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Article

Symmetry-like relation of relative entropy measure of quantum coherence

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Abstract: Quantum coherence is an important physical resource in quantum information science, and also as one of the most fundamental and striking features in quantum physics. In this paper, we obtain a symmetry-like relation of relative entropy measure $C_r(\rho)$ of coherence for n-partite quantum states ρ, which gives lower and upper bounds for $C_r(\rho)$. Meanwhile, we discuss the conjecture about the validity of the inequality $C_r(\rho) \leq C_{\ell_1}(\rho)$ for any state ρ. We observe that every mixture η of a state ρ satisfying $C_r(\rho) \leq C_{\ell_1}(\rho)$ and any incoherent state σ also satisfies the conjecture. We also note that if the von Neumann entropy is defined by the natural logarithm ln instead of $\log_2$, then the reduced relative entropy measure of coherence $\bar{C}_r(\rho) = -\rho_{\rm diag} \ln \rho_{\rm diag} + \rho \ln \rho$ satisfies the inequality $\bar{C}_r(\rho) \leq C_{\ell_1}(\rho)$ for any mixed state ρ.

Keywords: quantum coherence; measure; lower bound; upper bound

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1. Introduction

Quantum computing utilizes the superposition and entanglement of quantum states to operate and process information. Its most significant advantage lies in the parallelism of operations. To achieve efficient parallel computing in quantum computers, quantum coherence is essentially used. Quantum coherence arising from quantum superposition plays a central role in quantum mechanics and so becomes an important physical resource in quantum information and quantum computation [1]. It also plays an important role in a wide variety of research fields, such as quantum biology [2–7], nanoscale physics [8,9], and quantum metrology [10,11].

In 2014, Baumgratz et al. [12] proposed a framework to quantify coherence. In their seminal work, conditions that a suitable measure of coherence should satisfy have been put forward, including nonnegativity, the monotonicity under incoherent completely positive and trace preserving operations, the monotonicity under selective incoherent operations on average and the convexity under mixing of states. By introducing such a rigorous theoretical framework, a mass of properties and operations of quantification of coherence were discussed. Moreover, based on that framework, many coherence measures have been found, such as ℓ_1-norm of coherence and relative entropy of coherence [12], fidelity and trace norm distances for quantifying coherence [13], robustness of coherence [14], geometric measure of coherence [15], coherence of formation [16], relative quantum coherence[17], measuring coherence with entanglement concurrence [18], trace distance measure of coherence [19–21].

In addition, some research related to quantum coherence have been developed, including quantum coherence and quantum correlations [22–27], a uncertainly-like relation about coherence [28], distribution of quantum coherence in multipartite systems [29], quantum coherence over the noisy quantum channels [30], maximally coherent mixed states [31], ordering states with coherence measures [32], , coherence and path information [33], complementarity relations for quantum coherence [34], converting coherence to quantum correlations [35] and logarithmic coherence [36], quantum coherence and geometric quantum discord [37]. Recently, Guo and Cao [38] discussed the question of creating quantum correlation from a coherent state via incoherent quantum operations and obtained

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explicit interrelations among incoherent operations (IOs), maximally incoherent operations, genuinely incoherent operations and coherence breaking operations.

In this paper, we will discuss some inequalities on the measures of quantum coherence. The organization of this paper is as follows: In Section 2, we recall the framework of coherence measure and basic properties of quantum coherence. In Section 3, we establish lower and upper bounds for the relative entropy measure of coherence in a multipartite system. In Section 4, we will discuss the relation between $C_r(\rho)$ and $C_{\ell_1}(\rho)$ and verify the conjecture proposed in Rana et al. by a given d-dimensional maximally coherent mixed state. In Section 5, we will give our conclusion obtained in this paper.

2. Preliminaries

In this section, we give a review of some fundamental notions about quantification of coherence, such as incoherence states, incoherence operations, and the measures of coherence.

