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Dynamics of The Price Behavior in Stock Markets: A Statistical Physics Approach

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Abstract: We study in this paper the time evolution of stock markets using a statistical physics approach. We consider an ensemble of agents who sell or buy a good according to several factors acting on them: the majority of the neighbors, the market ambiance, the variation of the price and some specific measure applied at a given time. Each agent is represented by a spin having a number of discrete states q or continuous states, describing the tendency of the agent for buying or selling. The market ambiance is represented by a parameter T which plays the role of the temperature in physics: low T corresponds to a calm market, high T to a turbulent one. We show that there is a critical value of T , say T_c , where strong fluctuations between individual states lead to a disordered situation in which there is no majority: the numbers of sellers and buyers are equal, namely the market clearing. The specific measure, by the government or by economic organisms, is parameterized by H applied on the market at the time t_1 and removed at the time t_2 . We have used Monte Carlo simulations to study the time evolution of the price as functions of those parameters. In particular we show that the price strongly fluctuates near T_c and there exists a critical value H_c above which the boosting effect remains after H is removed. Our model replicates the stylized facts in finance (time-independent price variation), volatility clustering (time-dependent dynamics) and persistent effect of a temporary shock. The second party of the paper deals with the price variation using a time-dependent mean-field theory. By supposing that the sellers and the buyers belong to two distinct communities with their characteristics different in both intra-group and inter-group interactions, we find the price oscillation with time. Results are shown and discussed.

Keywords: econophysics; market dynamics; market networks; price variation; Monte Carlo simulations; mean-field theory; statistical physics models

1. Introduction

Statistical physics models treat large ensembles of particles interacting with each other via interactions of various kinds. Much has been achieved in the understanding of their properties in different situations [1]. In particular, collective behaviors such as those observed in phase transitions and in correlated dynamics have been demonstrated as consequences of the microscopic underlying interactions between particles, the space dimension and the system symmetry.

Social systems with a large number of members interacting with each other via some rules which are not always well defined due to the very nature of human beings. Nevertheless, we can see

macroscopic effects of a society via surveys, and behavior studies. This means that in spite of the fact that a human is not a particle, collective behaviors retain only common features, washing away individual particularities in the averaging process.

The interaction of an individual with his neighbors under the social ambiance (peace or unrest) has been used in many domains such as politics and sociology [2–7]. Quantitative sociodynamics using stochastic methods and models of social interaction processes has been investigated [8]. The validity of statistical laws in physics and social sciences has been examined [9]. In particular, the role of the interaction network in the emergence of diversity of behavior has been shown [10]. Using interacting social networks, social conflicts have been studied using models of statistical physics [11–13] with the mean-field theory and Monte Carlo (MC) simulations. It is noted that MC simulations have been used with success to study social phenomena [14].

We note that sociophysics models try to outline essential ingredients which govern macroscopic behavior of a society. Knowing these, one can predict a collective behavior in order to avoid undesired effects or to intervene to change the course of its evolution if necessary [7]. We will see an example in this paper.

Statistical physics used to study economic problems has been called "econophysics" [15]. As in sociophysics, the correspondence between physical quantities and parameters can be interpreted in economic terms. We will give some discussion in the next section on this point.

Econophysics is a research field using theories and methods developed by physicists in order to solve problems in economics. These methods usually include uncertainty or stochastic processes and nonlinear dynamics which are inherent to economic matters [16–18].

In this paper, we consider the problem of the evolution during the time of the price in a commodity market by examining the effects of several parameters. There is an enormous number of works dealing with the stock markets. We can mention a few of them: a book on the price theory and applications [19,20], stock market crashes [17], pricing policy [18], price fluctuations [21], stock market behaviors [22–24] and scaling behavior in the dynamics of an economic index [25].

Price variation is subject of many investigations trying to find rules to predict or to understand the market dynamics. These investigations used many methods. Let us mention a few of them. Hsu et al. [26] have used a probability model for describing rates of return based on the hypothesized phenomenon of a changing variance, Castanias [27] has studied the correlation between macroinformation and the variability of stock market prices, Cutler et al. [28] have investigated some causes which move the stock prices by estimating the fraction of the variance in aggregate stock returns that can be attributed to various kinds of news. Bak et al. [29] have constructed simple models of a stock market, and argued that the large variations may be due to a crowd effect, where agents imitate each other's behavior. Tse et al. [30] have constructed complex networks to study correlations between the closing prices for all US stocks that were traded over two periods of time (from July 2005 to August 2007; and from June 2007 to May 2009). Networks have been used in a work by Huang et al. [31] who employed a threshold method to construct China's stock correlation network and then study the network's structural properties and topological stability. They found that it follows a power-law model. Sun et al. [32] have used a multifractal analysis of Hang Seng index to study the variation of the Hong Kong stock market, Chang et al. [33] have used a fuzzy model to test with success on the Taiwan Electronic Shares from the Taiwan Stock Exchange. Song et al. [34] have investigated the evolution of worldwide stock markets using correlation structure and correlation-based graphs.

The above mentioned works show that there have been many models trying to find rules for price variation in stock markets. Large variations in stock prices happened so frequently that one can raise doubts about universal models which can be applied to many situations.

In this paper, we introduce a new model inspired from models of statistical physics [1] but with modifications where they should be for econophysics. Our model contributes to the family of already abundant stock price variation models, but as seen below, it gives rise to new features not seen before in the price variation. Our model uses an assembly of agents interacting with each other and with the

economic temperature as well as with economic measures taken by the government. This model is studied using Monte Carlo simulations and by the time-dependence mean-field theory.

The aim of the paper is to study the variation of the price of a good under the effects of the interaction strength between agents, the economic environment (economic crisis, economic agitation, ...) and boosting measures taken by the government.

We note that the change of price directly yields gross returns. From the price variation shown in this paper, one can deduce gross returns (namely before fees, taxes, inflation, ...) as seen below. Our objective is to study the price dynamics under different conditions and to identify which stylized facts our model replicates. The first stylized fact replicated by our model is the fact that the variation of the price at time t is independent of the price at time $(t - 1)$. This stylized fact, well-known in finance, is reproduced in our model as seen below. The second stylized fact concerns autocorrelated fluctuations such as volatility clustering explained below. This is found in our model as well (see section 3.5).

T. Lux has given a review [35] on stochastic models borrowed from statistical physics using microscopic interactions between a larger number of traders to deduce macroscopic regularities of the market, independent of microscopic details. We will show in this paper the macroscopic behavior after the time averaging does not depend on particularities of each agent but on a set of general parameters such as interaction strength between them and the economic environment. On the same line, R. Cont [36] has shown various statistical properties of asset returns with emphasis on properties common to a wide variety of markets and instruments. This analysis invalidates many of the common statistical approaches used to study financial data sets. We need therefore a general model which does not use empirical rules in the course of calculation. This motivates our present work.

As said above, a stylized fact in finance is the so-called "volatility clustering". Volatility clustering was suggested by Mandelbrot in 1963 from the observation that "large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes." [37] Many works, mostly empirical, have been done since then [38–42] to study the persistence of some tendencies due to an autocorrelation, namely variation at time t depends on the variation in the past. This is one of the objective of the present paper. We will show in this paper that a specific measure taken by the government or an economic organism to boost or to lower the market price can have a long-lasting effect. The absence of a model to explain why a temporary shock can generate persistent effects has motivated our present work. The reader is referred to see textbooks on macroeconomics for this point (see for example Refs. [43,44]). Our model presented in this paper explained this stylized fact as will be seen below.

This paper is organized as follows. Section 2 is devoted to the description of our model with emphasis put on the meaning of physical parameters in economic issues. The method of Monte Carlo simulation is described in this section. In section 3, we present the results of three models: the 3-state model, the 5-state model and the continuous model. Section 4 is devoted to a mean-field study. The model used in this section uses the assumption that the sellers and the buyers belong to two communities with different intra-group interactions and different inter-group interactions. This non symmetrical interactions give rise to the price oscillations in a region of market temperature. Mean-field results are shown and discussed. Concluding remarks are given in section 5.

