Discussion

The Triangle Wave Versus the Cosine: How Classical Systems Can Optimally Approximate EPR-B Correlations

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Abstract: The famous singlet correlations of a composite quantum system consisting of two spatially separated components exhibit notable features of two kinds. The first kind are striking *certainty relations*: perfect correlation and perfect anti-correlation in certain settings. The second kind are a number of symmetries, in particular, invariance under rotation, as well as invariance under exchange of components, parity, or chirality. In this note I investigate the class of correlation functions that can be generated by classical composite physical systems when we restrict attention to systems which reproduce the certainty relations exactly, and for which the rotational invariance of the correlation function is the manifestation of rotational invariance of the underlying classical physics. I call such correlation functions *classical EPR-B correlations*. It turns out that the other three (binary) symmetries can then be obtained "for free": they are exhibited by the correlation function, and can be imposed on the underlying physics by adding an underlying randomisation level. We end up with a simple probabilistic description of all possible classical EPR-B correlations in terms of a "spinning coloured disk" model, and a research programme: describe these functions in a concise analytic way.

Keywords: Singlet correlations, twisted Malus law, EPR-B experiments, local hidden variables, spinning coloured disk model, spinning coloured ball model, simulation models

1. The problem, in a picture

Just about every introduction to Bell's theorem contains the following picture.

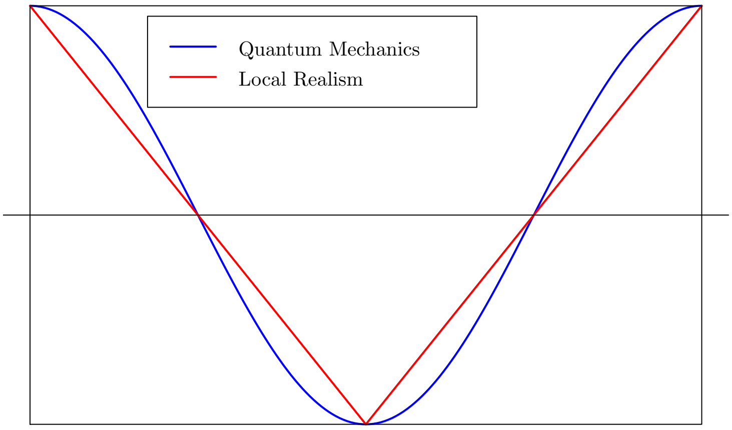

Figure 1. Correlation, between -1 and $+1$, plotted against angle between measurement directions, from $-\pi$ to π

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The accompanying text claims that the triangle wave is the prediction of local realism, the beautiful (negative) cosine curve is the prediction of quantum mechanics. However, the triangle wave is just one of many *possible* correlation functions allowed by local realism, but by no means the only one. In some sense it might be the best ... but in what sense? I am unaware of a decent mathematical answer to this question. The following notes make an attempt to describe concisely, all that local realism allows when some key features of the curve are insisted on. The picture shows the correlations obtained when two parties measure something called "spin", each on one of two correlated (or entangled) particles; they can measure in any direction in a certain plane; their measurement outcome is binary (spin up or spin down in the chosen direction). The x axis of the graph represents the angle between the two measurement directions.

In theory one could also measure in directions in 3D space. One can clearly ask the corresponding new mathematical question. That question was formulated, and partially answered, by Kent and Pitalúa-García (2014) [1]. The present paper was first posted to arXiv in 2013, at the same time as an arXiv preprint of the Kent and Pitalúa-García paper. As far as I know, there have been no further developments since then!

