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Article

The stability of a general sextic functional equation by fixed point theory

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Abstract: In this paper, we consider the generalized sextic functional equation $\sum_{i=0}^{7} {}_7C_i(-1)^{7-i}f(x+iy) = 0$. And by applying the fixed point theory in the sense of L. C\u adariu and V. Radu, we will discuss the stability of the solutions for this functional equation.

Keywords: sextic mapping; general sextic functional equation; fixed point theory method; generalized Hyers-Ulam stability

MSC: 34D10; 34A40; 34A45; 39B82; 41A30; 39B52

1. Introduction

The origin of concept for Ulam stability is an open problem, formulated by Ulam [14], concerning the stability of group homomorphisms. The first partial answer to Ulam's problem came within a year, when Hyers [6] proved a stability result for the additive Cauchy functional equation in Banach spaces. Since then, many mathematicians have dealt with this problem (cf. [5,13]).

The solution of the so-called generalized sextic (sixth order) functional equation

$$Df(x,y) := \sum_{i=0}^{7}(-1)^{7-i}{}_7C_i f(x+iy) = 0, \tag{1}$$

for all (x,y) in the domain of f, is called the generalized sextic (sixth order) mapping, where ${}_nC_i = \frac{n!}{i!(n-i)!}$. For example, if a_1, a_2, a_3, a_4, a_5, a_6, and a_7 are real constants, the mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a_1x^6 + a_2x^5 + a_3x^4 + a_4x^3 + a_5x^2 + a_6x + a_7$ is a solution of the generalized sextic(sixth order) functional equation (1).

In this paper, we will use a fixed point theory to prove that there exists only one exact solution F near suitable approximate solution f to functional equation (1) (ref. [2,3]). Specially, in Theorem 2 and Theorem 2, the exact solution F to functional equation (1) will be explicitly constructed from the approximate solution f within a reasonable distance. The advantage of this paper over other papers is that we first proved the uniqueness and existence of the exact solution, to the generalized sextic(sixth order) functional equation (1), from approximate solution within a reasonable distance. In fact, Lee-Jung[9] obtained similar results by using fixed point theory method for the general quartic functional equation. But, their method can not be generalized to quintic and sextic functional equation. In this paper we will use very technical calculation with useful lemmas to extend lee-Jung's method to quintic and sextic functional equation.

Unfortunately, we could not generalize our fixed point theory method to every n^{th} functional equation and we leave that as an open problem.

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2. Main results

We first recall the following Margolis and Diaz fixed point theorem, which is necessary to obtain the main results of this paper.

([4]) Suppose (X, d) is a complete generalized metric space, which means that the metric d may assume infinite values, and $J : X \to X$ is a strictly contractive mapping with the Lipschitz constant $0 < L < 1$. Then, for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = +\infty \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}$$

or there exists an integer $k \geq 0$ such that:

(i) $d(J^n x, J^{n+1} x) < +\infty$ for all $n \geq k$;
(ii) the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;
(iii) y^* is the unique fixed point of J in $Y := \{y \in X : d(J^k x, y) < +\infty\}$;
(iv) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

Throughout this paper, let V and W be real vector spaces, X a real normed space, and let Y be a real Banach space. For a given mapping $f : V \to W$, we use the following abbreviations

$$\begin{aligned} f_e(x) &:= \frac{f(x) + f(-x)}{2}, \\ f_o(x) &:= \frac{f(x) - f(-x)}{2}, \\ Df(x, y) &:= \sum_{i=0}^{7} {}_7C_i (-1)^{7-i} f(x + iy) \end{aligned}$$

for all $x, y \in V$.

Now, we will see useful Lemma for the proof of main theorem.

