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ON SOME INEQUALITIES FOR THE GENERALIZED EUCLIDEAN OPERATOR RADIUS

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ABSTRACT. There are many criterion to generalize the concept of numerical radius; one of the most recent interesting generalization is what so called the generalized Euclidean operator radius. Simply, it is the numerical radius of multivariable operators. In this work, several new inequalities, refinements and generalizations are established for this kind of numerical radius.

1. INTRODUCTION

Let $\mathscr{B}(\mathscr{H})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathscr{H}; \langle\cdot,\cdot\rangle)$ with the identity operator $1_{\mathscr{H}}$ in $\mathscr{B}(\mathscr{H})$. When $\mathscr{H} = \mathbb{C}^n$, we identify $\mathscr{B}(\mathscr{H})$ with the algebra $\mathfrak{M}_{n\times n}$ of n-by-n complex matrices. Then, $\mathfrak{M}^+_{n\times n}$ is just the cone of n-by-n positive semidefinite matrices.

For a bounded linear operator T on a Hilbert space $\mathscr{H}$, the numerical range $W(T)$ is the image of the unit sphere of $\mathscr{H}$ under the quadratic form $x \to \langle Tx, x\rangle$ associated with the operator. More precisely,

$$W(T) = \{\langle Tx, x\rangle : x \in \mathscr{H}, \|x\| = 1\}$$

Also, the numerical radius is defined to be

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \sup_{\|x\|=1} |\langle Tx, x\rangle| .$$

We recall that, the usual operator norm of an operator T is defined to be

$$\|T\| = \sup\{\|Tx\| : x \in H, \|x\| = 1\} .$$

It is well known that $w(\cdot)$ defines an operator norm on $\mathscr{B}(\mathscr{H})$ which is equivalent to operator norm $\|\cdot\|$. Moreover, we have

$$\frac{1}{2}\|T\| \le w(T) \le \|T\| \tag{1.1}$$

for any $T \in \mathscr{B}(\mathscr{H})$ and this inequality is sharp.

It is known that $w(A)$ is a norm on $\mathscr{B}(\mathscr{H})$, but it is not unitarily invariant. But the numerical radius norm is weakly unitarily invariant; i.e., $w(U^*TU) = w(T)$ for all unitary U. Also, let us don't miss the chance to mention the important property that $w(T) = w(T^*)$ and $w(T^*T) = w(TT^*)$ for every $T \in \mathscr{B}(\mathscr{H})$.

Denote $|T| = (T^*T)^{1/2}$ the absolute value of the operator T. Then we have

$$w(|T|) = \|T\|.$$

It's well known that the numerical radius is not submultiplicative, but it satisfies

$$w(TS) \le 4w(T)w(S)$$

for all $T, S \in \mathscr{B}(\mathscr{H})$. In particular if T, S commute, then

$$w(TS) \le 2w(T)w(S) .$$

Moreover, if T, S are normal then $w(\cdot)$ is submultiplicative, i.e.,

$$w(TS) \le w(T)w(S)$$

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In 2009, Popsecu [21] introduced the concept of Euclidean operator radius of an n-tuple $\mathbf{T} = (T_1, \cdots, T_n) \in \mathscr{B}(\mathscr{H})^n := \mathscr{B}(\mathscr{H}) \times \cdots \times \mathscr{B}(\mathscr{H})$. Namely, for $T_1, \cdots, T_n \in \mathscr{B}(\mathscr{H})$. The Euclidean operator radius of $T_1, \cdots, T_n$ is defined by

$$w_{\mathrm{e}}(T_1, \cdots, T_n) := \sup_{\|x\|=1} \left(\sum_{i=1}^{n} |\langle T_i x, x\rangle|^2 \right)^{1/2}.$$

Indeed, the Euclidean operator radius was generalized in [24] as follows:

$$w_p(T_1, \cdots, T_n) := \sup_{\|x\|=1} \left(\sum_{i=1}^{n} |\langle T_i x, x\rangle|^p \right)^{1/p}, \qquad p \geq 1.$$

If $p = 1$ then $w_1(T_1, \cdots, T_n)$ (also, it is denoted by $w_R(T_1, \cdots, T_n)$) is called the Rhombic numerical radius which have been studied in [5]. In an interesting case, $w_1(C, \cdots, C) = n \cdot w(C)$.

The Crawford number is defined to be

$$c(T) = \inf\{|\lambda| : \lambda \in W(T)\} = \inf_{\|x\|=1} |\langle Tx, x\rangle|.$$

Consequently, we define the generalized Crawford number as:

$$c_p(T_1, \cdots, T_n) := \inf_{\|x\|=1} \left(\sum_{i=1}^{n} |\langle T_i x, x\rangle|^p \right)^{1/p}, \qquad p \geq 1.$$

In case $p = 1$, the generalized Crawford number is called the Rhombic Crawford number and is denoted by $c_R(T_1, \cdots, T_n)$.

We note that in case $p = \infty$, the generalized Euclidean operator radius is defined as:

$$\begin{aligned} w_\infty(T_1, \cdots, T_n) &:= \sup_{\|x\|=1} \sum_{i=1}^{n} |\langle T_i x, x\rangle| - \inf_{\|x\|=1} \sum_{i=1}^{n} |\langle T_i x, x\rangle| \\ &= w_R(T_1, \cdots, T_n) - c_R(T_1, \cdots, T_n). \end{aligned}$$

Thus, the inequality

$$w_\infty(T_1, \cdots, T_n) \leq w_p(T_1, \cdots, T_n) \leq w_R(T_1, \cdots, T_n) \tag{1.2}$$

for all $p \in (1, \infty)$. This fact follows by Jensen's inequality applied for the function $h(p) = w_p(T_1, \cdots, T_n)$, which is log-convex and decreasing for all $p > 1$.

On the other hand, by employing the Jensen's inequality

$$\left(\frac{1}{n} \sum_{k=1}^{n} a_k \right)^p \leq \frac{1}{n} \sum_{k=1}^{n} a_k^p,$$

which holds for every finite positive sequence of real numbers $(a_k)_{k=1}^n$ and $p \geq 1$; by setting $a_k = |\langle T_k x, x\rangle|$ for all $(k = 1, 2, \cdots, n)$, we get

$$\sum_{k=1}^{n} |\langle T_k x, x\rangle| \leq n^{1-\frac{1}{p}} \left(\sum_{k=1}^{n} |\langle T_k x, x\rangle|^p \right)^{\frac{1}{p}}.$$

Taking the supremum over all unit vector $x \in \mathscr{H}$, one could get

$$w_R(T_1, \cdots, T_n) \leq n^{1-\frac{1}{p}} w_p(T_1, \cdots, T_n). \tag{1.3}$$

Combining the inequalities (1.2) and (1.3) we get

$$w_\infty(T_1, \cdots, T_n) \leq w_p(T_1, \cdots, T_n) \leq w_R(T_1, \cdots, T_n) \leq n^{1-\frac{1}{p}} w_p(T_1, \cdots, T_n). \tag{1.4}$$

More generally, in the power mean inequality

$$\left(\frac{1}{n} \sum_{k=1}^{n} a_k^p \right)^{\frac{1}{p}} \leq \left(\frac{1}{n} \sum_{k=1}^{n} a_k^q \right)^{\frac{1}{q}}, \qquad \forall p \leq q$$

if one chooses $a_k = |\langle T_k x, x\rangle|$ for all $(k = 1, 2, \cdots, n)$, then we have

$$\left(\frac{1}{n}\sum_{k=1}^{n}|\langle T_k x, x\rangle|^p\right)^{\frac{1}{p}} \le \left(\frac{1}{n}\sum_{k=1}^{n}|\langle T_k x, x\rangle|^q\right)^{\frac{1}{q}}.$$

Taking the supremum over all unit vector $x \in \mathscr{H}$, we get

$$w_p(T_1, \cdots, T_n) \le n^{\frac{1}{p}-\frac{1}{q}} w_q(T_1, \cdots, T_n), \qquad \forall q \ge p \ge 1. \tag{1.5}$$

Indeed, one can refine (1.3) by applying the Jensen's inequality

$$\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p \le \frac{1}{n}\sum_{k=1}^{n}a_k^p - \frac{1}{n}\sum_{k=1}^{n}\left|a_k - \frac{1}{n}\sum_{j=1}^{n}a_j\right|^p \qquad p \ge 2 \tag{1.6}$$

which obtained from more general result for superquadratic functions [1].

Thus, by setting $a_k = |\langle T_k x, x\rangle|$ in (1.6) we get

$$\begin{aligned}\left(\sum_{k=1}^{n}|\langle T_k x, x\rangle|\right)^p &\le n^{p-1}\sum_{k=1}^{n}|\langle T_k x, x\rangle|^p - n^{p-1}\sum_{k=1}^{n}\left||\langle T_k x, x\rangle| - \frac{1}{n}\sum_{j=1}^{n}|\langle T_j x, x\rangle|\right|^p \\ &\le n^{p-1}\sum_{k=1}^{n}|\langle T_k x, x\rangle|^p - n^{p-1}\sum_{k=1}^{n}\left||\langle T_k x, x\rangle| - \frac{1}{n}\sup_{\|x\|=1}\sum_{j=1}^{n}|\langle T_j x, x\rangle|\right|^p.\end{aligned}$$

Taking the supremum again over all unit vector $x \in \mathscr{H}$, we get

$$\begin{aligned}&\sup_{\|x\|=1}\left(\sum_{k=1}^{n}|\langle T_k x, x\rangle|\right)^p \\ &\le \sup_{\|x\|=1}\left\{n^{p-1}\sum_{k=1}^{n}|\langle T_k x, x\rangle|^p - n^{p-1}\sum_{k=1}^{n}\left||\langle T_k x, x\rangle| - \frac{1}{n}\sup_{\|x\|=1}\sum_{j=1}^{n}|\langle T_j x, x\rangle|\right|^p\right\} \\ &\le n^{p-1}\sup_{\|x\|=1}\sum_{k=1}^{n}|\langle T_k x, x\rangle|^p - n^{p-1}\inf_{\|x\|=1}\sum_{k=1}^{n}\left||\langle T_k x, x\rangle| - \frac{1}{n}\sup_{\|x\|=1}\sum_{j=1}^{n}|\langle T_j x, x\rangle|\right|^p \\ &= n^{p-1}w_p^p(T_1, \cdots, T_n) - n^{p-1}\inf_{\|x\|=1}\sum_{k=1}^{n}\left||\langle T_k x, x\rangle| - \frac{1}{n}w_R(T_1, \cdots, T_n)\right|^p,\end{aligned}$$

which gives

$$w_R^p(T_1, \cdots, T_n) \le n^{p-1}w_p^p(T_1, \cdots, T_n) - n^{p-1}\inf_{\|x\|=1}\sum_{k=1}^{n}\left||\langle T_k x, x\rangle| - \frac{1}{n}w_R(T_1, \cdots, T_n)\right|^p.$$

which refine the right hand side of (1.4). Clearly, all above mentioned inequalities generalize and refine some inequalities obtained in [20]. For recent inequalities, counterparts, refinements and other related properties concerning the generalized Euclidean operator radius the reader my refer to [5], [9] ,[12],[13], [21], [23] and[24].

