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Separation axioms interval-valued fuzzy soft topology via quasi-neighbourhood structure

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Abstract

In this study, we present the concept of interval-valued fuzzy soft point and then introduce the notions of neighborhood and quasi-neighbourhood of it in interval-valued fuzzy soft topological spaces. Separation axioms in interval-valued fuzzy soft topology, so-called q-T_i for $i = 0, 1, 2, 3, 4$, is introduced and some of its basic properties are also studied.

Keywords:interval-valued fuzzy soft set, interval-valued fuzzy soft topology, interval-valued fuzzy soft point, interval-valued fuzzy soft neighbourhood, interval-valued fuzzy soft quasi-neighbourhood, interval-valued fuzzy soft separation axioms.

1 Introduction

In 1999, Molodtsov[1] proposed a new mathematical approach known as soft set theory, for dealing with uncertainties and vagueness.Traditinal tools such as fuzzy sets[25] and rough sets[26], cannot clearly defined objects. Where the soft set theory is different from traditional tools for dealing with uncertainties. A soft set, defined by a collection of approximate descriptions of an object based on parameters by a given set-valued map. Maji et al.[3]initiated the research on both fuzzy sets and soft sets hybrid structures called fuzzy soft sets and presented a concept was subsequently discussed by many researchers. Different extensions of the classical fuzzy soft sets were introduced, such as generalized fuzzy soft sets[4], intuitionist fuzzy soft sets[5,6] , vague soft sets[7], interval-valued fuzzy soft sets[8] and interval valued intuitive fuzzy soft sets[9]. In particular, to alleviate some disadvantages of fuzzy soft sets, interval-valued fuzzy soft sets was introduced where no objective procedure is available to select the crisp membership degree of elements in a fuzzy soft sets. Tanya and Kandemir [10] started topological studies of fuzzy soft sets. They used classical concept of topology to construct a topological space over a fuzzy

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soft set and named it fuzzy soft topology. They also studied some fundamental topological properties for fuzzy soft topology, such as interior, closure, and base. Later Simsekler and Yuksel[11] studied fuzzy soft topological space in the case of Tanay and Kandemir[10]. But they established the concept of fuzzy soft topology over a fuzzy soft set with a set of fixed parameters and considered some topological concepts for fuzzy soft topological spaces such as base, subbase, neighbourhood, and Q-neighbourhood. Roy and Samanta [15] noted a new concept of fuzzy soft topology . They suggested the notion of fuzzy soft topology over an ordinary set by adding fuzzy soft subsets of it where everywhere parameter set is supposed to be fixed. Then in[12], they continued to study fuzzy soft topology and established a fuzzy soft point definition and various neighbourhood structures. Atmaca and Zorlutuna [16] were considering the concept of soft quasi-coincidence for fuzzy soft sets . By applying this new concept, they also studied the basic topological notions such as interior and closure for a fuzzy soft sets. The concept of product fuzzy soft topology and the boundary fuzzy soft topology have introduced by Zahedi et al.[13],[14] and some of its properties have been studied. They also suggested a new definition for fuzzy soft point and then,different neighbourhood structures. Separation axioms of fuzzy topological and fuzzy soft topological, it had been studied by many authors[18,19,21,22,23]. The aim of this work is to develop interval-valued fuzzy soft separation axioms. We start with preliminaries and then, give definition of interval-valued fuzzy soft point as a generalization of interval-valued fuzzy point and fuzzy soft point, both in order to create different neighborhood structures in interval-valued fuzzy soft topological space in sections 3 and 4.Finally, in section 5, the notion of separation axioms $q\text{-}T_i, i = 0, 1, 2, 3, 4$, in interval-valued fuzzy soft topology is introduced and some of its basic properties were also studied.

2 preliminaries

Throughout this paper X is the set of objects and E is the set of parameters. The set of all subset, of X is denoted by $P(X)$ and $A \subset E$. Shows a subset of E.

Definition 2.1. [1] *A pair (f, A) is called a soft set over X, where f is a mapping given by*

$$f : A \to P(X).$$

For any parameter $e \in A, f(e) \subset X$ may be considered as the set e-approximate elements of the soft set (f, A). In other words, the soft set is not a kind of set. but a parameterized family of subset of the set X.

Before introduce the notion of the interval-valued fuzzy soft sets, we give the concept of interval-valued fuzzy set.

Definition 2.2. [20] *An interval-valued fuzzy (IVF) set over X, is defined by the membership function $f : X \to int([0, 1])$, where $int([0, 1])$ denotes the set of all closed*

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subintervals of $[0,1]$. *Suppose that* $x \in X$. *Then* $f(x) = [f^-(x), f^+(x)]$ *is called the degree of membership of the element* $x \in X$, *where* $f^-(x)$ *and* f^+ *are the lower and upper degree of membership of* x *and* $0 < f^-(x) < f^+(x) < 1$.