2. Model and Method

2.1. Model

We recall a simple textbook static model of a market for one commodity (one can say one good). There is a population of agents facing an aggregate supply function $S(P)$. The decision of every agent i is a demand of good x_i . The price P is determined by market clearing (*i.e.* supply=demand)

$$\sum_i x_i = S(P) \quad (1)$$

122 Consider the inverse supply function $p(.) = S^{-1}(.)$. Market clearing rewrites

$$P = p\left(\sum_i x_i\right) \quad (2)$$

We assume a linear and increasing function p

$$p(X) = A + BX \quad (3)$$

123 where A and B are two positive real parameters.

124 The demand x_i of agent i can take a finite number of values. For example, $x_i \in \{-1, 0, 1\}$

This static game is repeated indefinitely (time is discrete: $t = 1, 2, \dots$):

$$P(t) = p\left(\sum_i x_{i,t}\right) \quad (4)$$

125 The decision $x_{i,t}$ of every agent i are revised according to some learning algorithm based on the
126 past values $P(t-1)$ and $x_{j,t-1}$ where the agents j are i 's neighbors. This algorithm has 2 main features:

- 127 - $x_{i,t}$ is increasing in the $x_{j,t-1}$ (agent i imitates his neighbors' past behavior)
- 128 - the effect of $P(t-1)$ on $x_{i,t}$ is twofold: a higher price decreases i 's demand (direct effect), a
129 higher price reveals that other agents buy a lot, which increases i 's demand (indirect informative effect,
130 due to i 's imitation of others' behavior). Hence, $x_{i,t}$ can be either increasing or decreasing in $P(t-1)$.
131 Since $P(t-1) = p(\sum_i x_{i,t-1})$, the demand $x_{i,t}$ can be increasing or decreasing in $\sum_i x_{i,t-1}$.

132 The practical implication of these remarks for numerical simulations is that the two cases ($x_{i,t}$
133 increasing or decreasing in $\sum_i x_{i,t-1}$) should be considered.

134 In the simulation, we translate the above rule into a model in statistical physics using a spin
135 language: a spin σ_i describes the state of an agent i . σ_i has several states, for example $\sigma_i = -1, 0, 1$.
136 Each state represents an action: let's define $\sigma_i = 1$ for buying action, $\sigma_i = -1$ for selling, and $\sigma_i = 0$ for
137 waiting. The action of agent i is guided by different mechanisms described in what follows.

138 We consider an ensemble of N agents.

The Hamiltonian for σ_i is given by

$$E_i(t) = \sigma_i(t) \left[-J \sum_j \sigma_j(t-1) \right] + a \sigma_i(t) [N(\text{up}, t-1) - N(\text{down}, t-1)] - H \sigma_i(t) \quad (5)$$

139 where $a > 0$. Let us explain each term:

140 i) The first term represents the sum of the influence on $\sigma_i(t)$ at the time t by his neighbors' attitudes
141 at the time $(t-1)$. For simplicity, we assume here all neighbors have the same interaction J with $\sigma_i(t)$.
142 This will not alter general aspects of the model. The agent imitates the majority of his neighbors

143 ii) In addition to the influence of neighbors given by the first term, $\sigma_i(t)$ also depends on the price
144 tendency given by the second term: let $N(\text{up}, t-1)$ ($N(\text{down}, t-1)$) be the number of the people who
145 wants to buy (sell) at the previous time $t-1$. The price is proportional to $N(\text{up}, t-1) - N(\text{down}, t-1)$,
146 namely

147 * if $N(\text{up}, t-1) > N(\text{down}, t-1)$, i. e. more people who buy, so the price is high (increasing), σ_i
148 may take the value -1 (sell), against the buying tendency of the first term, to take benefits of selling at
149 a high price

150 * if $N(\text{up}, t-1) < N(\text{down}, t-1)$, i. e. more people who sell, so the price is low (decreasing),
151 σ_i may take the value $+1$ (buy), against the imitation tendency of the first term, to take advantage of
152 buying at a low price

iii) The third term is a market-oriented measure to boost buying if H is positive, or to favor selling if H is negative. This measure can be applied for a lapse of time and is removed to leave the market evolve

Note that the decision of $\sigma_i(t)$ at a given T (to buy, to sell or to wait) depends on the balance of the three terms in Eq. (5). It is the total sum which matters.

Before describing the Monte Carlo method, let us discuss the correspondence between parameters from statistical physics and their meaning in economics. First of all, the temperature T in physics is an external parameter which acts on each particle in a way to make it disordered (thermal agitation): higher T higher disorder. Second, particles interact with each other via an interaction J which leads to some order or collective structure, in contrast with T which favors disorder. Therefore, there is a competition between T and J . Third, particles have various natures. We are interested here in particles defined by spins. A spin in statistical physics is an object characterized by a number of internal states. An Ising spin has two states up and down, a XY spin has two components, a Heisenberg spin has three components, ... In this paper we take a spin with one component of integer amplitude S but, unlike the Ising spin, it has $2S + 1$ states: $-S, -S + 1, -S + 2, \dots, 0, 1, 2, \dots, S + 1$. For $S = 1$, one has three states $M = -1, 0, 1$.

In sociophysics, spins are individuals in a society. Their internal states represent their opinion on a subject. Interactions between them need not to be symmetric, *i. e.* A interacts with B not in the same way as B interacts with A (see discussion in Refs.[11,13]).

In econophysics, a spin is an economic action of an agent (selling, buying or waiting, for example). The market temperature T means the economic environment resulting from many factors such as political situation, economic crisis, international conflicts, ... Low T means stability, peace. High T means unstable situation.

2.2. Monte Carlo Method

Let us now briefly describe the Monte Carlo algorithm used in this work (see details in Refs.[11, 13,45]):

- We generate a system of N individuals. Each of them has a random initial value among $-1, 0$ and 1 , with a chosen percentage of each kind randomly distributed
- We fix the "temperature" T which represents the market agitation, analogous to the temperature in physics: low (high) T corresponds to peaceful (agitated) economic environment
- At the time t we calculate $E_i(t)$ of the individual $\sigma_i(t)$, using the state of its neighbors at the previous time $(t - 1)$.
- We take randomly a new value of $\sigma_i(t)$ and calculate the energy difference $\Delta E_i = E_i(\text{new}) - E_i(t)$ between the new and the old states
 - if $\Delta E_i < 0$ the individual follows its neighbors' influence and the market tendency to take the new state for its state at t .
 - if $\Delta E_i > 0$ the individual changes only its state according to a probability $W = \exp(-\beta \Delta E_i)$ where $\beta = \frac{1}{k_B T}$, k_B being the Boltzmann factor. This step is done as follows: we take a random probability r between 0 and 1 (using a uniform random number generator). If $r > W$ then the new state for $\sigma_i(t)$ is accepted. Otherwise, $\sigma_i(t)$ takes back its old state.
- We take another $\sigma_i(t)$ and repeat the updating procedure described above until the states of all agents are updated. We say we achieve one Monte Carlo sweep (or step).
- We calculate $N(\text{up}, t - 1) = \sum_i \delta(\sigma_i - 1)$ and $N(\text{down}, t - 1) = \sum_i \delta(\sigma_i + 1)$ and memorize all $\sigma_i(t)$
- We start again another Monte Carlo sweep (namely for $t + 1$) by repeating from point 3. We do that for a large number of times to follow the time evolution of the system.