I recently became aware of an interesting, much earlier, paper by Toner and Bacon (2003) [2] which actually rather satisfactorily solves a somewhat related problem: how many bits of classical communication are needed for two classical agents, Alice and Bob, to reproduce the 3D quantum correlations? The answer turns out to be just one bit (on average) and the required communication protocol is "simple" according to the authors. Now, making use of the *detection loophole* to classically simulate quantum correlations is also a case of using, in effect, classical communication, namely by rejecting some runs: Alice and Bob are allowed to "try all over again" if either experiences an adverse event. One wants protocols such that each of the two parties ends up with an outcome and moreover both know that the other has an outcome too. It seems to me that the various problems – faking *all* the singlet correlations (either with settings in S^1 or with settings in S^2) and doing it with the detection loophole, or with classical communication, or without any cheating at all but just optimising a sensible expression of the numerical quality of the approximation – are quite subtly related, and that it should be possible to convert solutions of one problem to a solution of another.

The original version of this paper was written in 2013 and appeared on arXiv as https://arxiv.org/abs/1312.6403. It has been resting, waiting for new ideas, for five years.

2. The problem, formalised

According to quantum mechanics, it is in principle possible to arrange the following experiment. Alice and Bob are in their respective laboratories at distant locations, but have set up all kinds of practical arrangements in advance. In particular, they are at rest with respect to the same inertial frame of reference and they have set up accordingly synchronised clocks. They both possess some kind of random number generators and are able to simultaneously and independently choose angles α and β in the interval $[0, 2\pi)$ according to any desired probability distributions. They each input their chosen angle into a physical device in each of their laboratories and after a short time interval, the device responds with a binary output, which we shall code numerically as ± 1. The length of time between initiating the choice of random angle and output of ± 1 is so short that a signal travelling at the speed of light and carrying Alice's chosen angle from Alice's to Bob's lab could not arrive till after Bob's output is fixed, and vice-versa. We call the inputs *settings* and the outputs *outcomes*.

This can now be repeated independently as many times as one likes, say N times, resulting in synchronised lists, all of length N, of settings (angles) and outcomes (± 1) at the two locations. We call these N repetitions *runs*.

In an ideal experiment, the outcomes of each separate run (a pair of random numbers ± 1) are statistically independent from those of other runs and distributed according to the following conditional probability law (conditional on the chosen settings α and β):

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$$\Pr(++) \;=\; \Pr(--) \;=\; \tfrac{1}{4}\Big(1-\cos(\alpha-\beta)\Big),$$
$$\Pr(+-) \;=\; \Pr(-+) \;=\; \tfrac{1}{4}\Big(1+\cos(\alpha-\beta)\Big).$$

These joint probabilities are a manifestation of the *twisted Malus law*.

Notice that these joint probabilities possess a large number of symmetries: they are symmetric with respect to rotation, parity switch (exchange of outcome values ± 1), exchange of the two parties (Alice and Bob), and chirality switch (exchange of clockwise and anti-clockwise). They also reflect two "certainty" relations: at exactly equal settings, outcomes are opposite with probability one; at exactly opposed settings, outcomes are equal with probability one. As a consequence of the symmetry in outcome values, we also have a "complete randomness" property: each outcome separately, whatever the setting, is a symmetric Bernoulli trial with outcomes $\{-1,+1\}$.

3. Classical physical representation

If these outcomes were generated by a classical (i.e., local realist) physical model, we would be able to construct simultaneously defined random variables $A(\alpha)$ and $B(\beta)$, α, $\beta \in [0,2\pi)$, such that in one run of the experiment, all these random variables are realised simultaneously, and the actually observed outcomes are merely selected by the *independent* choice of settings α, $\beta \in [0,2\pi)$. It seems physically reasonable to assume that the joint probability distribution of the complete stochastic processes A, B satisfies the same symmetries: i.e., the symmetries observed in the twisted Malus law reflect underlying (physical, fundamental) model symmetries.

In fact, under measurability conditions, we can symmetrize a given model, converting a non-symmetric model to a fully symmetric one: this is because probabilistic mixing of stochastic processes with the same, given, marginal distributions, results in a new stochastic process with the same marginal distributions. I will only explicitly impose the symmetry under rotations. As we will see, because of the "certainty relations" which we also impose, the other symmetries in the correlation functions are automatically true. So whether or not the other symmetries are imposed on the underlying process makes no difference to the family of correlation functions which can be generated by the model.

