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ALMOST HYPERCOMPLEX MANIFOLDS WITH HERMITIAN-NORDEN METRICS AND 4-DIMENSIONAL INDECOMPOSABLE REAL LIE ALGEBRAS DEPENDING ON ONE PARAMETER

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Abstract. We study almost hypercomplex structure with Hermitian-Norden metrics on 4-dimensional Lie groups considered as smooth manifolds. All the basic classes of a classification of 4-dimensional indecomposable real Lie algebras depending on one parameter are investigated. There are studied some geometrical characteristics of the respective almost hypercomplex manifolds with Hermitian-Norden metrics.

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Introduction

Almost hypercomplex structure H on a $4n$-dimensional manifold $\mathcal{M}$ is a triad of anticommuting almost complex structures such that each of them is a composition of the other two structures. In [6, 7], this structure H is equipped with a metric structure of Hermitian-Norden type, generated by a pseudo-Riemannian metric g of neutral signature. In this case, one (resp., the other two) of the almost complex structures of H acts as an isometry (resp., act as anti-isometries) with respect to g in each tangent fibre. The metric g is Hermitian with respect to one of almost complex structures of H and g is a Norden metric regarding the other two almost complex structures of H. Then, there exist three (0,2)-tensors associated by H to the metric g – a Kähler form and two metrics of Hermitian-Norden type.

The derived manifold is called an almost hypercomplex manifold with Hermitian-Norden metrics. Its geometry is investigated in [6, 7, 10, 11, 13, 14, 17]. Let us remark that this type of manifolds are the only possible way to involve Norden-type metrics on almost hypercomplex manifolds. Similar structures and metrics on Lie groups considered as manifolds are studied in [1, 2, 8, 15, 18, 21].

The present paper is organized as follows. In Sect. 1, we present some definitions and facts about the almost hypercomplex manifold with Hermitian-Norden metrics. In the next Sect. 2, we construct almost hypercomplex structures with Hermitian-Norden metrics on 4-dimensional Lie groups considered as smooth manifolds. Moreover, we determine the belonging of the constructed almost hypercomplex manifolds with Hermitian-Norden metrics to the certain classes of the respective classifications. The last Sect. 3 is devoted to the study of some geometric characteristics of the considered manifolds.

1. Preliminaries

1.1. Almost hypercomplex manifolds with Hermitian-Norden metrics.

A $4n$-dimensional differentiable manifold $\mathcal{M}$ is called *an almost hypercomplex manifold* if it is equipped with *an almost hypercomplex structure* $H = (J_1, J_2, J_3)$ with the following properties:

$$J_\alpha = J_\beta \circ J_\gamma = -J_\gamma \circ J_\beta, \qquad J_\alpha^2 = -I,$$

for all cyclic permutations (α, β, γ) of $(1, 2, 3)$ and the identity I.

Let g be a neutral metric on $(\mathcal{M}, H)$ having the properties

$$g(\cdot,\cdot) = \varepsilon_\alpha g(J_\alpha\cdot, J_\alpha\cdot), \tag{1.1}$$

where

$$\varepsilon_\alpha = \begin{cases} 1, & \alpha = 1; \\ -1, & \alpha = 2; 3. \end{cases}$$

The associated 2-form g_1 and the associated neutral metrics g_2 and g_3 are determined by

$$g_\alpha(\cdot,\cdot) = g(J_\alpha\cdot,\cdot) = -\varepsilon_\alpha g(\cdot, J_\alpha\cdot). \tag{1.2}$$

Let us remark that here and further, α runs over the range $\{1, 2, 3\}$ unless otherwise is stated.

The obtained structure $(H, G) = (J_1, J_2, J_3; g, g_1, g_2, g_3)$ on $\mathcal{M}$ is called an *almost hypercomplex structure with Hermitian-Norden metrics* and manifold $(\mathcal{M}, H, G)$ is called an *almost hypercomplex manifold with Hermitian-Norden metrics* ([7]).

According to [7], the fundamental tensors of such a manifold are the following three $(0,3)$-tensors

$$F_\alpha(x,y,z) = g\big((\nabla_x J_\alpha)\,y, z\big) = \big(\nabla_x g_\alpha\big)\,(y,z)\,, \tag{1.3}$$

where ∇ is the Levi-Civita connection generated by g. These tensors have the following basic properties caused by the structures

$$F_\alpha(x,y,z) = -\varepsilon_\alpha F_\alpha(x,z,y) = -\varepsilon_\alpha F_\alpha(x,J_\alpha y,J_\alpha z). \tag{1.4}$$

The following relations between the tensors F_α are valid

$$\begin{aligned}
F_1(x,y,z) &= F_2(x,J_3y,z) + F_3(x,y,J_2z),\\
F_2(x,y,z) &= F_3(x,J_1y,z) + F_1(x,y,J_3z),\\
F_3(x,y,z) &= F_1(x,J_2y,z) - F_2(x,y,J_1z).
\end{aligned}$$

The corresponding Lee forms θ_α are determined by

$$\theta_\alpha(\cdot) = g^{kl}F_\alpha(e_k,e_l,\cdot) \tag{1.5}$$

where $\{e_1,e_2,\dots,e_{4n}\}$ is an arbitrary basis of $T_p\mathcal{M}$, $p\in\mathcal{M}$ and g^{ij} are the corresponding components of the inverse matrix of g.