2. Bounds for the generalized Euclidean operator radius

Lemma 1. *We have*

(1) *The Power-Mean inequality*

$$a^{\alpha}b^{1-\alpha} \le \alpha a + (1-\alpha)b \le (\alpha a^p + (1-\alpha)b^p)^{\frac{1}{p}} \tag{2.1}$$

for all $\alpha \in [0,1]$, $a, b \ge 0$ *and* $p \ge 1$.

(2) *The Power-Young inequality*

$$ab \le \frac{a^{\alpha}}{\alpha} + \frac{b^{\beta}}{\beta} \le \left(\frac{a^{p\alpha}}{\alpha} + \frac{b^{p\beta}}{\beta}\right)^{\frac{1}{p}} \tag{2.2}$$

for all $a, b \geq 0$ *and* $\alpha, \beta > 1$ *with* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ *and all* $p \geq 1$.

Lemma 2. (The McCarty inequality). *Let* $A \in \mathscr{B}(\mathscr{H})^+$, *then*

$$\langle Ax, x\rangle^p \leq \langle A^p x, x\rangle, \qquad p \geq 1 \tag{2.3}$$

for any unit vector $x \in \mathscr{H}$

Lemma 3. *If* $a, b > 0$, *and* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then for* $m = 1, 2, 3, \ldots,$

$$(a^{\frac{1}{p}} b^{\frac{1}{q}})^m + r_0^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq \left(\frac{a^r}{p} + \frac{b^r}{q}\right)^{\frac{m}{r}}, \ r \geq 1, \tag{2.4}$$

where $r_0 = min\left\{\frac{1}{p}, \frac{1}{q}\right\}$. *In particular, if* $p = q = 2$, *then*

$$(a^{\frac{1}{2}} b^{\frac{1}{2}})^m + \frac{1}{2^m}(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq 2^{\frac{-m}{r}} (a^r + b^r)^{\frac{m}{r}}.$$

For $m = 1$

$$(a^{\frac{1}{2}} b^{\frac{1}{2}}) + \frac{1}{2}(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2 \leq 2^{\frac{-1}{r}} (a^r + b^r)^{\frac{1}{r}}.$$

In 1994, Furuta [11] proved the following generalization of Kato's inequality (1.3)

$$\left|\left\langle T\left|T\right|^{\alpha+\beta-1} x, y\right\rangle\right|^2 \leq \left\langle \left|T\right|^{2\alpha} x, x\right\rangle \left\langle \left|T\right|^{2\beta} y, y\right\rangle \tag{2.5}$$

for any $x, y \in \mathscr{H}$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \geq 1$.

The inequality (2.5) was generalized for any $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$ by Dragomir in [8]. Indeed, as noted by Dragomir the condition $\alpha, \beta \in [0, 1]$ was assumed by Furuta to fit with the Heinz–Kato inequality, which reads:

$$|\langle Tx, y\rangle| \leq \|A^\alpha x\| \left\|B^{1-\alpha} y\right\|$$

for any $x, y \in \mathscr{H}$ and $\alpha \in [0, 1]$ where A and B are positive operators such that $\|Tx\| \leq \|Ax\|$ and $\|T^* y\| \leq \|By\|$ for any $x, y \in \mathscr{H}$.

In the same work [8], Dragomir provides a useful extension of Furuta's inequality, as follows:

$$|\langle DCBAx, y\rangle|^2 \leq \left\langle A^* \left|B\right|^2 Ax, x\right\rangle \left\langle D\left|C^*\right|^2 D^* y, y\right\rangle \tag{2.6}$$

for any $A, B, C, D \in \mathscr{B}(\mathscr{H})$ and any vectors $x, y \in \mathscr{H}$. The equality in (2.6) holds iff the vectors BAx and $C^* D^* y$ are linearly dependent in $\mathscr{H}$. For other closely related version of Kato's inequality see [2], [14], [15], [18], [19] and [22].

2.1. **Basic properties of the generalized Euclidean operator radius.** Moslehian et al. [20], mention without proofs the following properties of the generalized Euclidean operator radius:

1. $w_p(T_1, \cdots, T_n) = 0$ if and only if $T_k = 0$ for each $k = 1, \cdots, n$.
2. $w_p(\lambda T_1, \cdots, \lambda T_n) = |\lambda| \, w_p(T_1, \cdots, T_n)$.
3. $w_p(A_1 + B_1, \cdots, A_n + B_n) \leq w_p(A_1, \cdots, A_n) + w_p(B_1, \cdots, B_n)$.
4. $w_p(X^* T_1 X, \cdots, X^* T_n X) = \|X\| \, w_p(T_1, \cdots, T_n)$.

for every $T_k, A_k, B_k, X \in \mathscr{B}(\mathscr{H})$ $(1 \leq k \leq n)$ and every scalar $\lambda \in \mathbb{C}$.

Despite of the authors in [20], mentioned the above basic properties of the generalized Euclidean operator radius, but it seems they missed some other important properties, rather than they left these properties without proof. Sometimes, it's nice to elaborate the proof of these elementary facts. Because of that we are going to give a proof of each property. Clearly, the first two properites follows from the definition of the generalized Euclidean operator radius. In what follows, and as the classical sense we have the following properties:

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Let $T_1, \cdots, T_n, U \in \mathscr{B}(\mathscr{H})$ such that U is a unitary. Then, the following properties of the generalized Euclidean operator radius holds.

(1) The generalized Euclidean operator radius is weakly unitarily invariant i.e.,

$$w_p\left(U^*T_1U, \cdots, U^*T_nU\right) = w_p\left(T_1, \cdots, T_n\right).$$

(2) $w_p\left(T_1, \cdots, T_n\right) = w_p\left(T_1^*, \cdots, T_n^*\right)$.

(3) $w_p\left(T_1^*T_1, \cdots, T_n^*T_n\right) = w_p\left(T_1T_1^*, \cdots, T_nT_n^*\right)$.

Proof. (1) The first property follows since

$$\begin{aligned} w_p\left(U_1^*T_1U_1, \cdots, U_n^*T_nU_n\right) &:= \sup_{\|x\|=1}\left(\sum_{i=1}^n \left|\langle U_i^*T_iU_ix, x\rangle\right|^p\right)^{1/p} \\ &= \sup_{\|x\|=1}\left(\sum_{i=1}^n \left|\langle T_iU_ix, U_ix\rangle\right|^p\right)^{1/p} \\ &= \sup_{\|y\|=1}\left(\sum_{i=1}^n \left|\langle T_iy, y\rangle\right|^p\right)^{1/p} \\ &= w_p\left(T_1, \cdots, T_n\right). \end{aligned}$$

(2) By the definition of the generalized Euclidean operator radius we have

$$\begin{aligned} w_p\left(T_1, \cdots, T_n\right) := \sup_{\|x\|=1}\left(\sum_{i=1}^n \left|\langle T_ix, x\rangle\right|^p\right)^{1/p} &= \sup_{\|x\|=1}\left(\sum_{i=1}^n \left|\langle x, T_i^*x\rangle\right|^p\right)^{1/p} \\ &= w_p\left(T_1^*, \cdots, T_n^*\right). \end{aligned}$$

(3) Similarly, by definition we have

$$\begin{aligned} w_p\left(T_1T_1^*, \cdots, T_nT_n^*\right) := \sup_{\|x\|=1}\left(\sum_{i=1}^n \left|\langle T_iT_i^*x, x\rangle\right|^p\right)^{1/p} &= \sup_{\|x\|=1}\left(\sum_{i=1}^n \left|\langle x, T_iT_i^*x\rangle\right|^p\right)^{1/p} \\ &= w_p\left(T_1^*T_1, \cdots, T_n^*T_n\right). \end{aligned}$$

(4) Finally, employing the classical Minlowski inequality, i.e.,
we get

$$\begin{aligned} w_p\left(A_1+B_1, \cdots, A_n+B_n\right) &= \left(\sum_{i=}^n \left|\langle\left(A_i+B_i\right)x, x\rangle\right|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{i=}^n \left(\left|\langle A_ix, x\rangle\right| + \left|\langle B_ix, x\rangle\right|\right)^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{i=}^n \left|\langle A_ix, x\rangle\right|^p\right)^{\frac{1}{p}} + \left(\sum_{i=}^n \left|\langle B_ix, x\rangle\right|^p\right)^{\frac{1}{p}} \\ &= w_p\left(A_1, \cdots, A_n\right) + w_p\left(B_1, \cdots, B_n\right). \end{aligned}$$

which proves the last property.