For the Monte Carlo simulation, several remarks are in order:

- i) Stochastic process is used mainly by the random probability R taken at point 4 above,

ii) Monte Carlo simulation techniques are numerous. The algorithm implemented above follows the Metropolis algorithm. There exist many other methods. Depending on the problem, one has to choose the appropriate one among the simplest Metropolis algorithm, the cluster-flipping method, the histogram method, the Wang-Landau flat energy-histogram technique [45].

iii) For macroscopic properties of a system at equilibrium, we need to average physical quantities over a large number of microscopic states namely over a large number of Monte Carlo sweeps. However, to study dynamical properties of a system evolving with time, we have to record quantities of interest at each sweep and show them as functions of t . If the price increases or decreases while fluctuating, then the market is in a relaxation toward equilibrium. If the fluctuations of the price are around a mean value, then the market is at equilibrium. Note that we have to make a large number of runs to check the reproducibility of the main characteristics of the results. This is what we have done for the dynamics of the price shown below.

We note that the price $P(t)$ can be defined as

$$P(t) = a [N(\text{up}, t) - N(\text{down}, t)] + A \quad (6)$$

where a is a proportional constant and A is stable price determined by the market clearing, namely when $N(\text{up}, t) = N(\text{down}, t)$. The gross return is defined as

$$R(t) = [P(t) - P(t-1)] / P(t-1) \quad (7)$$

In practice, we can start the simulation at a given T with different initial conditions, for example with 70% of σ up (percentage of buyers), 30% of σ down (percentage of sellers), and leave the system evolve to the final state at that given T . We note that at the opening of the exchange market the initial percentages of buyers and sellers may strongly increase or decrease according to the economic temperature T . After the relaxation, the price fluctuates around a mean value. This period corresponds to a time-independent dynamics which is known as the "stylized fact of price informational efficiency". Both kinds of dynamics occur in our simulations as seen below.

3. Monte Carlo simulation results

We perform Monte Carlo simulations using a network of $N = 4 \times 12^3 = 6912$ agents on a lattice where each agent has 12 nearest neighbors (NN). We limit in the present paper the interactions between NN with the same interaction strength J as indicated in Eq. (5).

It is interesting to examine first the stability of the market as a function of T . By stability we mean that the interaction term in Eq. (5) dominates, namely the collective effect with correlation among agents is present. This occurs when T is lower than a value T_c beyond which agents are independent of each other because the agitation of T which breaks the correlation. This temperature T_c is called "transition temperature" which is seen by anomalies in various quantities such as the system energy E , the order parameter M , fluctuations of E and M called calorific capacity C_v and susceptibility χ in statistical physics. These quantities are defined as

$$E(t) = \sum_i E_i(t) \quad (8)$$

$$E = \langle E(t) \rangle \quad (9)$$

$$C_v = \frac{\langle E(t)^2 \rangle - \langle E(t) \rangle^2}{k_B T^2} \quad (10)$$

$$M = \langle M(t) \rangle = \langle \sum_i \sigma_i(t) \rangle \quad (11)$$

$$\chi = \frac{\langle M(t)^2 \rangle - \langle M(t) \rangle^2}{k_B T} \quad (12)$$

where $\langle \dots \rangle$ indicates the time average.

We show in Fig. 1 the above quantities as functions of T , taking $J = 1$, $a = 3$ and $H = 0$. These quantities show the role of T which allows us to choose appropriate temperature regions when examining the dynamics of the system. As seen in the figure, the energy changes the curvature and the order parameter falls to zero at $T_c = 6.6957$ (the residue of M after T_c is due to the size effect). The fluctuations are very strong, C_v and χ show a peak at T_c , indicating the transition from the low- T phase to high- T disordered phase. We will see below that it is in the critical region just below T_c that interesting dynamics occurs.

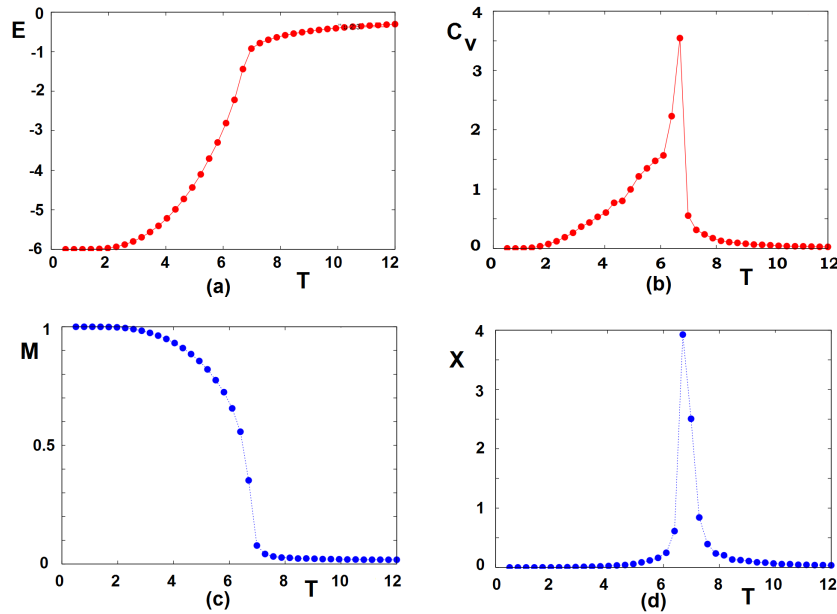


Figure 1. (a) Energy E , (b) specific heat C_v , (c) order parameter M , (d) susceptibility χ , versus T . Parameters in Eq. (5): $J = 1$, $a=3$, $H = 0$. See text for comments.

3.1. Case of more people who sell

We show in Fig. 2 the time evolution of the seller and buyer percentages S at some typical market temperature T for $J = 1$, $a = 3$ and $H = 0$. We use the initial condition: more people to sell than people to buy. At a low T , each agent imitates the majority of his surrounding neighbors who sells so that the price decreases and the system quickly comes to an equilibrium of percentages at that given T , as seen in Fig. 2a. Near T_c , the difference in the percentages is reduced as seen in Fig. 2b. Note the strong fluctuations of S which occur in the critical region of market temperature. For $T > T_c$ the two populations are equal (not shown), as expected. T

It is interesting to examine the effect of the boosting measure H . Taking the same system as that used in Fig. 2, we apply $H = 0.2$ at the time $t_1 = 400$ and remove it at $t_2 = 600$. A striking effect is observed in Fig. 3:

-if T is just below T_c , namely T is in the critical region, the boosting effect lasts for a long time after t_2 (see Fig. 3a).

-if T is far below T_c or well above T_c (Fig. 3b) the boosting measure goes away as soon as it is removed.

This effect is very interesting because it shows that boosting measures from government or economic organisms are efficient only in the turbulence zone just below T_c . We show in Fig. 4 the variation of the price at three typical temperatures: well below T_c , near T_c and above T_c . At low T the

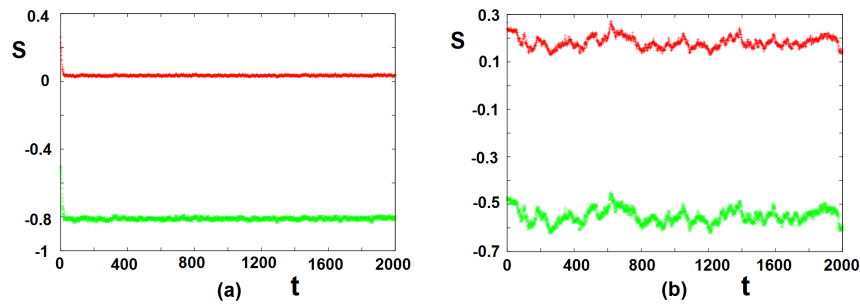


Figure 2. Time evolution of the percentages S of buyers (red) and sellers (green) at temperatures (a) $T = 5.513$ (well below T_c) and (b) $T = 6.692$ (just below T_c). Initial condition: 60% of agents who sell, 40% of agents who buy. Parameters used: $J = 1$, $H = 0$. See text for comments.

