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Stability and Boundedness Properties of a Rational Exponential Difference Equation

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Abstract

This article aims to discuss, the stability and boundedness character of the solutions of the rational equation of the form

$$y_{t+1} = \frac{\nu\epsilon^{-y_t} + \delta\epsilon^{-y_{t-1}}}{\mu + \nu y_t + \delta y_{t-1}}, \quad t \in N(0). \tag{1}$$

Here, $\epsilon > 1, \nu, \delta, \mu \in (0, \infty)$ and y_0, y_1 are taken as arbitrary non-negative reals and $N(a) = \{a, a+1, a+2, \cdots\}$. Relevant examples are provided to validate our results. The exactness is tested using MATLAB.

Key words: Boundedness, Equilibrium, Global asymptotic stability, Rational Equation.

AMS Classification 2000: 39A22.

1 Introduction

Difference equations involving geometrical and exponential functions have many applications in biology. Growth of a perennial grass, generally rely on the parameters biomass, like litter mass and soil nitrogen, was described by the difference equations

$$B_{t+1} = \mu N \frac{e^{\nu-\delta L_t}}{1+e^{\nu-\delta L_t}}, \; L_{t+1} = \frac{L_t^2}{L_t+d} + \mu s N \frac{e^{\nu-\delta L_t}}{1+e^{\nu-\delta L_t}}, s \in (0,1). \tag{2}$$

Here, the parameters B, L and N denote biomass, litter mass, soil nitrogen respectively, $\nu, \delta, \mu, d > 0$ are fixed. Oscillatory behaviour and chaotic climate of (2) was discussed in [22].

The boundedness, stability and periodic character of the solution obtained from exponential rational equation

$$y_{t+1} = \alpha + \beta y_{t-1} e^{-y_t}, \qquad t \in N(0)$$

was obtained by El-Metwally et all [13]. The population growth rate β and immigration rate α are positive reals with initial conditions y_0 and y_1.

Global asymptotic behavior and Boundedness behavior of the difference equations

$$y_{t+1} = \frac{\alpha + \beta e^{-y_t}}{\gamma + y_{t-1}},$$

and

$$y_{t+1} = \frac{\alpha e^{-(t y_t + (t-s) y_{t-s})}}{\beta + y_t + (t-s) y_{t-s}}, \qquad t \in N(0)$$

have been developed by Ozturk et all [19, 20], where $\alpha > 0$, $\beta > 0$ and $s \in N(1)$ and the y_{-j} for $j = 0, 1, 2, \cdots, s$ can be taken as reals.

For given reals δ, μ, d, s and $0 < a < 1$, the entity of periodical solution is given by the equation

$$y_{t+1} = \frac{\nu y_t^2}{y_t + \delta} + \mu \frac{e^{s-dy_t}}{1+e^{s-dy_t}}.$$

Authors in [3] discussed stability behaviour of the equation

$$y_{t+1} = \frac{\alpha e^{-y_t} + \beta e^{-y_{t-1}}}{\gamma + \alpha y_t + \beta y_{t-1}}, t \in N(0). \tag{3}$$

Here, the initial conditions are taken as arbitrary reals and α, β are positive numbers.

Difference equations are normally discrete version of differential equations

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which preserve symmetries. The role of difference equations are well established in the study of Lie theory. One can refer[10]-[12] for a detailed study on this aspect.

Qualitative properties of certain class of rational difference equations was analyzed in [8, 4]. Stability and bounded conditions of the equation $y_{t+1} = f(y_t)g(y_{t-s})$ was developed in [9] and the qualitative behavior of $y_t = \dfrac{f(y_{t-1}, \ldots, y_{t-s})}{g(y_{t-1}, \ldots, y_{t-s})}, t \in N(0)$ was studied in [2]. For a detailed study on the theory and applications of the relevant topic, one can refer [1], [5]-[7], [14]-[18], [21].

In this paper, we extend the theory to (3) and establish new conditions for stability and other behaviors of the equations (1) for $\epsilon > 1$. MATLAB is used to test the exactness of the behavior of the solutions.

2 Preliminaries

Definition 2.1. *[8] Let $f : I \times I \to I$, $I \in \mathbb{R}$, be a continuous function and $y_0, y_{-1} \in I$ be given values. Then, for*

$$y_{t+1} = g(y_t, y_{t-1}), t \in N(1) \tag{4}$$

$\bar{y}$ is called equilibrium of (4) if $f(\bar{y}, \bar{y}) = \bar{y}$.