Let us note that, according to (1.1), $(\mathcal{M},J_1,g)$ is an almost Hermitian manifold whereas the manifolds $(\mathcal{M},J_2,g)$ and $(\mathcal{M},J_3,g)$ are almost complex manifolds with Norden metric. The basic classes of these two types of manifolds are given in [5] and [3], respectively. In the case of the lowest dimension, $\dim\mathcal{M}=4$, the four basic classes of almost Hermitian manifolds with respect to J_1 are restricted to two: the class of the almost Kähler manifolds $\mathcal{W}_2(J_1)$ and the class of the Hermitian manifolds $\mathcal{W}_4(J_1)$, determined by:

$$\begin{aligned}
&\mathcal{W}_2(J_1):\ \mathop{\mathfrak{S}}_{x,y,z}\big\{F_1(x,y,z)\big\}=0;\\
&\mathcal{W}_4(J_1):\ F_1(x,y,z)=\frac{1}{2}\{g(x,y)\theta_1(z)-g(x,J_1y)\theta_1(J_1z)\\
&\qquad\qquad -g(x,z)\theta_1(y)+g(x,J_1z)\theta_1(J_1y)\}\,,
\end{aligned} \tag{1.6}$$

where $\mathfrak{S}$ is the cyclic sum by three arguments. The basic classes of the 4-dimensional almost Norden manifolds ($\alpha=2$ or 3) are determined as follows:

$$\begin{aligned}
&\mathcal{W}_1(J_\alpha):\ F_\alpha(x,y,z)=\frac{1}{4}\big\{g(x,y)\theta_\alpha(z)+g(x,J_\alpha y)\theta_\alpha(J_\alpha z)\\
&\qquad\qquad +g(x,z)\theta_\alpha(y)+g(x,J_\alpha z)\theta_\alpha(J_\alpha y)\big\};\\
&\mathcal{W}_2(J_\alpha):\ \mathop{\mathfrak{S}}_{x,y,z}\big\{F_\alpha(x,y,J_\alpha z)\big\}=0,\qquad \theta_\alpha=0;\\
&\mathcal{W}_3(J_\alpha):\ \mathop{\mathfrak{S}}_{x,y,z}\big\{F_\alpha(x,y,z)\big\}=0.
\end{aligned} \tag{1.7}$$

The Nijenhuis tensor in terms of the covariant derivatives of J_α and the corresponding (0,3)-tensor for J_α are defined by

$$\begin{aligned}
N_\alpha(x,y) &= (\nabla_x J_\alpha)\,J_\alpha y - (\nabla_y J_\alpha)\,J_\alpha x + (\nabla_{J_\alpha x} J_\alpha)\,y - (\nabla_{J_\alpha y} J_\alpha)\,x,\\
N_\alpha(x,y,z) &= g\,(N_\alpha(x,y),z)\,.
\end{aligned} \tag{1.8}$$

Moreover, the following properties of N_α are valid ([12]):

$$N_\alpha(x,y,z) = N_\alpha(x,J_\alpha y,J_\alpha z) = N_\alpha(J_\alpha x,y,J_\alpha z) = -N_\alpha(J_\alpha x,J_\alpha y,z). \tag{1.9}$$

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Let $R = [\nabla, \nabla] - \nabla_{[\ ,\ ]}$ be the curvature (1,3)-tensor of ∇ and let the corresponding curvature $(0,4)$-tensor with respect to g be denoted by the same letter:

$$R(x,y,z,w) = g(R(x,y)z,w). \tag{1.10}$$

The following properties of R are well-known:

$$\begin{aligned} &R(x,y,z,w) = -R(y,x,z,w) = -R(x,y,w,z),\\ &R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0. \end{aligned} \tag{1.11}$$

The Ricci tensor ρ and the scalar curvature τ for R as well as their associated quantities ρ^*, τ^* and τ^{**} are defined by:

$$\rho(y,z) = g^{ij}R(e_i,y,z,e_j), \qquad \rho^*(y,z) = g^{ij}R(e_i,y,z,J_\alpha e_j),$$

$$\tau = g^{ij}\rho(e_i,e_j), \qquad \tau^* = g^{ij}\rho^*(e_i,e_j), \qquad \tau^{**} = g^{ij}\rho^*(e_i,J_\alpha e_j).$$

Every non-degenerate 2-plane μ with a basis $\{x,y\}$ with respect to g in T_pM, $p \in M$, has the following sectional curvature

$$k(\mu;p) = \frac{R(x,y,y,x)}{g(x,x)g(y,y) - g(x,y)^2}.$$

A 2-plane μ is said to be holomorphic (resp., totally real) if $\mu = J_\alpha\mu$ (resp., $\mu \perp J_\alpha\mu \neq \mu$ with respect to g) holds.

