f*.

The quasi–arithmetic means are averaging aggregation functions that can be considered as the modern definition of mean value in the sense of Cauchy [20]. So, they may be viewed as the solution for the following problem.

Problem II. Constructing an averaging aggregation function $A \in \mathcal{A}$ such that $Min \leq A \leq Max$.

Clearly, the class of quasi-arithmetic means includes the most commonly useful averaging aggregation functions, i.e., the arithmetic mean and the geometric mean. The below example provides some well-known instances of quasi-arithmetic means generated by means of different formulas for f.

Example 10. *In Eq. (4),*

1. *if $f(x) = x$ then $AM_f(\mathbf{x}_{(n)}) = AM(\mathbf{x}_{(n)}) = \frac{1}{n}\sum_{i=1}^{n} x_i$ (arithmetic mean),*
2. *if $f(x) = x^2$ then $AM_f(\mathbf{x}_{(n)}) = QM(\mathbf{x}_{(n)}) = (\frac{1}{n}\sum_{i=1}^{n} {x_i}^2)^{1/2}$ (quadratic mean mean),*
3. *if $f(x) = \log x$ then $AM_f(\mathbf{x}_{(n)}) = GM(\mathbf{x}_{(n)}) = (\prod_{i=1}^{n} x_i)^{1/n}$ (geometric mean),*
4. *if $f(x) = \frac{1}{x}$ then $AM_f(\mathbf{x}_{(n)}) = HM(\mathbf{x}_{(n)}) = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}}$ (harmonic mean),*
5. *if $f(x) = e^{\alpha x}$ where $0 \neq \alpha \in \mathbb{R}$ then $AM_f(\mathbf{x}_{(n)}) = EM_\alpha(\mathbf{x}_{(n)}) = \frac{1}{\alpha}\ln(\frac{1}{n}\sum_{i=1}^{n} e^{\alpha x_i})$ (exponential mean).*

The following result shows the relation between quasi-arithmetic means generated by two functions f and g.

Theorem 1. *([12]) Let $f, g : [0,1] \to \mathbb{R}$ be continuous and strictly monotonic functions where g is also increasing. Then*

1. *$AM_f \leq AM_g$ if and only if $g \circ f^{-1}$ is convex.*
2. *$AM_f = AM_g$ if and only if $g \circ f^{-1}$ is linear, i.e., $g(x) = af(x) + b$ where $a, b \in \mathbb{R}$ and $a \neq 0$.*

Moreover, the next theorem talks about an axiomatization of quasi-arithmetic mean as an n-ary aggregation function.

Theorem 2. *([8]) The function $F : [0,1]^n \to \mathbb{R}$ is symmetric, continuous, strictly increasing, idempotent, and bisymmetric if and only if there is a continuous and strictly monotonic function $f : [0,1] \to \mathbb{R}$ such that $F := AM_f$ is the quasi-arithmetic mean generated by f.*

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4. Composite Aggregation Functions

Another class of constructing methods to aggregation operators is composition over some given aggregation functions. The main objective for applying composition technique to aggregation functions is to give an alternative aggregating approach when we face multi-source data. So, such composition technique can be addressed the following problem:

Problem III. Constructing an aggregation function $A \in \mathcal{A}$ to combine the input data that have been merged by different techniques.

Proposition 4. *([6]) Let $B : [0,1]^m \to [0,1]$ and $A_i : [0,1]^n \to [0,1]$ for $i \in \{1, \cdots, m\}$ are some aggregation functions where $m, n \in \mathbb{N}/\{1\}$. Then, the composite function $C_{B;A_1,\cdots,A_m} : [0,1]^n \to [0,1]$ defined by*

$$C_{B;A_1,\cdots,A_m}(x_1, \cdots, x_n) = B(A_1(x_1, \cdots, x_n), \cdots, A_m(x_1, \cdots, x_n)) \tag{5}$$

is an aggregation function if and only if B is an idempotent (averaging) aggregation operator.

Note that, in Eq. (5) each x_i is aggregated by each of the inner aggregation functions A_i, so totally m-times.