Let H be a d-dimensional Hilbert space, $D(H)$ denote the set of all density operators on H. Fixed an orthonormal basis $e=\{|e_i\rangle\}_{i=1}^d$ for H, a state $\rho\in D(H)$ is said to be incoherent with respect to (w.r.t.) the basis e, if $\langle e_i|\rho|e_j\rangle=0(i\neq j)$. Otherwise, it is said to be coherent w.r.t. e.

Let $\mathcal{I}(e)$ be a set of all incoherent states of the system described by H, that is,

$$\mathcal{I}(e)=\left\{\rho\in D(H):\langle e_i|\rho|e_j\rangle=0(i\neq j)\right\}.$$

When $\rho=\sum_{ij}\rho_{ij}|e_i\rangle\langle e_j|$, we define

$$\rho_{e\text{-diag}}=\sum_{i=1}^{d}\rho_{ii}|e_i\rangle\langle e_i|,$$

called the diagonal state of ρ w.r.t. the basis e, it is always an incoherent state w.r.t. e, i.e., $\rho_{e-\text{diag}}\in\mathcal{I}(e)$.

By definition, a state ρ is incoherent w.r.t e if and only if it has a diagonal-matrix representation w.r.t. e, i.e.

$$\rho=\sum_{i=1}^{d}ł_i|e_i\rangle\langle e_i|,$$

where $ł_i\geq 0$ are eigenvalues of ρ and $\sum_{i=1}^{d}ł_i=\text{tr}\rho=1$; it is coherent w.r.t. e if and only if it can not be written as a diagonal matrix under this basis.

According to [39], a linear map $\mathcal{E}$ on $B(H)$ is a completely positive and trace preserving (CPTP) map if and only if there exists a set of operators $\{K_n\}_{n=1}^m\subset B(H)$ with $\sum_{n=1}^m K_n^\dagger K_n=I_d$ such that

$$\mathcal{E}(T)=\sum_{n=1}^{m}K_nTK_n{}^\dagger,\ \ \forall T\in B(H).$$

These operators $K_1,\ldots,K_m$ are called the Kraus operators of $\mathcal{E}$.

A CPTP map $\mathcal{E}$ on $B(H)$ is said to be an *incoherent operation (IO)* it has Kraus operators $K_1,\ldots,K_m$ such that if for all $n=1,2,\ldots,m$, it holds that whenever

$$K_n\rho K_n{}^\dagger\in\text{tr}\left(K_n\rho K_n{}^\dagger\right)\mathcal{I}(e),\ \ \forall\rho\in\mathcal{I}(e).$$

In this case, we call $\{K_n\}_{n=1}^m$ a set of incoherent Kraus operators of $\mathcal{E}$.

In order to measure coherence, Baumgratz *et al.* [12] presented the following four defining conditions for a coherence measure C^e:

(A_1) $C^e(\rho)\geq 0$, $\forall\rho\in D(H)$; and $C^e(\rho)=0$ iff $\rho\in\mathcal{I}(e)$.

(A_2) $C^e(\mathcal{E}(\rho))\leq C^e(\rho)$ for any incoherent operation $\mathcal{E}$ and any state $\rho\in D(H)$.

(A_3) $C^e(\rho) \geq \sum_n p_n C^e(\rho_n)$ for any incoherent operation $\mathcal{E}$ with a set of incoherent Kraus operators $\{K_n\}$ and any state $\rho \in D(H)$ where $\rho_n = {p_n}^{-1} K_n \rho K_n^\dagger$ with $p_n = \mathrm{tr}(K_n \rho K_n^\dagger) \neq 0$.

(A_4) $\sum_i p_i C^e(\rho_i) \geq C^e(\sum_i p_i \rho_i)$ for any ensemble $\{p_i, \rho_i\}$.

It was proved in [12] that the relative-entropy measure $C_r^e(\rho)$ and the ℓ_1-norm measure $C_{\ell_1}^e(\rho)$ of coherence satisfy these defining conditions.