2. Four-dimensional indecomposable real Lie algebras and almost hypercomplex structures with Hermitian-Norden metrics

Let L be a simply connected 4-dimensional real Lie group with corresponding Lie algebra $\mathfrak{l}$. A standard hypercomplex structure on $\mathfrak{l}$ is defined as in [20]:

$$\begin{array}{llll} J_1e_1 = e_2, & J_1e_2 = -e_1, & J_1e_3 = -e_4, & J_1e_4 = e_3;\\ J_2e_1 = e_3, & J_2e_2 = e_4, & J_2e_3 = -e_1, & J_2e_4 = -e_2;\\ J_3e_1 = -e_4, & J_3e_2 = e_3, & J_3e_3 = -e_2, & J_3e_4 = e_1, \end{array} \tag{2.1}$$

where $\{e_1,e_2,e_3,e_4\}$ is a basis of $\mathfrak{l}$.

We introduce a pseudo-Euclidian metric g of neutral signature for $x(x^1,x^2,x^3,x^4)$, $y(y^1,y^2,y^3,y^4) \in \mathfrak{l}$:

$$g(x,y) = x^1y^1 + x^2y^2 - x^3y^3 - x^4y^4.$$

According to (1.1) and (1.2), the metric g generates an almost hypercomplex structure with Hermitian-Norden metrics on $\mathfrak{l}$. Then, (L,H,G) is an almost hypercomplex manifold with Hermitian-Norden metrics.

A classification of real 4-dimensional indecomposable Lie algebras is given for instance in [16] and it can be found easily in [19] and [4]. The twelve basic classes are described by the non-zero Lie brackets with respect to $\{e_1,e_2,e_3,e_4\}$. Five of the basic classes are determined by real parameters – two classes use two parameters and three classes use one parameter. Our purpose is to investigate how the basic geometrical properties of the manifolds under study depend on these parameters.

In [9], we study both the basic classes of the considered classification depending on two parameters. Now, we focus our investigations on the following basic classes $\mathfrak{g}_{4,2}$, $\mathfrak{g}_{4,9}$ and $\mathfrak{g}_{4,11}$ which depend on one real parameter:

$$\mathfrak{g}_{4,2}: \ [e_1,e_4] = ae_1, \quad [e_2,e_4] = e_2, \quad [e_3,e_4] = e_2 + e_3, \quad (a \neq 0); \tag{2.2}$$

$$\begin{array}{llll} \mathfrak{g}_{4,9}: & [e_1,e_4]=(b+1)e_1, & [e_2,e_3]=e_1, & \\ & [e_2,e_4]=e_2, & [e_3,e_4]=be_3, & (-1<b\leq 1); \end{array} \tag{2.3}$$

$$\begin{array}{llll} \mathfrak{g}_{4,11}: & [e_1,e_4]=2ce_1, & [e_2,e_3]=e_1, & \\ & [e_2,e_4]=ce_2-e_3, & [e_3,e_4]=e_2+ce_3, & (c>0). \end{array} \tag{2.4}$$

Let us note that further, the indices i, j, k, l run over the range $\{1,2,3,4\}$.

2.1. **The class $\mathfrak{g}_{4,2}$.** Let us consider a manifold (L,G,H) with corresponding Lie algebra from $\mathfrak{g}_{4,2}$.

Using (1.1), (1.3), (2.1), (2.2) and the Koszul equality for the considered basis

$$2g\left(\nabla_{e_i}e_j,e_k\right)=g\left([e_i,e_j],e_k\right)+g\left([e_k,e_i],e_j\right)+g\left([e_k,e_j],e_i\right),$$

we obtain the basic components $(F_\alpha)_{ijk}=F_\alpha(e_i,e_j,e_k)$ of F_α. The non-zero of them are determined by the following ones and properties (1.4)

$$\begin{aligned} &(F_1)_{113}=(F_2)_{112}=-\tfrac{1}{2}(F_3)_{111}=a,\\ &(F_1)_{223}=2(F_1)_{323}=2(F_1)_{413}=(F_2)_{314}=(F_2)_{322}\\ &\qquad=\tfrac{1}{2}(F_2)_{222}=2(F_2)_{223}=-2(F_2)_{412}=(F_3)_{234}\\ &\qquad=(F_3)_{313}=-(F_3)_{422}=2(F_3)_{334}=-2(F_3)_{213}=1. \end{aligned} \tag{2.5}$$