Remark 9. *In reality, Eq. (5) can be seen as a decision making situation involving m decision makers where each A_i correspond to the ith expert and operator B to the head or mentor of the group.*

For example, if we take B in (5) as the weighted arithmetic mean WAM with weighting vector $\mathbf{w}_{(m)} = (w_1, \cdots, w_m)$, then the function $C_{B;A_1,\cdots,A_m} := \sum_{i=1}^{m} w_i A_i$ is the convex combination of aggregation functions $A_1, \cdots, A_m$. As another simple example if $B := \max$ or $B := \min$, then $C_{B;A_1,\cdots,A_m} := \max(A_1, \cdots, A_m)$ and $C_{B;A_1,\cdots,A_m} := \min(A_1, \cdots, A_m)$, respectively.

4.1. Composition Over Different Source of Data

To aggregate a list of data coming from two or more different sources into a single output by using different aggregation functions that are defined according to the type of each source of input data, an alternative construction method is proposed in [10] to solve the following problem:

Problem IV. Constructing an aggregation function $A \in \mathcal{A}$ to combine two different types of input data.

Proposition 5. *([10]) Let $B : [0,1]^2 \to [0,1]$, $A_1 : [0,1]^n \to [0,1]$ and $A_2 : [0,1]^m \to [0,1]$ be some aggregation functions where $m, n \in \mathbb{N}/\{1\}$. Then, the composite function $D_{B;A_1,A_2} : [0,1]^{n+m} \to [0,1]$ given by*

$$D_{B;A_1,A_2}(x_1, \cdots, x_{n+m}) = B(A_1(x_1, \cdots, x_n), A_2(x_{n+1}, \cdots, x_{n+m})) \tag{6}$$

is an aggregation operator that is called double aggregation function.

For example, let in Proposition 5, B be the median operator Med. Then the function $D_{B;A_1,A_2}$ given by (6) is the average value of aggregation functions A_1 and A_2, i.e., $D_{B;A_1,A_2} := \frac{A_1+A_2}{2}$ is the arithmetic mean of functions A_1 and A_2.

Remark 10. *The main difference between aggregation functions given in (5) and (6) is in the domains of inner aggregation functions. In (6), each input x_j from the initial list of data is aggregated by only one of the inner aggregation functions A_i. However, in (5) each input x_j is aggregated by each of the inner aggregation functions A_i. Thus, Proposition 5 allows us to aggregate a list of data coming from two or more different sources into a single output based on different aggregation functions that are defined according to each source of data.*

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The Proposition 5 can be generalized for aggregation of m lists of inputs $\mathbf{x_1}, \cdots, \mathbf{x_m}$ by an aggregation function $D_{B;A_1,\cdots,A_m} : [0,1]^n \to [0,1]$ given by $D_{B;A_1,\cdots,A_m}(\mathbf{x_1}, \cdots, \mathbf{x_m}) = B(A_1(\mathbf{x_1}), \cdots, A_m(\mathbf{x_m}))$ where for each $i \in \{1, \cdots, m\}$: $\mathbf{x_i} \in [0,1]^{n_i}$, $\sum_{i=1}^{m} n_i = n$ and the outer and inner aggregation functions are defined by mappings $B : [0,1]^m \to [0,1]$ and $A_i : [0,1]^{n_i} \to [0,1]$.

4.2. Composition Over Sub-groups of Data

An interesting composition method of aggregation functions was recently proposed in [21]. This method can help to handle the problem of partial agreement (not necessarily the full agreement) when α number of total n criteria or decision makers are sufficient to reach consensus. The aim of this method is to overcome the following difficulty.

Problem V. Constructing an alternative aggregating approach in multi-criteria decision-making problems with n arguments where any possible list of $\alpha \leq n$ arguments (not necessarily all n arguments) can affect the final decision at the consensus level α.

Before giving the explanation of this method, first let $X = \{x_1, \cdots, x_n\}$ be the universal set of n elements and $C_{n,\alpha}$ stands for α-combination operator of n where $C^i_{n,\alpha}\{x_1, \cdots, x_n\}$ represents the ith α-combination from the set X. We apply the operator $C_{n,\alpha}$ for the index set $I = \{1, \cdots, n\}$ to cut off I into the different subsets $I_i \subseteq I$ with cardinality $| I_i |= \alpha$ such that $C_{n,\alpha}(I) = \{I_1, \cdots, I_k\}$ where for $i = 1, \cdots, k$: I_i shows the ith α-combination of I. By using the permutation operator $\sigma^* : \{1, \cdots, k\} \to \{1, \cdots, k\}$, we rearrange the collection $C_{n,\alpha}(I)$ to provide a lexicographical order on $C_{n,\alpha}(I)$.