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It remains to prove $w_p(X^*T_1X,\cdots,X^*T_nX)=\|X\|\,w_p(T_1,\cdots,T_n)$. Form the definition of the generalized Euclidean operator radius, we have

$$\begin{aligned}
w_p(X^*T_1X,\cdots,X^*T_nX) &:= \sup_{\|x\|=1}\left(\sum_{i=1}^n |\langle X^*T_iXx,x\rangle|^p\right)^{1/p}\\
&= \sup_{\|x\|=1}\left(\sum_{i=1}^n |\langle T_iXx,Xx\rangle|^p\right)^{1/p}\\
&\le \sup_{\|x\|=1}\left(\sum_{i=1}^n \|X\|^2|\langle T_ix,x\rangle|^p\right)^{1/p}\\
&= \|X\|^2\sup_{\|x\|=1}\left(\sum_{i=1}^n |\langle T_ix,x\rangle|^p\right)^{1/p}\\
&= \|X\|^2 w_p(T_1^*,\cdots,T_n^*).
\end{aligned}$$

as required. □

Proposition 1. *Let $T_1,\cdots,T_n\in\mathscr{B}(\mathscr{H})$ and f,g are nonnegative continuous functions defined on $[0,\infty)$ satisfying that $f(t)g(t)=t$ $(t\ge 0)$. Then,*

$$w_r^r(T_1,\cdots,T_n)\le w_p(f^r(|T_1|),\cdots,f^r(|T_n|))\,w_q(g^r(|T_1^*|),\cdots,g^r(|T_n^*|)) \tag{2.7}$$

for all $r\ge 2$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof.

$$\begin{aligned}
|\langle T_ix,x\rangle|^r &\le \|f(|T_i|)x\|^r\,\|g(|T_i^*|)x\|^r\\
&= \left\langle f^2(|T_i|)x,x\right\rangle^{\frac{r}{2}}\left\langle g^2(|T_i^*|)x,x\right\rangle^{\frac{r}{2}}\\
&\le \langle f^r(|T_i|)x,x\rangle\langle g^r(|T_i^*|)x,x\rangle \qquad \text{(by the McCarthy inequality)}
\end{aligned}$$

Taking the sum over all i from 1 to n we get

$$\begin{aligned}
\sum_{i=1}^n|\langle T_ix,x\rangle|^r &\le \sum_{I=1}^n\langle f^r(|T_i|)x,x\rangle\langle g^r(|T_i^*|)x,x\rangle\\
&\le \left(\sum_{I=1}^n\langle f^r(|T_i|)x,x\rangle^p\right)^{\frac{1}{p}}\left(\sum_{I=1}^n\langle g^r(|T_i^*|)x,x\rangle^q\right)^{\frac{1}{q}} \qquad \text{(by the Hölder inequality)}
\end{aligned}$$

Taking the supremum over all vectors $x\in\mathscr{H}$ such that $\|x\|=1$, we get the desired result. □

Corollary 1. *Let $T_1,\cdots,T_n\in\mathscr{B}(\mathscr{H})$ and f,g are nonnegative continuous functions defined on $[0,\infty)$ satisfying that $f(t)g(t)=t$ $(t\ge 0)$. Then,*

$$w_r^r(T_1,\cdots,T_n)\le w_{\mathrm{e}}(f^r(|T_1|),\cdots,f^r(|T_n|))\,w_{\mathrm{e}}(g^r(|T_1^*|),\cdots,g^r(|T_n^*|)) \tag{2.8}$$

for all $r\ge 2$.

Proof. Setting $p=q=2$ in (2.7). □

Proposition 2. *Let $A_i,B_i,C_i,D_i\in\mathscr{B}(\mathscr{H})$ $(i=1,\cdots,n)$. Then,*

$$\begin{aligned}
&w_e^2(D_1C_1B_1A_1,\cdots,D_nC_nB_nA_n)\\
&\qquad\le w_p\left(A_1^*|B_1|^2A_1,\cdots,A_n^*|B_n|^2A_n\right)w_q\left(D_1|C_1^*|^2D_1^*,\cdots,D_n|C_n^*|^2D_n^*\right)
\end{aligned} \tag{2.9}$$

Proof. Let $y = x$ in (2.6), we get

$$\begin{aligned}
&\sup_{\|x\|=1} \sum_{i=1}^{n} \left|\langle D_i C_i B_i A_i x, x\rangle\right|^2 \\
&\le \sup_{\|x\|=1} \sum_{i=1}^{n} \left\langle A_i^* \left|B_i\right|^2 A_i x, x \right\rangle \left\langle D_i \left|C_i^*\right|^2 D_i^* x, x \right\rangle \\
&\le \sup_{\|x\|=1} \left(\sum_{i=1}^{n} \left\langle A_i^* \left|B_i\right|^2 A_i x, x \right\rangle^p \right)^{\frac{1}{p}} \sup_{\|x\|=1} \left(\sum_{i=1}^{n} \left\langle D_i \left|C_i^*\right|^2 D_i^* x, x \right\rangle^q \right)^{\frac{1}{q}} \\
&= w_p \left(A_1^* \left|B_1\right|^2 A_1, \cdots, A_n^* \left|B_n\right|^2 A_n \right) w_q \left(D_1 \left|C_1^*\right|^2 D_1^*, \cdots, D_n \left|C_n^*\right|^2 D_n^* \right),
\end{aligned}$$

which is gives the desired result. □

Corollary 2. *Let $B_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$. Then,*

$$w_e^2 \left(B_1^2, \cdots, B_n^2\right) \le w_p \left(\left|B_1\right|^2, \cdots, \left|B_n\right|^2\right) w_q \left(\left|B_1\right|^2, \cdots, \left|B_n\right|^2\right) \tag{2.10}$$

for all $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. In particular, for $p = q = 2$ we have

$$w_{\mathrm{e}} \left(B_1^2, \cdots, B_n^2\right) \le w_{\mathrm{e}} \left(\left|B_1\right|^2, \cdots, \left|B_n\right|^2\right) \tag{2.11}$$

Proof. Setting $A_i = U_i$, $D_i = U_i^*$ (U_i are unitaries for all $i = 1, \cdots, n$) and $C_i = B_i$ in the previous result. Then

$$w_{\mathrm{e}}^2 \left(U_1^* B_1^2 U_1, \cdots, U_n^* B_n^2 U_n\right) \le w_p \left(U_1^* \left|B_1\right|^2 U_1, \cdots, U_n^* \left|B_n\right|^2 U_n\right) w_q \left(U_1^* \left|B_1^*\right|^2 U_1, \cdots, U_n^* \left|B_n^*\right|^2 U_n\right).$$

Since $w_p(\cdot)$ is weakly unitarily invariant and

$$w_q \left(\left|B_1^*\right|^2, \cdots, \left|B_n^*\right|^2\right) = w_q \left(\left|B_1\right|^2, \cdots, \left|B_n\right|^2\right).$$

Thus, the desired result is obtained. □

Corollary 3. *Let $T_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$, $\alpha, \beta \ge 0$ such that $\alpha + \beta \ge 1$. Then,*

$$w_{\mathrm{e}}^2 \left(T_1 \left|T_1\right|^{\alpha+\beta-1}, \cdots, T_n \left|T_n\right|^{\alpha+\beta-1}\right) \le w_p \left(\left|T_1\right|^{2\alpha}, \cdots, \left|T_n\right|^{2\alpha}\right) w_q \left(\left|T_1^*\right|^{2\beta}, \cdots, \left|T_n^*\right|^{2\beta}\right) \tag{2.12}$$

for all $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let U_i be unitaries for all $i = 1, \cdots, n$, setting $D_i = U_i$, $B = 1_{\mathscr{H}}$, $C = \left|T_i\right|^{\beta}$ and $A_i = \left|T_i\right|^{\alpha}$ for all $\alpha, \beta \ge 0$ such that $\alpha + \beta \ge 1$ in (2.9), then we have

$$D_i C_i B_i A_i = U_i \left|T_i\right|^{\beta} \left|T_i\right|^{\alpha} = U_i \left|T_i\right| \left|T_i\right|^{\alpha+\beta-1} = T_i \left|T_i\right|^{\alpha+\beta-1},$$

also, we have $A_i^* \left|B_i\right|^2 A_i = \left|T_i\right|^{2\alpha}$ and $D_i \left|C_i^*\right|^2 D_i^* = U_i \left|T_i\right|^{2\beta} U_i^* = \left|T_i\right|^{2\beta}$ for all $i = 1, \cdots, n$. □

Corollary 4. *Let $T_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$. Then,*

$$w_{\mathrm{e}}^2 \left(T_1, \cdots, T_n\right) \le w_p \left(\left|T_1\right|, \cdots, \left|T_n\right|\right) w_q \left(\left|T_1^*\right|, \cdots, \left|T_n^*\right|\right) \tag{2.13}$$

for all $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Setting $\alpha = \beta = \frac{1}{2}$ in (2.12). □

Corollary 5. *Let $T_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$, $\alpha, \beta \ge 0$ such that $\alpha + \beta \ge 1$. Then,*

$$w_{\mathrm{e}}^2 \left(T_1 \left|T_1\right|, \cdots, T_n \left|T_n\right|\right) \le w_p \left(\left|T_1\right|^2, \cdots, \left|T_n\right|^2\right) w_q \left(\left|T_1\right|^2, \cdots, \left|T_n\right|^2\right) \tag{2.14}$$

for all $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. In particular, for $p = q = 2$ we have

$$w_{\mathrm{e}}^2 \left(T_1 \left|T_1\right|, \cdots, T_n \left|T_n\right|\right) \le w_{\mathrm{e}}^2 \left(\left|T_1\right|^2, \cdots, \left|T_n\right|^2\right) \tag{2.15}$$

Proof. Setting $\alpha = \beta = 1$ in (2.12) and use the properties of $w_p(\cdot)$. □

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2.2. **Inequalities for the generalized Euclidean operator radius.**

Theorem 1. *Let* $A_i, B_i, C_i, D_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$*. Then,*

$$w_R(D_1C_1B_1A_1, \cdots, D_nC_nB_nA_n) \le \left\| \sum_{i=1}^{n} \frac{1}{2p}\left[\left(A_i^* |B_i|^2 A_i\right)^{(1-\gamma)pr} + \left(A_i^* |B_i|^2 A_i\right)^{\gamma pr}\right] + \frac{1}{2q}\left[\left(D_i |C_i^*|^2 D_i^*\right)^{\gamma qr} + \left(D_i |C_i^*|^2 D_i^*\right)^{(1-\gamma)qr}\right]\right\| \tag{2.16}$$

for all $r \ge 1$, $p, q > 1$ *and* $\gamma \in [0,1]$ *such that* $p\gamma \ge 2$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $y = x$ in (2.6), then we have