Using (1.5) and (2.5), we establish that the only non-zero basic components $(\theta_\alpha)_i=(\theta_\alpha)(e_i)$ of the corresponding Lee forms are

$$\begin{gathered} (\theta_1)_2=(\theta_2)_3=-(\theta_3)_4=1,\\ (\theta_1)_3=-\tfrac{1}{2}(\theta_3)_1=a+1,\quad (\theta_2)_2=a+3. \end{gathered} \tag{2.6}$$

2.2. **The class $\mathfrak{g}_{4,9}$.** In this subsection we focus our investigations on a manifold (L,G,H) with corresponding Lie algebra from $\mathfrak{g}_{4,9}$.

By similar way as in the previous subsection, we obtain the following results for (L,G,H) in this case:

$$\begin{gathered} (F_1)_{113}=(F_2)_{314}=(F_3)_{313}=b+\tfrac{1}{2},\\ (F_2)_{112}=b+\tfrac{3}{2},\quad (F_3)_{111}=-2(b+1),\\ (F_3)_{212}=-\tfrac{1}{3}(F_1)_{223}=-\tfrac{1}{2}(F_2)_{211}=-\tfrac{1}{4}(F_2)_{222}=-\tfrac{1}{2}(F_3)_{122}=-\tfrac{1}{2},\\ (\theta_1)_3=b+2,\quad (\theta_2)_2=2(b+2),\quad (\theta_3)_1=-3(b+1), \end{gathered} \tag{2.7}$$

and we calculate the rest nonzero components of $(F_\alpha)_{ijk}$ using (1.4).

2.3. **The class $\mathfrak{g}_{4,11}$.** Now, we focus our investigations on a manifold (L,G,H) with corresponding Lie algebra from $\mathfrak{g}_{4,11}$.

By similar way as in the previous two subsections, we obtain the following results for (L, G, H) in the present case:

$$\begin{gathered}
(F_1)_{223} = (F_2)_{314} = (F_3)_{313} = c + \tfrac{1}{2}, \quad (F_1)_{113} = 2c - \tfrac{1}{2} \\
(F_2)_{112} = 2c + \tfrac{1}{2}, \quad (F_3)_{234} = c - \tfrac{1}{2}, \quad (F_3)_{111} = -2(F_2)_{222} = -4c \\
(F_1)_{323} = (F_2)_{211} = (F_3)_{122} = (F_3)_{334} = 1, \\
(F_2)_{214} = -\tfrac{1}{2}(F_2)_{322} = (F_3)_{213} = -1, \\
(\theta_1)_2 = (\theta_2)_3 = -\tfrac{1}{2}(\theta_3)_4 = 1, \\
(\theta_1)_3 = 3c, \quad (\theta_2)_2 = 5c + 1, \quad (\theta_3)_1 = -6c,
\end{gathered} \tag{2.8}$$

and we calculate the rest nonzero components of $(F_\alpha)_{ijk}$ using (1.4).

We generalize the obtained results in the three relevant classes by the following

Theorem 2.1. *Let (L, H, G) be a 4-dimensional almost hypercomplex manifold with Hermitian-Norden metrics. Then, the manifold (L, H, G), corresponding to the different classes of 4-dimensional Lie algebras $\mathfrak{g}_{4,2}$, $\mathfrak{g}_{4,9}$ and $\mathfrak{g}_{4,11}$, belongs to a certain class regarding J_α given in the following table:*

Lie algebra	*Parameter*	J_1	J_2	J_3
$\mathfrak{g}_{4,2}$	$a = 1$	$\mathcal{W}_4$	$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$	$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$
	$a \neq 0;\ a \neq 1$	$\mathcal{W}_2 \oplus \mathcal{W}_4$	$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$	$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$
$\mathfrak{g}_{4,9}$	$b = 1$	$\mathcal{W}_4$	$\mathcal{W}_1 \oplus \mathcal{W}_2$	$\mathcal{W}_1 \oplus \mathcal{W}_2$
	$-1 < b < 1$	$\mathcal{W}_2 \oplus \mathcal{W}_4$	$\mathcal{W}_1 \oplus \mathcal{W}_2$	$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$
$\mathfrak{g}_{4,11}$	$c > 0$	$\mathcal{W}_2 \oplus \mathcal{W}_4$	$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$	$\mathcal{W}_1 \oplus \mathcal{W}_2$

Moreover, we have:

- *for each $a \neq 0$, (L, H, G) does not belong to neither of $\mathcal{W}_0$, $\mathcal{W}_2$ for J_1; $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_1 \oplus \mathcal{W}_2$, $\mathcal{W}_1 \oplus \mathcal{W}_3$, $\mathcal{W}_2 \oplus \mathcal{W}_3$ for J_2; $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_1 \oplus \mathcal{W}_2$, $\mathcal{W}_1 \oplus \mathcal{W}_3$, $\mathcal{W}_2 \oplus \mathcal{W}_3$ for J_3;*
- *for each $-1 < b \leq 1$, (L, H, G) does not belong to neither of $\mathcal{W}_0$, $\mathcal{W}_2$ for J_1; $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$ for J_2; $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_1 \oplus \mathcal{W}_3$, $\mathcal{W}_2 \oplus \mathcal{W}_3$ for J_3;*
- *for each $c > 0$, (L, H, G) does not belong to neither of $\mathcal{W}_0$, $\mathcal{W}_2$, $\mathcal{W}_4$ for J_1; $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_1 \oplus \mathcal{W}_2$, $\mathcal{W}_1 \oplus \mathcal{W}_3$, $\mathcal{W}_2 \oplus \mathcal{W}_3$ for J_2; $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$ for J_3.*

Proof. Using the results in (2.5), (2.6), (2.7), (2.8) and the classification conditions (1.6), (1.7) for dimension 4, we establish the truthfulness of the assertion in each case. □

3. Some geometric characteristics of the considered manifolds

In this section we determine some geometric characteristics of the manifolds (L, G, H) considered in the previous section and we investigate the corresponding geometric properties in relation with the real parameters of the considered three classes of Lie algebras.

Firstly, we consider a manifold (L, G, H) with corresponding Lie algebra from $\mathfrak{g}_{4,2}$. Using (1.1), (1.8), (2.1) and (2.2), we obtain the basic components $(N_\alpha)_{ijk} = N_\alpha(e_i, e_j, e_k)$ of N_α. The non-zero of them are determined by the following ones and properties (1.9)

$$(N_1)_{132} = (N_2)_{123} = a - 1, \quad (N_2)_{122} = (N_3)_{122} = 1. \tag{3.1}$$

We calculate the basic components $R_{ijkl} = R(e_i, e_j, e_k, e_l)$ of R, using (1.1), (1.10), (2.1) and (2.2). The non-zero of them are determined by the following ones and properties (1.11)

$$\begin{gathered} R_{1212} = -R_{1313} = -a, \quad R_{1213} = -\tfrac{a}{2}, \quad R_{1414} = a^2, \\ R_{2323} = R_{2424} = \tfrac{5}{4}, \quad R_{2434} = 1, \quad R_{3434} = -\tfrac{1}{4}. \end{gathered} \tag{3.2}$$

We obtain the basic components $\rho_{jk} = \rho(e_j, e_k)$, $(\rho^*_\alpha)_{jk} = \rho^*_\alpha(e_j, e_k)$, as well as the values of τ, τ^*_α, τ^{**}_α and $k_{ij} = k(e_i, e_j)$ as follows:

$$\begin{gathered} \rho_{11} = a(a+2), \quad \rho_{22} = a + \tfrac{5}{2}, \quad \rho_{23} = \rho_{32} = \tfrac{a}{2} + 1, \\ \rho_{33} = -(a + \tfrac{3}{2}), \quad \rho_{44} = -(a^2 + \tfrac{3}{2}), \\ (\rho^*_1)_{12} = -(\rho^*_1)_{21} = (\rho^*_2)_{13} = (\rho^*_2)_{31} = (\rho^*_3)_{11} = a, \\ (\rho^*_1)_{13} = -(\rho^*_1)_{31} = -(\rho^*_2)_{12} = -(\rho^*_2)_{21} = \tfrac{a}{2}, \\ (\rho^*_1)_{24} = -(\rho^*_1)_{42} = -(\rho^*_2)_{34} = -(\rho^*_2)_{43} = -1, \\ (\rho^*_1)_{34} = -(\rho^*_1)_{43} = \tfrac{1}{4}, \quad (\rho^*_2)_{24} = (\rho^*_2)_{42} = (\rho^*_3)_{23} = (\rho^*_3)_{32} = \tfrac{5}{4}, \\ (\rho^*_3)_{14} = (\rho^*_3)_{41} = -a^2, \quad (\rho^*_3)_{44} = -2, \\ \tau = 2a^2 + 4a + \tfrac{11}{2}, \quad \tau^*_1 = \tau^*_2 = 0, \quad \tau^*_3 = a + 2 \\ \tau^{**}_1 = 2a + \tfrac{1}{2}, \quad \tau^{**}_2 = 2a + \tfrac{5}{2}, \quad \tau^{**}_3 = 2a^2 + \tfrac{5}{2}, \\ k_{12} = k_{13} = a, \quad k_{14} = a^2, \quad k_{23} = k_{24} = \tfrac{5}{4} \quad k_{34} = \tfrac{1}{4}. \end{gathered} \tag{3.3}$$