The definitions of these measures are as follows:

$$C_r^e(\rho) = \min_{\sigma \in \mathcal{I}(e)} S(\rho \parallel \sigma) = S(\rho_{e\text{-diag}}) - S(\rho), \tag{1}$$

where $S(\rho \parallel \sigma)$ is the quantum relative entropy and $S(\rho) = -\mathrm{tr}(\rho \log \rho)$ is the von Neumann entropy, and

$$C_{\ell_1}^e(\rho) = \sum_{i \neq j} |\langle e_i|\rho|e_j\rangle| = \|\rho - \rho_{e\text{-diag}}\|_{\ell_1}. \tag{2}$$

Notably, for a bipartite quantum system AB, the reference basis for $H_{AB} = H_A \otimes H_B$ can be taken as a local basis:

$$e_{AB} := e_A \otimes e_B = \{|e_i\rangle \otimes |f_k\rangle | i = 1, 2, \ldots, d_A, k = 1, 2, \ldots, d_B\},$$

where $e_A = \{|e_i\rangle\}_{i=1}^{d_A}$ and $e_B = \{|f_k\rangle\}_{k=1}^{d_B}$ are the orthonormal bases for H_A and H_B, respectively. In this case, every ρ^{AB} of AB has the following representation:

$$\rho^{AB} = \sum_{i,j=1}^{d_A} \sum_{k,l=1}^{d_B} \rho_{i,j,k,l} |e_i\rangle\langle e_j| \otimes |f_k\rangle\langle f_l|.$$

We call the state

$$\rho^{AB}_{e_{AB}\text{-diag}} = \sum_{i=1}^{d_A} \sum_{k=1}^{d_B} \rho_{i,i,k,k} |e_i\rangle\langle e_i| \otimes |f_k\rangle\langle f_k|$$

the diagonal state of ρ^{AB} w.r.t. the basis e_{AB}. A state $\rho^{AB} \in D(H_{AB})$ is said to be incoherent w.r.t. e_{AB} if and only if $\rho^{AB} = \rho^{AB}_{e_{AB}\text{-diag}}$, i.e.,

$$\left\langle e_i | f_k | \rho^{AB} | e_j | f_l \right\rangle = 0((i,k) \neq (j,l)).$$

A state which is not incoherent w.r.t. e_{AB} is said to be coherent w.r.t. e_{AB}.

In next section, we will derive some inequalities, which give lower and upper bounds for the relative-entropy measure of coherence of multi-partite states.

3. Lower and upper bounds for the relative-entropy measure of coherence

Xi *et al.* [27] proved that for any bipartite quantum state ρ^{AB}, the relative-entropy measure of coherence obeys some uncertainty-like relation by using the properties of relative entropy, which reads

$$C_r^{e_{AB}}(\rho^{AB}) \geq C_r^{e_A}(\rho^A) + C_r^{e_B}(\rho^B), \tag{3}$$

where $\rho^A = \mathrm{tr}_B \rho^{AB}$, $\rho^B = \mathrm{tr}_A \rho^{AB}$. Afterwards, Liu *et al.* [28] proved that any tripartite pure state ρ^{ABC} satisfies

$$C_r^{e_{ABC}}(\rho^{ABC}) \geq C_r^{e_{AB}}(\rho^{AB}) + C_r^{e_{AC}}(\rho^{AC}), \tag{4}$$

where $\rho^{AB} = \mathrm{tr}_C \rho^{ABC}$ and $\rho^{AC} = \mathrm{tr}_B \rho^{ABC}$, provided that

$$\lambda S(\rho^{AB}_{e\text{-diag}}) \leq S(\rho^{AB}), (1 - \lambda) S(\rho^{AC}_{e\text{-diag}}) \leq S(\rho^{AC}) \tag{5}$$

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for some $0 \leq \lambda \leq 1$. Combining Eqs. (3) and (4), the following inequality was derived in [28]:

$$C_r^{e_{ABC}}(\rho^{ABC}) \geq \frac{4}{3}\left(C_r^{e_A}(\rho^A) + C_r^{e_B}(\rho^B) + C_r^{e_C}(\rho^C)\right) \tag{6}$$

for a pure state ρ^{ABC} satisfying the condition (5).