$$\begin{aligned}
&\sum_{i=1}^{n} |\langle D_iC_iB_iA_i x, x\rangle| \\
&= \sum_{i=1}^{n} \left\langle A_i^* |B_i|^2 A_i x, x\right\rangle^{\frac{1}{2}} \left\langle D_i |C_i^*|^2 D_i^* x, x\right\rangle^{\frac{1}{2}} \\
&\le \frac{1}{2}\sum_{i=1}^{n}\left[\left\langle A_i^* |B_i|^2 A_i x, x\right\rangle^{1-\gamma} \left\langle D_i |C_i^*|^2 D_i^* x, x\right\rangle^{\gamma}\right. \qquad \left(\text{since } \sqrt{ab} \le \frac{a^\gamma b^{1-\gamma} + a^{1-\gamma}b^\gamma}{2}\right) \\
&\qquad \left. + \left\langle A_i^* |B_i|^2 A_i x, x\right\rangle^{\gamma} \left\langle D_i |C_i^*|^2 D_i^* x, x\right\rangle^{1-\gamma}\right] \\
&\le \frac{1}{2}\sum_{i=1}^{n}\left\{\left(\left\langle A_i^* |B_i|^2 A_i x, x\right\rangle^{(1-\gamma)p} + \left\langle A_i^* |B_i|^2 A_i x, x\right\rangle^{\gamma p}\right)^{\frac{1}{p}}\right. \qquad (\text{ by Hölder inequality}) \\
&\qquad \left. \times \left(\left\langle D_i |C_i^*|^2 D_i^* x, x\right\rangle^{\gamma q} + \left\langle D_i |C_i^*|^2 D_i^* x, x\right\rangle^{(1-\gamma)q}\right)^{\frac{1}{q}}\right\} \\
&\le \frac{1}{2p}\sum_{i=1}^{n}\left(\left\langle \left(A_i^* |B_i|^2 A_i\right)^{(1-\gamma)p} x, x\right\rangle + \left\langle \left(A_i^* |B_i|^2 A_i\right)^{\gamma p} x, x\right\rangle\right) \qquad (\text{ by AM-GM}) \\
&\qquad + \frac{1}{2q}\sum_{i=1}^{n}\left(\left\langle \left(D_i |C_i^*|^2 D_i^*\right)^{\gamma q} x, x\right\rangle + \left\langle \left(D_i |C_i^*|^2 D_i^*\right)^{(1-\gamma)q} x, x\right\rangle\right) \\
&= \sum_{i=1}^{n}\left\langle\left\{\frac{1}{2p}\left[\left(A_i^* |B_i|^2 A_i\right)^{(1-\gamma)p} + \left(A_i^* |B_i|^2 A_i\right)^{\gamma p}\right]\right.\right. \\
&\qquad \left.\left. + \frac{1}{2q}\left[\left(D_i |C_i^*|^2 D_i^*\right)^{\gamma q} + \left(D_i |C_i^*|^2 D_i^*\right)^{(1-\gamma)q}\right]\right\} x, x\right\rangle.
\end{aligned}$$

Taking the supremum over all unit vector $x \in \mathscr{H}$ we get the required result. □

Corollary 6. *Let* $A_i, B_i, C_i, D_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$*. Then,*

$$\begin{aligned}
&w_R(D_1C_1B_1A_1, \cdots, D_nC_nB_nA_n) \\
&\le \left\| \sum_{i=1}^{n}\left\{\frac{1}{4}\left[\left(A_i^* |B_i|^2 A_i\right)^{2(1-\gamma)} + \left(A_i^* |B_i|^2 A_i\right)^{2\gamma}\right] + \left[\left(D_i |C_i^*|^2 D_i^*\right)^{2\gamma} + \left(D_i |C_i^*|^2 D_i^*\right)^{2(1-\gamma)}\right]\right\}\right\|
\end{aligned} \tag{2.17}$$

for all $\gamma \in [0,1]$

Proof. Setting $p = q = 2$ and $r = 1$ in (2.16). □

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Corollary 7. *Let $T_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$. Then,*

$$w_R\left(T_1 |T_1|^{\alpha+\beta-1}, \cdots, T_n |T_n|^{\alpha+\beta-1}\right) \leq \left\| \sum_{i=1}^{n} \left\{ \frac{1}{2p}\left[|T_i|^{2\alpha(1-\gamma)pr} + |T_i|^{2\alpha\gamma pr}\right] + \frac{1}{2q}\left[|T_i^*|^{2\beta\gamma qr} + |T_i^*|^{2\beta(1-\gamma)qr}\right] \right\} \right\| \tag{2.18}$$

for all $r \geq 1$, $p, q > 1$ and $\gamma \in [0, 1]$ such that $p\gamma \geq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, we have

$$w_R\left(T_1 |T_1|^{\alpha+\beta-1}, \cdots, T_n |T_n|^{\alpha+\beta-1}\right) \leq \frac{1}{4}\left\| \sum_{i=1}^{n} \left\{ \left[|T_i|^{4\alpha(1-\gamma)r} + |T_i|^{4\alpha\gamma r}\right] + \left[|T_i^*|^{4\beta\gamma r} + |T_i^*|^{4\beta(1-\gamma)r}\right] \right\} \right\| \tag{2.19}$$

Proof. The proof is similar to the inequality (2.12) by employing (2.16). □

Remark 1. *Setting $\gamma = 1$, $\alpha = \beta = \frac{1}{2}$ and $r = 2$ in (2.19), so that we have*

$$w_R(T_1, \cdots, T_n) \leq \frac{1}{4}\left\| \sum_{i=1}^{n} \left\{ \left[1_{\mathscr{H}} + |T_i|^4\right] + \left[|T_i^*|^4 + 1_{\mathscr{H}}\right] \right\} \right\| \tag{2.20}$$

Corollary 8. *Let $T_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$. Then,*

$$w_R\left(T_1 |T_1|^{\alpha+\beta-1}, \cdots, T_n |T_n|^{\alpha+\beta-1}\right) \leq \frac{1}{4}\left\| \sum_{i=1}^{n} \left[|T_i|^{\frac{8}{3}\alpha} + |T_i|^{\frac{4}{3}\alpha} + |T_i^*|^{\frac{8}{3}\beta} + |T_i^*|^{\frac{4}{3}\beta}\right] \right\| \tag{2.21}$$

Proof. Setting $p = q = 2$, $r = 1$ and $\gamma = \frac{1}{3}$ in (2.19). □

Corollary 9. *Let $T_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$. Then,*

$$w_R\left(T_1 |T_1|^{\frac{1}{2}}, \cdots, T_n |T_n|^{\frac{1}{2}}\right) \leq \frac{1}{4}\left\| \sum_{i=1}^{n} \left(|T_i|^2 + |T_i^*|^2 + |T_i| + |T_i^*|\right) \right\|$$

Proof. Setting $\alpha = \beta = \frac{3}{4}$ in (2.21). □

Theorem 2. *Let $A_i, B_i, C_i, D_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$. Then,*

$$w_p^p(D_1C_1B_1A_1, \cdots, D_nC_nB_nA_n) \leq \frac{1}{2}\left\| \sum_{i=1}^{n} \left[(1-\gamma) A_i^* |B_i|^2 A_i + \gamma D_i |C_i^*|^2 D_i^*\right]^p + \left[\gamma A_i^* |B_i|^2 A_i + (1-\gamma) D_i |C_i^*|^2 D_i^*\right]^p \right\| \tag{2.22}$$

for all $\gamma \in [0, 1]$ and $p \geq 1$. In particular, we have

$$w_p^p(D_1C_1B_1A_1, \cdots, D_nC_nB_nA_n) \leq \frac{1}{2^p}\left\| \sum_{i=1}^{n} \left(A_i^* |B_i|^2 A_i + D_i |C_i^*|^2 D_i^*\right)^p \right\| \tag{2.23}$$

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Proof. Let $y = x$ in (2.6), then we have

$$
\begin{aligned}
& w_p^p\left(D_1C_1B_1A_1, \cdots, D_nC_nB_nA_n\right) \\
& = \sum_{i=1}^n \left|\left\langle D_iC_iB_iA_ix, x\right\rangle\right|^p \\
& = \sum_{i=1}^n \left(\left\langle A_i^* \left|B_i\right|^2 A_ix, x\right\rangle^{\frac{1}{2}} \left\langle D_i \left|C_i^*\right|^2 D_i^*x, x\right\rangle^{\frac{1}{2}} \right)^p \\
& \le \frac{1}{2^p} \sum_{i=1}^n \left[\left(\left\langle A_i^* \left|B_i\right|^2 A_ix, x\right\rangle^{(1-\gamma)} \left\langle D_i \left|C_i^*\right|^2 D_i^*x, x\right\rangle^{\gamma} \right) \right. \qquad \left(\text{since } \sqrt{ab} \le \frac{a^\gamma b^{1-\gamma} + a^{1-\gamma}b^\gamma}{2}\right) \\
& \qquad \left. + \left(\left\langle A_i^* \left|B_i\right|^2 A_ix, x\right\rangle^{\gamma} \left\langle D_i \left|C_i^*\right|^2 D_i^*x, x\right\rangle^{(1-\gamma)} \right) \right]^p \\
& \le \frac{1}{2} \sum_{i=1}^n \left[\left((1-\gamma) \left\langle A_i^* \left|B_i\right|^2 A_ix, x\right\rangle + \gamma \left\langle D_i \left|C_i^*\right|^2 D_i^*x, x\right\rangle \right)^p \right. \qquad (\text{by AM-GM inequality}) \\
& \qquad \left. + \left(\gamma \left\langle A_i^* \left|B_i\right|^2 A_ix, x\right\rangle + (1-\gamma) \left\langle D_i \left|C_i^*\right|^2 D_i^*x, x\right\rangle \right)^p \right] \\
& = \frac{1}{2} \sum_{i=1}^n \left\langle \left[(1-\gamma) A_i^* \left|B_i\right|^2 A_i + \gamma D_i \left|C_i^*\right|^2 D_i^* \right] x, x\right\rangle^p \\
& \qquad + \frac{1}{2} \sum_{i=1}^n \left\langle \left[\gamma A_i^* \left|B_i\right|^2 A_i + (1-\gamma) D_i \left|C_i^*\right|^2 D_i^* \right] x, x\right\rangle^p \\
& \le \frac{1}{2} \sum_{i=1}^n \left\langle \left[(1-\gamma) A_i^* \left|B_i\right|^2 A_i + \gamma D_i \left|C_i^*\right|^2 D_i^* \right]^p x, x\right\rangle \qquad (\text{by McCarthy inequality}) \\
& \qquad + \frac{1}{2} \sum_{i=1}^n \left\langle \left[\gamma A_i^* \left|B_i\right|^2 A_i + (1-\gamma) D_i \left|C_i^*\right|^2 D_i^* \right]^p x, x\right\rangle \\
& = \frac{1}{2} \left\langle \sum_{i=1}^n \left\{ \left[(1-\gamma) A_i^* \left|B_i\right|^2 A_i + \gamma D_i \left|C_i^*\right|^2 D_i^* \right]^p \right\}, \right. \\
& \qquad \left. + \left[\gamma A_i^* \left|B_i\right|^2 A_i + (1-\gamma) D_i \left|C_i^*\right|^2 D_i^* \right]^p x, x \right\rangle \Big].
\end{aligned}
$$

Taking the supremum over all unit vector $x \in \mathscr{H}$ we get the required result. The particular case is obtained by setting $\gamma = \frac{1}{2}$ in (2.22). □

Corollary 10. *Let $T_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$. Then, for all $\alpha, \beta \ge 0$ such that $\alpha + \beta \ge 1$ we have*

$$
w_p^p\left(T_1\left|T_1\right|^{\alpha+\beta-1}, \cdots, T_n\left|T_n\right|^{\alpha+\beta-1}\right) \le \frac{1}{2^p}\left\|\sum_{i=1}^n \left(\left|T_i\right|^{2\alpha} + \left|T_i^*\right|^{2\beta}\right)^p\right\| \tag{2.24}
$$

for all $p \ge 1$.