Definition 2.2. *[8] Let $p = \dfrac{\partial g}{\partial u}(\bar{y}, \bar{y})$ and $q = \dfrac{\partial g}{\partial v}(\bar{y}, \bar{y})$ denote the partial derivatives of $f(u, v)$ evaluated at an equilibrium $\bar{y}$ of (1). Then the equation*

$$y_{t+1} = py_t + qy_{t-1}, t \in N(0) \tag{5}$$

is called the linearized equation associated with (1) about the equilibrium point $\bar{y}$.

The auxillary equation of (5) is the equation

$$\epsilon^2 - p\epsilon - q = 0 \tag{6}$$

with characteristic roots $\epsilon_{\pm} = \dfrac{p \pm \sqrt{p^2 + 4q}}{2}$.

Theorem 2.3. *(Linearized stability)[8]*

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(i) If two roots of (6) *are in the region* $|\epsilon| < 1$, *then we have an equilibrium* $\bar{y}$ *of* (1) *which is asymptotically and locally stable.*

(ii) If at least one of the roots of (6) is in the region $|\epsilon| < 1$, *then the equilibrium* $\bar{y}$ *of (1) is unstable.*

(iii) The two roots of (6) *will lie in the open region* $|\epsilon| < 1$ *if and only if*

$$|p| < 1 - q < 2. \tag{7}$$

This locally asymptotically stable equilibrium point $\bar{y}$ *is called a sink.*

(iv) The magnitude of one of the two roots of (6) is more than unity if and only if

$$|1 - q| > |p| \text{ and } |q| > 1. \tag{8}$$

This equilibrium point $\bar{y}$ *is called a repeller.*

(v) The absolute value of one of the roots of (6) *is more than unity and the other has absolute value less than unity if and only if*

$$|p| > |1 - q| \text{ and } p^2 + 4q > 0. \tag{9}$$

and this unstable equilibrium point $\bar{y}$ *is called a saddle point.*

(vi) If a root of (6) has absolute value unity, then $|p| = |1 - q|$ *or* $q = -1$ *and* $|p| \leq 2$. *Conversely, if* $|p| = |1-q|$ *or* $q = -1$ *and* $|p| \leq 2$ *then we get one root whose absolute value is equal to unity and hence we get the equilibrium point* $\bar{y}$, *which is non hyperbolic.*

3 Main Results

Here, we discuss the existence, uniqueness and stability of the equation (1).

Theorem 3.1. *The equation* (1) *has a unique positive equilibrium solution.*

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Proof. Let $F(y) = \dfrac{(\nu+\delta)\epsilon^{-y}}{\mu+(\nu+\delta)y} - y$ and $F(0) = \dfrac{\nu+\delta}{\mu} > 0$.
Then, from our assumption, $\lim_{y\to\infty} F(y) = -\infty$.
This gives us that (1) has equilibrium $\bar{y}$.

$$F'(y) = -\frac{(\nu+\delta)\epsilon^{-y}\Big(\mu\ln\epsilon + (\nu+\delta)(y\ln\epsilon+1)\Big)}{(\mu+(\nu+\delta)y)^2} - 1 < 0$$

which implies that F is decreasing. Hence, the equilibrium $\bar{y}$ is unique. □

Theorem 3.2. *Equation* (1) *has the following properties.*
(i) Every positive solution of equation (1) *is bounded.*
(ii) The unique equilibrium point $\bar{y} > 0$ *of the equation* (1) *is bounded.*

Proof. (i) Let $\{y_t\}$ satisfies equation (1) and

$$0 < y_{t+1} = \frac{\nu\epsilon^{-y_t} + \delta\epsilon^{-y_{t-1}}}{\mu+\nu y_t+\delta y_{t-1}} < \frac{\nu+\delta}{c}.$$

Hence (i) is true.
(ii) Similarly

$$0 < \bar{y} = \frac{\nu\epsilon^{-\bar{y}} + \delta\epsilon^{-\bar{y}}}{\mu+\nu\bar{y}+\delta\bar{y}} < \frac{\nu+\delta}{\mu}.$$

Hence (ii) is true. □

Example 3.3. *For* $\nu = 3, \delta = 5, \mu = 2$ *and* $\epsilon = 3, y_{-1} = 4, y_0 = 2.5,$ *we get* $y_{21} = 0.7862 < 4$.