For any extended aggregation functions A, B we can define an aggregation function $F_{\alpha;B,A} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ by the following theorem.

Theorem 3. *([21]) Let $A, B : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ be two (extended) aggregation functions. The function $F_{\alpha;B,A} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ given by*

$$F^{(n)}_{\alpha;B,A}(x_1, \cdots, x_n) = B^{(k)}\Big(A^{(\alpha)}(C^{\sigma^*(1)}_{n,\alpha}\{x_1, \cdots, x_n\}), \cdots, A^{(\alpha)}(C^{\sigma^*(k)}_{n,\alpha}\{x_1, \cdots, x_n\})\Big) : n \in \mathbb{N} \quad (7)$$

is an (extended) aggregation operator called combination operator-based aggregation function of degree α where $\alpha \in \{1, 2, \cdots, n\}$ and $k = \frac{n!}{\alpha!(n-\alpha)!}$ is the binomial coefficient.

In Theorem 3, for any $1 \leq i \leq k$ we have the sequence $C^{\sigma^*(i)}_{n,\alpha}\{x_1, \cdots, x_n\} = (x_{\iota_{i,\sigma(1)}}, \cdots, x_{\iota_{i,\sigma(\alpha)}})$ that traverses the ith α-combination of the set $\{x_1, \cdots, x_n\}$ in the lexicographical order of $C_{n,\alpha}(I)$ such that for any j, $\iota_{i,\sigma(j)}$ is strictly smaller than $\iota_{i,\sigma(j+1)}$.

Thus, we first apply the operator A for aggregating any α-element selection of the set $\{x_1, \cdots, x_n\}$ and then use the operator B to combine these obtained values into an unique output.

Example 11. *Take the aggregation functions $B := \max$ and $A := \min$. If $n = 4$ and $\alpha = 3$, then the aggregation operator $F^{(4)}_{3;\max,\min} : [0,1]^4 \to [0,1]$ defined by*

$$F^{(4)}_{3;\max,\min}(x_1, x_2, x_3, x_4) = \max\Big(\min(x_1, x_2, x_3), \min(x_1, x_2, x_4), \min(x_1, x_3, x_4), \min(x_2, x_3, x_4)\Big)$$

first combines pessimistically any 3-member selection over the alternatives x_1, x_2, x_3 and x_4 and then merge them optimistically to get the overall result.

By changing the aggregation operators A, B we can get different classes of aggregation operator $F^{(n)}_{\alpha;B,A}$. For example, let $n = 3$ and $\alpha = 2$. If $B := \max$ and $A := \min$, then

$$F^{(3)}_{2;\max,\min}(x_1, x_2, x_3) = \max\Big(\min(x_1, x_2), \min(x_1, x_3), \min(x_2, x_3)\Big) = Med(x_1, x_2, x_3)$$

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If $A = B := AM$, then

$$F^{(3)}_{2;AM,AM}(x_1, x_2, x_3) = \frac{1}{3}(\frac{x_1 + x_2}{2} + \frac{x_1 + x_3}{2} + \frac{x_2 + x_3}{2}) = \frac{\sum_{i=1}^{3} x_i}{3} = AM(x_1, x_2, x_3)$$

If $A = B := \Pi$, then

$$F^{(3)}_{2;\Pi,\Pi}(x_1, x_2, x_3) = (x_1x_2) \cdot (x_1x_2) \cdot (x_2x_3) = \Pi_{i=1}^{3} x_i^2$$

That are not, of course, new operators.

5. Aggregating of Weighted Input

An important factor to handle aggregating problem of input data with different importance is weights. Indeed, the correspondence weighting vector $\mathbf{w}_{(n)} = (w_1, \cdots, w_n)$ to such input values $x_1, \cdots, x_n$ can be understood as the vector of cardinality of each input x_i where $w_i > 0$ for $i = 1, \cdots, n$. If $\sum_{i=1}^{n} w_i = 1$ then $\mathbf{w}_{(n)}$ is called the normal weighting vector.