Theorem 3.1. *Let (L, H, G) be an almost hypercomplex manifold with Hermitian-Norden metrics and Lie algebra from the class $\mathfrak{g}_{4,2}$. The following characteristics of the manifold are valid:*

1. *(L, H, G) is integrable for J_1 if and only if $a = 1$;*
2. *Every (L, H, G) is non-flat;*
3. *Every (L, H, G) has a positive scalar curvature;*
4. *Every (L, H, G) is $*$-scalar flat w.r.t. J_1 and J_2;*
5. *(L, H, G) is $*$-scalar flat w.r.t. J_3 if and only if $a = -2$;*
6. *(L, H, G) is $**$-scalar flat w.r.t. J_1 if and only if $a = -\frac{1}{4}$;*
7. *(L, H, G) is $**$-scalar flat w.r.t. J_2 if and only if $a = -\frac{5}{4}$;*
8. *Every (L, H, G) has a positive $**$-scalar curvature w.r.t. J_3;*
9. *(L, H, G) has a positive basic holomorphic sectional curvatures w.r.t. J_1 and J_2 if and only if $a > 0$;*
10. *Every (L, H, G) has a positive basic holomorphic sectional curvatures w.r.t. J_3;*
11. *(L, H, G) has a positive basic totally real sectional curvatures if and only if $a > 0$.*

Proof. By virtue of (3.1), (3.2) and (3.3), we establish the truthfullness of the statements. □

Now we focus our study on a manifold (L, G, H) with corresponding Lie algebra from $\mathfrak{g}_{4,9}$. By similar way as for $\mathfrak{g}_{4,2}$, we obtain the following results for (L, G, H)

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in this case:

$$
\begin{gathered}
(N_1)_{132} = (N_3)_{132} = b - 1, \\
R_{1221} = R_{2323} = b + \tfrac{3}{4}, \quad R_{1234} = -\tfrac{b}{2}, \\
R_{1313} = b^2 + b - \tfrac{1}{4}, \quad R_{1324} = \tfrac{1}{2}, \quad R_{1414} = (b+1)^2, \\
R_{1423} = \tfrac{1}{2}(b+1), \quad R_{2424} = 1, \quad R_{3443} = b^2, \\
\rho_{11} = 2b^2 + 4b + \tfrac{3}{2}, \quad \rho_{22} = 2b + \tfrac{5}{2}, \\
\rho_{33} = -2b^2 - 2b - \tfrac{1}{2}, \quad \rho_{44} = -2b^2 - 2b - 2, \\
(\rho^*_1)_{12} = \tfrac{3}{2}(b + \tfrac{1}{2}), \quad (\rho^*_1)_{34} = b(b + \tfrac{1}{2}), \quad (\rho^*_2)_{13} = b^2 + 2b + \tfrac{1}{4}, \\
(\rho^*_2)_{24} = b + \tfrac{3}{2}, \quad (\rho^*_3)_{14} = -(b^2 + \tfrac{5}{2}b + \tfrac{1}{2}), \quad (\rho^*_3)_{23} = \tfrac{3}{2}(b + \tfrac{1}{6}), \\
\tau = 6b^2 + 10b + \tfrac{13}{2}, \quad \tau^*_1 = \tau^*_2 = \tau^*_3 = 0, \\
\tau^{**}_1 = \tau^{**}_2 = 2(b^2 + b + \tfrac{3}{4}), \quad \tau^{**}_3 = 2(b^2 + 3b + \tfrac{7}{4}), \\
k_{12} = k_{23} = b + \tfrac{3}{4}, \quad k_{13} = b^2 + b - \tfrac{1}{4}, \\
k_{14} = (b+1)^2, \quad k_{24} = 1, \quad k_{34} = b^2
\end{gathered}
\tag{3.4}
$$

and we calculate the rest nonzero components of $(N_\alpha)_{ijk}$ and R_{ijkl} using (1.9) and (1.11), respectively.

