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On fixed point for derivative of the interval-valued functions

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Abstract

In this article, we show the existence of fixed point for the derivative of interval-valued functions. Meanwhile, the fixed point inquiry will utilize the common fixed point methods under the condition of compatibility of the hybrid composite mappings in the sense of the Hausdorff metric. Some examples are given to support the usability of the result of this research.

Keywords : *common fixed point theorem, set-valued maps, compatible mappings, differentiable maps, interval-valued functions.*
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1 Introduction

Suppose $f, g : [0, 1] \longrightarrow [0, 1]$ are two continuous mappings which are commutative mappings, i.e., $f(g(t)) = g(f(t))$ for each $t \in [0, 1]$. In 1954, E. Dyer conjectured that f and g must have a common fixed point in $[0, 1]$(i.e. $t_0 = f(t_0) = g(t_0)$) for some $t_0 \in [0, 1]$. For certain functions, f and g the conjecture has been verified but in general, the question of the existence of this common fixed point remains open [2]. In 1967, W.M. Boyce [3] replied in his paper that Dyer's question is negative as well as an answer from Husein [9] and Singh [22]. However, many researchers are curious about conjecture. In the early seventies, a researcher named Geral Jungck [12] introduces a common fixed point through the commuting mapping in the more general on metric spaces. Since then the common fixed point theory has grown rapidly, involving not only a pair of two mappings but also more than it (see [17]). Even research involving the set-valued mapping also developed [4, 14]. By the concepts of commutativity and compatibility between the function and its derivatives show that the derivative function of the real-valued function has a fixed point [19].

On progress, the composition mappings discussed not only between fellow of single-valued mappings or set-valued mappings but also the combination. The latter form of the composition is called the hybrid composite mappings (mixed compositions with single-valued and set-valued mappings). Since then many authors have studied common fixed point theorems for the hybrid composite mappings in different ways ([1, 6, 10, 11, 23] and references therein).

In 1977, Itoh [11] introduces "commute" term of hybrid composite functions, namely $fFx \subseteq Ffx$, where f is a single-valued mapping and F is a set-valued mapping. By this properties, they have proven common fixed point theorems for F and f in topological vector spaces. In 1982, Fisher [5] has proved common fixed point theorems for commuting mappings in the sense of the other (i.e. $fFx = Ffx$) in metric spaces. Then Imdad [10], mentioned the properties $fFx \subseteq Ffx$ as "quasi-commute" to distinguish with the latter term. In addition, in the same article, they introduce the property of "weakly commute" and "slightly commute" of hybrid composite mappings. The concepts of weak, quasi and slight commutativity between a set-valued mapping and a single-valued function do not imply each of the other two (see [10]). Whereas two commuting mappings F and f are weakly commuting, but in general two weakly commuting mappings do not commute as it is shown in Example 1 of [20].

In 1993, Jungck [13] introduced the concept of "δ-compatible" mappings in metric spaces and proved some common fixed point theorems for δ-compatible mappings. These results are a modification of common fixed point theorems in the article [10] and [21]. In 1989, Kaneko [15] also introduced the same things but used Hausdorff metric and proved common fixed point theorems with the concept.

Motivated by the results mentioned above, in this article, we introduced common fixed point theorems for hybrid composite mappings which involving gh-derivative of the interval-valued function [25] under the condition of compatibility; even though, it does not explicitly. Also, we give some examples to illustrate the main results in this article. In fact, our main results create the corresponding result given by some authors.

Suppose (X, d) and (Y, ρ) are two metric spaces. Then we use the notation $\mathcal{P}_0(X)$ (resp. $\mathcal{B}(X)$, $\mathcal{CB}(X)$, $\mathcal{K}(X)$ and $\mathcal{KC}(X)$) as the family of all non-empty (resp. bounded, closed-bounded, compact and compact-convex) subsets of X.

The function $\delta(A, B)$ for each $A, B \in \mathcal{B}(X)$ is defined by

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}.$$

It follows easily from the definition

$$\delta(A, B) = \delta(B, A) \geq 0$$

for all $A, B \in \mathcal{B}(X)$.

Suppose A and B are two subsets of a metric space (X, d). Then the Hausdorff distance between A and B is a distance function $H : \mathcal{P}_0(X) \times \mathcal{P}_0(X) \longrightarrow \mathbb{R}^+$ which is defined as

$$H(A, B) = \max\{d(A, B), d(B, A)\},$$

where $d(A, B) = \sup_{a \in A} d(a, B)$ and $d(a, B) = \inf_{b \in B} d(a, b)$. It is clear that $d(A, B) \neq d(B, A)$. The Hausdorff distance H is a metric on the family $\mathcal{CB}(X)$ which is called Hausdorff metric. The metric space $(\mathcal{CB}(X), H)$ is said to be complete if X is a complete metric space.

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Suppose that $A = \{a\}$, $B = \{b\} \subset X$ and $C \in \mathcal{CB}(X)$. The Hausdorff metric for these sets are defined by the formula

(i) $H(A,B) = H(\{a\},\{b\}) = d(a,b)$

(ii) $H(\{x\},C) = d(x,C)$.