Thus, the aggregation function over weighted data contributes to solve the following problem.

Problem VI. Constructing an aggregation operators $A_{\mathbf{w}} \in \mathcal{A}$ that permit to consider different weights of the sources or data where in fact,

$$A_{\mathbf{w}}(x_1, \cdots, x_n) = A(\underbrace{x_1, \cdots, x_1}_{w_1}, \cdots, \underbrace{x_n, \cdots, x_n}_{w_n})$$

To aggregate a list of weighted input values, there are two main methods: (1) first using the weighted aggregation operators defined naturally based on the weighting vector, such as weighted arithmetic mean or (2) applying some techniques like weighted quasi-arithmetic means and weighted rule based on Fagin and Wimmerse's approach [23] to produce a weighted aggregation operator based on unweighted one.

However, in both methodologies the identification of weights is an interesting topic. In practice, there is no unique strategy to find the associated weighting vector $\mathbf{w}_{(n)}$. Sometimes, these weights are given by the decision makers or mentor involved in the decision making problem based on their knowledge, information and past experiences. The weighting vector can be also determined by a fuzzy linguistic quantifier function $Q : [0, 1] \to [0, 1]$ based on the formulation $w_i = Q(\frac{i}{n}) - Q(\frac{i-1}{n})$ for all $i = 1, \cdots, n$ where the definition of Q may be changed from one case to another one [24–29]. For example, the case "Q:=most", where "most" is interpreted as 60% of all data, may be defined by

$$Q_{most}(z) = \begin{cases} 0 & \text{if } z \le 0.2 \\ \frac{z-0.2}{0.4} & \text{if } 0.2 < z < 0.6 \\ 1 & \text{if } z \ge 0.6 \end{cases} \tag{8}$$

that means if at least 60% of some elements satisfy a property, then most of them certainly (to degree 1) satisfy it, when less than 20% of them satisfy it, then most of them certainly do not satisfy it (satisfy to degree 0). If between 20% and 60% of them satisfy it, more of them satisfy it, computed by the given formula.

5.1. Weighted Aggregation Functions

The weighted arithmetic mean defined by

$$WAM(x_1, \cdots, x_n) = \sum_{i=1}^{n} w_i x_i$$

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and weighted geometric mean defined by

$$WGM(x_1, \cdots, x_n) = \prod_{i=1}^{n} {x_i}^{w_i}$$

are the most commonly used operators to compute the aggregating of weighted input such that the weighting vector $\mathbf{w}_{(n)} = (w_1, \cdots, w_n)$, where $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$, shows the importance degrees of x_is.

Ordered weighted average (*OWA*) operator [25,30], which is calculated based on the arithmetic mean, and ordered weighted geometric (*OWG*) operator [26], which is formulated based on the geometric mean, are other two important aggregation operators for weighted input where the position/order of input has the weight rather than their sources.

The ordered weighted average (*OWA*) operator is defined by

$$OWA(x_1, \cdots, x_n) = \sum_{i=1}^{n} w_i x_{\sigma(i)}$$

where σ is a permutation on $\{1, \cdots, n\}$ and $x_{\sigma(i)}$ is the ith largest element among $x_1, \cdots, x_n$. Indeed, *OWA* opeartor is a revised version of *WAM* operator where the reordering step of the *OWA* operator is carried out to assign the weight w_i to the ith place/location not ith value. Similarly the ordered weighted geometric (*OWG*) operator is defined by

$$OWG(x_1, \cdots, x_n) = \prod_{i=1}^{n} {x_{\sigma(i)}}^{w_i}$$

Remark 11. *Not that*

- *If* $w_1 = 1$ *and* $w_i = 0$ *else, then* $OWA := Max$ *and* $OWG := Max$;
- *If* $w_n = 1$ *and* $w_i = 0$ *else, then* $OWA := Min$ *and* $OWG := Min$;
- *If* $w_i = \frac{1}{n}$, *then* $OWA := AM$ *and* $OWG := GM$;
- *If n is odd,* $w_{\frac{n+1}{2}} = 1$ *and* $w_i = 0$ *else, then* $OWA := Med$ *and* $OWG := Med$;
- *If n is even,* $w_{\frac{n}{2}} = w_{\frac{n}{2}+1} = \frac{1}{2}$, *and* $w_i = 0$ *else, then* $OWA := Med$.