Proof. Let U_i be unitaries for all $i = 1, \cdots, n$, setting $D_i = U_i$, $B = 1_{\mathscr{H}}$, $C = \left|T_i\right|^\beta$ and $A_i = \left|T_i\right|^\alpha$ for all $\alpha, \beta \ge 0$ such that $\alpha + \beta \ge 1$ in (2.23), then we have

$$
D_iC_iB_iA_i = U_i\left|T_i\right|^\beta\left|T_i\right|^\alpha = U_i\left|T_i\right|\left|T_i\right|^{\alpha+\beta-1} = T_i\left|T_i\right|^{\alpha+\beta-1},
$$

also, we have $A_i^*\left|B_i\right|^2 A_i = \left|T_i\right|^{2\alpha}$ and $D_i\left|C_i^*\right|^2 D_i^* = U_i\left|T_i\right|^{2\beta}U_i^* = \left|T_i\right|^{2\beta}$ for all $i = 1, \cdots, n$. □

Corollary 11. *Let $T_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$. Then, we have*

$$
w_p^p\left(T_1\left|T_1\right|, \cdots, T_n\left|T_n\right|\right) \le \frac{1}{2^p}\left\|\sum_{i=1}^n \left(\left|T_i\right|^2 + \left|T_i^*\right|^2\right)^p\right\| \tag{2.25}
$$

Proof. Setting $\alpha = \beta = \frac{1}{2}$ in (2.24). □

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Theorem 3. *Let $A_i, B_i, C_i, D_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$. Then, for $m \in \mathbb{N}$ and $r, p \geq m \geq 1$,*

$$w_p^p\left(D_1C_1B_1A_1, \cdots, D_nC_nB_nA_n\right) \leq \frac{n^{1-\frac{m}{r}}}{2^{\frac{m}{r}}}\left\|\sum_{i=1}^{n}\left(\left(A_i^*\left|B_i\right|^2A_i\right)^{\frac{rp}{m}} + \left(D_i\left|C_i^*\right|^2D_i^*\right)^{\frac{rp}{m}}\right)\right\|^{\frac{m}{r}} - \inf_{\|x\|=1}\xi_{p,m}(x), \tag{2.26}$$

where

$$\xi_{p,m}(x) := 2^{-m}\sum_{i=1}^{n}\left(\left\langle\left(A_i^*\left|B_i\right|^2A_i\right)^{\frac{p}{m}}x, x\right\rangle^{\frac{m}{2}} - \left\langle\left(D_i\left|C_i^*\right|^2D_i^*\right)^{\frac{p}{m}}x, x\right\rangle^{\frac{m}{2}}\right)^2.$$

Proof. Let $x \in \mathscr{H}$ be a unit vector. Let $y = x$ in (2.6) then we have

$$\begin{aligned}
&\sum_{i=1}^{n}\left|\left\langle D_iC_iB_iA_ix, x\right\rangle\right|^p \\
&= \sum_{i=1}^{n}\left\langle A_i^*\left|B_i\right|^2A_ix, x\right\rangle^{\frac{p}{2}}\left\langle D_i\left|C_i^*\right|^2D_i^*x, x\right\rangle^{\frac{p}{2}} \\
&= \sum_{i=1}^{n}\left(\left\langle A_i^*\left|B_i\right|^2A_ix, x\right\rangle^{\frac{p}{2m}}\left\langle D_i\left|C_i^*\right|^2D_i^*x, x\right\rangle^{\frac{p}{2m}}\right)^m \\
&\leq \sum_{i=1}^{n}\left(\left\langle\left(A_i^*\left|B_i\right|^2A_i\right)^{\frac{p}{m}}x, x\right\rangle^{\frac{1}{2}}\left\langle\left(D_i\left|C_i^*\right|^2D_i^*\right)^{\frac{p}{m}}x, x\right\rangle^{\frac{1}{2}}\right)^m \qquad \text{(by McCarthy inequality)} \\
&\leq \sum_{i=1}^{n}\left(\frac{1}{2}\left\langle\left(A_i^*\left|B_i\right|^2A_i\right)^{\frac{p}{m}}x, x\right\rangle^r + \left\langle\left(D_i\left|C_i^*\right|^2D_i^*\right)^{\frac{p}{m}}x, x\right\rangle^r\right)^{\frac{m}{r}} \qquad \text{(by (2.4))} \\
&\quad - 2^{-m}\sum_{i=1}^{n}\left(\left\langle\left(A_i^*\left|B_i\right|^2A_i\right)^{\frac{p}{m}}x, x\right\rangle^{\frac{m}{2}} - \left\langle\left(D_i\left|C_i^*\right|^2D_i^*\right)^{\frac{p}{m}}x, x\right\rangle^{\frac{m}{2}}\right)^2 \\
&\leq \sum_{i=1}^{n}\left(\frac{1}{2}\left\langle\left(A_i^*\left|B_i\right|^2A_i\right)^{\frac{rp}{m}}x, x\right\rangle + \left\langle\left(D_i\left|C_i^*\right|^2D_i^*\right)^{\frac{rp}{m}}x, x\right\rangle\right)^{\frac{m}{r}} \qquad \text{(by McCarthy inequality)} \\
&\quad - 2^{-m}\sum_{i=1}^{n}\left(\left\langle\left(A_i^*\left|B_i\right|^2A_i\right)^{\frac{p}{m}}x, x\right\rangle^{\frac{m}{2}} - \left\langle\left(D_i\left|C_i^*\right|^2D_i^*\right)^{\frac{p}{m}}x, x\right\rangle^{\frac{m}{2}}\right)^2 \\
&\leq \frac{n^{1-\frac{m}{r}}}{2^{\frac{m}{r}}}\sum_{i=1}^{n}\left(\frac{1}{2}\left\langle\left(A_i^*\left|B_i\right|^2A_i\right)^{\frac{rp}{m}}x, x\right\rangle + \left\langle\left(D_i\left|C_i^*\right|^2D_i^*\right)^{\frac{rp}{m}}x, x\right\rangle\right)^{\frac{m}{r}} \qquad \text{(by concavity of } t^{\frac{m}{r}}\text{)} \\
&\quad - 2^{-m}\sum_{i=1}^{n}\left(\left\langle\left(A_i^*\left|B_i\right|^2A_i\right)^{\frac{p}{m}}x, x\right\rangle^{\frac{m}{2}} - \left\langle\left(D_i\left|C_i^*\right|^2D_i^*\right)^{\frac{p}{m}}x, x\right\rangle^{\frac{m}{2}}\right)^2
\end{aligned}$$

□

Theorem 4. *Let $T_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$, $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 1$, $m \in \mathbb{N}$ and $r, p \geq m \geq 1$. Then*

$$w_p^p\left(T_1\left|T_1\right|^{\alpha+\beta-1}, \cdots, T_n\left|T_n\right|^{\alpha+\beta-1}\right) \leq \frac{n^{1-\frac{m}{r}}}{2^{\frac{m}{r}}}\left\|\sum_{i=1}^{n}\left(\left|T_i\right|^{\frac{2\alpha rp}{m}} + \left|T_i^*\right|^{\frac{2\beta rp}{m}}\right)\right\|^{\frac{m}{r}} - \inf_{\|x\|=1}\psi_{p,m,\alpha,\beta}(x), \tag{2.27}$$

where

$$\psi_{p,m,\alpha,\beta}(x) := 2^{-m}\sum_{i=1}^{n}\left(\left\langle\left|T_i\right|^{\frac{2\alpha p}{m}}x, x\right\rangle^{\frac{m}{2}} - \left\langle\left|T_i^*\right|^{\frac{2\beta p}{m}}x, x\right\rangle^{\frac{m}{2}}\right)^2.$$

for all $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 1$.

Proof. Let U_i be unitaries for all $i = 1, \cdots, n$, setting $D_i = U_i$, $B = 1_{\mathscr{H}}$, $C = |T_i|^{\beta}$ and $A_i = |T_i|^{\alpha}$ for all $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 1$ in (2.26), then we have

$$D_i C_i B_i A_i = U_i |T_i|^{\beta} |T_i|^{\alpha} = U_i |T_i| |T_i|^{\alpha+\beta-1} = T_i |T_i|^{\alpha+\beta-1},$$

also, we have $A_i^* |B_i|^2 A_i = |T_i|^{2\alpha}$ and $D_i |C_i^*|^2 D_i^* = U_i |T_i|^{2\beta} U_i^* = |T_i|^{2\beta}$ for all $i = 1, \cdots, n$. □

Corollary 12. *Let $T_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$, $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 1$, $m \in \mathbb{N}$ and $r, p \geq m \geq 1$. Then*

$$w_p^p \left(T_1 |T_1|^{\alpha+\beta-1}, \cdots, T_n |T_n|^{\alpha+\beta-1}\right) \leq \frac{1}{2} \left\| \sum_{i=1}^n \left(|T_i|^{2\alpha p} + |T_i^*|^{2\beta p} \right) \right\| - \inf_{\|x\|=1} \psi_{p,1,\alpha,\beta}(x), \tag{2.28}$$

where

$$\psi_{p,1,\alpha,\beta}(x) := \frac{1}{2} \sum_{i=1}^n \left(\left\langle |T_i|^{2\alpha p} x, x \right\rangle^{\frac{1}{2}} - \left\langle |T_i^*|^{2\beta p} x, x \right\rangle^{\frac{1}{2}} \right)^2.$$

for all $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 1$.

Proof. Setting $m = r = 1$ in (2.27). □

Corollary 13. *Let $T_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$. Then, for $m \in \mathbb{N}$ and $r, p \geq m \geq 1$,*

$$w_p^p (T_1, \cdots, T_n) \leq \frac{n^{1-\frac{m}{r}}}{2^{\frac{m}{r}}} \left\| \sum_{i=1}^n \left(|T_i|^{\frac{rp}{m}} + |T_i^*|^{\frac{rp}{m}} \right) \right\|^{\frac{m}{r}} - \inf_{\|x\|=1} \psi_{p,m,\frac{1}{2},\frac{1}{2}}(x), \tag{2.29}$$

where

$$\psi_{p,m,\frac{1}{2},\frac{1}{2}}(x) := 2^{-m} \sum_{i=1}^n \left(\left\langle |T_i|^{\frac{p}{m}} x, x \right\rangle^{\frac{m}{2}} - \left\langle |T_i^*|^{\frac{p}{m}} x, x \right\rangle^{\frac{m}{2}} \right)^2.$$

for all $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 1$.