t	0	1	2	3	4	5	6	7
Y(t)	4.0007	2.5000	0.0086	0.2267	2.6777	0.3632	0.1382	1.4022
t	8	9	10	11	12	13	14	15
Y(t)	0.7160	0.2164	0.7436	0.9879	0.3712	0.4575	0.9832	0.5587
t	16	17	18	19	20	21	22	23
Y(t)	0.3866	0.7842	0.7219	0.4291	0.5995	**0.7862**	0.5237	0.5059

Table-1

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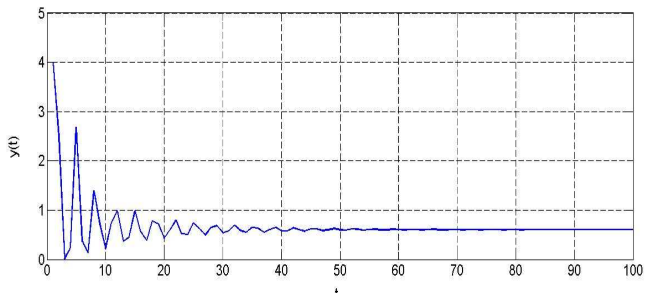

Figure - 1

Theorem 3.4. *Let* $\delta > \nu$ *and*

$$\delta\epsilon^{-\frac{(2\delta\mu-\nu\mu)\ln\epsilon}{\nu(\nu-\delta)(\ln\epsilon+2)-2\delta^2\ln\epsilon}} < \left(\mu + \frac{((\nu+\delta)\ln\epsilon-\delta)(2\delta\mu-\nu\mu)\ln\epsilon}{\nu(\nu-\delta)(\ln\epsilon+2)-2\delta^2\ln\epsilon}\right)\ln\epsilon. \quad (10)$$

Then, the equilibrium point $\bar{y} > 0$ *of (1) is locally asymptotically stable.*

Proof. From Definition 2.2, we get the linearized equation and the characteristic equation associated with (1) about $\bar{y}$ is

$$y_{t+1} + \frac{\nu(\epsilon^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\epsilon} y_t + \frac{\delta(\epsilon^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\epsilon} y_{t-1} = 0, n \in N(0) \quad (11)$$

and

$$\mu^2 + \frac{\nu(\epsilon^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\epsilon}\mu + \frac{\delta(\epsilon^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\epsilon} = 0, \quad (12)$$

respectively. From Theorem 2.3 we obtain

$$\left|-\frac{\nu(\epsilon^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\epsilon}\right| < 1 + \frac{\delta(\epsilon^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\epsilon} < 2. \quad (13)$$

We derive

$$\epsilon^{-\bar{y}} < \frac{(\mu+(\nu+\delta)\bar{y})\ln\epsilon - \delta\bar{y}}{b}, \quad (14)$$

$$\epsilon^{-\bar{y}} < \frac{(\mu+(\nu+\delta)\bar{y})\ln\epsilon}{\nu-\delta} - \bar{y} \quad (15)$$

and

$$\epsilon^{-\bar{y}} < \frac{2(\mu+(\nu+\delta)\bar{y})\ln\epsilon - \nu\bar{y}}{\nu}. \quad (16)$$

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(15)⇒ $$(\nu-\delta)\bar{y}<(\mu+(\nu+\delta)\bar{y})\ln\epsilon+(\delta-\nu)\epsilon^{-\bar{y}}.$$

Substituting (16), we get

$$\bar{y}<\frac{(2\delta\mu-\nu\mu)\ln\epsilon}{\nu(\nu-\delta)(\ln\epsilon+2)-2\delta^2\ln\epsilon}.$$

Again substituting in (14), we get

$$\delta\epsilon^{-\frac{(2\delta\mu-\nu\mu)\ln\epsilon}{\nu(\nu-\delta)(\ln\epsilon+2)-2\delta^2\ln\epsilon}}<\left(\mu+\frac{((\nu+\delta)\ln\epsilon-\delta)(2\delta\mu-\nu\mu)\ln\epsilon}{\nu(\nu-\delta)(\ln\epsilon+2)-2\delta^2\ln\epsilon}\right)\ln\epsilon.$$

□

Example 3.5. *For $\nu=3,\delta=5,\mu=2,\epsilon=3$ and condition (10) of the Theorem 3.4 does not hold, then every positive equilibrium solution of (1) is not locally asymptotically stable.*