But, the main disadvantage of *OWA* and *OWG* operators is ignoring the importance of given arguments $x_1, \cdots, x_n$ for calculating the aggregated value. That is why they were extended into the induced *OWA* (or *IOWA*) operator in [27], and induced *OWG* (or *IOWG*) operator in [4], respectively. The *IOWA* operator, introduced by Yager and Filev [27], and the *IOWG* operator, given by Xu and Da [4], are two important extensions of operators *OWA* and *OWG*, respectively, to bridge the gap of weighted input arguments that is not mentioned in *OWA* and *OWG*.

The *IOWA* operator is defined by

$$IOWA(\langle u_1, x_1 \rangle, \cdots, \langle u_n, x_n \rangle) = \sum_{j=1}^{n} w_j y_j \tag{9}$$

and *IOWG* operator is defined by

$$IOWG(\langle u_1, x_1 \rangle, \cdots, \langle u_n, x_n \rangle) = \prod_{j=1}^{n} {y_j}^{w_j} \tag{10}$$

where y_j is the value of x_i that has the jth largest u_i; and u_i in $\langle u_i, x_i \rangle$ is referred to as the weight of variable x_i. The weights $w_1, \cdots, w_n$ such that $\sum_{i=1}^{n} w_i = 1$ are the associated weights to the *IOWA* and *IOWG* operators. Here, the reordering step of x_is is carried out by the variable u_i rather than the value of x_i, using in *OWA* and *OWG*. Therefore, the collection $x_1, \cdots, x_n$ is reordered as $\langle \max\{u_i\}, y_1\rangle \geq \cdots \geq \langle \min\{u_i\}, y_n\rangle$.

However, these extensions have the inherent limitations from *OWA* operator and *OWG* operator, concerning the determination of associated weighting vector $\mathbf{w}_{(n)}$ for *IOWA* and *IOWG* operators.

More extensions of operators *OWA*, *OWG*, *IOWA* and *IOWG* have been discussed to aggregating data with fuzzy and vague information [31–37].

The weighted minimum and the weighted maximum are also commonly used classes of aggregation operators, discussed firstly in [38], dealing with objects having non-negative weights $w_1, \cdots, w_n$ such that $\max_{i=1}^n w_i = 1$. Using the concept of possibility and necessity of fuzzy events, Dubois and Prade [38] proposed the following operators, so-called the weighted disjunction and weighted conjunction, to interpretation the weighted maximum and weighted minimum:

$$WMax(x_1, \cdots, x_n) = \max_{i=1}^{n} \min(w_i, x_i) \tag{11}$$

and

$$WMin(x_1, \cdots, x_n) = \min_{i=1}^{n} \min(1 - w_i, x_i) \tag{12}$$

where $\max_{i=1}^n w_i = 1$.

5.2. Weighted Quasi–Arithmetic Means

By taking the weighted arithmetic mean $WAM = \sum_{i=1}^n w_i x_i$, where $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$, in Eq. (4) we get the *weighted quasi-arithmetic mean* as below:

$$WAM_{f,w}(x_1, \cdots, x_n) = f^{-1}\Big(\sum_{i=1}^{n} w_i f(x_i)\Big) \tag{13}$$

Obviously, if $f(x) = x$ or $f(x) = \log x$ then we have the weighted arithmetic mean *WAM* and the weighted geometric mean *WGM*, respectively.

Theorem 4. *([8]) The function $F : [0,1]^n \to \mathbb{R}$ is symmetric, continuous, strictly increasing, idempotent, and bisymmetric if and only if there is a continuous and strictly monotonic function $f : [0,1] \to \mathbb{R}$ and real numbers $w_1, \cdots, w_n \in [0,1]$ where $\sum_{i=1}^n w_i = 1$ such that $F := WAM_{f,w}$ is the weighted quasi-arithmetic mean generated by f.*