259 H effect is not significant, at high T the effect disappears when H is removed. Near T_c , the boosting
 260 effect remains unchanged after the removal of H .

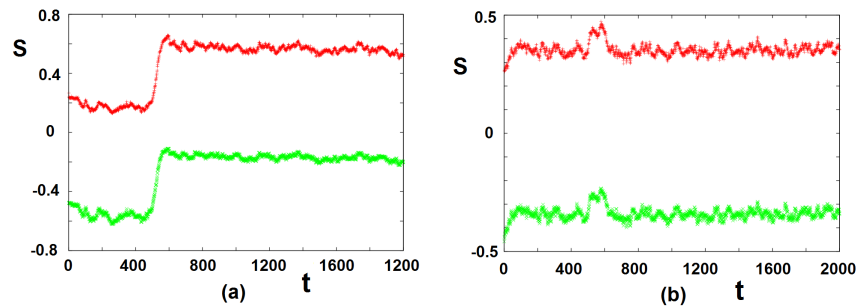


Figure 3. Initial condition 60% of agents who sell, 40% of agents who buy: effect of H on the time evolution of buyer and seller percentages under the boosting measure $H = 0.2$ applied between $t_1 = 400$ and $t_2 = 600$, at temperatures (a) $T = 6.692$ (just below T_c) and (b) $T = 7.872$ (above T_c). Parameters: $J = 1$, $a=3$. See text for comments.

261 The time evolution of the price P is shown in Fig. 4 at several market temperatures with and
 262 without a boosting measure. These curves result from the percentage variations of buyers and sellers
 263 shown in Fig. 3. A discussion was given there.

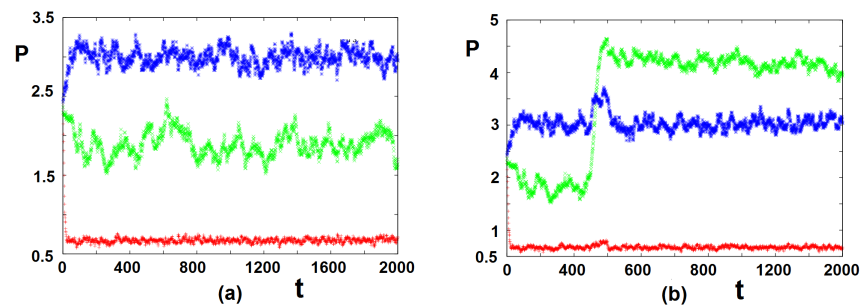


Figure 4. Time evolution of price P with initial condition 60% sellers, 40% buyers. (a) Price variation when $H = 0$: red data are for $T = 5.513$, green for $T = 6.692$ (just below T_c), blue for $T = 7.872$ (after market clearing) (b) Effect of $H = 0.2$ applied between $t = 400$ and 600 . The variation of the price P is shown: red data are for $T = 5.513$, green for $T = 6.692$, blue for $T = 7.872$. Parameters: $J = 1$, $a = 3$, and the market clearing price A is fixed arbitrarily at 3, see Eq. (a6).

At this stage it is interesting to examine the effect of the amplitude of H . We have seen in Fig. 4b the jump in the price at $H = 0.2$. We show in Fig. 5a the case when $H = 0.05$. We see here that the long lasting jump remains. However when $H = 0.02$ (Fig. 5b), that jump disappears. It means that there is a critical value H_c (between 0.02 and 0.05) above which the long-lasting jump sets in.

The above result is a very important finding: to boost the price for a long lasting, the measure has to be strong enough and taken in the critical region of the market temperature (crisis, unrest) in a short lapse of time.

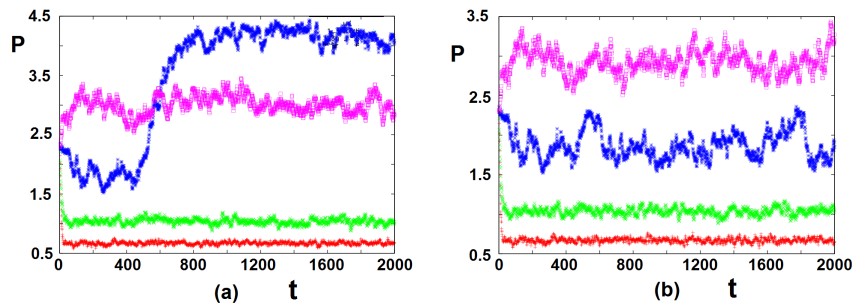


Figure 5. Time evolution of price P with initial condition 60% of agents who sell, 60% of agents who buy. Red, green and blue symbols are for $T = 6.103$ (well below T_c), $T = 6.692$ (just below T_c) and $T = 7.282$ (a) $H = 0.05$; (b) $H = 0.02$. See text for comments.

3.2. Case of more people who buy

Let us take the case of more people who want to buy than people who want to sell. We start the simulation with for example 60% buyers and 40% sellers among the agents. In this case, at low T each agent imitates the majority of his neighbors, leading to more people to buy. As a consequence, the price increases. We show in Fig. 6 the time evolution of the percentages of buyers and sellers at several market temperatures. One observes strong fluctuations in the critical region.

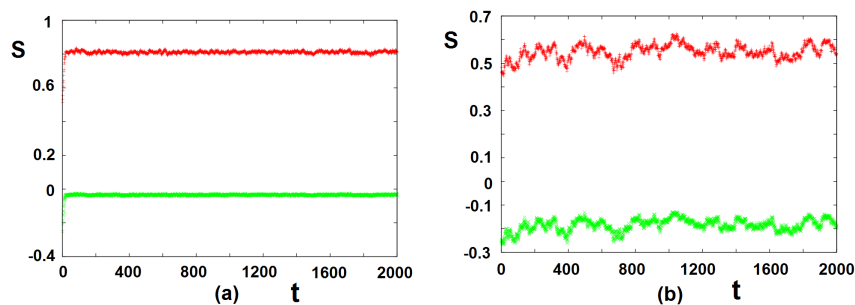


Figure 6. Time evolution of buyer and seller percentages with $H = 0$ and initial condition: 60% of agents who buy, 40% of agents who sell. Red symbols are buyers, green symbols are sellers (a) $T = 5.513$ (well below T_c) (b) $T = 6.692$ (just below T_c). See text for comments.

The price P as a function of time is shown in Fig. 7 for three typical temperatures below, near and above T_c . Figure 7a shows the price when $H = 0$. One notes that for $T < T_c$ the price is higher than the market clearing price fixed at 3. Figure 7b shows the effect of H applied to reduce the price (with a negative value $H = -0.2$). As seen, the effect of H is not significant at $T \ll T_c$ and $T > T_c$. However just below T_c , one observes a striking effect of H : a strong fall of price and this fall remains unchanged after the removal of H . This feature has been found above for the case of more people to sell than people to buy: we found a long-lasting price jump just below T_c . As the previous case, H should be stronger than a critical value for a long-lasting jump to occur.

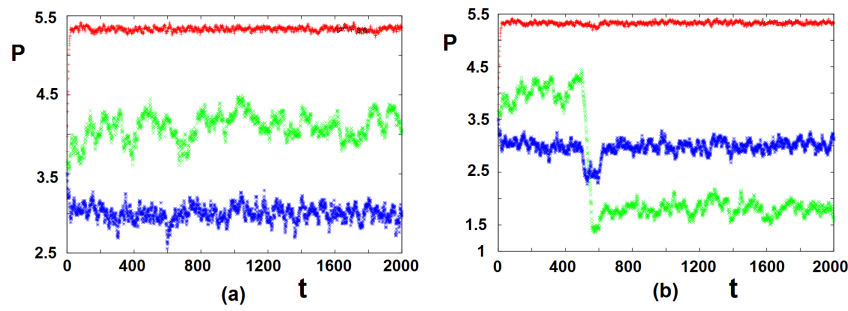


Figure 7. Time evolution of price with initial condition 60% of agents who buy, 40% of agents who sell. Red, green and blue symbols are for $T = 5.513$ (well below T_c), $T = 6.692$ (just below T_c) and $T = 7.872$ (a) $H = 0$; (b) Effect of $H = -0.2$. The market clearing price is fixed at $A = 3$. See text for comments.