I already used the difficult word measurability and I need now to pay this tricky topic some further attention. So far we have just assumed the existence of a single probability space on which are defined two indexed families of binary random variables $A(\alpha)$ and $B(\beta)$. It follows that joint probability distributions of any finite number of these random variables are also well defined. The random function $A(\alpha), \alpha \in [0,2\pi)$ takes values in $\{-1,+1\}$ but might in principle be extraordinarily irregular. However, I would like to argue that it can at least be approximated by a piece-wise constant function making only a finite number of changes of value: in other words, $\{\alpha \in [0,2\pi) : A(\alpha) = 1\}$ is a finite union of intervals, and so of course too is its complement $\{\alpha \in [0,2\pi) : A(\alpha) = -1\}$; and the same for B.

Here is one argument, heuristic to be sure. First of all, in my opinion there is no loss of *physical* generality in assuming that any realisation of the set $\{\alpha \in [0,2\pi) : A(\alpha) = 1\}$ is a Borel measurable subset of $[0,2\pi)$. Here is a mathematical argument for this physical claim. If one is prepared to reject the axiom of choice, it is possible to axiomatically demand that all subsets of the real line are measurable. The mathematical existence of non-measurable sets is apparently a matter of mathematical taste, it refers to how large or small we want our abstract mathematical universe to be. Note that with the axiom of choice, though they exist mathematically, non-measurable sets cannot be constructed or computed or exhibited in any sense.

So let's assume the realisations of the just mentioned sets are at least nice enough that they are Borel measurable. Now, any bounded measurable subset of real numbers can be arbitrarily well approximated by finite unions of intervals in the following sense: for any $\epsilon > 0$ one can find an

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approximating set (a finite union of intervals) such that the set-theoretic difference between the set being approximated and its approximation can be covered by a countable set of intervals the sum of whose lengths is at most ϵ.

These ideas lead to me to propose that (up to an arbitrarily good approximation, in a precise mathematical sense which has yet to be determined), the sample paths of A and B are very regular indeed: only a finite number of jumps between ± 1.

An alternative approach would be to only consider a finite number of different angles altogether: for instance, only angles which are expressed as a whole numbers of degrees, minutes, and seconds. Instead of continuous rotational symmetry we have discrete rotational symmetry. There are now no measurability or regularity issues at all. The analysis we are going to make in the regular, continuous index case, can also be made in the discrete index case, subject to the obvious modifications. We will remark later on how the results would be modified.

So let us suppose that (perhaps after an initial approximation), the stochastic processes A and B are such that the sets of angles where they take their possible values ± 1 are finite unions of intervals. It can be shown from this that the angles at which the value jumps between ± 1 , and the number of those angles, are random variables (they can be written as limits of functions of finitely many coordinates).

Think of the sample paths of A and B as two functions on the unit circle. Because of one of the "certainty relations" between A and B it follows that the sample path of B is identical to the path of A after rotation through an angle π. It also follows from the other that the sample path of B is identical to the negative of the path of A. Thus the two sample paths are determined completely by the path of A on the first half of the circle, $[0, \pi)$. The negative of the same path is repeated, for A, on $[\pi, 2\pi)$; and the path of B is the path of A shifted (rotated) a distance π.

Let us suppose that the joint probability distribution of A and B is invariant under rotation, just as the correlation function is. We can write the joint probability distribution of the processes A and B as a probabilistic mixture over the numbers of jumps of each process. It can be argued (details in a later to be written appendix?) that any such probability distribution can be built up as follows, according to what I call the randomised spinning coloured disk model.