Theorem 3.2. *Let (L, H, G) be an almost hypercomplex manifold with Hermitian-Norden metrics and Lie algebra from the class $\mathfrak{g}_{4,9}$. The following characteristics of the manifold are valid:*

1. *Every (L, H, G) is integrable for J_2;*
2. *Every (L, H, G) is non-flat;*
3. *Every (L, H, G) has a positive scalar curvature;*
4. *Every (L, H, G) is $*$-scalar flat;*
5. *Every (L, H, G) has a positive $**$-scalar curvature w.r.t. J_1 and J_2;*
6. *(L, H, G) is $**$-scalar flat w.r.t. J_3 if and only if $b = \frac{\sqrt{2}-3}{2}$;*
7. *(L, H, G) has positive basic holomorphic sectional curvatures w.r.t. J_1 and J_3 if and only if $b \in \left(-\frac{3}{4}; 0\right) \cup (0; 1]$;*
8. *(L, H, G) has positive basic holomorphic sectional curvatures w.r.t. J_2 if and only if $b \in \left(\frac{\sqrt{2}-1}{2}; 1\right]$;*
9. *(L, H, G) has positive basic totally real sectional curvatures w.r.t. J_1 and J_3 if and only if $b \in \left(\frac{\sqrt{2}-1}{2}; 1\right]$;*
10. *(L, H, G) has positive basic totally real sectional curvatures w.r.t. J_2 if and only if $b \in \left(-\frac{3}{4}; 0\right) \cup (0; 1]$.*

Proof. By virtue of (3.4), we establish the truthfulness of the statements. □

Herein, we continue our study on a manifold (L, G, H) with corresponding Lie algebra from $\mathfrak{g}_{4,11}$. By similar way as previous ones, we obtain the following results for (L, G, H) in this case:

$$\begin{gathered}
(N_1)_{132} = (N_2)_{123} = c - 1, \quad (N_2)_{144} = 3c, \quad (N_1)_{133} = -(N_2)_{122} = -1, \\
R_{2434} = -R_{1213} = 2R_{1423} = -4R_{1234} = 4R_{1324} = 2c, \\
R_{1221} = R_{1313} = 2c^2 - \tfrac{1}{4}, \quad R_{2424} = -R_{3434} = c^2 - 1, \\
R_{1414} = 4c^2, \quad R_{2323} = c^2 + \tfrac{7}{4}, \quad R_{1224} = R_{1334} = \tfrac{1}{2}, \\
\rho_{11} = 8c^2 - \tfrac{1}{2}, \quad \rho_{22} = -\rho_{33} = 4c^2 + \tfrac{1}{2}, \quad \rho_{23} = 4c, \quad \rho_{44} = -6c^2 + 2, \\
(\rho_1^*)_{12} = 2c^2 + \tfrac{c}{2} - \tfrac{1}{4}, \quad (\rho_1^*)_{34} = c^2 + \tfrac{c}{2} - 1, \quad (\rho_1^*)_{13} = (\rho_2^*)_{34} = 2c - \tfrac{1}{2}, \\
(\rho_2^*)_{13} = 2c^2 + \tfrac{3c}{2} - \tfrac{1}{4}, \quad (\rho_2^*)_{24} = c^2 + \tfrac{3c}{2} - 1, \\
(\rho_1^*)_{24} = (\rho_2^*)_{12} = -(2c + \tfrac{1}{2}), \quad (\rho_3^*)_{23} = c^2 + \tfrac{7}{4} \\
(\rho_3^*)_{11} = -(\rho_3^*)_{44} = 4c, \quad (\rho_3^*)_{22} = (\rho_3^*)_{33} = -1, \\
\tau = 18c^2 - \tfrac{3}{2}, \quad \tau_1^* = \tau_2^* = 0, \quad \tau_3^* = 8c, \\
\tau_1^{**} = \tau_2^{**} = 6c^2 - \tfrac{5}{2}, \quad \tau_3^{**} = 10c^2 + \tfrac{7}{2}, \\
k_{12} = k_{13} = 2c^2 - \tfrac{1}{4}, \quad k_{14} = 4c^2, \quad k_{23} = c^2 + \tfrac{7}{4}, \quad k_{24} = k_{34} = c^2 - 1
\end{gathered} \tag{3.5}$$

and we calculate the rest nonzero components of $(N_\alpha)_{ijk}$ and R_{ijkl} using (1.9) and (1.11), respectively.

Theorem 3.3. *Let (L, H, G) be an almost hypercomplex manifold with Hermitian-Norden metrics and Lie algebra from the class $\mathfrak{g}_{4,11}$. The following characteristics of the manifold are valid:*

(1) *Every (L, H, G) is integrable for J_3;*
(2) *Every (L, H, G) is non-flat;*
(3) *(L, H, G) is scalar flat if and only if $c = \frac{\sqrt{3}}{6}$;*
(4) *Every (L, H, G) is $*$-scalar flat w.r.t. J_1 and J_2;*
(5) *Every (L, H, G) has a positive $*$-scalar curvature w.r.t. J_3;*
(6) *(L, H, G) is $**$-scalar flat w.r.t. J_1 and J_2 if and only if $c = \frac{\sqrt{15}}{6}$;*
(7) *Every (L, H, G) has a positive $**$-scalar curvature w.r.t. J_3;*
(8) *(L, H, G) has positive (resp., negative) basic holomorphic sectional curvatures w.r.t. J_1 and J_2 if and only if $c > 1$ (resp., $0 < c < 1$);*
(9) *Every (L, H, G) has positive basic holomorphic sectional curvatures w.r.t. J_3;*
(10) *(L, H, G) has positive basic totally real sectional curvatures w.r.t. J_1 and J_2 if and only if $c > 1$;*
(11) *(L, H, G) has positive (resp., negative) basic totally real sectional curvatures w.r.t. J_3 if and only if $c > 1$ (resp., $0 < a < 1$).*