3.3. Model of 5 individual states

Let us consider the 5-state model: $-2, -1, 0, 1, 2$. Negative values express two degrees of selling desire (strong and moderate), while positive values express two degrees of buying desire. The market-temperature effect is shown in Fig. 8 where one recognizes the critical value $T_c = 21.051$ for parameters $J = 1, a=6, H = 0$. Note that the susceptibility χ expresses the fluctuations of the order parameter M .

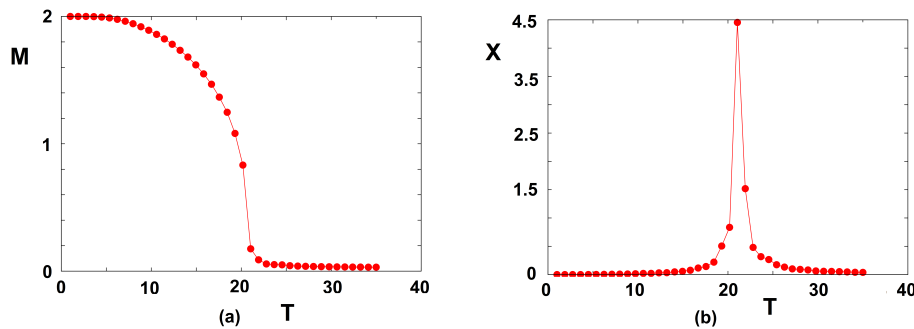


Figure 8. Model of 5 individual states (a) Order parameter M , (b) susceptibility, versus T . Parameters in Eq. (5): $J = 1, a=6, H = 0$. See text for comments.

The case of less people to buy (40%) than people to sell is initially started, namely the initial price is lower than the market clearing price. The time evolution of the percentages of buyers and sellers are shown in Fig. 9. At low T , agents follow mainly the majority of their neighbors' desire so that the percentage of sellers is higher than that of buyers (Fig. 9a). At T_c there are strong fluctuations seen in the agent percentages and in the price. At $T > T_c$, the price is stabilized at the market clearing. These features are common with the 3-state case above.

We examine now the effect of boosting measure H . We show in Fig. 10 the time evolution of the price for several values of H . Fig. 10a shows the case of a strong value of H , namely $H = 0.6$, at several T . As seen, the boosting measure lasts only during its application except when T is near T_c where the effect seems to last forever (green curve). When we decrease H down to $H = 0.4$, the same effect is still observed (not shown). At $H = 0.35$ the boosting falls down when H is removed (Fig. 10b). Thus, there is a critical value between $H = 0.35$ and 0.4 above which the boosting effect is long-lasting.

The same feature is found when we start with more buyers, namely the initial price at $T < T_c$ is higher than the market clearing price. The specific measure applied to reduce the price has a long-lasting effect only in the region just below T_c if $|H|$ is larger than a critical value between 0.35 and 0.40 .

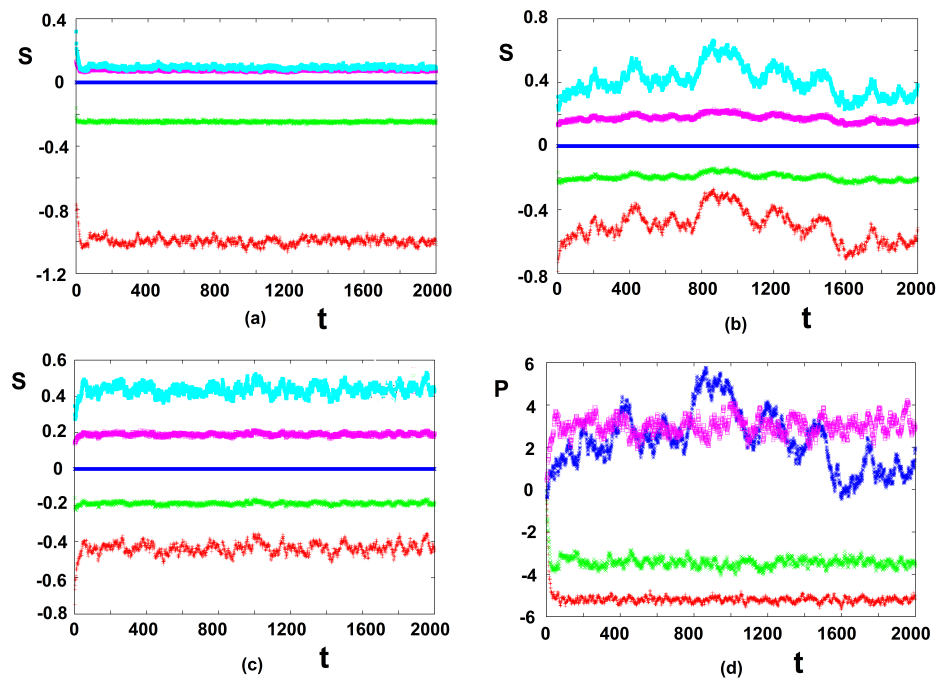


Figure 9. (a)-(b)-(c): Time evolution of buyer and seller percentages with $H = 0$ and initial condition 60% of agents who sell and 40% of agents who buy, at (a) $T = 19.308 < T_c$, (b) $T = 21.051 = T_c$, (c) $T = 22.795 > T_c$. Color code: red and green=sellers, sky blue and magenta=buyers, blue=no action. (d) Price versus time at four temperatures $T_1 = 17.764 < T_c$ (red), $T = 19.308 < T_c$ (green), $T = 21.051 = T_c$ (blue), $T_4 = 22.795 > T_c$ (magenta). See text for comments.

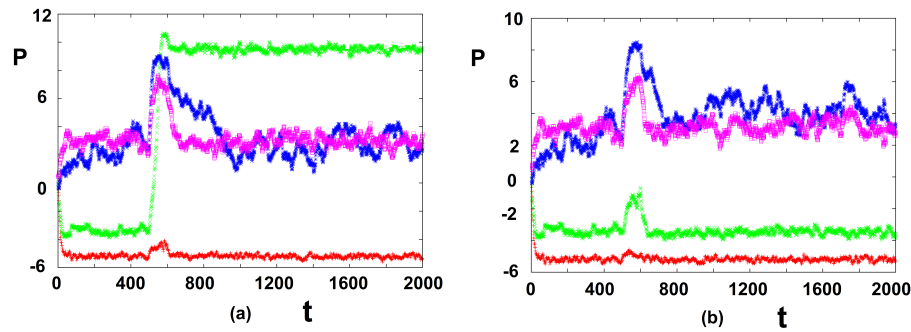


Figure 10. Effect of the boosting measure H : time evolution of price with initial condition 60% of agents who sell and 40% of agents who buy, for (a) $H = 0.6$; (b) $H = 0.35$. Color code: red for $T = 17.564$, green for $T = 19.308$ (just below T_c , blue is for $T = 21.051$, magenta is for $T = 22.795$. Note that $T_c = 21.144$. See text for comments.

3.4. Continuous model

Let us consider the case where each agent has a continuous spectrum of states going from a strong selling desire $S = -1$ to a strong buying desire $S = 1$ in a continuous manner: $S \in [-1, 1]$. The determination of T_c by examining the order parameter M and the susceptibility χ gives $T_c = 3.513$.

We show in Fig. 11 the case of more sellers than buyers initially. As in the previous cases, the price is low for $T < T_c$ and reaches the market clearing value for $T \geq T_c$. We have as before strong fluctuations of the price at and near T_c (see Fig. 11d).