Proof. Setting $\alpha = \beta = \frac{1}{2}$ in (2.27). □

Corollary 14. *Let $T_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$. Then, for $m \in \mathbb{N}$ and $r, p \geq 1$,*

$$w_p^p (T_1, \cdots, T_n) \leq \frac{n^{1-\frac{1}{r}}}{2^{\frac{1}{r}}} \left\| \sum_{i=1}^n \left(|T_i|^{rp} + |T_i^*|^{rp} \right) \right\|^{\frac{1}{r}} - \inf_{\|x\|=1} \psi_{p,1,\frac{1}{2},\frac{1}{2}}(x), \tag{2.30}$$

where

$$\psi_{p,1,\frac{1}{2},\frac{1}{2}}(x) := \frac{1}{2} \sum_{i=1}^n \left(\langle |T_i|^p x, x \rangle^{\frac{1}{2}} - \langle |T_i^*|^p x, x \rangle^{\frac{1}{2}} \right)^2.$$

for all $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 1$.

Proof. Setting $\alpha = \beta = \frac{1}{2}$ and $m = 1$ in (2.29). □

Corollary 15. *Let $T_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$. Then, for $m \in \mathbb{N}$ and $p \geq m \geq 1$,*

$$w_p^p (T_1, \cdots, T_n) \leq \frac{1}{2} \left\| \sum_{i=1}^n \left(|T_i|^p + |T_i^*|^p \right) \right\| - \inf_{\|x\|=1} \psi_{p,m,\frac{1}{2},\frac{1}{2}}(x), \tag{2.31}$$

where

$$\psi_{p,m,\frac{1}{2},\frac{1}{2}}(x) := 2^{-m} \sum_{i=1}^n \left(\left\langle |T_i|^{\frac{p}{m}} x, x \right\rangle^{\frac{m}{2}} - \left\langle |T_i^*|^{\frac{p}{m}} x, x \right\rangle^{\frac{m}{2}} \right)^2.$$

Proof. Setting $m = r$ in (2.29). □

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Corollary 16. *Let $T_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$. Then, for $m \in \mathbb{N}$ and $r, p \geq m \geq 1$, we have*

$$w_p^p (T_1 |T_1|, \cdots, T_n |T_n|) \leq \frac{n^{1-\frac{m}{r}}}{2^{\frac{m}{r}}} \left\| \sum_{i=1}^n \left(|T_i|^{\frac{2rp}{m}} + |T_i^*|^{\frac{2rp}{m}} \right) \right\|^{\frac{m}{r}} - \inf_{\|x\|=1} \psi_{p,m,1,1}(x), \tag{2.32}$$

where

$$\psi_{p,m,1,1}(x) := 2^{-m} \sum_{i=1}^n \left(\left\langle |T_i|^{\frac{2p}{m}} x, x \right\rangle^{\frac{m}{2}} - \left\langle |T_i^*|^{\frac{2p}{m}} x, x \right\rangle^{\frac{m}{2}} \right)^2.$$

for all $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 1$.

Proof. Setting $\alpha = \beta = 1$ in (2.27). □

Corollary 17. *Let $B_i, C_i, U_i, V_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$ such that U_i, V_i unitaries. Then, for $m \in \mathbb{N}$ and $r, p \geq m \geq 1$, we have*

$$\begin{aligned} & w_p^p (U_1^* C_1 B_1 U_1, \cdots, U_n^* C_n B_n U_n) \\ & \leq \frac{n^{1-\frac{m}{r}}}{2^{\frac{m}{r}}} \left\| \sum_{i=1}^n \left(\left(U_i^* |B_i|^2 U_i \right)^{\frac{rp}{m}} + \left(U_i^* |C_i^*|^2 U_i \right)^{\frac{rp}{m}} \right) \right\|^{\frac{m}{r}} - \inf_{\|x\|=1} \xi_{p,m}(x), \end{aligned} \tag{2.33}$$

where

$$\xi_{p,m}(x) := 2^{-m} \sum_{i=1}^n \left(\left\langle \left(U_i^* |B_i|^2 U_i \right)^{\frac{p}{m}} x, x \right\rangle^{\frac{m}{2}} - \left\langle \left(U_i^* |C_i^*|^2 U_i \right)^{\frac{p}{m}} x, x \right\rangle^{\frac{m}{2}} \right)^2.$$

Proof. Setting $A_i = U_i$ and $D_i = U_i^*$ in (2.26) and using the fact that . □

Corollary 18. *Let $A_i, D_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$. Then, for $m \in \mathbb{N}$ and $r, p \geq m \geq 1$,*

$$w_p^p (D_1 A_1, \cdots, D_n A_n) \leq \frac{n^{1-\frac{m}{r}}}{2^{\frac{m}{r}}} \left\| \sum_{i=1}^n \left((A_i^* A_i)^{\frac{rp}{m}} + (D_i D_i^*)^{\frac{rp}{m}} \right) \right\|^{\frac{m}{r}} - \inf_{\|x\|=1} \xi_{p,m}(x), \tag{2.34}$$

where

$$\xi_{p,m}(x) := 2^{-m} \sum_{i=1}^n \left(\left\langle (A_i^* A_i)^{\frac{p}{m}} x, x \right\rangle^{\frac{m}{2}} - \left\langle (D_i D_i^*)^{\frac{p}{m}} x, x \right\rangle^{\frac{m}{2}} \right)^2.$$

Proof. Setting $B_i = C_i = 1_{\mathscr{H}}$ for all $i = 1, \cdots, n$ in (2.26). □

Corollary 19. *Let $A_i, D_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$. Then, for $m \in \mathbb{N}$ and $r, p \geq m \geq 1$,*

$$w_p^p (A_1^* A_1, \cdots, A_n^* A_n) \leq n^{1-\frac{m}{r}} \left\| \sum_{i=1}^n (A_i^* A_i)^{\frac{rp}{m}} \right\|^{\frac{m}{r}}. \tag{2.35}$$

In particular, for $m = r$

$$w_p^p (A_1^* A_1, \cdots, A_n^* A_n) \leq \left\| \sum_{i=1}^n (A_i^* A_i)^p \right\|.$$

Proof. Setting $D_i = A_i^*$ for all $i = 1, \cdots, n$ in (2.34). □

3. Upper and Lower bounds for the generalized Euclidean operator radius

In this section we provide some lower and upper bounds for the product of the generalized Euclidean operator radius. In order to to prove our results we need to recall the the following Hölder type inequality, which reads:

$$\left(\sum_{j=1}^n |x_j y_j|^r \right)^{\frac{1}{r}} \leq \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |y_j|^q \right)^{\frac{1}{q}} \tag{3.1}$$

for all complex numbers x_j, y_j $(1 \leq j \leq n)$ and all $p, q, r \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

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Theorem 5. *Let $D_i, C_i, B_i, A_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$, $r \geq 1$ and $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then*

$$\begin{aligned}
&\frac{1}{n^{2r-1}} \left\| \sum_{i=1}^{n} D_i C_i B_i A_i \right\|^{2r} \\
&\le w_p^r \left(A_1^* |B_1|^2 A_1, \cdots, A_n^* |B_n|^2 A_n \right) w_q^r \left(D_1 |C_1^*|^2 D_1^*, \cdots, D_n |C_n^*|^2 D_n^* \right) \qquad (3.2)\\
&\le \max\left\{\frac{r}{p}, \frac{r}{q}\right\} \left\| \sum_{i=1}^{n} \left(A_i^* |B_i|^2 A_i \right)^p + \left(D_i |C_i^*|^2 D_i^* \right)^q \right\| - \inf_{\|x\|=1} \lambda(x, y),
\end{aligned}$$

where,

$$\lambda(x, y) := \min\left\{\frac{r}{p}, \frac{r}{q}\right\} \left(\sqrt{\sum_{i=1}^{n} \left\langle A_i^* |B_i|^2 A_i x, y \right\rangle^p} - \sqrt{\sum_{i=1}^{n} \left\langle D_i |C_i^*|^2 D_i^* x, y \right\rangle^q} \right)^2$$

Proof. Let $x, y \in \mathscr{H}$. Applying inequality (3.1) and the convexity of t^{2r}, we have

$$\begin{aligned}
&\frac{1}{n^{2r-1}} \left| \left\langle \left(\sum_{i=1}^{n} D_i C_i B_i A_i \right) x, y \right\rangle \right|^{2r} \\
&= \frac{1}{n^{2r-1}} \left| \sum_{i=1}^{n} \left\langle (D_i C_i B_i A_i) x, y \right\rangle \right|^{2r} \\
&\le \frac{1}{n^{2r-1}} \left(\sum_{i=1}^{n} \left| \left\langle (D_i C_i B_i A_i) x, y \right\rangle \right| \right)^{2r} \\
&\le \left(\sum_{i=1}^{n} \left| \left\langle (D_i C_i B_i A_i) x, y \right\rangle \right|^{2r} \right) \qquad \text{(by Jensen's inequality)} \\
&\le \left(\sum_{i=1}^{n} \left(\left\langle A_i^* |B_i|^2 A_i x, y \right\rangle \left\langle D_i |C_i^*|^2 D_i^* x, y \right\rangle \right)^r \right) \qquad \text{(by (2.6))} \\
&\le \left(\sum_{i=1}^{n} \left\langle A_i^* |B_i|^2 A_i x, y \right\rangle^p \right)^{\frac{r}{p}} \left(\sum_{i=1}^{n} \left\langle D_i |C_i^*|^2 D_i^* x, y \right\rangle^q \right)^{\frac{r}{q}} \qquad (3.3) \\
&\le \frac{r}{p} \sum_{i=1}^{n} \left\langle A_i^* |B_i|^2 A_i x, y \right\rangle^p + \frac{r}{q} \sum_{i=1}^{n} \left\langle D_i |C_i^*|^2 D_i^* x, y \right\rangle^q \qquad \text{by (2.4)} \\
&\quad - \min\left\{\frac{r}{p}, \frac{r}{q}\right\} \left(\sqrt{\sum_{i=1}^{n} \left\langle A_i^* |B_i|^2 A_i x, y \right\rangle^p} - \sqrt{\sum_{i=1}^{n} \left\langle D_i |C_i^*|^2 D_i^* x, y \right\rangle^q} \right)^2 \\
&\le \frac{r}{p} \sum_{i=1}^{n} \left\langle \left(A_i^* |B_i|^2 A_i \right)^p x, y \right\rangle + \frac{r}{q} \sum_{i=1}^{n} \left\langle \left(D_i |C_i^*|^2 D_i^* \right)^q x, y \right\rangle \qquad \text{(by McCarthy inequality)} \\
&\quad - \min\left\{\frac{r}{p}, \frac{r}{q}\right\} \left(\sqrt{\sum_{i=1}^{n} \left\langle A_i^* |B_i|^2 A_i x, y \right\rangle^p} - \sqrt{\sum_{i=1}^{n} \left\langle D_i |C_i^*|^2 D_i^* x, y \right\rangle^q} \right)^2 \\
&\le \max\left\{\frac{r}{p}, \frac{r}{q}\right\} \left\| \sum_{i=1}^{n} \left(A_i^* |B_i|^2 A_i \right)^p + \left(D_i |C_i^*|^2 D_i^* \right)^q \right\| \qquad \text{(properties of max)} \\
&\quad - \min\left\{\frac{r}{p}, \frac{r}{q}\right\} \left(\sqrt{\sum_{i=1}^{n} \left\langle A_i^* |B_i|^2 A_i x, y \right\rangle^p} - \sqrt{\sum_{i=1}^{n} \left\langle D_i |C_i^*|^2 D_i^* x, y \right\rangle^q} \right)^2
\end{aligned}$$