Suppose $\mathcal{I}(\mathbb{R}) = \{[a,b] \mid a,b \in \mathbb{R}, a < b\}$. In [18], R.E. Moore et al. introduced an absolute value of the interval. The absolute value of an interval $I = [a,b]$ is the real number.

$$|I| = |[a,b]| = \max\{|a|,|b|\}.$$

The Hausdorff distance function on $\mathcal{I}(\mathbb{R})$ is a metric defined as $H : \mathcal{I}(\mathbb{R}) \times \mathcal{I}(\mathbb{R}) \longrightarrow [0,\infty)$ by

$$H(I,J) = \max\{|a-c|,|b-d|\}, \tag{1}$$

where $I = [a,b]$ and $J = [c,d]$. The pair $(\mathcal{I}(\mathbb{R}),H)$ is a complete and separable metric space.

Suppose U,V is subsets of $\mathbb{R}^k$. In 1967, M. Hukuhara [8] introduced the Hukuhara difference (h-difference) defined $U \overset{h}{-} V = W \Longleftrightarrow U = V + W$ for each $U,V,W \in \mathcal{KC}(\mathbb{R}^k)$. An important properties of the Hukuhara difference is that $U \overset{h}{-} U = \{\Theta\}$ and $(U+V) \overset{h}{-} V = U$. The Hukuhara difference is unique, but it does not always exists.

L. Stefanini [24] introduced the concept of a more general Hukuhara difference which called generalization Hukuhara difference (gh-difference) and it is defined as follows:

Definition 1 *[24] Let $U,V \in \mathcal{KC}(\mathbb{R}^k)$. The gh-difference of two sets U and V defined as $U \overset{gh}{-} V = W$ if it satisfies (**a**) $U = V + W$ or (**b**) $V = U + (-1)W$.*

It is also possible that $U = V + W$ and $V = U + (-1)W$ hold simultaneously. In that case part (a) of the gh-difference is equivalent to the h-difference. Thus the gh-difference is a generalization of the h-difference. The gh-difference always exists for any two intervals in $\mathcal{I}(\mathbb{R})$.

Proposition 1 ([*24*]) *Suppose $I = [x^-,x^+]$ and $J = [y^-,y^+]$ are intervals in $\mathcal{I}(\mathbb{R})$. The gh-difference of two intervals I and J always exists and*

$$I \overset{gh}{-} J = [x^-,x^+] \overset{gh}{-} [y^-,y^+] = [z^-,z^+] \tag{2}$$

where $z^- = \min\{(x^- - y^-),(x^+ - y^+)\}$ and $z^+ = \max\{(x^- - y^-),(x^+ - y^+)\}$.

The mapping $F : X \longrightarrow \mathcal{P}_0(Y)$ is called ***set-valued functions*** where the maps $F(x) \in \mathcal{P}_0(Y)$ for each $x \in X$. The function $f : X \longrightarrow Y$ is said to be ***selection*** of F

if $f(x) \in F(x)$ for all $x \in X$. We say that a point $z \in X$ is a fixed point of F if $z \in F(z)$.

Base on the gh-difference, the definition of the derivative for the interval-valued functions is as follows:

Definition 2 ([25]) *Let $F : [a, b] \longrightarrow \mathcal{I}(\mathbb{R})$ be a set-valued function and suppose $t_0, t_0 + h \in (a, b)$. The gh-derivative $F'_{gh}(t_0) \in \mathcal{I}(\mathbb{R})$ defined as*

$$F'_{gh}(t_0) = \lim_{h \to 0} \frac{F(t_0 + h) \overset{gh}{-} F(t_0)}{h}. \tag{3}$$

If $F'_{gh}(t_0) \in \mathcal{I}(\mathbb{R})$ exists and satisfies equation (3), then F is said to be generalized Hukuhara differentiable (gh-differentiable) at the point $t_0 \in (a, b)$. The derivatives F'_{gh} is called generalized Hukuhara derivatives.

Theorem 1 *If interval-valued functions $F : [a, b] \longrightarrow \mathcal{I}(\mathbb{R})$ is a gh-differentiable at a point $p \in [a, b]$ then F is continuous at $p \in [a, b]$.*

Proof.

$$\begin{aligned}
\lim_{x \to p} F(x) \overset{gh}{-} F(p) &= \lim_{x \to p} \left[\frac{F(x) \overset{gh}{-} F(p)}{(x - p)} (x - p) \right] \\
&= \left[\lim_{x \to p} \frac{F(x) \overset{gh}{-} F(p)}{(x - p)} \right] \left[\lim_{x \to p} (x - p) \right] \\
&= F'_{gh}(p) \cdot 0 = 0.
\end{aligned}$$

So F is continuous at the point $p \in [a, b]$. ■

Theorem 2 ([25]) *Let $F : [a, b] \longrightarrow \mathcal{I}(\mathbb{R})$ be an interval-valued functions and $F(x) = [f(x), g(x)]$, where $f, g : [a, b] \longrightarrow \mathbb{R}$. F is gh-differentiable on (a, b) if and only if f and g are differentiable on (a, b) and*

$$F'_{gh}(x) = [\min\{f'(x), g'(x)\}, \max\{f'(x), g'(x)\}],$$

for all $x \in (a, b)$.

This means that

$$F'_{gh}(x) = \begin{cases} [f'(x), g'(x)] & \text{if } f'(x) < g'(x), \\ [g'(x), f'(x)] & \text{if } f'(x) > g'(x) \end{cases}$$

for all $x \in (a, b)$.

An important class of interval-valued functions is defined by

$$F(t) = f(t) \cdot I,$$

where $I \in \mathcal{I}(\mathbb{R})$ and f is a real function of a real variable.