Let θ be a real constant such that $0 < \theta < \frac{\pi}{4}$ and $\cos(3\theta) = \frac{-17}{21\sqrt{21}}$. Let $\varphi : V^2 \to [0, \infty)$ be a function for which there exists a constant $0 < L < 1$ such that

$$\varphi(2x, 2y) \leq (4\sqrt{21}\cos\theta - 14) L \varphi(x, y) \tag{2}$$

for all $x, y \in V$. Then $1 < 4\sqrt{21}\cos\theta - 14 < 2$,

$$\frac{(4\sqrt{21}\cos\theta - 14)^3 + 42(4\sqrt{21}\cos\theta - 14)^2 + 336(4\sqrt{21}\cos\theta - 14)}{512} = 1 \tag{3}$$

and the equality

$$\lim_{n\to\infty} \frac{1}{512^n} \sum_{i=0}^{n} {}_nC_i \left(\sum_{j=0}^{i} {}_iC_j 42^{i-j} 336^j \varphi_e(2^{3n-i-j} x, 2^{3n-i-j} y) \right) = 0 \tag{4}$$

holds for all $x, y \in V$.

Proof. When $0 < \theta < \frac{\pi}{4}$ and $\cos 3\theta = -\frac{17}{21\sqrt{21}}$, it is not difficult to see that $1.74837 < 3\theta < 1.7484$ and $0.83493 < \cos\theta < 0.834925$ in the trigonometric function table. So $1.3 < 4\sqrt{21}\cos\theta - 14 < 1.4$.

We can also obtain the equality (3) by the following calculation :

$$\begin{aligned}
&\frac{336(4\sqrt{21}\cos\theta-14)}{512}+\frac{42(4\sqrt{21}\cos\theta-14)^2}{512}+\frac{(4\sqrt{21}\cos\theta-14)^3}{512}\\
&=\frac{1344\sqrt{21}\cos\theta-4704}{512}+\frac{14112\cos^2\theta-4704\sqrt{21}\cos\theta+8232}{512}\\
&\quad+\frac{1344\sqrt{21}\cos^3\theta-14112\cos^2\theta+2352\sqrt{21}\cos\theta-2744}{512}\\
&=\frac{336\sqrt{21}(4\cos^3\theta-3\cos\theta)+784}{512}\\
&=\frac{336\sqrt{21}\cos 3\theta+784}{512}\\
&=\frac{336\sqrt{21}\times\frac{-17}{21\sqrt{21}}+784}{512}\\
&=1.
\end{aligned}$$

And, to obtain the equality (4), by (3) we obtain the following calculation :

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{512^n}\sum_{i=0}^{n}{}_n\mathrm{C}_i\left(\sum_{j=0}^{i}{}_i\mathrm{C}_j42^{i-j}336^j\varphi_e(2^{3n-i-j}x,2^{3n-i-j}y)\right)\\
&\le\lim_{n\to\infty}\frac{1}{512^n}\sum_{i=0}^{n}{}_n\mathrm{C}_i\left(\sum_{j=0}^{i}{}_i\mathrm{C}_j42^{i-j}336^j(4\sqrt{21}\cos\theta-14)^{2n-i-j}L^{2n-i-j}\varphi_e(2^nx,2^ny)\right)\\
&\le\lim_{n\to\infty}\frac{1}{512^n}\sum_{i=0}^{n}{}_n\mathrm{C}_i(4\sqrt{21}\cos\theta-14)^{2n-2i}\left(\sum_{j=0}^{i}{}_i\mathrm{C}_j42^{i-j}(4\sqrt{21}\cos\theta-14)^{i-j}336^j\right)\times\\
&\qquad\times\varphi_e(2^nx,2^ny)\\
&=\lim_{n\to\infty}\frac{1}{512^n}\sum_{i=0}^{n}{}_n\mathrm{C}_i(4\sqrt{21}\cos\theta-14)^{2n-2i}\Big(42(4\sqrt{21}\cos\theta-14)+336\Big)^i\varphi_e(2^nx,2^ny)\\
&\le\lim_{n\to\infty}\frac{1}{512^n}\Big((4\sqrt{21}\cos\theta-14)^2+42(4\sqrt{21}\cos\theta-14)+336\Big)^n\varphi_e(2^nx,2^ny)\\
&=\lim_{n\to\infty}L^n\left(\frac{(4\sqrt{21}\cos\theta-14)^3+42(4\sqrt{21}\cos\theta-14)^2+336(4\sqrt{21}\cos\theta-14)}{512}\right)^n\times\\
&\qquad\times\varphi_e(x,y)\\
&=\lim_{n\to\infty}L^n\varphi_e(x,y)=0,\quad\text{for all}\quad x,y\in V.
\end{aligned}$$