For simplicity, we use ρ^X_{diag} and $C_r(\rho^X)$ to denote $\rho^X_{e_X\text{-diag}}$ and $C_r^{e_X}(\rho^X)$, respectively.

Given a quantum state ρ^{AB} of the system AB, we know from Eq. (1) that

$$\begin{aligned}
& C_r(\rho^{AB}) - C_r(\rho^A) - C_r(\rho^B) \\
= \; & S(\rho^{AB}_{\text{diag}}) - S(\rho^{AB}) - S(\rho^A_{\text{diag}}) + S(\rho^A) - S(\rho^B_{\text{diag}}) + S(\rho^B) \\
\leq \; & S(\rho^A_{\text{diag}}) + S(\rho^B_{\text{diag}}) + S(\rho^A_{\text{diag}}) - S(\rho^B_{\text{diag}}) - S(\rho^{AB}) - S(\rho^A) + S(\rho^B) \\
\leq \; & S(\rho^A) + S(\rho^B).
\end{aligned}$$

Thus

$$C_r(\rho^{AB}) \leq C_r(\rho^A) + C_r(\rho^B) + S(\rho^A) + S(\rho^B).$$

Combing this with Eq. (3), we have

$$\frac{1}{2}\left(2C_r(\rho^A) + 2C_r(\rho^B)\right) \leq C_r(\rho^{AB}) \leq C_r(\rho^A) + C_r(\rho^B) + S(\rho^A) + S(\rho^B). \tag{7}$$

For any tripartite quantum state ρ^{ABC}, according to the super-additivity inequality (3), we have

$$C_r(\rho^{ABC}) \geq C_r(\rho^A) + C_r(\rho^{BC}),$$

$$C_r(\rho^{ABC}) \geq C_r(\rho^B) + C_r(\rho^{AC}),$$

$$C_r(\rho^{ABC}) \geq C_r(\rho^C) + C_r(\rho^{AB}).$$

By finding the sums of two sides of the inequalities above, we obtain

$$C_r(\rho^{ABC}) \geq \frac{1}{3}\left(C_r(\rho^{AB}) + C_r(\rho^{BC}) + C_r(\rho^{AC}) + C_r(\rho^A) + C_r(\rho^B) + C_r(\rho^C)\right). \tag{8}$$

On the other hand, using definition (1) yields that

$$\begin{aligned}
& C_r(\rho^{ABC}) - C_r(\rho^{AB}) - C_r(\rho^{AC}) - C_r(\rho^{BC}) \\
= \; & S(\rho^{ABC}_{\text{diag}}) - S(\rho^{ABC}) - S(\rho^{AB}_{\text{diag}}) + S(\rho^{AB}) \\
& - S(\rho^{BC}_{\text{diag}}) + S(\rho^{BC}) - S(\rho^{AC}_{\text{diag}}) + S(\rho^{AC}) \\
= \; & \left[S(\rho^{ABC}_{\text{diag}}) + S(\rho^B_{\text{diag}}) - S(\rho^{AB}_{\text{diag}}) - S(\rho^{BC}_{\text{diag}})\right] \\
& + \left[S(\rho^{AC}) - S(\rho^{AC}_{\text{diag}})\right] + \left[S(\rho^{AB}) + S(\rho^{BC}) - S(\rho^{ABC}) - S(\rho^B_{\text{diag}})\right] \\
\leq \; & S(\rho^A) + S(\rho^B) + S(\rho^B) + S(\rho^C) - S(\rho^B_{\text{diag}}) - S(\rho^{ABC}) \\
\leq \; & S(\rho^A) + S(\rho^B) + S(\rho^C),
\end{aligned}$$