4. The randomised spinning coloured disk

Pick (at random) an even number $k \geq 0$ and angles $0 < \theta_1 < \cdots < \theta_k < \pi$. Colour the $k + 1$ segments $(0, \theta_1)$, (θ_1, θ_2), $\ldots$, (θ_k, π) "black", "white", $\ldots$, "black" . If $k = 0$ there is just one segment $(0, \pi)$, and it is coloured black. Colour $(\pi, 2\pi)$ in complementary way: $(\pi, \pi + \theta_1)$, $\ldots$, $(\pi + \theta_k, 2\pi)$ are coloured "white", "black", $\ldots$, "white". The colour assigned to end-points does not matter. We have now coloured the entire unit circle with our two colours "white" and "black" except that we didn't determine the colours of the finitely many points on the boundaries between intervals of fixed colour. Each point is opposite a point of the opposite colour, so the total length of white segments and the total length of black segments are equal. Now give the coloured unit circle a random rotation chosen uniformly between 0 and 2π. Define $A(\alpha) = \pm 1$ according to whether the colour of the randomly rotated circle at point α is black or white. Define B as the rotation of A through the angle π. A and B aren't defined at finitely many angles, but we don't have to worry about this. The probability is zero that A is undefined at any particular given point.

If we choose k at random according to an arbitrary probability distribution over the even non-negative integers, and then $0 < \theta_1 < \ldots \theta_k < \pi$ according to an arbitrary joint distribution given k, and finally choose a rotation of the coloured circle completely at random, we have defined two stochastic processes A and B, except at finitely many points, such that the joint probability distribution of A and B is invariant under rotation and possesses the desired "certainty relations". My mathematical claim is that this recipe generates the class of all possible processes A and B subject to rotation invariance, certainty relations, and regularity of sample paths (finitely many sign changes). It therefore generates all of these process's correlation functions. By adding to these correlation functions also all possible limits (in some precise sense) of such functions one also includes correlations of processes which can be obtained as limits (in some precise sense) of regular ones.

Marginally, each $A(\alpha)$ and $B(\beta)$ represents a symmetric Bernoulli trial, each is a symmetric Rademacher random variable. The joint distribution of the two stochastic processes A, B is not necessarily invariant under any of chirality switch, parity switch, or component exchange. However, the joint distribution of $A(\alpha), B(\beta)$ for any fixed pair of angles (α, β) does possess all these invariances. This joint probability distribution consists of four probabilities adding to one. Since its two margins are symmetric Bernoulli trials, the joint distribution is completely determined by the single number $\Pr(A(\alpha) = B(\beta)) = \Pr(A(\alpha) \neq A(\beta)$ which is the probability that the number of colour switches on the randomly rotated coloured unit circle between angles α and β is odd. This probability can only depend on the *absolute value* of the difference between these angles, hence is invariant under parity, party and chirality switch.

The raw product moment between $A(\alpha)$ and $B(\beta)$ is the expectation of its product, and is easily seen to be equal to $2\Pr(A(\alpha) = B(\beta)) - 1$. Within our local realist model, $B(\beta) = -A(\beta)$ so that is the same as $2\Pr(A(\alpha) \neq A(\beta)) - 1$. If we write $\delta = \beta - \alpha$ then we obtain from this the correlation function $\rho(\gamma) = 2\Pr(A(\gamma) \neq A(0)) - 1$.

5. Computation

The arguments given so far show that the set of all possible correlation functions ρ is the set of all convex combinations (including continuous combinations) of correlation functions corresponding to some even number $k \geq 0$ and some $0 < \theta_1 < \ldots \theta_k < \pi$. To each k and $\theta_1, \ldots, \theta_k$ there corresponds a colouring of the unit circle. This colouring determines stochastic processes A and B. Their correlation function ρ can be described in terms of the colouring of the unit circle as follows: pick a point uniformly at random on the unit circle. Then $\rho(\gamma)$ is the probability that the uniform random point has the opposite colour (black or white) to that of the point at a distance γ clockwise around the circle from the first chosen point.