If $p : [0,1] \to R^+$ is a positive-valued function, then the n-ary function $M : [0,1]^n \to [0,1]$ defined by

$$M(x_1, \cdots, x_n) = \frac{\sum_{i=1}^n p(x_i) x_i}{\sum_{i=1}^n p(x_i)} \tag{14}$$

is an averaging aggregation function, called *mixture operator*, that is a generalization of *WAM* [22,39]. However, the monotonicity of mixture operators are not clear. There are some sufficient conditions ensuring the monotonicity of such operators discussed in [22]. If $p : I \to R^+$ is a non–decreasing differentiable function, then the next two conditions:

1. $p(x) \geq p'(x) \cdot l(I)$ for all $x \in I$ where $l(I)$ is the length of the interval I;
2. $p(x) \geq p'(x) \cdot (x - \inf I)$ for all $x \in I$.

For example, if $p : [0,1] \to R^+$ is given by $g(x) = x + \frac{3}{4}$ then the function $M : [0,1]^2 \to [0,1]$ defined by (14) is an idempotent aggregation function.

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By using (13), the *quasi-mixture operator* generated by f with weight function p is the aggregation function $M_{f,p} : [0,1]^n \to [0,1]$ defined by

$$M_{f,p}(x_1, \cdots, x_n) = f^{-1}\Big(\frac{\sum_{i=1}^n p(x_i) f(x_i)}{\sum_{i=1}^n p(x_i)}\Big). \tag{15}$$

We can also consider the weighted version of given method in Theorem 3 by using weighted quasi-arithmetic mean formula in (13) as the following.

Definition 6. *Let for any $n \in \mathbb{N}$, $\mathbf{w}_{(n)} = (w_1, \cdots, w_n) \in [0,1]^n$ be an n-dimensional weighting vector for a list of n arguments $x_1 \cdots, x_n$ such that $\sum_{i=1}^n w_i = 1$.*
In order to determine the relevant weights for all k possible α-combinations of $\{x_1 \cdots, x_n\}$, where $k = \binom{n}{\alpha}$, the $k \times \alpha$ weighting matrix U_α is constructed where for each $i \in \{1, \cdots, k\}$ the row vector $\mathbf{u}_i = (u_{i,1}, \cdots, u_{i,\alpha}) \in [0,1]^\alpha$ from the matrix U_α is defined by

$$\mathbf{u}_i := \frac{\mathbf{w}_i}{\sum_{j=1}^{\alpha} w_{\iota_{i,\sigma(j)}}} \tag{16}$$

if $\sum_{j=1}^{\alpha} w_{\iota_{i,\sigma(j)}} \neq 0$ and else $\mathbf{u}_i = \mathbf{0}$ such that $\mathbf{w}_i = (w_{\iota_{i,\sigma(1)}}, \cdots, w_{\iota_{i,\sigma(\alpha)}}) \in [0,1]^\alpha$ is a subsequence of $\mathbf{w}_{(n)} = (w_1, \cdots, w_n)$ corresponding to the ith α-combination of $x_1, \cdots, x_n$.

The next theorem provides a new class of the weighted quasi-arithmetic mean approach so-called **w**-*weighted quasi-arithmetic mean combination operator-based aggregation functions*.

Theorem 5. *([21]) Suppose that $\mathbf{w}_{(n)} = (w_1, \cdots, w_n) \in \mathbb{I}^n$ be an n-dimensional rational weighting vector. Let $A, B : \bigcup_{n\in\mathbb{N}} \mathbb{I}^n \to \mathbb{I}$ be (extended) aggregation functions where B is idempotent and continuous and $A := WAM_f$ by means of a continuous monotone function $f : [0,1] \to [-\infty, +\infty]$. If U_α is the weighting matrix, then function $WF_{\alpha;B,f} : \bigcup_{n\in\mathbb{N}} \mathbb{I}^n \to \mathbb{I}$ given by*

$$WF^{(n)}_{\alpha;B,f}\langle(x_1, \cdots, x_n), \mathbf{w}_{(n)}\rangle = B^{(k)}\Big(WAM_f\langle C^{\sigma^*(1)}_{n,\alpha}\{x_1, \cdots, x_n\}, \mathbf{u}_1\rangle, \cdots, WAM_f\langle C^{\sigma^*(k)}_{n,\alpha}\{x_1, \cdots, x_n\}, \mathbf{u}_k\rangle\Big); n \in \mathbb{N} \tag{17}$$

is an idempotent continuous n-array function that is called **w**-*weighted quasi arithmetic mean combination operator-based aggregation function.*