since $S(\rho^{ABC}_{\text{diag}}) + S(\rho^B_{\text{diag}}) - S(\rho^{AB}_{\text{diag}}) - S(\rho^{BC}_{\text{diag}}) \leq 0$ (strong subadditivity) and $S(\rho^{AC}) - S(\rho^{AC}_{\text{diag}}) \leq 0$. This shows that

$$C_r(\rho^{ABC}) \leq C_r(\rho^{AB}) + C_r(\rho^{AC}) + C_r(\rho^{BC}) + S(\rho^A) + S(\rho^B) + S(\rho^C). \tag{9}$$

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Combining Eqs. (8) and (9) gives

$$\begin{aligned}&\frac{1}{3}\left(C_r(\rho^{AB})+C_r(\rho^{AC})+C_r(\rho^{BC})+C_r(\rho^{A})+C_r(\rho^{B})+C_r(\rho^{C})\right)\\&\le C_r(\rho^{ABC})\\&\le C_r(\rho^{AB})+C_r(\rho^{AC})+C_r(\rho^{BC})+S(\rho^{A})+S(\rho^{B})+S(\rho^{C}).\end{aligned} \tag{10}$$

As a generalization of inequalities (7) and (10), we can prove the following inequalities for any n-partite state, which give lower and upper bounds for the relative-entropy measure of coherence.

Theorem 3.1. *For any n-partite quantum state $\rho^{A_1\cdots A_n}$, it holds that*

$$\frac{1}{n}\left(\sum_{i=1}^{n}C_r(\mathrm{tr}_{A_i}\rho^{A_1\cdots A_n})+\sum_{i=1}^{n}C_r(\rho^{A_i})\right)\le C_r(\rho^{A_1\cdots A_n})\le\sum_{i=1}^{n}C_r(\mathrm{tr}_{A_i}\rho^{A_1\cdots A_n})+\sum_{i=i}^{n}S(\rho^{A_i}). \tag{11}$$

Proof. To prove that the first inequality in Eq. (11) holds, we know from Eq. (3) that

$$C_r(\rho^{A_1\cdots A_n})\ge C_r(\rho^{A_1})+C_r(\mathrm{tr}_{A_1}\rho^{A_1\cdots A_n}),$$

$$C_r(\rho^{A_1\cdots A_n})\ge C_r(\rho^{A_2})+C_r(\mathrm{tr}_{A_2}\rho^{A_1\cdots A_n}),$$

$$\vdots$$

$$C_r(\rho^{A_1\cdots A_n})\ge C_r(\rho^{A_n})+C_r(\mathrm{tr}_{A_n}\rho^{A_1\cdots A_n}),$$

and consequently,

$$C_r(\rho^{A_1\cdots A_n})\ge\frac{1}{n}\left(\sum_{i=1}^{N}C_r(\mathrm{tr}_{A_i}\rho^{A_1\cdots A_n})+\sum_{i=1}^{n}C_r(\rho^{A_i})\right).$$

Next, let us prove that the second inequality in (11) holds by using mathematical induction. Firstly, we know from Eq. (7) that the desired inequality holds for $n=2$ and any bipartite state. Secondly, we assume the second inequality in (11) holds for $n=N-1$ and any $N-1$-partite state. Then for any N-partite state $\rho^{A_1\cdots A_N}$, we have

$$\begin{aligned}C_r(\rho^{A_1\cdots A_N}) &= C_r(\rho^{A_1\cdots A_{N-2}(A_{N-1}A_N)})\\&\le \sum_{i=1}^{N-2}C_r(\mathrm{tr}_{A_i}\rho^{A_1\cdots A_{N-2}(A_{N-1}A_N)})+C_r(\mathrm{tr}_{A_{N-1}A_N}\rho^{A_1\cdots A_N})\\&\quad+\sum_{i=1}^{N-2}S(\rho^{A_i})+S(\rho^{A_{N-1}A_N}).\end{aligned}$$