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2 Preliminary

Definition 3 *Suppose (X, d) is a metric space, $E \subset X$, $F : E \to \mathcal{B}(X)$ is a set-valued mapping and $f : E \to X$ is single-valued mapping.*

*(i). F and f are said to **quasi commute** if $fFx \subseteq Ffx$ for each $x \in E$*

*(ii). F and f are said to **commute** if $fFx = Ffx$ for each $x \in E$*

*(iii). F and f are said to **slightly commute** if $fFx \in \mathcal{B}(X)$ for each $x \in E$ and $\delta(fFx, Ffx) \leq \max\{\delta(fx, Fx), diam(Fx)\}$*

*(iv). F and f are said to **weakly commute** if $fFx \in \mathcal{B}(X)$ for each $x \in E$ and $\delta(fFx, Ffx) \leq \max\{\delta(fx, Fx), diam(fFx)\}$.*

The concepts of weak, quasi and slight commutativity between a set-valued mapping and a single-valued mapping do not imply each of the other two [10]. While two commuting mappings F and f are weakly commuting, but in general two weakly commuting mappings do not commute as it is shown in Example 1 of [20].

Let $F, G : X \to \mathcal{B}(X)$ be a set-valued mapping and $f, g : X \to X$ be a single-valued mapping. For each $x, y \in X$, we follow the following notation

$$\mathcal{M}(F, f) = \max\{d(fx, fy), \delta(fx, Fx), \delta(fy, Fy), \delta(fx), Fy, \delta(fy, Fx)\}. \tag{4}$$

and

$$\mathcal{N}(F, f) = \max\{d(fx, fy), d(fx, Fx), d(fy, Fy), \frac{1}{2}[d(fx, Fy) + d(fy, Fx)]\}. \tag{5}$$

and

$$\mathcal{M}(F, G, f, g) = \max\{d(fx, gy), \delta(fx, Gy), \delta(gy, Fx)\}. \tag{6}$$

B. Fisher [5] proved the following result

Theorem 3 ([5]) *Suppose (X, d) is a complete metric space, $F : X \to \mathcal{B}(X)$ is a set-valued mapping and $f : X \to X$ is a single-valued mapping satisfying the inequality*

$$\delta(Fx, Fy) \leq c\mathcal{M}(F, f)$$

for all $x, y \in X$, where $0 \leq c < 1$. If

(A). f is continuous,

(B). $F(X) \subseteq f(X)$, and

(C). F and f are commute,

then F and f have a unique common fixed point.

In [6], B. Fisher also proofs the same as Theorem 3 with assumes the continuity of F in X instead of the continuity of f (see [7]).

Generalization of the Theorem 3 has been resulted by M. Imdad et al [10] as follows

Theorem 4 ([*10*]) *Suppose* (X,d) *is a complete metric space,* $F : X \to \mathcal{B}(X)$ *is a set-valued mapping and* $f : X \to X$ *is a single-valued mapping satisfying the inequality*

$$\delta(Fx, Fy) \leq \psi\mathcal{M}(F, f)$$

for all $x, y \in X$*, where* $\psi : [0,\infty) \to [0,\infty)$ *is a nondecreasing, right continuous and* $\psi(t) < t$*, for all* $t > 0$ *. If*

(A). f *is continuous,*

(B). $F(X) \subseteq f(X)$*,*

(C). F *and* f *are weakly commute, and*

(D). There exists $x_0 \in X$ *such that* $\sup\{\delta(Fx_n, Fx_1) : n = 0, 1\cdots\} < +\infty$*,*

then F *and* f *have a unique common fixed point.*

The other generalizations that involving quasi commute and slightly commute can be seen in [10].

Theorem 5 ([*10*]) *Suppose* (X,d) *is a complete metric space,* $F : X \to \mathcal{B}(X)$ *is a set-valued mapping and* $f : X \to X$ *is a single-valued mapping satisfying the inequality*

$$\delta(Fx, Fy) \leq \psi\mathcal{M}(F, f)$$

for all $x, y \in X$*, where* $\psi : [0,\infty) \to [0,\infty)$ *is a non-decreasing, right continuous and* $\psi(t) < t$*, for all* $t > 0$ *. If*

(A). F *or* f *is continuous,*

(B). $F(X) \subseteq f(X)$*,*

(C). F *and* f *are slightly commute, and*

(D). There exists $x_0 \in X$ *such that* $\sup\{\delta(Fx_n, Fx_1) : n = 0, 1\cdots\} < +\infty$*,*

then F *and* f *have a unique common fixed point.*

In this context, Kaneko and Sessa introduce "compatibility" term for F and f defined as follows:

Definition 4 ([*15*]) *Suppose* (X,d) *is a metric space,* $F : X \to \mathcal{CB}(X)$ *is a set-valued mapping and* $f : X \to X$ *is a single-valued mapping.* F *and* f *are* ***compatible*** *if* $fFx \in \mathcal{CB}(X)$ *and* $H(Ffx_n, fFx_n) \to 0$ *whenever* $\{x_n\}$ *is sequence in* X *such that* $fx_n \to t \in B \in \mathcal{CB}(X)$ *and* $Fx_n \to B \in \mathcal{CB}(X)$*.*

Base on that notion, they proved the following result.