We show in Fig. 12 the effect of boosting measure H applied between $t_1 = 400$ and $t_2 = 600$ (in MC-step time unit). The percentages of sellers and buyers make a jump at near T_c . This jump lasts

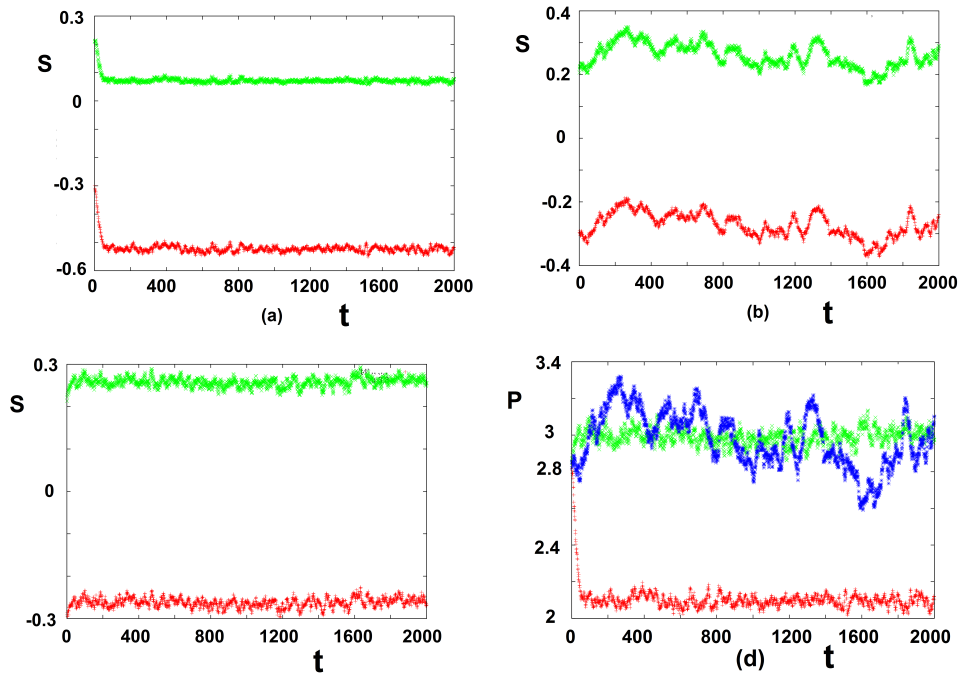


Figure 11. (a)-(b)-(c): Time evolution of seller (red) and buyer (green) percentages S with $H = 0$ using the initial condition 60% of agents who sell, 40% of agents who buy at (a) $T_1 = 3.256 \ll T_c$, (b) $T_2 = 3.564$ just above T_c , (c) $T_3 = 3.974 \gg T_c$. (d) Price P versus time t at temperatures T_1 (red), T_2 just above T_c (blue), T_3 (green). Market clearing price is fixed at $a = 3$. See text for comments.

even after the removal of H . At other temperatures far from T_c , the H effect is insignificant and barely seen in Figs. 12a and Figs. 12b, during the application of H .

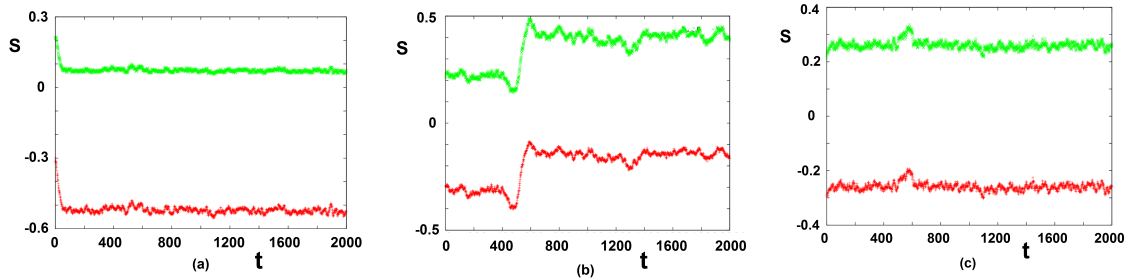


Figure 12. Effect of the boosting measure H : time evolution of seller (red) and buyer (green) percentages with initial condition 60% of agents who sell and 40% of agents who buy for $H = 0.1$ at (a) $T = 3.256 \ll T_c$, (b) $T = 3.462$ just below T_c , (c) $T = 3.974 \gg T_c$. Color code: red for sellers, green for buyers. Note the jump observed at $T = 3.462$. See text for comments.

The effect of H on the price reflects the variations of seller and buyer percentages as shown in Fig. 13 for three values of H : 0.06, 0.025 and 0.02. At each value of H four temperatures are examined, two below and two above T_c . While for $H = 0.06$ and 0.025 a long-lasting jump of the price is seen at T just below T_c , the case of $H = 0.02$ does not show such a jump. The critical value of H is thus between 0.02 and 0.025.

3.5. Summary and discussion

Let us summarize some main results of our model treated by Monte Carlo simulations:

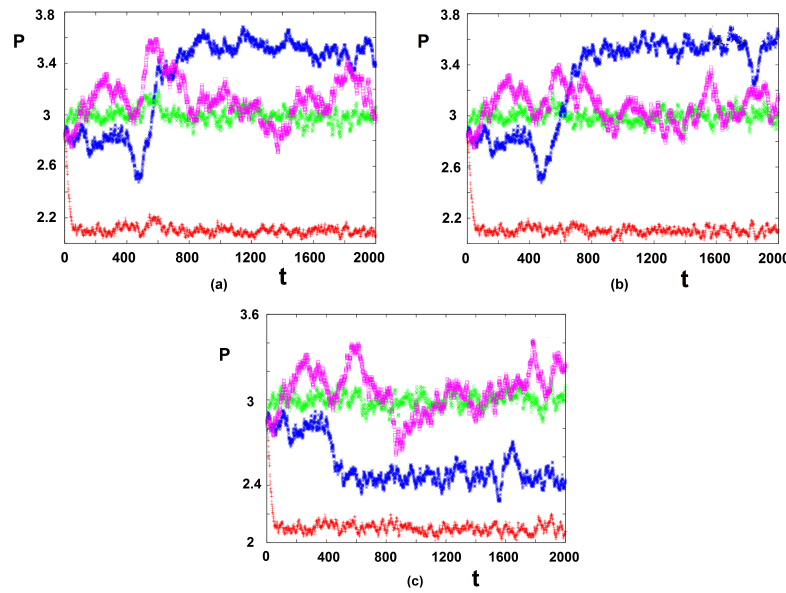


Figure 13. Effect of the boosting measure H on the price, with initial condition 60% of agents who sell and 40% of agents who buy for (a) $H = 0.06$, (b) $H = 0.025$, (c) $H = 0.02$. Color code for temperatures: red is for $T = 3.256 \ll T_c$, blue is for $T = 3.462$ just below T_c , magenta is for $T = 3.564$ just above T_c , green is for $T = 3.974 \gg T_c$. See text for comments.

- The results show that the main features do not depend on the number of individual states q : we have similar qualitative results for $q = 3, 5$ and ∞ .
- We have shown the primordial role of the economic temperature T : there is a critical value T_c corresponding to the market clearing point. At and above T_c the two populations buyers and sellers are equal, the price is stabilized.
- In the region just below T_c the fluctuations are very strong, namely strong exchange of stock market shares. In this region, volatility is much higher. More generally, different temperatures correspond to different volatilities, and infrequent changes in the temperature generate volatility clustering (a well known stylized fact). The long autocorrelation occurs in this "critical" region is well-known in statistical physics as the critical-slowing-down phenomenon occurring near the second-order phase transition [1].
- It is also in the region below T_c that under a temporary shock the price can have a persistent effect if the shock is strong enough.

The above results are interesting because they come from a single Hamiltonian model without approximation.