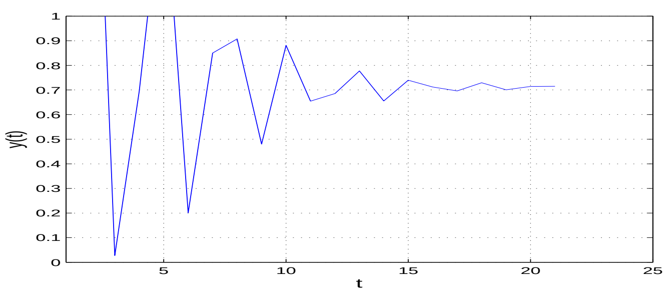

Figure - 2

Theorem 3.6. *(i). Equilibrium solution $\bar{y}$ is nonrepeller.*
(ii). Equilibrium solution $\bar{y}$ is not a saddle point.
(iii). Equilibrium solution $\bar{y}$ is a nonhyperbolic point when $a\leq 2b$.

Proof. (i). From (12) and from Theorem 2.3 (iv), we get

$$\left|\frac{\delta(\epsilon^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\epsilon}\right|>1 \tag{17}$$

and

$$\left|\frac{\nu(\epsilon^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\epsilon}\right|<\left|1-\frac{\delta(\epsilon^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\epsilon}\right|. \tag{18}$$

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Substituting (17) in (18) we get $\nu < 0$ which contradicts our assumption that $\nu > 0$. Thus the equilibrium solution $\bar{y}$ is nonrepeller.

(ii). From (12) and from Theorem 2.3 (v), we get

$$(\mu + (\nu + \delta)\bar{y}) \ln \epsilon > \frac{-\nu^2(\epsilon^{-\bar{y}} + \bar{y})}{4\delta} \tag{19}$$

and

$$\left| \frac{\nu(\epsilon^{-\bar{y}} + \bar{y})}{(\mu + (\nu + \delta)\bar{y}) \ln \epsilon} \right| > \left| 1 - \frac{\delta(\epsilon^{-\bar{y}} + \bar{y})}{(\mu + (\nu + \delta)\bar{y}) \ln \epsilon} \right|. \tag{20}$$

Substituting (19) in (20) we get $4\nu\delta > \nu^2 + 4\delta^2$.

This is not possible since $\nu > 0$ and $\delta > 0$ are constants. Therefore, no equilibrium solution is a saddle point.

(iii). From (12) and from Theorem 2.3 (vi), we get

$$(\mu + (\nu + \delta)\bar{y}) \ln \epsilon = -\delta(\epsilon^{-\bar{y}} + \bar{y}) \tag{21}$$

and

$$\left| \frac{\nu(\epsilon^{-\bar{y}} + \bar{y})}{(\mu + (\nu + \delta)\bar{y}) \ln \epsilon} \right| \leq 2. \tag{22}$$

Substituting (21) in (22) we get $\nu < 2b$. □

Example 3.7. *For $\nu = \delta = \mu$ and for $\epsilon = 2$ we get equilibrium solution ≈ 0.65 which is a nonhyperbolic point.*

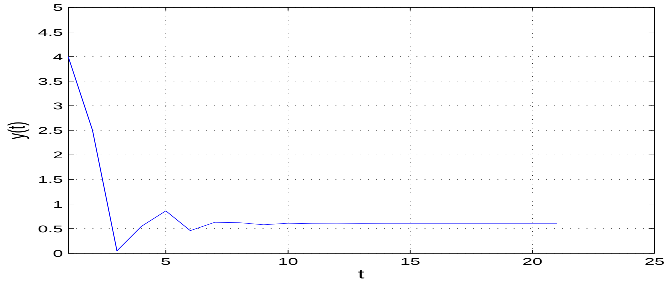

Figure - 3

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4 Conclusion

In this paper, we discuss the different characters like periodic, stability and boundedness of the solutions of the rational exponential difference equation (1). Earlier results exist for similar type of difference equation when the independent variable is raised as a power of e. Here we have generalized the results when the independent variable is raised to any $\epsilon > 1$. Earlier results are available only on the study of the stability of the solutions but, we have analyzed more characters like boundedness and the asymptotic behavior of solutions of the equation (1) which is new in the literature. Suitable examples are provided to validate our results and they are verified with MATLAB.

Author contributions

All authors contributed equally and the all authors have read and agreed to the present version of the manuscript.

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Conflicts of interest

The authors declare no conflict of interest.

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