In the following main theorem, we will prove the generalized Hyers-Ulam stability of the functional equation (1) by using the direct method.

Let θ, L, φ be as in Lemma prop2. If $f: V\to Y$ is a mapping satisfying $f(0)=0$ and the inequality

$$\|Df(x,y)\|\le\varphi(x,y),\quad\text{for all}\quad x,y\in V, \tag{5}$$

then there exists the unique solution mapping $F: V\to Y$ of (1) such that

$$\|f(x)-F(x)\|\le\frac{\Phi(x)}{1-L} \tag{6}$$

for all $x\in V$, where

$$\Phi(x):=\frac{9\varphi_e(-6x,2x)+56\varphi_e(-x,x)+392\varphi_e(-2x,x)+1008\varphi_e(-3x,x)}{4096}.$$

In particular, F is represented by

$$F(x) = \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{i} {}_nC_{i\,i}C_j \left(\frac{(-84)^{i-j}1344^j}{4096^n} f_e(2^{3n-i-j}x) + \frac{(-42)^{i-j}336^j}{512^n} f_o(2^{3n-i-j}x) \right) \tag{7}$$

41 for all $x \in V$.

Proof. Let S be the set of all functions $g : V \to Y$ with $g(0) = 0$. We introduce a generalized metric on S by

$$d(g,h) = \inf\left\{K \in R_+ \middle| \|g(x) - h(x)\| \le K\Phi(x) \text{ for all } x \in V\right\}.$$

It is not difficult to show that (S,d) is a complete generalized metric space (see [3, Theorem 2.5] or the proof of [8, Theorem 3.1]). Now we consider the mapping $J : S \to S$, which is defined by

$$Jg(x) := \frac{4032g(2x)}{8192} - \frac{1344g(-2x)}{8192} - \frac{420g(4x)}{8192} + \frac{252g(-4x)}{8192} + \frac{9g(8x)}{8192} - \frac{7g(-8x)}{8192}$$

42 for all $x \in V$.

And, by using the oddness and the evenness of g_o and g_e, and ${}_nC_{i-1} + {}_nC_i = {}_{n+1}C_i$, due to mathematical induction we can get

$$J^n g(x) = \sum_{i=0}^{n} {}_nC_i \sum_{j=0}^{i} {}_iC_j \left(\frac{(-84)^{i-j}1344^j}{4096^n} g_e(2^{3n-i-j}x) + \frac{(-42)^{i-j}336^j}{512^n} g_o(2^{3n-i-j}x) \right)$$

43 holds for all $n \in N$ and $x \in V$.

Let $g,h \in S$ and let $K \in [0,\infty]$ be an arbitrary constant with $d(g,h) \le K$. From the definition of d and (3), we have