4. Mean-field theory

The mean-field theory (MFT) is a very popular method in statistical physics. Its principle consists in taking into account only the average values of the neighbors acting on a spin, neglecting real-time fluctuations of each neighbor. In doing so, the calculation of thermodynamic properties is simple and the main features can be easily obtained.

4.1. Mean-Field Model

In this paper, we use a time-dependent MFT, namely a spin at the time t interacts with its neighbors of the same community in their state at $t - 1$ and interacts with the average value of the other community at $t - 1$. We recall that a spin $\sigma_i(t)$ describes the state of an agent i at t . The

time-dependent MFT has been used to study social conflicts [11–13]. We follow the same method in the present paper.

In order to make our problem of price variation richer, we consider in this section a more general situation which consists of introducing the following hypotheses:

- The buyers and the sellers belong to two distinct communities. The intra-community interactions are different, namely J_1 for the buyer community and J_2 for the seller. Note that in Eq. (5), we have only J for both. This two-community hypothesis expresses the fact that in reality the moral of willing to sell or to buy is different. Sellers and buyers do not share the same enthusiasm.
- The interaction between the communities 1 and 2, namely $K_{1,2}$, does not need to be equal to $K_{2,1}$. This hypothesis means that "you want to sell, but I do not want to buy". So in general $K_{1,2} \neq K_{2,1}$. They are different in magnitude and in sign.

For our present problem with the hypotheses described above, the energy E_i of an agent of group i ($i = 1, 2$) at time t is written as

$$E_1 = s_1(t) \{-j_1 s_1(t-1) - k_{12} s_2(t-1) + a[s_1(t-1) - s_2(t-1)]\} \quad (13)$$

$$E_2 = s_2(t) \{-j_2 s_2(t-1) - k_{21} s_1(t-1) + a[s_1(t-1) - s_2(t-1)]\} \quad (14)$$

where we introduce the lag time in the above equations by letting the preference s_1 and s_2 at time t interact with the averages s_1 and s_2 evaluated at an earlier time $t-1$. Here time is measured in units of the delay time.

The value of an individual in each group at the time t is given by the MFT equations

$$s_1(t) = \frac{\sum_{s=-M_1}^{M_1} s e^{\{s[j_1 s_1(t-1) + k_{12} s_2(t-1) - a(s_1(t-1) - s_2(t-1))]\}}}{\sum_{s=-M_1}^{M_1} e^{\{s[j_1 s_1(t-1) + k_{12} s_2(t-1) - a(s_1(t-1) - s_2(t-1))]\}}} \quad (15)$$

$$s_2(t) = \frac{\sum_{s=-M_2}^{M_2} s e^{\{s[j_2 s_2(t-1) + k_{21} s_1(t-1) - a(s_1(t-1) - s_2(t-1))]\}}}{\sum_{s=-M_2}^{M_2} e^{\{s[j_2 s_2(t-1) + k_{21} s_1(t-1) - a(s_1(t-1) - s_2(t-1))]\}}} \quad (16)$$

where $j_n = J_n/T$ and $k_{n,m} = K_{n,m}/T$ for $n, m = 1, 2$. We use units such that $k_B = 1$. The sums on the right hand sides of the above equations lead to the Brillouin function (p. 292 in Ref. [1]):

$$B(x, y, z, j, k, l, M) = (M + \frac{1}{2}) \coth[(M + \frac{1}{2})(jx + ky + lz)] - \frac{1}{2} \coth[\frac{1}{2}(jx + ky + lz)] \quad (17)$$

Equations (15)-(16) can be written as:

$$s_1(t) = B(s_1(t-1), s_2(t-1), j_1, k_{12}, M_1, a) \quad (18)$$

$$s_2(t) = B(s_2(t-1), s_1(t-1), j_2, k_{21}, M_2, a) \quad (19)$$

For a given set of $(J_1, J_2, K_{12}, K_{2,1}, a)$ at the market temperature T , we can solve numerically these equations iteratively as a function of t : taking input values (initial condition) for $s_1(t=0)$ and $s_2(t=0)$, we calculate $s_1(t=1)$ and $s_2(t=1)$. Using these values we calculate $s_1(t=2)$ and $s_2(t=2)$ and so on. Note that as in the model for MC simulations, the MFT model has two separate communities: the initial value of buyers $s_1(t=0)$ should be positive and that of $s_2(t=0)$ should be negative, but in the MFT each group starts with 100%. The constant a in the case of two-group MFT model is more important than in the MC model as will be seen below.

As the time evolves, s_1 and s_2 change. We have to bear in mind that in the present MFT hypotheses, the average value $\langle s_i \rangle$ ($i = 1, 2$) means the percentage of the community i with the following

convention: positive $\langle s_i \rangle$ represents the percentage of people who want to buy, negative $\langle s_i \rangle$ represents that of people who want to sell, as we have used in MC model below Eq. (5).

4.2. Mean-Field Results

For simplicity, we choose $M_1 = M_2 = 1$, namely $s = -1, 0, 1$ in Eqs. (15)-(16) (three-state model). Let us show now the results of the MFT.

Before the interaction between two communities is turned on, the will strength of the group i depends on the intra-group interaction J_i . Choosing (J_1, J_2) , we plot the will of each groups against the market temperature T in Fig. 14. The values $\langle s_i \rangle$ in Fig. 14 represents the will degree of group i . The higher T the lower $|\langle s_i \rangle|$ meaning that less people of the group want to sell or buy. At the market temperature T_i^c the group i becomes disordered, namely no market orientation (buying or selling). Due to the difference of J_1 and J_2 , we see that $T_1^c \neq T_2^c$.

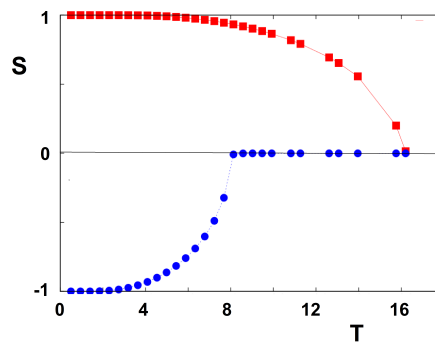


Figure 14. Willing to buy or to sell of two groups buyers (red squares) and and sellers (blue circles) versus market temperature, when there is no inter-group interaction. Intra-group interactions $J_1 = 1$ and $J_2 = 0.5$ are used. Above $T_1^c \simeq 16.1$, group 1 does not want to buy, and above $T_2^c \simeq 8.0$ group 2 does not want to sell. See text for comments.

We turn on now the interaction between the two groups. For the parameters $J_1 = 1$, $J_2 = 0.5$, $K_{12} = 1$, $K_{21} = -0.5$ and $a = 5$ used in Fig. 15, the two groups are both ordered below $T^{c1} \simeq 5.01$. They both become disordered above $T^{c2} \simeq 12.10$. This is the market clearance point. Between these two temperatures, the systems are dynamically not stable. We show the time evolutions of $\langle s_1 \rangle$ and $\langle s_2 \rangle$ in Fig. 15 at several market temperatures. Depending on T , the dynamic behaviors of $\langle s_1 \rangle$ and $\langle s_2 \rangle$ are different:

- At low T namely $T < T^{c1}$, each group stays in their initial willingness (selling or buying) as shown in Fig. 15a
- At a temperature between T^{c1} and T^{c2} , the position of each group oscillates as shown in Figs. 15b and 15c. The lower T , the larger the period of oscillation (see Ref. [11])
- At T near T^{c2} , the two groups oscillate at the market opening but tend to the market clearance as seen in Fig. 15d.

We show in Fig. 16 the variation of the price corresponding to the temperatures of Figs. 15c and 15d. The price at the market clearance has been arbitrarily fixed to 3. The price oscillates for the market temperature between T^{c1} and T^{c2} . For $T > T^{c2} \simeq 12.10$, the price decays to the market clearing price.