Theorem 6 ([*15*]) *Suppose* (X,d) *is a complete metric space,* $F : X \to \mathcal{CB}(X)$ *is a set-valued mapping, and* $f : X \to X$ *is a single-valued mapping satisfying the inequality*

$$H(Fx, Fy) \leq c\mathcal{N}(F, f)$$

for all $x, y \in X$*, where* $0 \leq c < 1$*. If*

(A). F and f are continuous,

(B). $F(X) \subseteq f(X)$, and

(C). F and f are compatible,

then there exists a point $z \in X$ such that $f(z) \in F(z)$.

By the compatibility, Kaneko showed the result as follows:

Lemma 1 ([15]) *Let $F : X \to \mathcal{CB}(X)$ and $f : X \to X$ be a compatible. If $fw \in Fw$ for some $w \in X$, then $Ffw = fFw$.*

In the discussion of the main results, we shall make frequent use of the following Lemmas.

Lemma 2 ([16]) *If $A, B \in \mathcal{K}(X)$ and $a \in A$, then one can choose $b \in B$ such that $d(a, b) \leq H(A, B)$.*

Lemma 3 *(Lemma 1 [22]) Let $\psi : [0, \infty) \longrightarrow [0, \infty)$ be a real function such that non-decreasing, right continuous. For every $t > 0$ and $\psi < t$ if and only if*

$$\lim_{n\to\infty} \psi^n(t) = 0.$$

3 Main Result

Some of the results will improve the previous results. In this result, we found that Lemma 1 also holds to conversely provide the mapping values of compact sets.

Lemma 4 *Suppose (X, d) is a metric space and $F : X \to \mathcal{K}(X)$ is a continuous. If there exists $f : X \to X$ that is continuous on X such that $fz \in Fz$ for some $z \in X$, then the hybrid composite mappings F and f are compatible.*

Proof. Since f is continuous and $Fx \in \mathcal{K}(X)$ for each $x \in X$, we have $fFx \in \mathcal{K}(X)$ for all $x \in X$. Let $\{x_n\}$ be a sequence on X such that the sequence $Fx_n \longrightarrow K \in \mathcal{K}(X)$ and the sequence $fx_n \longrightarrow z \in K$. In this case, we choose $z \in X$ such that $fz \in Fz$. Since F and f are continuous, we obtained

$$\begin{aligned}\lim_{n\to\infty} H(Ffx_n, fFx_n) &\leq \lim_{n\to\infty} [H(Ffx_n, Fz) + H(Fz, \{fz\}) + H(\{fz\}, fFx_n)]\\ &= H(Fz, Fz) + H(Fz, \{fz\}) + H(\{fz\}, fK)\\ &= 0.\end{aligned}$$

By Definition 4, Pairs F and f are proved as compatible. ■

By using the Lemma 4 we obtain the theorem as follows :

Theorem 7 *Let (X,d) be a complete metric space, $F : X \to \mathcal{K}(X)$ be a continuous. Suppose there exists $f : X \to X$ is continuous on X such that $F(X) \subseteq f(X)$ and satisfying the inequality*

$$H(Fx,Fy) \leq c\mathcal{N}(F,f) \tag{7}$$

for all $x,y \in X$, where $0 \leq c < 1$. Then the pair F and f are compatible if and only if $fz \in Fz$ for some $z \in X$.

Proof. We use a proof method from [15]. Let $x_0 \in X$ be an arbitrary. Since $F(X) \subseteq f(X)$, we choose the point $x_1 \in X$ such that $fx_1 \in Fx_0$. If $c = 0$, then

$$d(fx_1, Fx_1) \leq H(Fx_0, Fx_1) = 0.$$

Since Fx_1 is compact (hence closed), we obtain $fx_1 \in Fx_1$.

Now we assume $c \neq 0$. By Lemma 2 for each $\epsilon = \frac{1}{\sqrt{c}} > 1$ there exists a point $y_1 \in Fx_1$ such that

$$d(y_1, Fx_1) \leq H(Fx_1, Fx_0) < \epsilon H(Fx_1, Fx_0).$$

Choose $x_2 \in X$ such that $y_1 = fx_2 \in Fx_1$ and so on. In general, if $x_n \in X$ there exists $x_{n+1} \in X$ such that $y_n = fx_{n+1} \in Fx_n$ and

$$d(y_n, fx_n) < \epsilon H(Fx_n, Fx_{n-1})$$

for each $n \geq 1$. By the inequality (7) we have

$$\begin{aligned}
d(fx_{n+1}, fx_n) &< \epsilon H(Fx_n, Fx_{n-1}) \leq \frac{c}{\sqrt{c}}\mathcal{N}(F,f) = \sqrt{c}\mathcal{N}(F,f) \\
&< \sqrt{c}\max\{d(fx_n, fx_{n-1}), d(fx_n, Fx_n), d(fx_{n-1}, Fx_{n-1}), \\
&\qquad \frac{1}{2}[d(fx_n, Fx_{n-1}) + d(fx_{n-1}, Fx_n)]\}. \\
&< \sqrt{c}\max\{d(fx_n, fx_{n-1}), d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_n), \\
&\qquad \frac{1}{2}d(fx_{n-1}, fx_{n+1})]\}. \\
&< \sqrt{c}\max\{d(fx_n, fx_{n-1}), d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_n), \\
&\qquad \frac{1}{2}[d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})]\}. \\
&= \sqrt{c}\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\}. \\
&= \sqrt{c}d(fx_{n-1}, fx_n)
\end{aligned}$$