$$\begin{aligned}
\|Jg(x) - Jh(x)\| \le & \frac{4032}{8192}\|g(2x) - h(2x)\| + \frac{1344}{8192}\|g(-2x) - h(-2x)\| \\
& + \frac{420}{8192}\|g(4x) - h(4x)\| + \frac{252}{8192}\|g(-4x) - h(-4x)\| \\
& + \frac{9}{8192}\|g(8x) - h(8x)\| + \frac{7}{8192}\|g(-8x) - h(-8x)\| \\
\le & K\left(\frac{336}{512}\Phi(2x) + \frac{42}{512}\Phi(4x) + \frac{1}{512}\Phi(8x)\right) \\
\le & K\left(\frac{336(4\sqrt{21}\cos\theta - 14)L}{512} + \frac{42(4\sqrt{21}\cos\theta - 14)^2L^2}{512}\right. \\
& \left. + \frac{(4\sqrt{21}\cos\theta - 14)^3L^3}{512}\right)\Phi(x) \\
\le & K\left(\frac{336(4\sqrt{21}\cos\theta - 14)}{512} + \frac{42(4\sqrt{21}\cos\theta - 14)^2}{512}\right. \\
& \left. + \frac{(4\sqrt{21}\cos\theta - 14)^3}{512}\right)L\Phi(x) \\
\le & LK\Phi(x)
\end{aligned}$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \le Ld(g,h)$$

44 for any $g,h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L.

Now, after long and tedious calculation, we have

$$f(x)-Jf(x)=\frac{1}{4096}(Df_e(-6x,2x)+8Df_e(-x,x)+56Df_e(-2x,x)+112Df_e(-3x,x)) \\ +\frac{1}{512}(Df_o(-6x,2x)+6Df_o(-x,x)+42Df_o(-2x,x)+112Df_o(-3x,x)).$$

And, by (5) we obtain

$$\begin{aligned}&\|f(x)-Jf(x)\| \\ &\le \frac{1}{4096}\|Df_e(-6x,2x)+8Df_e(-x,x)+56Df_e(-2x,x)+112Df_e(-3x,x)\| \\ &\quad+\frac{1}{512}\|Df_o(-6x,2x)+6Df_o(-x,x)+42Df_o(-2x,x)+112Df_o(-3x,x)\| \\ &\le \frac{9\varphi_e(-6x,2x)+56\varphi_e(-x,x)+392\varphi_e(-2x,x)+1008\varphi_e(-3x,x)}{4096} \\ &=\Phi(x)\end{aligned}$$

for all $x\in V$. It means that $d(f,Jf)\le 1<\infty$ by the definition of d and due to Proposition 2 the sequence $\{J^n f\}$ converges to the unique fixed point $F:V\to Y$ of J in the set $T=\{g\in S\ :\ d(f,g)<\infty\}$ which implies (7). Moreover, by Proposition 2, we have

$$d(f,F)\le\frac{1}{1-L}d(f,Jf)\le\frac{1}{1-L}$$

which implies (6).

Also, by the equality (4), since one has

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{4096^n}\left\|\sum_{i=0}^{n}{}_nC_i\left(\sum_{j=0}^{i}{}_iC_j(-84)^{i-j}1344^j Df_e(2^{3n-i-j}x,2^{3n-i-j}y)\right)\right\| \\ &\le\lim_{n\to\infty}\frac{1}{4096^n}\sum_{i=0}^{n}{}_nC_i\left(\sum_{j=0}^{i}{}_iC_j84^{i-j}1344^j\varphi_e(2^{3n-i-j}x,2^{3n-i-j}y)\right) \\ &\le\lim_{n\to\infty}\frac{1}{512^n}\sum_{i=0}^{n}{}_nC_i\left(\sum_{j=0}^{i}{}_iC_j42^{i-j}336^j\varphi_e(2^{3n-i-j}x,2^{3n-i-j}y)\right)=0\end{aligned}$$

and

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{512^n}\left\|\sum_{i=0}^{n}{}_nC_i\left(\sum_{j=0}^{i}{}_iC_j(-42)^{i-j}336^j Df_o(2^{3n-i-j}x,2^{3n-i-j}y)\right)\right\| \\ &\le\lim_{n\to\infty}\frac{1}{512^n}\sum_{i=0}^{n}{}_nC_i\left(\sum_{j=0}^{i}{}_iC_j42^{i-j}336^j\varphi_e(2^{3n-i-j}x,2^{3n-i-j}y)\right)=0,\end{aligned}$$