The following R script calculates (by Monte Carlo integration) and graphs a sample of 12 such correlation functions, all with $k = 4$, but with various values of θ_1 to θ_4.

```
oneplot <- function() {
   if (nswitch%%2 == 0)  times <- c(times,1)
   timesplus <- c(times,times+1,times+2,times+3)
   count <- function(t,d) sum(timesplus > t & timesplus <= t+d)
   points <- seq(from=0,to=1,by=0.01)
   numbers <- outer(data,points,Vectorize(count))
   corr <- 2*(apply(numbers%%2,2,sum)/1000)-1
   correlation <- c(corr[100:1],corr)
   difference <- pi*(c(points,1+points[2:101]))
   plot(difference,correlation,
       type="l",bty="n",ann=FALSE,xaxt="n",yaxt="n")
   lines(c(0,2*pi,2*pi,0,0),c(+1,+1,-1,-1,+1))
   abline(h=0)
   lines(difference,cos(difference),col="blue")
   lines(c(0,pi,2*pi),c(+1,-1,+1),col="red")
}
nswitch <- 4
set.seed(11091951)
par(mfrow=c(3,4),oma=c(0,0,0,0),mar=c(0,0,0,0))
for( i in (1:12) ) {
   times <- sort(runif(nswitch))
   data <- 2*runif(1000)
   oneplot()
}
```

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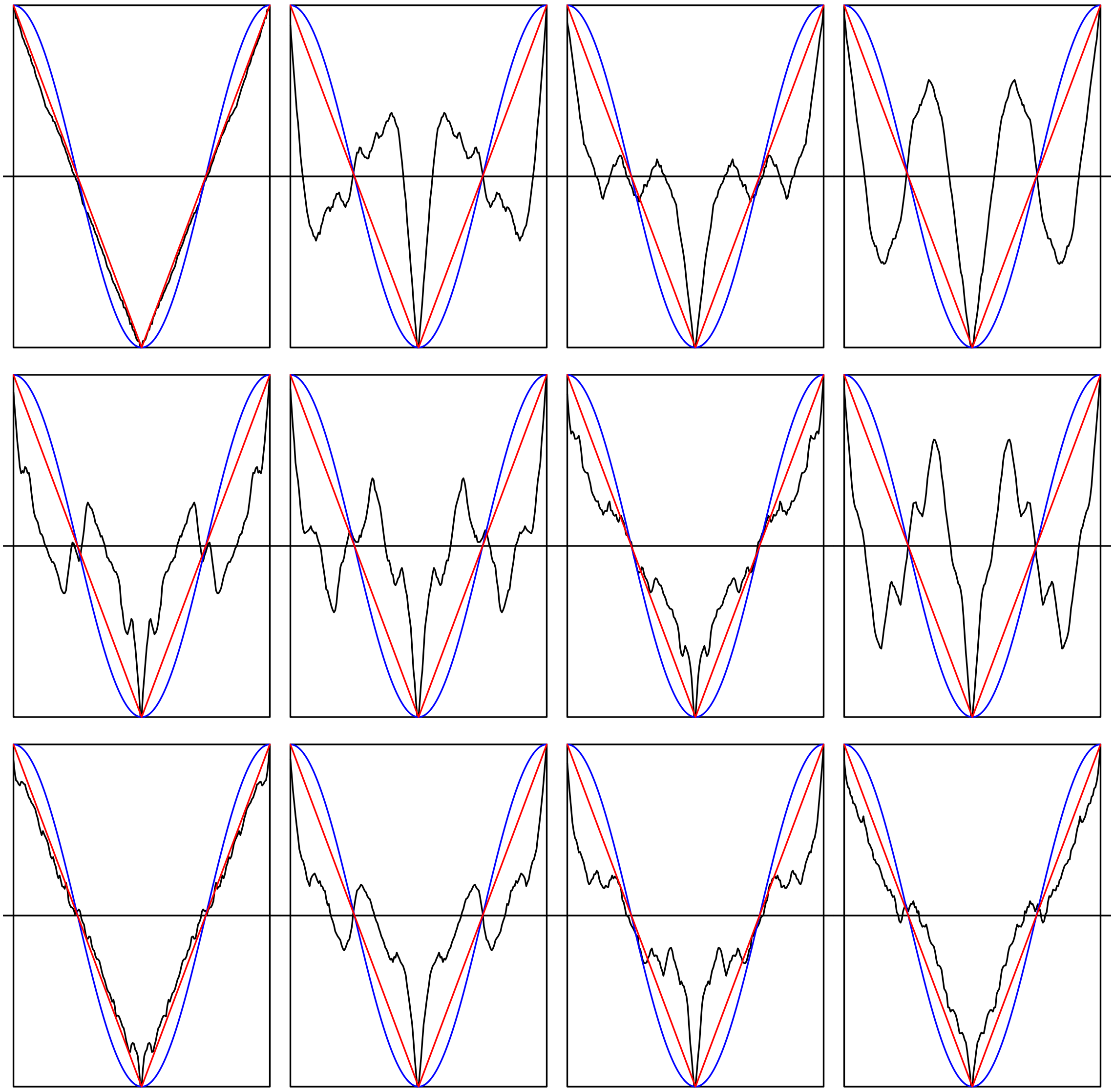