for each $n \in \mathbb{N}$. Since $\sqrt{c} < 1$, the sequence $\{fx_n\}$ is a Cauchy sequence on the complete metric space X. Therefore, it converges to a point $z \in X$. The sequence of sets $\{Fx_n\}$ is also Cauchy sequence on the complete metric space $(\mathcal{K}(X), H)$, hence it converges to a set $K \in \mathcal{K}(X)$. As a result

$$d(z, K) \leq d(z, fx_n) + d(fx_n, K) \leq d(z, fx_n) + H(Fx_{n-1}, K).$$

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Since $d(z, fx_n) \to 0$ and $H(Fx_{n-1}, K) \to 0$ as $n \to \infty$, $d(z, K) = 0$ implies $z \in K$ since set K is a compact set (hence closed set). By the compatibility of F and f, we obtain

$$\begin{aligned} d(fz, Fz) &= \lim_{n\to\infty} d(fz, Fz) \leq \lim_{n\to\infty} [d(fz, ffx_n) + d(ffx_n, Fz)] \\ &\leq \lim_{n\to\infty} [d(fz, ffx_n) + H(fFx_n, Fz)] \\ &\leq \lim_{n\to\infty} [d(fz, ffx_n) + H(fFx_n, Ffx_n) + H(Ffx_n, Fz)] \\ &= d(fz, fz) + H(Fz, Fz) \\ &= 0. \end{aligned}$$

So $fz \in Fz$ since Fz is a compact set (hence closed set).

Conversely, if $fz \in Fz$ for some $z \in X$, then the pair F and f are compatible by Lemma 4. ■

Remark 1 *Theorem 7 is a special occurrence of results obtained by H Kaneko and S Sessa [15] on metric subspace $\mathcal{K}(X) \subset \mathcal{CB}(X)$. We obtain the equivalence between compatibility of the hybrid composite mapping F and f with conditions $fz \in Fz$ for some $z \in X$. Of course, all of the requirements in Theorem 6 should be satisfied.*

The following main result modifies Theorem 5 by substituting compatibility with respect to Hausdorff metric on $\mathcal{K}(X)$ for slight commutativity. This result improves Theorem 6 in finding common fixed point of the hybrid composite mapping.

Theorem 8 *Let (X, d) be a complete metric space, $F : X \to \mathcal{K}(X)$ be a set-valued mapping and $f : X \to X$ be a single-valued mapping satisfying the inequality*

$$H(Fx, Fy) \leq \psi\mathcal{N}(F, f) \tag{8}$$

for all $x, y \in X$, where $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing, right continuous and $\psi(t) < t$, for all $t > 0$. If

(A). F and f is continuous,

(B). $F(X) \subseteq f(X)$,

(C). F and f are compatible, and

(D). There exists $x_0 \in X$ such that $\sup\{H(Fx_n, Fx_1) : n = 0, 1 \cdots\} < +\infty$,

then F and f have a unique common fixed point.

Proof. This proof is the same as Theorem 5 in [10] ■

Example 1 *Let $X = [0, 3]$ with usual metric. Let $F(x) = [0, x^2]$ and $f(x) = 2x^2 - 1$ for each $x \in [0, 3]$. F and f are clearly continuous on X and $F(X) = F([0, 3]) = [0, 9] \subset f([0, 3]) = [-1, 17] = f(X)$. If the sequence $x_n \to 1$, then $Fx_n \to [0, 1] = K$ and $fx_n \to 1 \in K$. We know that $Ffx_n = [0, (2x_n^2 - 1)^2]$ and $fFx_n = [-1, 2x_n^4 - 1]$, we obtain*

$$H(Ffx_n, fFx_n) = |4x_n^4 - 6x_n^2 + 2| \to 0$$

since $x_n \to 1$. *It is clear* $\sup\{H(Fx_n, Fx_1 : n = 0, 1, \cdots\} = 9 < +\infty$. *Since* F *and* f *are continuous, we have*

$$\lim_{x_n \to 1} Ffx_n = F(1) = [0,1] = K, \quad \textit{and} \quad \lim_{x_n \to 1} fx_n = 1 = f(1).$$

On the other hand, $1 = f(1) \in F(1) = K$.

Remark 2 *Simple examples above prove that the condition of the continuity of both mappings* F *and* f *is necessary for Theorem 8 other than the conditions presented in above examples. However, in general, the common fixed point theorem for hybrid composite mappings obtained by researchers required sufficient continuity of either* F *or* f *is continuous. In our opinion, it can be used if the set* K *is a singleton.*

Now we come to the main result a discussion of the existence of fixed point for the derivative of an interval-valued function. In this case, the derivative of the interval-valued function follows the concept introduced by L. Stefanini and B. Bede [25].

Theorem 9 *Let* $F : [a, b] \to \mathcal{I}(\mathbb{R})$ *be continuously gh-differentiable on* (a, b) *such that there exists* $f : [a, b] \to \mathbb{R}$ *and* $fx \in F'_{gh}(x)$ *for all* $x \in [a, b]$ *satisfying the inequality*

$$H(Fx, Fy) \leq \psi \mathcal{N}(F, f)$$

for all $x, y \in [a, b]$*, where* $\psi : [0, \infty) \to [0, \infty)$ *is a non-decreasing, right continuous, and* $\psi(t) < t$*, for all* $t > 0$ *. If*

(A). $F([a, b]) \subseteq f([a, b])$*,*

(B). F *and* f *are compatible, and*

(C). There exists $x_0 \in [a, b]$ *such that* $\sup\{H(Fx_n, Fx_1) : n = 0, 1 \cdots\} < +\infty$*,*

then the gh-derivative F'_{gh} *has a unique fixed point.*

Proof. From hypothesis (D), suppose $H(Fx_s, Fx_t) \leq H(Fx_s, Fx_1) + H(Fx_t, Fx_1) \leq M$ so that

$$\sup\{H(Fx_s, Fx_t) : s, t = 0, 1, 2 \cdots\} = M < +\infty. \tag{9}$$

By Lemma 3, let there be $N \in \mathbb{N}$ such that

$$\psi^N M < \varepsilon \tag{10}$$

for each $\varepsilon > 0$.