for all $x,y\in V$, we obtain

$$\begin{aligned}DF(x,y)=\lim_{n\to\infty}\Bigg(\sum_{i=0}^{n}{}_nC_i\sum_{j=0}^{i}{}_iC_j\Bigg(&\frac{(-84)^{i-j}1344^j}{4096^n}Df_e(2^{3n-i-j}x) \\ &+\frac{(-42)^{i-j}336^j}{512^n}Df_o(2^{3n-i-j}x,2^{3n-i-j}y)\Bigg)\Bigg)=0,\end{aligned}$$

for all $x,y\in V$.

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Therefore, F is the unique solution of the functional equation (1) with (6). Finally se see that if F is a solution of the sextic functional equation (1) with $F(0) = 0$, then we can derive that F is a fixed point of J from the equality

$$\begin{aligned}
&F(x) - JF(x) \\
&= \frac{1}{4096}(DF_e(-6x, 2x) + 8DF_e(-x, x) + 56DF_e(-2x, x) + 112DF_e(-3x, x)) \\
&+ \frac{1}{512}(DF_o(-6x, 2x) + 6DF_o(-x, x) + 42DF_o(-2x, x) + 112DF_o(-3x, x)).
\end{aligned}$$

47 Let θ be a real constant such that $0 < \theta < \frac{\pi}{4}$ and $\cos(3\theta) = \frac{637}{77\sqrt{77}}$. Let $\varphi : V^2 \to [0, \infty)$ be a function for which there exists a constant $0 < L < 1$ such that

$$L\varphi(2x, 2y) \geq (8\sqrt{77}\cos\theta + 28)\varphi(x, y) \tag{8}$$

for all $x, y \in V$. Then we have $97 < 8\sqrt{77}\cos\theta + 28 < 97.8$,

$$\frac{84}{(8\sqrt{77}\cos\theta + 28)} + \frac{1344}{(8\sqrt{77}\cos\theta + 28)^2} + \frac{4096}{(8\sqrt{77}\cos\theta + 28)^3} = 1 \tag{9}$$

and the equality

$$\lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \sum_{j=0}^{i} {}_iC_j 84^j 1344^{i-j} 4096^{n-i} \varphi_e\left(\frac{x}{2^{3n-i-j}}, \frac{y}{2^{3n-i-j}}\right) = 0 \tag{10}$$

48 holds for all $x, y \in V$.

Proof. When $0 < \theta < \frac{\pi}{4}$ and $\cos(3\theta) = \frac{637}{77\sqrt{77}}$, it is not difficult to see that $0 < 3\theta < 0.33997$ and $97 < 8\sqrt{77}\cos\theta + 28 < 97.8$ in the trigonometric function table. Also we obtain the equality (9) from the following calculation :

$$\begin{aligned}
&4096 + 1344(8\sqrt{77}\cos\theta + 28) + 84(8\sqrt{77}\cos\theta + 28)^2 - (8\sqrt{77}\cos\theta + 28)^3 \\
&= 4096 + 10752\sqrt{77}\cos\theta + 37632 + 413952\cos^2\theta + 37632\sqrt{77}\cos\theta + 65856 \\
&\quad - 39424\sqrt{77}\cos^3\theta - 413952\cos^2\theta - 18816\sqrt{77}\cos\theta - 21952 \\
&= -9856\sqrt{77}(4\cos^3\theta - 3\cos\theta) + 81536 \\
&= -9856\sqrt{77}\cos(3\theta) + 81536 = 0.
\end{aligned}$$