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Taking the supremum over $x, y \in \mathscr{H}$ with $\|x\| = \|y\| = 1$, then the left and right hand side follows immediately the middle term of the inequality follows by (3.3), and thus the desired result is obtained. □

Corollary 20. *Let* $D_i, C_i, B_i, A_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$, $r \geq 1$. *Then*

$$\begin{aligned} &\frac{1}{n}\left\|\sum_{i=1}^{n} D_i C_i B_i A_i\right\|^2 \\ &\leq w_{\mathrm{e}}\left(A_1^* |B_1|^2 A_1, \cdots, A_n^* |B_n|^2 A_n\right) w_{\mathrm{e}}\left(D_1 |C_1^*|^2 D_1^*, \cdots, D_n |C_n^*|^2 D_n^*\right) \\ &\leq \frac{1}{2}\left\|\sum_{i=1}^{n}\left\{\left(A_i^* |B_i|^2 A_i\right)^2 + \left(D_i |C_i^*|^2 D_i^*\right)^2\right\}\right\| - \inf_{\|x\|=\|y\|=1} \lambda(x, y), \end{aligned} \tag{3.4}$$

where

$$\lambda(x, y) := \frac{1}{2}\left(\sqrt{\sum_{i=1}^{n}\left\langle A_i^* |B_i|^2 A_i x, y\right\rangle^2} - \sqrt{\sum_{i=1}^{n}\left\langle D_i |C_i^*|^2 D_i^* x, y\right\rangle^2}\right)^2$$

Proof. Setting $p = q = 2$ and $r = 1$ in (3.2) we get the desired result. □

Corollary 21. *Let* $D_i, A_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$, $r \geq 1$ *and* $p, q \geq 1$ *with* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. *Then*

$$\begin{aligned} \frac{1}{n^{2r-1}}\left\|\sum_{i=1}^{n} D_i A_i\right\|^{2r} &\leq w_p^r\left(A_1^* A_1, \cdots, A_n^* A_n\right) w_q^r\left(D_1 D_1^*, \cdots, D_n D_n^*\right) \\ &\leq \max\left\{\frac{r}{p}, \frac{r}{q}\right\}\left\|\sum_{i=1}^{n}\left(A_i^* A_i\right)^p + \left(D_i D_i^*\right)^q\right\| - \inf_{\|x\|=1} \lambda(x, y), \end{aligned} \tag{3.5}$$

where,

$$\lambda(x, y) := \min\left\{\frac{r}{p}, \frac{r}{q}\right\}\left(\sqrt{\sum_{i=1}^{n}\left\langle A_i^* |B_i|^2 A_i x, y\right\rangle^p} - \sqrt{\sum_{i=1}^{n}\left\langle D_i |C_i^*|^2 D_i^* x, y\right\rangle^q}\right)^2$$

Proof. Setting $B_i = C_i = 1_{\mathscr{H}}$ in (3.2) we get the required result. □

Corollary 22. *Let* $D_i, A_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$, $r \geq 1$ *and* $p, q \geq 1$ *with* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. *Then*

$$\begin{aligned} \frac{1}{n^{2r-1}}\left\|\sum_{i=1}^{n} A_i^* A_i\right\|^{2r} &\leq w_p^r\left(A_1^* A_1, \cdots, A_n^* A_n\right) w_q^r\left(A_i A_i^*, \cdots, A_i A_i^*\right) \\ &\leq \max\left\{\frac{r}{p}, \frac{r}{q}\right\}\left\|\sum_{i=1}^{n}\left(A_i^* A_i\right)^p + \left(A_i A_i^*\right)^q\right\| - \inf_{\|x\|=1} \lambda(x, y), \end{aligned} \tag{3.6}$$

where,

$$\lambda(x, y) := \min\left\{\frac{r}{p}, \frac{r}{q}\right\}\left(\sqrt{\sum_{i=1}^{n}\left\langle A_i^* A_i x, y\right\rangle^p} - \sqrt{\sum_{i=1}^{n}\left\langle A_i A_i^* x, y\right\rangle^q}\right)^2$$

Proof. Setting $D_i = A_i$ and $B_i = C_i = 1_{\mathscr{H}}$ in (3.5). □

Corollary 23. *Let* $D_i, A_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$, $r \geq 1$ *and* $p, q \geq 1$ *with* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. *Then*

$$\frac{1}{n}\left\|\sum_{i=1}^{n} A_i^2\right\|^2 \leq w_{\mathrm{e}}^2\left(A_1^* A_1, \cdots, A_n^* A_n\right) \leq \frac{1}{2}\left\|\sum_{i=1}^{n}\left(A_i^* A_i\right)^2 + \left(A_i A_i^*\right)^2\right\| - \inf_{\|x\|=\|y\|=1} \lambda(x, y), \tag{3.7}$$

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where,

$$\lambda(x,y) := \frac{1}{2}\left(\sqrt{\sum_{i=1}^{n}\langle A_i^*A_ix,y\rangle^2}-\sqrt{\sum_{i=1}^{n}\langle A_iA_i^*x,y\rangle^2}\right)^2$$

Proof. Setting $p=q=2$ and $r=1$ in (3.6). □

Theorem 6. *Let $T_i\in\mathscr{B}(\mathscr{H})$ $(i=1,\cdots,n)$, $r\geq 1$, $p,q\geq 1$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ and $\alpha,\beta\geq 0$ such that $\alpha+\beta\geq 1$. Then*

$$\frac{1}{n^{2r-1}}\left\|\sum_{i=1}^{n}T_i|T_i|^{\alpha+\beta-1}\right\|^{2r}\leq w_p^r\left(|T_1|^{2\alpha},\cdots,|T_n|^{2\alpha}\right)w_q^r\left(|T_1^*|^{2\beta},\cdots,|T_n^*|^{2\beta}\right)$$
$$\leq\max\left\{\frac{r}{p},\frac{r}{q}\right\}\left\|\sum_{i=1}^{n}|T_i|^{2p\alpha}+|T_i^*|^{2q\beta}\right\|-\inf_{\|x\|=\|y\|=1}\lambda(x,y), \tag{3.8}$$

where,

$$\lambda(x,y) := \min\left\{\frac{r}{p},\frac{r}{q}\right\}\left(\sqrt{\sum_{i=1}^{n}\left\langle|T_i|^{2\alpha}x,y\right\rangle^p}-\sqrt{\sum_{i=1}^{n}\left\langle|T_i^*|^{2\beta}x,y\right\rangle^q}\right)^2$$

Proof. Let U_i be unitaries for all $i=1,\cdots,n$, setting $D_i=U_i$, $B=1_{\mathscr{H}}$, $C=|T_i|^{\beta}$ and $A_i=|T_i|^{\alpha}$ for all $\alpha,\beta\geq 0$ such that $\alpha+\beta\geq 1$ in (3.2), then we have

$$D_iC_iB_iA_i=U_i|T_i|^{\beta}|T_i|^{\alpha}=U_i|T_i||T_i|^{\alpha+\beta-1}=T_i|T_i|^{\alpha+\beta-1},$$

also, we have $A_i^*|B_i|^2A_i=|T_i|^{2\alpha}$ and $D_i|C_i^*|^2D_i^*=U_i|T_i|^{2\beta}U_i^*=|T_i|^{2\beta}$ for all $i=1,\cdots,n$. □

Corollary 24. *Let $T_i\in\mathscr{B}(\mathscr{H})$ $(i=1,\cdots,n)$, $r\geq 1$ and $p,q\geq 1$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. Then*

$$\frac{1}{n^{2r-1}}\left\|\sum_{i=1}^{n}T_i\right\|^{2r}\leq w_p^r(|T_1|,\cdots,|T_n|)\,w_q^r(|T_1^*|,\cdots,|T_n^*|)$$
$$\leq\max\left\{\frac{r}{p},\frac{r}{q}\right\}\left\|\sum_{i=1}^{n}|T_i|^p+|T_i^*|^q\right\|-\inf_{\|x\|=\|y\|=1}\lambda(x,y), \tag{3.9}$$

where,

$$\lambda(x,y) := \min\left\{\frac{r}{p},\frac{r}{q}\right\}\left(\sqrt{\sum_{i=1}^{n}\langle|T_i|x,y\rangle^p}-\sqrt{\sum_{i=1}^{n}\langle|T_i^*|x,y\rangle^q}\right)^2$$

Proof. Setting $\alpha=\beta=\frac{1}{2}$ in (3.8). □

Corollary 25. *Let $T_i\in\mathscr{B}(\mathscr{H})$ $(i=1,\cdots,n)$, $\alpha,\beta\geq 0$ such that $\alpha+\beta\geq 1$. Then*

$$\frac{1}{n}\left\|\sum_{i=1}^{n}T_i|T_i|^{\alpha+\beta-1}\right\|^{2}\leq w_e\left(|T_1|^{2\alpha},\cdots,|T_n|^{2\alpha}\right)w_e\left(|T_1^*|^{2\beta},\cdots,|T_n^*|^{2\beta}\right)$$
$$\leq\frac{1}{2}\left\|\sum_{i=1}^{n}|T_i|^{4\alpha}+|T_i^*|^{4\beta}\right\|-\inf_{\|x\|=\|y\|=1}\lambda(x,y), \tag{3.10}$$

where,

$$\lambda(x,y) := \frac{1}{2}\left(\sqrt{\sum_{i=1}^{n}\left\langle|T_i|^{2\alpha}x,y\right\rangle^2}-\sqrt{\sum_{i=1}^{n}\left\langle|T_i^*|^{2\beta}x,y\right\rangle^2}\right)^2$$