Example 12. *Let $B : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ be the arithmetic mean AM.*
If $f := x$ then

$$WF_{\alpha;AM,f}\langle(x_1, \cdots, x_n), \mathbf{w}_{(n)}\rangle = \frac{1}{k}\sum_{i=1}^{k}\sum_{j=1}^{\alpha} u_{i,j} \cdot x_{\iota_{i,\sigma(j)}}$$

If $f := \log x$ then

$$WF_{\alpha;AM,f}\langle(x_1, \cdots, x_n), \mathbf{w}_{(n)}\rangle = \frac{1}{k}\sum_{i=1}^{k}\prod_{j=1}^{\alpha} x_{\iota_{i,\sigma(j)}}{}^{u_{i,j}}$$

where for each $i = 1, \cdots, k$: $u_{i,j} = \frac{w_{\iota_{i,\sigma(j)}}}{\sum_{j=1}^{\alpha} w_{\iota_{i,\sigma(j)}}}$ and $\sum_{j=1}^{\alpha} w_{\iota_{i,\sigma(j)}} \neq 0$.

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5.3. Weighted Rule

How to modify an unweighted scoring rule to apply for arguments with different importance degrees motivated authors in [23] to present a technique for extending an n-ary rule to its weighted version where the proposed method answers the following question.

Problem VII. Constructing aggregation operators $A_{\mathbf{w}} \in \mathcal{A}$ based on a convex combination of unweighted aggregation operators.

Theorem 6. *([23]) For every unweighted rule f there exists a weighted rule that is based on f; compatible; and locally linear. If σ is a permutation that orders the weights as follows: $w_{\sigma(1)} \geq \cdots \geq w_{\sigma(n)}$ and $w_{n+1} = 0$, then*

$$f_w(x_1, \cdots, x_n) = \sum_{i=1}^{n} i \cdot (w_{\sigma(i)} - w_{\sigma(i+1)}) \cdot f(x_{\sigma(1)}, \cdots, x_{\sigma(i)}) \tag{18}$$

is a weighted rule based on f.

Then, they proved that if an unweighted rule f is continuous (or monotonic, strictly monotonic, idempotent), then the corresponding weighted rule f_w is continuous (monotonic, strictly monotonic, idempotent) as well. They also showed that $\sum_{i=1}^{n} i \cdot (w_{\sigma(i)} - w_{\sigma(i+1)}) = \sum_{i=1}^{n} w_i = 1$ where $w_{\sigma(1)} \geq \cdots \geq w_{\sigma(n)}$ and $w_{n+1} = 0$.

Corollary 1. *Corresponding to any aggregation function A there exists a weighted aggregation function A_w based on the weights $w_1, \cdots, w_n$ defined by*

$$A_w(x_1, \cdots, x_n) = \sum_{i=1}^{n} i \cdot (w_{\sigma(i)} - w_{\sigma(i+1)}) \cdot A(x_{\sigma(1)}, \cdots, x_{\sigma(i)}) \tag{19}$$

where $w_{\sigma(1)} \geq \cdots \geq w_{\sigma(n)}$ and $w_{n+1} = 0$.

For example, if $A := \min$ or $A := \max$ then the weighted minimum and the weighted maximum can be defined as below, respectively.

$$\min_w(x_1, \cdots, x_n) = \sum_{i=1}^{n} i \cdot (w_{\sigma(i)} - w_{\sigma(i+1)}) \cdot \min(x_{\sigma(1)}, \cdots, x_{\sigma(i)}) \tag{20}$$

and

$$\max_w(x_1, \cdots, x_n) = \sum_{i=1}^{n} i \cdot (w_{\sigma(i)} - w_{\sigma(i+1)}) \cdot \max(x_{\sigma(1)}, \cdots, x_{\sigma(i)}) \tag{21}$$