Let $x_0 \in [a, b]$ be arbitrary. Since $F([a, b]) \subseteq f([a, b])$, we choose the point $x_1 \in [a, b]$ such that $y_1 = fx_1 \in Fx_0$. In general, if $x_n \in X$ there exists $x_{n+1} \in X$ such that $y_n = fx_n \in Fx_{n-1}$. By applying inequality (8) to term $H(Fx_m, Fx_n)$ we have for $m, n \geq N$:

$$\begin{aligned}
H(Fx_m, Fx_n) &\leq \psi \max\{d(fx_m, fx_n), d(fx_n, Fx_n), d(fx_m, Fx_m), \\
&\qquad \frac{1}{2}[d(fx_n, Fx_m) + d(fx_m, Fx_n)]\} \\
&\leq \psi \max\{H(Fx_{m-1}, Fx_{n-1}), H(Fx_{n-1}, Fx_n), H(Fx_{m-1}, Fx_m), \\
&\qquad \frac{1}{2}[H(Fx_{n-1}, Fx_m) + H(Fx_{m-1}, Fx_n)]\} \\
&\leq \psi \max\{H(Fx_{m-1}, Fx_{n-1}), H(Fx_{n-1}, Fx_n), H(Fx_{m-1}, Fx_m), \\
&\qquad \frac{1}{2}[H(Fx_{n-1}, Fx_{m-1}) + H(Fx_{m-1}, Fx_m)], \\
&\qquad \frac{1}{2}[H(Fx_{m-1}, Fx_{n-1}) + H(Fx_{n-1}, Fx_n)]\} \\
&= \psi \max\{H(Fx_{m-1}, Fx_{n-1}), H(Fx_{n-1}, Fx_n), H(Fx_{m-1}, Fx_m)\}
\end{aligned} \tag{11}$$

By iterating (11) above as much as N times, we deduce for each $m, n > N$ as follows:

$$\begin{aligned}
H(Fx_m, Fx_n) &\leq \psi \max\{H(Fx_r, Fx_s), H(Fx_r, Fx_t), H(Fx_s, Fx_k) : \\
&\qquad m-1 \leq r; t \leq n; n-1 \leq s; k \leq m\} \\
&\leq \psi^2 \max\{H(Fx_r, Fx_s), H(Fx_r, Fx_t), H(Fx_s, Fx_k) : \\
&\qquad m-2 \leq r; t \leq n; n-2 \leq s; k \leq m\} \leq \cdots \\
&\leq \psi^N \max\{H(Fx_r, Fx_s), H(Fx_r, Fx_t), H(Fx_s, Fx_k) : \\
&\qquad m-N \leq r; t \leq n; n-N \leq s; k \leq m\} \\
&\leq \psi^N M < \varepsilon,
\end{aligned} \tag{12}$$

by inequality (10). The sequence of sets $\{Fx_n\}$ is a Cauchy sequence on the complete metric spaces $(\mathcal{I}(\mathbb{R}), H)$. Therefore, it converges to an interval $J \in \mathcal{I}(\mathbb{R})$. It is clear the sequence of single-valued functions $\{fx_n\}$ is also a Cauchy sequence on a complete metric space $\mathbb{R}$ hence converges to a point $z \in \mathbb{R}$. We get

$$|z - J| \leq |z - fx_n| + |fx_n - J| \leq |z - fx_n| + H(Fx_{n-1}, J), \tag{13}$$

as $n \to \infty$, $|z - J| = 0$. This means, $z \in J$ since $J \in \mathcal{I}(\mathbb{R})$. By compatibility of F and f, we obtain

$$\lim_{n\to\infty} H(Ffx_n, fFx_n) = 0. \tag{14}$$

By using inequality (8), we have

$$
\begin{aligned}
H(Ffx_{n+1}, Fx_n) &\le \psi \max\{d(f^2x_{n+1}, fx_n), d(f^2x_{n+1}, Ffx_{n+1}), d(fx_n, Fx_n),\\
&\qquad \frac{1}{2}[d(f^2x_{n+1}, Fx_n) + d(fx_n, Ffx_{n+1})\}\\
&\le \psi \max\{d(fFx_n, fx_n), d(fFx_n, Ffx_{n+1}), d(fx_n, Fx_n),\\
&\qquad \frac{1}{2}[d(fFx_n, Fx_n) + d(fx_n, Ffx_{n+1})\}\\
&\le \psi \max\{d(fFx_n, Ffx_n) + d(Ffx_n, Fx_n) + d(Fx_n, fx_n),\\
&\qquad d(fFx_n, Ffx_n) + d(Ffx_n), fx_n) + d(fx_n, Ffx_{n+1}),\\
&\qquad d(fx_n, Fx_n), \frac{1}{2}[d(fFx_n, Fx_n) + d(fx_n, Ffx_{n+1})\}\\
&\le \psi \max\{d(fFx_n, Ffx_n) + d(Ffx_n, Fx_n) + d(Fx_n, fx_n),\\
&\qquad d(fFx_n, Ffx_n) + d(Ffx_n, fx_n) + d(fx_n, Ffx_{n+1})\}\\
&\le \psi \max\{H(fFx_n, Ffx_n) + H(Ffx_n, Fx_n) + d(Fx_n, fx_n),\\
&\qquad H(fFx_n, Ffx_n) + d(Ffx_n, fx_n) + d(fx_n, Ffx_{n+1})\}\\
&\le \psi \max\{H(fFx_n, Ffx_n) + H(Ffx_n, Fx_n) + d(Fx_n, fx_n),\\
&\qquad H(fFx_n, Ffx_n) + H(Ffx_n, Fx_{n-1}) + H(Fx_{n-1}, Ffx_{n+1})\}
\end{aligned}
$$