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And by (8) and (9) we have

$$\begin{aligned}
&\sum_{i=0}^{n} {}_nC_i \sum_{j=0}^{i} {}_iC_j 84^j 1344^{i-j} 4096^{n-i} \varphi_e\left(\frac{x}{2^{3n-i-j}}, \frac{y}{2^{3n-i-j}}\right)\\
&\leq \sum_{i=0}^{n} {}_nC_i \sum_{j=0}^{i} {}_iC_j 84^j \left(\frac{1344L}{8\sqrt{77}\cos\theta+28}\right)^{i-j} 4096^{n-i} \varphi_e\left(\frac{x}{2^{3n-2i}}, \frac{y}{2^{3n-2i}}\right)\\
&= \sum_{i=0}^{n} {}_nC_i \left(84 + \frac{1344L}{8\sqrt{77}\cos\theta+28}\right)^{i} 4096^{n-i} \varphi_e\left(\frac{x}{2^{3n-2i}}, \frac{y}{2^{3n-2i}}\right)\\
&\leq \sum_{i=0}^{n} {}_nC_i \left(84 + \frac{1344L}{8\sqrt{77}\cos\theta+28}\right)^{i} \left(\frac{4096L^2}{(8\sqrt{77}\cos\theta+28)^2}\right)^{n-i} \varphi_e\left(\frac{x}{2^{n}}, \frac{y}{2^{n}}\right)\\
&\leq \left(84 + \frac{1344L}{8\sqrt{77}\cos\theta+28} + \frac{4096L^2}{(8\sqrt{77}\cos\theta+28)^2}\right)^{n} \left(\frac{L}{8\sqrt{77}\cos\theta+28}\right)^{n} \varphi_e(x,y)\\
&\leq \left(\frac{84}{8\sqrt{77}\cos\theta+28} + \frac{1344}{(8\sqrt{77}\cos\theta+28)^2} + \frac{4096}{(8\sqrt{77}\cos\theta+28)^3}\right)^{n} L^n \varphi_e(x,y)\\
&= L^n \varphi_e(x,y)
\end{aligned}$$

49 for all $x, y \in V$. Therefore, by taking the limit we complete the proof of (10).

Let θ, L, φ be as in Lemma 2. If $f : V \to Y$ be a mapping such that the inequality (5) holds for all $x, y \in V$, then there exists the unique solution $F : V \to Y$ of (1) satisfying the inequality

$$\|f(x) - F(x)\| \leq \frac{\Psi(x)}{1-L} \tag{11}$$

for all $x \in V$, where

$$\Psi(x) := 2\varphi_e(\frac{-3x}{4}, \frac{x}{4}) + 14\varphi_e(\frac{-x}{8}, \frac{x}{8}) + 98\varphi_e(\frac{-x}{4}, \frac{x}{8}) + 224\varphi_e(\frac{-3x}{8}, \frac{x}{8}).$$

In particular, F is represented by

$$\begin{aligned}
F(x) = \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \sum_{j=0}^{i} {}_iC_j \Bigg[& 42^j(-336)^{i-j} 512^{n-i} f_o\left(\frac{x}{2^{3n-i-j}}\right)\\
& + 84^j(-1344)^{i-j} 4096^{n-i} f_e\left(\frac{x}{2^{3n-i-j}}\right)\Bigg],
\end{aligned} \tag{12}$$

50 for all $x \in V$.

Proof. As we did in the proof of Theorem 2, we let S as the set of all functions $g : V \to Y$ with $g(0) = 0$ and we define a generalized metric on S by

$$d(g,h) = \inf\left\{K \in R_+ \middle| \|g(x) - h(x)\| \leq K\Psi(x) \text{ for all } x \in V\right\}.$$

We now consider the mapping $J : S \to S$ defined by

$$\begin{aligned}
Jg(x) := 63g\left(\frac{x}{2}\right) + 21g\left(-\frac{x}{2}\right) - 840g\left(\frac{x}{4}\right) - 504g\left(-\frac{x}{4}\right)\\
+ 2304g\left(\frac{x}{8}\right) + 1792g\left(-\frac{x}{8}\right)
\end{aligned}$$

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for all $x \in V$. Then, as in the proof of Theorem 2, by mathematical induction we obtain that

$$J^n g(x) = \sum_{i=0}^{n} {}_n\mathrm{C}_i \sum_{j=0}^{i} {}_i\mathrm{C}_j \left(42^j (-336)^{i-j} 512^{n-i} g_o\left(\frac{x}{2^{3n-i-j}}\right) \right. \\ \left. + 84^j (-1344)^{i-j} 4096^{n-i} g_e\left(\frac{x}{2^{3n-i-j}}\right) \right)$$

51 holds for all $n \in \mathbb{N}$ and $x \in V$.

Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g,h) \le K$. From the definition of d and (9), we have

$$\begin{aligned}
&\|Jg(x) - Jh(x)\| \\
&\le 63 \left\| g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right) \right\| + 21 \left\| g\left(-\frac{x}{2}\right) - h\left(-\frac{x}{2}\right) \right\| \\
&\quad + 840 \left\| g\left(\frac{x}{4}\right) - h\left(\frac{x}{4}\right) \right\| + 504 \left\| g\left(-\frac{x}{4}\right) - h\left(-\frac{x}{4}\right) \right\| \\
&\quad + 2304 \left\| g\left(\frac{x}{8}\right) - h\left(\frac{x}{8}\right) \right\| + 1792 \left\| g\left(-\frac{x}{8}\right) - h\left(-\frac{x}{8}\right) \right\| \\
&\le 84K\Psi\left(\frac{x}{2}\right) + 1344K\Psi\left(\frac{x}{4}\right) + 4096K\Psi\left(\frac{x}{8}\right) \\
&\le \frac{84LK\Psi(x)}{8\sqrt{77}\cos\theta + 28} + \frac{1344L^2K\Psi(x)}{(8\sqrt{77}\cos\theta + 28)^2} + \frac{4096L^3K\Psi(x)}{(8\sqrt{77}\cos\theta + 28)^3} \\
&\le \left(\frac{84}{8\sqrt{77}\cos\theta + 28} + \frac{1344}{(8\sqrt{77}\cos\theta + 28)^2} + \frac{4096}{(8\sqrt{77}\cos\theta + 28)^3} \right) LK\Psi(x) \\
&\le LK\Psi(x)
\end{aligned}$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \le Ld(g,h)$$

52 for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L.

Moreover, by the definition of $Df(x,y)$, with long and tedious calculation, we have

$$\begin{aligned}
&f(x) - Jf(x) \\
&= \left[Df_e\left(\frac{-3x}{4}, \frac{x}{4}\right) + 8Df_e\left(\frac{-x}{8}, \frac{x}{8}\right) + 56Df_e\left(\frac{-x}{4}, \frac{x}{8}\right) + 112Df_e\left(\frac{-3x}{8}, \frac{x}{8}\right) \right] \\
&\quad + \left[Df_o\left(\frac{-3x}{4}, \frac{x}{4}\right) + 6Df_o\left(\frac{-x}{8}, \frac{x}{8}\right) + 42Df_o\left(\frac{-x}{4}, \frac{x}{8}\right) + 112Df_o\left(\frac{-3x}{8}, \frac{x}{8}\right) \right].
\end{aligned}$$

And, by (5) we obtain

$$\begin{aligned}
&\|f(x) - Jf(x)\| \\
&\le \left\| Df_e\left(\frac{-3x}{4}, \frac{x}{4}\right) + 8Df_e\left(\frac{-x}{8}, \frac{x}{8}\right) + 56Df_e\left(\frac{-x}{4}, \frac{x}{8}\right) + 112Df_e\left(\frac{-3x}{8}, \frac{x}{8}\right) \right\| \\
&\quad + \left\| Df_o\left(\frac{-3x}{4}, \frac{x}{4}\right) + 6Df_o\left(\frac{-x}{8}, \frac{x}{8}\right) + 42Df_o\left(\frac{-x}{4}, \frac{x}{8}\right) + 112Df_o\left(\frac{-3x}{8}, \frac{x}{8}\right) \right\| \\
&\le 2\varphi_e\left(\frac{-3x}{4}, \frac{x}{4}\right) + 14\varphi_e\left(\frac{-x}{8}, \frac{x}{8}\right) + 98\varphi_e\left(\frac{-x}{4}, \frac{x}{8}\right) + 224\varphi_e\left(\frac{-3x}{8}, \frac{x}{8}\right) \\
&= \Psi(x)
\end{aligned}$$

for all $x \in V$.