Remark 12. *Using discussed constructing methods in previous sections as the underlying formula, we can provide some more new classes of aggregation operators based on the given technique in (18).*
For example, combining formulas given by (7) and (20), we can get a weighted version of Example 11 as below:

$$F^{(n)}_{\alpha;\max,\min_w}(x_1, \cdots, x_n) = \max_{i=1}^{k} \left\{ \sum_{kj=1}^{\alpha} j \cdot (\mathbf{u}_{\sigma(j)} - \mathbf{u}_{\sigma(j+1)}) \cdot \min(x_{\sigma(1)}, \cdots, x_{\sigma(j)}) \right\} \tag{22}$$

Example 13. *Reconsider Example 11 where* $B := \max$ *and* $A := \min_w$. *Take the weighting vector* $\mathbf{w} = (w_1, w_2, w_3, w_4)$ *such that* $w_i \in [0,1], \sum_{i=1}^4 w_i = 1, w_1 \geq w_2 \geq w_3 \geq w_4$ *and* $w_5 = 0$. *Then, using (22), the aggregation operator* $F^{(4)}_{3;\max,\min_w} : [0,1]^4 \to [0,1]$ *can be defined as below:*

$$F^{(4)}_{3;\max,\min_w}(x_1,x_2,x_3,x_4) = \max\Big((\frac{w_1}{\sum_{i=1}^3 w_i} - \frac{w_2}{\sum_{i=1}^3 w_i})x_1 + 2(\frac{w_2}{\sum_{i=1}^3 w_i} - \frac{w_3}{\sum_{i=1}^3 w_i}) \min\{x_1,x_2\}$$
$$+3\frac{w_3}{\sum_{i=1}^3 w_i}\min\{x_1,x_2,x_3\}, (\frac{w_1}{\sum_{i=1,i\neq 3}^4 w_i} - \frac{w_2}{\sum_{i=1,i\neq 3}^4 w_i})x_1 + 2(\frac{w_2}{\sum_{i=1,i\neq 3}^4 w_i} - \frac{w_4}{\sum_{i=1,i\neq 3}^4 w_i})\min\{x_1,x_2\}$$
$$+3\frac{w_4}{\sum_{i=1,i\neq 3}^4 w_i}\min\{x_1,x_2,x_4\}, (\frac{w_1}{\sum_{i=1,i\neq 2}^4 w_i} - \frac{w_3}{\sum_{i=1,i\neq 2}^4 w_i})x_1 + 2(\frac{w_3}{\sum_{i=1,i\neq 2}^4 w_i} - \frac{w_4}{\sum_{i=1,i\neq 2}^4 w_i})\min\{x_1,x_3\}$$
$$+3\frac{w_4}{\sum_{i=1,i\neq 2}^4 w_i}\min\{x_1,x_3,x_4\}, (\frac{w_2}{\sum_{i=2}^4 w_i} - \frac{w_3}{\sum_{i=2}^4 w_i})x_2 + 2(\frac{w_3}{\sum_{i=2}^4 w_i} - \frac{w_4}{\sum_{i=2}^4 w_i})\min\{x_2,x_3\}$$
$$+3\frac{w_4}{\sum_{i=2}^4 w_i}\min\{x_2,x_3,x_4\}\Big)$$

Conclusion

In decision situations, aggregating for reaching to consensus is one of the most important steps before taking the final action. This means that the used aggregating method affects the final solution. Different aggregating techniques have been discussed in literature, however the most useful one is aggregation functions. In this paper, we have reviewed properties and classification of aggregation functions. Some construction methods to generate new aggregation operators based on the existing ones have been also discussed. Based on their characteristics, these techniques are classified into the following three groups: (1) transformation (which can produce linear transformation and dual of aggregation functions), (2) composition (that can cope with the problem of multi-source data) and (3) weighted rule of aggregation operators (which permits data with different importance degrees). Especially, we have focused on the applications and consensus problems that may be handled by each of these methods. We also briefly discussed the differences between them.

Moreover, to develop new aggregation operators in future research, this overview gives an insight to the researchers. It is observed that, however, to date many efforts have been made to develop aggregation functions theory, there are some gaps that need to be discussed by researchers in the future. For instance, there exist situations with multi-source data where some of the input information is dynamic or not complete that means the aggregating progress over each source can be dynamic or partial (such as negotiation process in management level). Composition of aggregation functions over different sub-groups of data where there are some incomplete/missing information may be the solution.

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