since $f^2x_{n+1} \in fFx_n$ and ψ are non-decreasing. By compatibility of F and f, as $n \longrightarrow \infty$ we obtain

$$
\begin{aligned}
H(Fz, J) &\le \psi \max\{0 + H(Fz, J) + d(J, z), 0 + 2H(Fz, J)\}\\
&\le \psi \max\{H(Fz, J), 2H(Fz, J)\}\\
&\le 2\psi H(Fz, J)
\end{aligned}
$$

and hence $H(Fz, J) = 0$ since $\psi(t) < t$ for all $t > 0$. This means $Fz = J$. By compatibility of F and f and Since F is continuously differentiable on $[a, b]$ (hence continuous), we have

$$
\lim_{n\to\infty} H(Fz, fJ) = \lim_{n\to\infty} H(Ffx_n, fFx_n) = 0. \tag{15}
$$

So $Fz = fJ$. Since $z \in J$, $f(z) \in f(J)$, consequently

$$
f(z) \in F(z) = f(J) = J. \tag{16}
$$

Since $f \in F'_{gh}$ and F'_{gh} is continuous, the function f is continuous. Of course, the sequence $\{f^2x_n\}$ converges to the point fz and the sequence of set $\{fFx_n\}$ converges to a set fJ. Since the limit

$$
\lim_{n\to\infty} H(Ffx_n, fJ) \le \lim_{n\to\infty} [H(Ffx_n, fFx_n) + H(fFx_n, fJ)] = 0, \tag{17}
$$

the sequence of set $\{Ffx_n\}$ also converges to a set fJ.

Since $f^2x_{n+1} \in fFx_n$ and using inequality (8), we get

$$
\begin{aligned}
|f^2x_{n+1} - fx_{n+1})| &\le H(fFx_n, Fx_n) \le H(fFx_n, Ffx_n) + H(Ffx_n, Fx_n)\\
&\le H(fFx_n, Ffx_n) + \psi \max\{|f^2x_n - fx_n|, |f^2x_n - Ffx_n|,\\
&\qquad |fx_n - Fx_n|, \frac{1}{2}[|f^2x_n - Fx_n| + |fx_n, Ffx_n|]\}.
\end{aligned}
$$

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As $n \to \infty$, it follows from C (compatibility) and equality (16) that

$$\begin{aligned}
|fz - z| &\le 0 + \psi \max\{|fz - z|, |fz - fJ|, |z - J|, \frac{1}{2}[|fz - J| + |z - Fz|]\} \\
&\le \psi \max\{|fz - z|, 0, 0, \frac{1}{2}[0 + 0]\} \\
&\le \psi |fz - z|,
\end{aligned}$$

which implies $z = fz$. In other words, z is a fixed point of f. It follows from (16) we obtain $z = fz \in F(z)$ and hence z is also a fixed point of F'_{gh} since $z = fz \in F'_{gh}(z)$.

Suppose F and f have another common fixed point u. By inequality (8), we have that

$$\begin{aligned}
H(Fz, Fu) &\le \psi max\{|fz - fu|, |fz - Fz|, |fu - Fu|, \frac{1}{2}[|fz - Fu| + |fu - Fz|]\}. \\
&\le \psi max\{H(Fz, Fu), 0, 0, \frac{1}{2}[H(Fz, Fu) + H(Fu, Fz)]\}. \\
&= \psi max\{H(Fz, Fu)\}. \\
&= \psi H(Fz, Fu)
\end{aligned}$$

which implies $z = u$ (unique) since $d(z, u) \le H(Fz, Fu) = 0$. Hence, z is a unique fixed point of F'_{gh}. This completes the proof. ■

Remark 3 *Usually, common fixed point theorems on the hybrid composite mapping contain at least two mappings in its hypothesis. This study found only one mapping in its hypothesis. While another mapping automatically obtained from the mapping that is given (Theorem 9). In addition, the continuity of function is not stated explicitly in its hypothesis. Thus this result renews some of the results reached by previous researchers.*

Example 2 *Let $X = [0, 2]$ be a usual metric. Let $F(x) = [(x^2 - x), x]$ for all $x \in [0, 2]$. It is clear F is gh-differentiable on $(0, 2)$ with derivative*

$$F'_{gh}(x) = \begin{cases} [(2x - 1), x] & \text{if } 0 \le x \le 1, \\ [x, (2x - 1)] & \text{if } 1 \le x \le 2. \end{cases}$$

We choose $f(x) = (2x - 1) \in F'_{gh}(x)$ for all $x \in [0, 2]$, it appears that

$$FX = F([0, 2]) = [-\frac{1}{4}, 1] \cup [1, 2] = [-\frac{1}{4}, 2] \subset [-1, 2] = f([0, 2]) = fX.$$

This means that the condition in part (A) is satisfied.