It means that $d(f, Jf) \leq 1 < \infty$ by the definition of d. Therefore according to Proposition 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S | d(f,g) < \infty\}$, which is represented by (12) for all $x \in V$.

We also due to Proposition 2.1 obtain that

$$d(f, F) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{1}{1-L},$$

which implies (11).

Now, since by (10) we have

$$\begin{aligned}
&\lim_{n\to\infty} \left\| \sum_{i=0}^{n} {}_nC_i \sum_{j=0}^{i} {}_iC_j 84^j (-1344)^{i-j} 4096^{n-i} Df_e\left(\frac{x}{2^{3n-i-j}}, \frac{y}{2^{3n-i-j}}\right) \right\| \\
&\leq \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \sum_{j=0}^{i} {}_iC_j 84^j 1344^{i-j} 4096^{n-i} \varphi_e\left(\frac{x}{2^{3n-i-j}}, \frac{y}{2^{3n-i-j}}\right) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} \left\| \sum_{i=0}^{n} {}_nC_i \sum_{j=0}^{i} {}_iC_j 42^j (-336)^{i-j} 512^{n-i} Df_o\left(\frac{x}{2^{3n-i-j}}, \frac{y}{2^{3n-i-j}}\right) \right\| \\
&\leq \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \sum_{j=0}^{i} {}_iC_j 42^j 336^{i-j} 512^{n-i} \varphi_e\left(\frac{x}{2^{3n-i-j}}, \frac{y}{2^{3n-i-j}}\right) \\
&\leq \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \sum_{j=0}^{i} {}_iC_j 84^j 1344^{i-j} 4096^{n-i} \varphi_e\left(\frac{x}{2^{3n-i-j}}, \frac{y}{2^{3n-i-j}}\right) = 0
\end{aligned}$$

for all $x, y \in V$, due to the equality (12) we obtain

$$\begin{aligned}
DF(x,y) &= \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \sum_{j=0}^{i} {}_iC_j 42^j (-336)^{i-j} 512^{n-i} Df_o\left(\frac{x}{2^{3n-i-j}}, \frac{y}{2^{3n-i-j}}\right) \\
&+ \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \sum_{j=0}^{i} {}_iC_j 84^j (-1344)^{i-j} 4096^{n-i} Df_e\left(\frac{x}{2^{3n-i-j}}, \frac{y}{2^{3n-i-j}}\right) = 0
\end{aligned}$$

which conclude that F is a solution of the sextic functional equation (1).

Finally we see that if F is a solution of the sextic functional equation (1), then the equality

$$\begin{aligned}
&F(x) - JF(x) \\
&= \left[DF_e\left(\frac{-3x}{4}, \frac{x}{4}\right) + 8DF_e\left(\frac{-x}{8}, \frac{x}{8}\right) + 56DF_e\left(\frac{-x}{4}, \frac{x}{8}\right) + 112DF_e\left(\frac{-3x}{8}, \frac{x}{8}\right)\right] \\
&+ \left[DF_o\left(\frac{-3x}{4}, \frac{x}{4}\right) + 6DF_o\left(\frac{-x}{8}, \frac{x}{8}\right) + 42DF_o\left(\frac{-x}{4}, \frac{x}{8}\right) + 112DF_o\left(\frac{-3x}{8}, \frac{x}{8}\right)\right].
\end{aligned}$$

implies that F is a fixed point of J.

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