Proof. Setting $p=q=2$ and $r=1$ in (3.8). □

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Corollary 26. *Let* $T_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$. *Then*

$$\frac{1}{n}\left\|\sum_{i=1}^{n} T_i\right\|^2 \leq w_{\mathrm{e}}\left(|T_1|, \cdots, |T_n|\right) w_{\mathrm{e}}\left(|T_1^*|, \cdots, |T_n^*|\right) \leq \frac{1}{2}\left\|\sum_{i=1}^{n} T_i^*T_i + T_iT_i^*\right\| - \inf_{\|x\|=\|y\|=1} \lambda(x, y), \tag{3.11}$$

where,

$$\lambda(x, y) := \frac{1}{2}\left(\sqrt{\sum_{i=1}^{n} \langle |T_i| x, y\rangle^2} - \sqrt{\sum_{i=1}^{n} \langle |T_i^*| x, y\rangle^2}\right)^2$$

Proof. Setting $\alpha = \beta = \frac{1}{2}$ in (3.10). □

We note that, in 2005, Kittaneh in [16] proved that

$$\frac{1}{4}\|T^*T + TT^*\| \leq w^2(T) \leq \frac{1}{2}\|T^*T + TT^*\| \tag{3.12}$$

for Hilbert space operator $T \in \mathscr{B}(\mathscr{H})$. These inequalities are sharp. These inequalities were also reformulated and generalized in [10] but in terms of the Cartesian decomposition.

In 2009, Popescu [21] proved that

$$\frac{1}{2\sqrt{n}}\left\|\sum_{k=1}^{n} T_kT_k^*\right\|^{\frac{1}{2}} \leq w_{\mathrm{e}}(T_1, \cdots, T_n) \leq \left\|\sum_{k=1}^{n} T_kT_k^*\right\|^{\frac{1}{2}} \tag{3.13}$$

As noted in [20], and as a special case of (3.13); if $A = B + iC$ is the Cartesian decomposition of A, then

$$w_{\mathrm{e}}^2(B, C) = \sup_{\|x\|=1}\left\{|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2\right\} = \sup_{\|x\|=1} |\langle Ax, x\rangle|^2 = w^2(A).$$

But since $A^*A + AA^* = 2\left(B^2 + C^2\right)$, then we have

$$\frac{1}{16}\|T^*T + TT^*\| \leq w^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|. \tag{3.14}$$

It should be noted here that, the case when $n = 2$, was also studied by Dragomir in [9] where he obtained some very interesting results regading Euclidean operator radius of two operators $w_{\mathrm{e}}(T_1, T_2)$.

Next, we give a generalization of (3.12) and refine (indeed improve) (3.13) (and thus (3.14)) to the generalized Euclidean operator radius.

Theorem 7. *Let* $T_k \in \mathscr{B}(\mathscr{H})$ $(k = 1, \cdots, n)$. *Then*

$$\frac{1}{2^{p+1}n^{p-1}}\left\|\sum_{k=1}^{n} T_k^*T_k + T_kT_k^*\right\|^p \leq w_{2p}^p(T_1, \cdots, T_n) \leq \frac{1}{2^p}\left\|\sum_{k=1}^{n} \left(T_k^*T_k + T_kT_k^*\right)^p\right\| \tag{3.15}$$

for all $p \geq 1$.

Proof. Let $B_k + iC_k$ be the Cartesian decomposition of T_k for all $k = 1, \cdots, n$. As in the proof of (3.12) in [16], we have

$$\begin{aligned}
|\langle T_kx, x\rangle|^{2p} &= \left(\langle B_kx, x\rangle^2 + \langle C_kx, x\rangle^2\right)^p \\
&\geq \frac{1}{2^p}\left(|\langle B_kx, x\rangle| + |\langle C_kx, x\rangle|\right)^{2p} \\
&\geq \frac{1}{2^p}|\langle B_kx, x\rangle + \langle C_kx, x\rangle|^{2p} \\
&= \frac{1}{2^p}|\langle B_k \pm C_kx, x\rangle|^{2p}.
\end{aligned}$$

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Summing over k and then taking the supremum over all unit vector $x \in \mathscr{H}$, we get

$$\begin{aligned} w_{2p}^{p}\left(T_1,\cdots,T_n\right) &\geq \frac{1}{2^p} \sup_{\|x\|=1} \sum_{k=1}^{n} \left|\langle B_k \pm C_k x, x\rangle\right|^{2p} \\ &\geq \frac{1}{2^p}\frac{1}{n^{p-1}} \sup_{\|x\|=1} \left(\sum_{k=1}^{n} \left|\langle B_k \pm C_k x, x\rangle\right|^{2}\right)^p \qquad \text{(by Jensen's inequality)} \\ &= \frac{1}{2^p}\frac{1}{n^{p-1}} \left\|\sum_{k=1}^{n} \left(B_k \pm C_k\right)^2\right\|^p. \end{aligned}$$

Thus,

$$\begin{aligned} 2w_{2p}^{p}\left(T_1,\cdots,T_n\right) &\geq \frac{1}{2^p}\frac{1}{n^{p-1}} \left\|\sum_{k=1}^{n} \left(B_k + C_k\right)^2\right\|^p + \frac{1}{2^p}\frac{1}{n^{p-1}} \left\|\sum_{k=1}^{n} \left(B_k - C_k\right)^2\right\|^p \\ &\geq \frac{1}{2^p}\frac{1}{n^{p-1}} \left\|\sum_{k=1}^{n} \left(B_k + C_k\right)^2 + \sum_{k=1}^{n} \left(B_k - C_k\right)^2\right\|^p \\ &= \frac{1}{2^p}\frac{1}{n^{p-1}} \left\|\sum_{k=1}^{n} \left\{\left(B_k + C_k\right)^2 + \left(B_k - C_k\right)^2\right\}\right\|^p \\ &= \frac{1}{n^{p-1}} \left\|\sum_{k=1}^{n} B_k^2 + C_k^2\right\|^p \\ &= \frac{1}{n^{p-1}} \left\|\sum_{k=1}^{n} \frac{T_k^*T_k + T_kT_k^*}{2}\right\|^p \\ &= \frac{1}{2^p n^{p-1}} \left\|\sum_{k=1}^{n} T_k^*T_k + T_kT_k^*\right\|^p, \end{aligned}$$

and hence,

$$w_{2p}^{p}\left(T_1,\cdots,T_n\right) \geq \frac{1}{2^{p+1} n^{p-1}} \left\|\sum_{k=1}^{n} T_k^*T_k + T_kT_k^*\right\|^p,$$

which proves the left hand side of the inequality in (3.15).

To prove the second inequality, for every unit vector $x \in \mathscr{H}$ we have

$$\begin{aligned} \sum_{k=1}^{n} \left|\langle T_k x, x\rangle\right|^{2p} &= \sum_{k=1}^{n} \left(\langle B_k x, x\rangle^2 + \langle C_k x, x\rangle^2\right)^p \\ &\leq \sum_{k=1}^{n} \left(\langle B_k^2 x, x\rangle + \langle C_k^2 x, x\rangle\right)^p \\ &= \sum_{k=1}^{n} \left\langle \left(B_k^2 + C_k^2\right) x, x\right\rangle^p, \end{aligned}$$

which implies that

$$\begin{aligned} \sup_{\|x\|=1} \sum_{k=1}^{n} \left|\langle T_k x, x\rangle\right|^{2p} = w_{2p}^{p}\left(T_1,\cdots,T_1\right) &\leq \sup_{\|x\|=1} \sum_{k=1}^{n} \left\langle \left(B_k^2 + C_k^2\right) x, x\right\rangle^p \\ &= \left\|\sum_{k=1}^{n} \left(B_k^2 + C_k^2\right)^p\right\| \\ &= \frac{1}{2^p} \left\|\sum_{k=1}^{n} \left(T_k^*T_k + T_kT_k^*\right)^p\right\|, \end{aligned}$$

which proves the right hand side of (3.15). □

Remark 2. *Clearly, by setting $n = 1$ in (3.15) we recapture (3.12).*

A very interesting case of (3.15) is considered in the following corollary.

Corollary 27. *Let $T, S \in \mathscr{B}(\mathscr{H})$. Then*

$$\frac{1}{2^{2p}} \|T^*T + TT^* + S^*S + SS^*\|^p \le w_{2p}^p (T, S) \le \frac{1}{2^p} \|(T^*T + TT^*)^p + (S^*S + SS^*)^p\| \tag{3.16}$$

for all $p \ge 1$.

Proof. Setting $n = 2$ in (3.15). □

Remark 3. *In particular, setting $p = 1$ in (3.16) we get*

$$\frac{1}{4} \|T^*T + TT^* + S^*S + SS^*\| \le w_{\mathrm{e}} (T, S) \le \frac{1}{2} \|T^*T + TT^* + S^*S + SS^*\| .$$

Moreover, if we choose $T = S$, then

$$\frac{1}{2} \|T^*T + TT^*\| \le w_{\mathrm{e}} (T, T) \le \|T^*T + TT^*\| .$$

Remark 4. *A lower and upper bounds for the Rhombic numerical radius could be deduced as follows: In (1.4) the inequality holds for any $p \ge 1$. Setting $p = 2q$, then (1.4) reduces to*

$$w_{2q} (T_1, \cdots, T_n) \le w_R (T_1, \cdots, T_n) \le n^{1-\frac{1}{2q}} w_{2q} (T_1, \cdots, T_n) .$$

which implies that

$$w_{2q}^q (T_1, \cdots, T_n) \le w_R^q (T_1, \cdots, T_n) \le n^{q-\frac{1}{2}} w_{2q}^q (T_1, \cdots, T_n) . \tag{3.17}$$

Combining the inequalities (3.17) with (3.15) we get

$$\begin{aligned}\frac{1}{2^{2q+1} n^{q-1}} \left\| \sum_{k=1}^n T_k^* T_k + T_k T_k^* \right\|^q &\le w_{2q}^q (T_1, \cdots, T_n) \\ &\le w_R^q (T_1, \cdots, T_n) \\ &\le n^{q-\frac{1}{2}} w_{2q}^q (T_1, \cdots, T_n) \\ &\le \frac{n^{q-\frac{1}{2}}}{2^q} \left\| \sum_{k=1}^n (T_k^* T_k + T_k T_k^*)^q \right\|\end{aligned}$$

for any $q \ge \frac{1}{2}$.

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