Before closing this section we discuss on the effect of a . As seen in Eqs. (13)-(14), a is in competition with the j and k terms. For a given set of $(j_1, j_2, k_{12}, k_{21})$, the value of a determines the qualitative dynamics of the system. We have seen an example for $a = 5$ shown above. We have calculated for other values of a . The results show that larger a will enlarge the temperature region of oscillation behavior, and below a value depending on $(j_1, j_2, k_{12}, k_{21})$, the oscillation disappears.

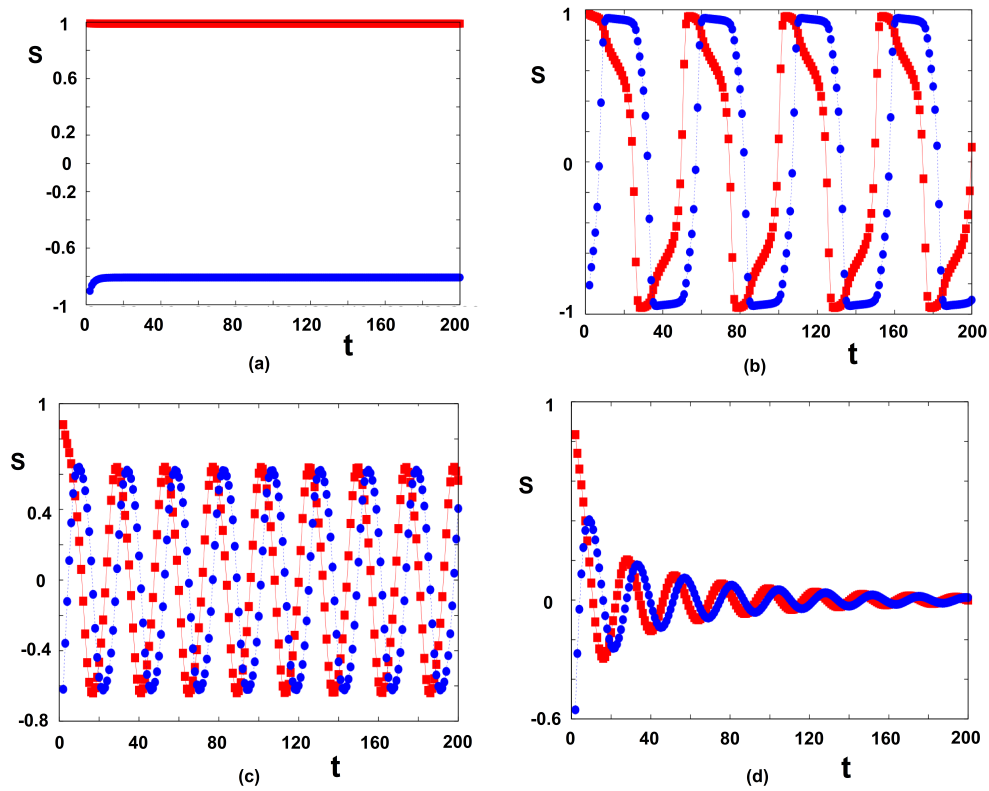


Figure 15. The percentages of buyers (red) and sellers (blue) at several temperatures for the intra-group and inter-group interactions $J_1 = 1$, $J_2 = 0.5$, $K_{12} = 1$, $K_{21} = -0.5$ and $a = 5$: (a) $T = 4.98$, (b) $T = 6.78$, (c) $T = 10.82$, (d) $T = 12.62$. See text for comments.

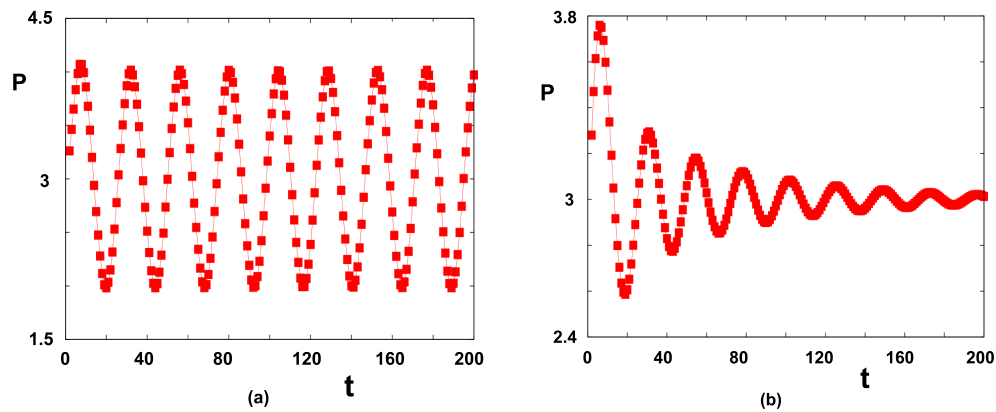


Figure 16. Price variation versus time for (a) $T = 10.82$, (b) $T = 12.62$. See text for comments.

4.3. Summary and discussion

Since the mean-field theory is based on the principle to replace each individual by a common mean value, fluctuations are neglected in the calculation. This theory cannot explain the nature of the phase transition and overestimates the ordering at low dimensions in statistical physics [1,45]. However, the mean-field theory can account for the essential qualitative features which are sufficient for economic issues. We used this theory in this paper for that purpose.

We have seen in the previous section that the Monte Carlo method takes into account these fluctuations which play an important role near the critical economic temperature T_c . In mean-field

theory, there is no random process as in Monte Carlo simulations: the price at time t depends on the price at $(t - 1)$ because the solution is obtained by iteration. So, if we consider only one community composed of sellers and buyers as what we did in Monte Carlo simulations, we will not have random fluctuations of the price with time. We have therefore introduced two interacting communities with no need to have inter-group symmetrical interaction. As a consequence, we have seen that the price can oscillate below T_c with large amplitudes, namely when the economic agitation is high. Price oscillations are decayed to the market clearing for $T > T_c$.

To conclude this section on the mean-field theory, we emphasize that the model used here is a simplest model aiming at showing the non trivial price oscillation in a region of market temperature. Such a regular oscillation stems certainly from the simplification of the model. We believe however that such a price oscillation bears an important feature of the market reality.

5. Conclusion

In this work, we have studied the variation of the price of a good. This price is determined by the behavior of sellers and buyers. An agent decides to buy or to sell according to a probability which takes into account the influence of his neighbors, the price level, the economic temperature and a specific measure in favor of buying or selling applied during a lapse of time.

We assimilated each agent to a spin possessing several internal states representing from selling desire to buying desire. The influence of neighbors is expressed by the majority imitation. The price is defined as proportional to the difference between seller and buyer numbers which is a function of time. The economic environment is expressed by a parameter T which plays the role of the temperature in physics. The higher T the more agitation. The specific measure H to boost or to lower the price is applied during a lapse of time by the government or an economic organism.s

Monte Carlo simulations have been performed on a population of agents with the above-mentioned parameters. The results are interesting, showing in particular strong fluctuations of the price in the critical region of T . There are two stylized facts that the simulation can produce: random variation of the price as observed in finance, and volatility clustering due to the autocorrelation related to the initial condition at the opening of the market.

Another striking finding observed in the simulation is the persistent effect of a temporary shock H such as a boosting measure applied during a short lapse of time. This phenomenon is observed only in the region of economic turbulence (near T_c) with H strong enough (above a critical value H_c). The boosting effect lasts long time after the removal of the measure. These results may be useful for the government and/or economic organisms to choose the right moment to intervene to modify the market tendency.

The second party of the paper deals with the price variation using the mean-field theory. Unlike the Monte Carlo model, we suppose here that the sellers and buyers belong to two distinct communities. Interesting results on the price oscillation are found when we break the symmetry of the inter-group interactions, namely $J_1 \neq J_2$ and $K_{12} \neq K_{21}$. We believe that this work paves the way for future more realistic models using statistical physics approach.

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