Proof. By virtue of (3.5), we establish the truthfulness of the statements. □

References

[1] M. L. Barberis, *Hypercomplex structures on four-dimensional Lie groups*. Proc. Amer. Math. Soc. **128** (4) (1997) 1043–1054.

[2] A. Fino, G. Grantcharov, *Properties of manifolds with skewsymmetric torsion and special holonomy*. Adv. Math. **189** (2004) 439–450.

[3] G. Ganchev, A. Borisov, *Note on the almost complex manifolds with a Norden metric*. C. R. Acad. Bulg. Sci. **39** (1986) 31–34.

[4] R. Ghanam, G. Thompson, *Minimal matrix representations of four-dimensional Lie algebras*. Bull. Malays. Math. Sci. Soc 2 **36** (2) (2013) 343–349.
[5] A. Gray, L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*. Ann. Mat. Pura Appl. **CXXIII** (IV) (1980) 35–58.
[6] K. Gribachev, M. Manev, *Almost hypercomplex pseudo-Hermitian manifolds and a 4-dimensional Lie group with such structure*. J. Geom. **88** (1-2) (2008) 41–52.
[7] K. Gribachev, M. Manev, S. Dimiev, *On the almost hypercomplex pseudo-Hermitian manifolds*. In: Trends in Complex Analysis, Differential Geometry and Mathematical Physics, eds. S. Dimiev and K. Sekigawa (World Sci. Publ., Singapore, River Edge, NJ, 2003) 51–62.
[8] H. Manev, *Matrix Lie groups as 3-dimensional almost contact B-metric manifolds*. Facta Univ. Ser. Math. Inform., **30** (3) (2015) 341–351.
[9] H. Manev, *Almost hypercomplex manifolds with Hermitian-Norden metrics and 4-dimensional indecomposable real Lie algebras depending on two parameters*. arXiv:1909.06329.
[10] M. Manev, *Tangent bundles with Sasaki metric and almost hypercomplex pseudo-Hermitian structure*. In: Topics in Almost Hermitian Geometry and Related Fields. (Y. Matsushita, E. Garcia-Rio, H. Hashimoto, T. Koda and T. Oguro, eds.) World Sci. Publ., Singapore (2005) 170–185.
[11] M. Manev, *A connection with parallel torsion on almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics*. J. Geom. Phys. **61** (1) (2011) 248–259.
[12] M. Manev, *On geometry of manifolds with some tensor structures and metrics of Norden type*. doi:10.13140/RG.2.2.33038.05446, arXiv:1706.05505 (2017).
[13] M. Manev, K. Gribachev, *A connection with parallel totally skew-symmetric torsion on a class of almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics*. Int. J. Geom. Methods Mod. Phys. **8** (1) (2011) 115–131.
[14] M. Manev, K. Sekigawa, *Some four-dimensional almost hypercomplex pseudo-Hermitian manifolds*. In: Contemporary Aspects of Complex Analysis, Differential Geometry and Mathematical Physics (S. Dimiev and K. Sekigawa, eds.) World Sci. Publ., Singapore (2005) 174–186.
[15] M. Manev, V. Tavkova, *Lie groups as 3-dimensional almost paracontact almost paracomplex Riemannian manifolds*, J. Geom. **110:43** (2019).
[16] G.M. Mubarakzyanov, *On solvable Lie algebras*. Izv. Vyssh. Uchebn. Zaved. Mat. **1** (1963) 114–123.
[17] G. Nakova, H. Manev, *Holomorphic submanifolds of some hypercomplex manifolds with Hermitian and Norden metrics*. C. R. Acad. Bulgare Sci. **70** (1) (2017) 29–40.
[18] G. Ovando, *Invariant complex structures on solvable real Lie groups*. Manuscripta Math. **103** (2000) 19–30.
[19] J. Patera, R.T. Sharp, P. Winternitz, H. Zassenhaus, *Invariants of real low dimension Lie algebras*. J. Math. Phys. **17** (1976) 986–994.
[20] A. Sommese, *Quaternionic manifolds*. Math. Ann. **212** (1975) 191–214.
[21] S. Zamkovoy, G. Nakova. *The decomposition of almost paracontact metric manifolds in eleven classes revisited*. J. Geom. **109:18** (2018)