By using Eq. (3), we know that $C_r(\mathrm{tr}_X\eta)\le C_r(\eta)$. Thus,

$$C_r(\mathrm{tr}_{A_{N-1}A_N}\rho^{A_1\cdots A_N})\le C_r(\mathrm{tr}_{A_{N-1}}\rho^{A_1\cdots A_N})+C_r(\mathrm{tr}_{A_N}\rho^{A_1\cdots A_N}).$$

Combining the fact that

$$S(\rho^{A_{N-1}A_N})\le S(\rho^{A_{N-1}})+S(\rho^{A_N}),$$

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we get that

$$\begin{aligned} C_r(\rho^{A_1\cdots A_N}) &\leq \sum_{i=1}^{N-2} C_r(\mathrm{tr}_{A_i}\rho^{A_1\cdots A_{N-2}(A_{N-1}A_N)}) + C_r(\mathrm{tr}_{A_{N-1}}\rho^{A_1\cdots A_N}) \\ &\quad + C_r(\mathrm{tr}_{A_N}\rho^{A_1\cdots A_N}) + \sum_{i=1}^{N-2} S(\rho^{A_i}) + S(\rho^{A_{N-1}}) + S(\rho^{A_N}) \\ &= \sum_{i=1}^{N} C_r(\mathrm{tr}_{A_i}\rho^{A_1\cdots A_N}) + \sum_{i=1}^{N} S(\rho^{A_i}). \end{aligned}$$

Thus, the validity of the second inequality in Eq. (11) is proved. The proof is completed.

4. The relation between $C_r(\rho)$ and $C_{\ell_1}(\rho)$

In this section, we discuss the relation between $C_r(\rho)$ and $C_{\ell_1}(\rho)$. Rana et al. found that the inequality

$$C_r(\rho) \leq C_{\ell_1}(\rho) \tag{12}$$

holds for any qubit mixed state [36, Proposition 1] and any pure state [36, Proposition 3]. However, for arbitrary state, they used an inequality between Holevo information and trace norm, and then proved that the upper bound of the relative entropy measure of coherence:

$$C_r(\rho) \leq C_{\ell_1}(\rho)\log_2 d. \tag{13}$$

Moreover, they conjectured the inequality (12) holds for all states ρ. Rana et al. [36] proved that for any state ρ, it holds that

$$C_r(\rho) \leq \begin{cases} C_{\ell_1}(\rho), & \text{if } C_{\ell_1}(æ) \geq 1; \\ C_{\ell_1}(\rho)\log_2 \mathrm{e}, & \text{if } C_{\ell_1}(æ) < 1. \end{cases} \tag{14}$$

Thus, if we redefine the von Neumann entropy as $\bar{S}(\rho) = -\mathrm{tr}(\rho \ln \rho)$, then the resulted relative entropy measure of coherence reads

$$\bar{C}_r(\rho) = \bar{S}(\rho_{\mathrm{diag}}) - \bar{S}(\rho) = \frac{1}{\log_2 \mathrm{e}} C_r(\rho). \tag{15}$$

This leads to the following inequality:

$$\bar{C}_r(\rho) \leq C_{\ell_1}(\rho), \ \ \forall \rho \in D(H). \tag{16}$$

About the original conjecture (12), we have the following.

Proposition 4.1. *Let ρ satisfy Eq. (12). Then every mixture $\eta := t\rho + (1-t)\sigma (0 \leq t \leq 1)$ of ρ and any incoherent state σ satisfies (12).*

Proof. The convexity of C_r implies that

$$\begin{aligned} C_r(\eta) &\leq tC_r(\rho) + (1-t)C_r(\sigma) \\ &= tC_r(\rho) \\ &\leq tC_{\ell_1}(\rho) \\ &= C_{\ell_1}(t\rho + (1-t)\sigma) \\ &= C_{\ell_1}(\eta). \end{aligned}$$

The proof is completed.