If the sequence $x_n \to 1$, then $Fx_n \to [0, 1] = K$ and $fx_n \to 1 \in K$. First, we start with the formula $Ffx_n = [(2x_n - 1)^2 - (2x_n - 1), (2x_n - 1)]$ and $fFx_n = [2(x_n^2 - x_n) - 1, (2x_n - 1)]$, we obtain

$$H(Ffx_n, fFx_n) = |2x_n^2 - 4x_n + 2| \to 0$$

since $x_n \to 1$. Thus F and f are compatible. It is clear that $\sup\{H(Fx_n, Fx_1) : n = 0, 1, \cdots\} = 3 < +\infty$. Since F is continuously gh-differentiable on $(0, 2)$, then implies that F and f are continuous on $(0, 2)$ (see Theorem 1). Hence we have

$$\lim_{x_n \to 1} Ffx_n = F(1) = [0, 1] = K, \quad and \quad \lim_{x_n \to 1} fx_n = 1 = f(1).$$

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Consequently, $1 = f(1) \in F(1) = K$*. Since* $f(x) \in F'_{gh}(x)$ *for all* $x \in [0,2]$*, we obtain* $1 = f(1) \in F'_{gh}(1) = 1$*. In other words, the point* $z = 1$ *is a unique fixed point of* F'_{gh}*.*

If $f \in F$, then we have the following.

Corollary 1 *Let* $F : [a,b] \to \mathcal{I}(\mathbb{R})$ *be a continuously gh-differentiable on* (a,b) *such that there exists* $f : [a,b] \to \mathbb{R}$ *and* $fx \in F(x)$ *for all* $x \in [a,b]$*. If the function* f *and the derivative* F'_{gh} *satisfies the inequality*

$$H(F'_{gh}x, F'_{gh}y) \leq \psi \mathcal{N}(F'_{gh}, f)$$

for all $x, y \in [a,b]$*, where* $\psi : [0,\infty) \to [0,\infty)$ *is a nondecreasing, right continuous, and* $\psi(t) < t$*, for all* $t > 0$ *and satisfies the condition*

(A). $F'([a,b]) \subseteq f([a,b])$*,*

(B). F' *and* f *are compatible, and*

(C). There exists $x_0 \in [a,b]$ *such that* $\sup\{H(Fx_n, Fx_1) : n = 0, 1 \cdots\} < +\infty$*,*

then F*,* F'_{gh}*, and* f *have a unique fixed point.*

Example 3 *Let* $X = [-2,2]$ *with usual metric. Let* $F : [-2,2] \longrightarrow \mathcal{I}(\mathbb{R})$ *with the formula*

$$F(x) = \begin{cases} [x, (x - \sin(x + \frac{1}{2}))] & \text{if } -2 \leq x \leq -\frac{1}{2}, \\ [(x - \sin(x + \frac{1}{2})), x] & \text{if } -\frac{1}{2} \leq x \leq 2. \end{cases}$$

It is clear that F *is gh-differentiable on* $(-2,2)$ *by Theorem 2 with derivative*

$$F'_{gh}(x) = \begin{cases} [1, (1 - \cos x)] & \text{if } -2 \leq x \leq -1,570 \text{ or } 1,570 \leq x \leq 2, \\ [(1 - \cos x), 1] & \text{if } -1,570 \leq x \leq 1,570. \end{cases}$$

If we choose $f(x) = x \in F(x)$ *for all* $x \in [-2,2]$*, then we obtain*

$$F'_{gh}X = F'_{gh}([-2,2]) = [1,(1-\cos 2)] \cup [(1-\cos 2), 1] = [1, 1.4] \subset [-2,2] = f([-2,2]) = fX.$$

This means the condition in part (A) is satisfied. If the sequence $x_n \to 1$*, then* $F'_{gh}(x_n) \to \{1\} = K$ *and* $fx_n \to 1 \in K$*. First, we start with the formula* $F'_{gh}fx_n = F'_{gh}(x_n) = [(1 - \cos x_n), 1]$ *and* $fF'_{gh}(x_n) = f([(1 - \cos x_n), 1]) = [(1 - \cos x_n), 1]$*, we obtain*

$$H(F'_{gh}fx_n, fF'_{gh}x_n) = \max\{|(1 - \cos x_n) - (1 - \cos x_n)|, |1 - 1|\} = 0 \to 0.$$

Thus F *and* f *are compatible. Since* F *is continuously gh-differentiable on* $(-2,2)$*, this implies that* F *and* f *are continuous on* $(-2,2)$ *(see Theorem 1). Hence we have*

$$\lim_{x_n \to 1} F'_{gh}fx_n = F'_{gh}(1) = \{1\}, \quad \text{and} \quad \lim_{x_n \to 1} fx_n = 1 = f(1).$$

Consequently, $1 = f(1) \in F'_{gh}(1) = \{1\}$*. Since* $f(x) \in F(x)$ *for all* $x \in [-2,2]$*, we obtain* $1 = f(1) \in F(1) = 1$*. In other words, the point* $z = 1$ *is a unique common fixed point of* F'_{gh}, F *and* f*.*

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