Figure 2. 12 random classical EPR-B correlations (in black), with coloured disks of 10 segments. Notice that some exceed the triangle wave (red) at some points where the triangle wave is positive.

The reader is invited to replace the assignment `nswitch <- 4` (“`nswitch` gets the value 4”) by assignments of other even numbers and look at the results.

6. Conclusion: what next?

This note is a research proposal rather than a report of definitive results. There are two main directions to explore, mathematically. One direction concerns the investigation of regularity conditions and approximation. I think it is mathematically important to tidy up the details but I do not think this direction is physically or metaphysically interesting. Suppose we only considered measurement angles which were whole numbers of degrees, minutes and seconds. Then there are no regularity issues at all and the representation of all classical physical EPR-B models through a random spinning disk model is mathematically precise: the lengths of the segments of the disk of different colours are restricted to be whole numbers of seconds; and at the end, we only investigate the correlation functions at whole numbers of seconds. Thus we finish up looking at a slightly smaller class of correlation functions, and we only look at them at on the very fine lattice of “whole second” angles.

The other direction is more interesting. Let’s accept the class of correlation functions arising from the spinning disk model. Can we analytically describe this class of functions, or the topological closure of this class of functions (according to a convenient but meaningful topology) in an alternative succinct way? For instance, is there an elegant description of the characteristic functions of these correlation functions?

An intriguing possible direction involving indeed the Fourier transform is suggested by some lecture slides by Steve Gull, going back now 35 years. On his "MaxEnt 2009" web-page http://www.mrao.cam.ac.uk/~steve/maxent2009/, under the heading *Quantum Acausality and Bell's Theorem*, Steve writes

> Many years ago (about 1984), I used to give a Mathematical Physics course to the Part II students. I illustrated the quantum paradox covered by Bell's theorem by showing that you can't program two independently running computers to mimic the results of spin measurements on two spin-1/2 particles in a singlet state. I believe this demonstration is actually better than Bell's original argument.

His slides can be found at http://www.mrao.cam.ac.uk/~steve/maxent2009/images/bell.pdf. They sketch a lovely proof of Bell's theorem using the fact that the Fourier transform of the correlation function ρ has to equal the expected squared absolute value of the Fourier transform of the random function A. The actual correlation function only has three non-zero Fourier coefficients. However the Fourier transform of any realisation of A must have infinitely many non-zero coefficients, since otherwise it could not have any jumps. Since their absolute values get squared before averaging, there is no way that all but three can vanish.

I have two ideas where to go next. Firstly, if we insist that the correlation function not only has the symmetries we want, but is also monotone decreasing (between 0 and π) that will further narrow the possibilities. Maybe there is only one left – the triangle wave of my first picture? Classical correlations can exceed quantum correlations but, it seems, only at the cost of oscillations. Secondly, we can express the L_2 distance between two curves in terms of the L_2 distance between their Fourier transforms: might that give us a way to show that the triangle wave is the closest approximation to the cosine?

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Conflicts of Interest: The author declares no conflict of interest

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