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Next, we verify this proposition by a class of maximally coherent mixed states [31], which are defined as

$$\rho_p = p|\psi_d\rangle\langle\psi_d| + \frac{1-p}{d}\mathbb{I}_d,$$

where $p \in (0,1]$, and $|\psi_d\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle$ is the maximally coherent state in the computational basis, $\mathbb{I}_d$ is the d-dimensional identity operator. Proposition 4.1 says that $C_r(\rho_p) \leq C_{\ell_1}(\rho_p)$. Next, we compute the difference $C_{\ell_1}(\rho_p) - C_r(\rho_p)$ and prove that it is always nonnegative for all $0 \leq p \leq 1$.

Through a computation, we know that the eigenvalues of ρ_p are $\frac{1+(d-1)p}{d}$ and $\frac{1-p}{d}$ with multiplicity $d-1$. By definition (1) and (2), we otbain

$$C_{\ell_1}(\rho_p) = (d-1)p,$$

$$C_r(\rho_p) = \log_2 d + \frac{1+(d-1)p}{d}\log_2\frac{1+(d-1)p}{d} + \frac{(d-1)(1-p)}{d}\log_2\frac{1-p}{d}.$$

For the given d, we have

$$\begin{aligned} f(p) &:= C_{\ell_1}(\rho_p) - C_r(\rho_p) \\ &= (d-1)p - \log_2 d + \frac{1+(d-1)p}{d}\log_2\frac{1+(d-1)p}{d} \\ &\quad + \frac{(d-1)(1-p)}{d}\log_2\frac{1-p}{d}. \end{aligned}$$

Then, the first-order derivative of $f(p)$ is

$$\begin{aligned} \frac{\mathrm{d}f(p)}{\mathrm{d}p} &= d-1-\frac{d-1}{d}\log_2\frac{1+(d-1)p}{d} + \frac{d-1}{d}\log_2\frac{1-p}{d} \\ &= d-1+\frac{d-1}{d}\log_2\frac{1-p}{1+(d-1)p}. \end{aligned}$$

Thus, $\frac{\mathrm{d}f(p)}{\mathrm{d}p} = 0$ if and only if $p = \frac{2^d-1}{2^d+d-1}$. When $0 < p \leq p_0$, we have $\frac{\mathrm{d}f(p)}{\mathrm{d}p} > 0$ and so $f(p)$ is increasing and $f(p) \geq f(0) = 0$; when $p_0 < p \leq 1$, we have $\frac{\mathrm{d}f(p)}{\mathrm{d}p} < 0$ and so $f(p)$ is decreasing and $f(p) \geq f(1) = d-1-\log_2 d > 0$. So, $f(p) = C_{\ell_1}(\rho_p) - C_r(\rho_p) > 0$, i.e. $C_{\ell_1}(\rho_p) > C_r(\rho_p) > 0$ for all $p \in (0,1]$.

5. Conclusions

In this paper, based on a relative entropy measure inequality of coherence for bipartite states, we have obtained a symmetry-like relation of relative entropy of coherence for n-partite quantum states. Our result implies that if the whole quantum system is coherent, at least one of its local subsystems is coherent; and that if the whole quantum system is incoherent, its local subsystems are incoherent. Meanwhile, we have observed that every mixture $\eta := t\rho + (1-t)\sigma (0 \leq t \leq 1)$ of a state ρ satisfying $C_r(\rho) \leq C_{\ell_1}(\rho)$ and any incoherent state σ also satisfies the conjecture $C_r(\eta) \leq C_{\ell_1}(\eta)$. We have also noted that if the von Neumann entropy is defined by the natural logarithm ln instead of $\log_2$, then the reduced relative-entropy measure of coherence $\bar{C}_r(\rho) = -\rho_{\mathrm{diag}} \ln \rho_{\mathrm{diag}} + \rho \ln \rho$ satisfies the inequality $\bar{C}_r(\rho) \leq C_{\ell_1}(\rho)$ for any mixed state ρ.

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