Yang et al.[8] suggested the concept of interval-valued fuzzy soft set by combined of interval-valued fuzzy set and soft set as below.

Definition 2.3. [8] *An interval-valued fuzzy soft*$(IVFS)$ *set, denoted by* f_E *or* (f,E) *over* X, *is defined by the mapping* $f : E \to \mathcal{IVF}(X)$, *where* $\mathcal{IVF}(X)$ *is the set of all interval-valued fuzzy set over* X. *For any* $e \in E$, $f(e)$ *can be written as an interval-valued fuzzy set such that* $f(e) = \{\langle x, [f_e^-(x), f_e^+(x)]\rangle : x \in X\}$ *where* $f_e^-(x) and f_e^+(x)$ *are the lower and upper degrees of membership,of* x *with respect to* e, *respectively, where* $0 \leq f_e^-(x) \leq f_e^+(x) \leq 1$.

Note that$\mathcal{IVFS}(X,E)$ shows the set of all $IVFS$-set over X.

Definition 2.4. [8] *Let* f_A *and* g_B *be two* $IVFS$*-sets over*X. *We say:*

1. f_A *is an interval-valued fuzzy soft subset of* g_B, *denoted by* $f_A \tilde{\leq} g_B$, *if and only if:*

(i) $A \leq B$,

(ii) *For all* $e \in A$, $f_e^-(x) \leq g_e^-(x)$ *and* $f_e^+(x) \leq g_e^+(x), \forall x \in X$.

2. $f_A = g_B$ *if and only if* $f_A \tilde{\leq} g_B$ *and* $g_A \tilde{\leq} f_B$.

3. *The union of two* $IVFS$ *sets* f_A *and* g_B, *denoted by* $f_A \tilde{\vee} g_B$, *is the*$IVFS$ *set* $(f \vee g, C)$, *where* $C = A \cup B$ *and for all* $e \in C$, *we have*

$$(f \vee g)_e(x) = \begin{cases} [f_e^-(x), f_e^+(x)], & e \in A - B \\ [g_e^-(x), g_e^+(x)], & e \in B - A, \\ [max(f_e^-(x), g_e^-(x), max(f_e^+(x), g_e^+(x)] & e \in A \cap B. \end{cases}$$

for all $x \in X$.

4. *The intersection of two* $IVFS$ *sets*f_A *and* g_B,*denoted by* $f_A \tilde{\wedge} g_B$, *is the*$IVFS$ *set* $(f \wedge g, C)$,*where* $C = A \cap B$ *and for all* $e \in C$, *we have* $(f \wedge g)_e(x) = [min f_e^-(x), g_e^-(x), min f_e^+(x), g_e^+(x)]$ *for all* $x \in X$.

5. *The complement of* $IVFS$ *set* f_A *is denoted by* $f_A^c(x)$ *where for all* $e \in A$ *we have* $f_e^c(x) = [1 - f_e^+(x), 1 - f_e^-(x)]$.

Definition 2.5. [8] *Let* f_E *be an* $IVFS$ *set.*

1. *The interval-valued fuzzy soft set* f_E *is called null interval-valued fuzzy soft set, denoted by* $\emptyset_E$, *if* $f_e^-(x) = f_e^+(x) = 0$, *for all* $x \in X, e \in E$.

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2. *The interval-valued fuzzy soft set f_E is called absolute interval-valued fuzzy soft set, denoted by X_E, if $f_e^-(x) = f_e^+(x) = 1$, for all $x \in X, e \in E$.*

Motivated by definition of soft mapping, discussed in [27], we define the concept of $IVFS$ mapping as the following:

Definition 2.6. *Suppose f_A is an $IVFS$ set over X_1 and g_B is an $IVFS$ set over X_2 where $A \subseteq E_1$ and $B \subseteq E_2$. If $\Phi_u : X_1 \to X_2$ and $\Phi_p : E_1 \to E_2$ are two mappings, then*

1. *The map $\Phi : \mathcal{IVFS}(X_1, E_1) \to \mathcal{IVFS}(X_2, E_2)$ is called an IVFS-map from X_1 to X_2 and for any $y \in X_2$ and $\varepsilon \in B \subseteq E_2$, The lower image and the upper image of f_A under Φ is the $IVFS\Phi(f_A)$ over X_2, respectively, defined as below:*

$$[\Phi(f^-)](\varepsilon)(y) = \begin{cases} sup_{x\in\Phi_u{}^{-1}(y)}[sup_{e\in\Phi_p{}^{-1}\cap A} f^-(e)](x), & if \Phi_p^{-1}(\varepsilon)\cap A \neq \Phi and \Phi_u^{-1}(y) \neq \Phi \\ 0, & otherwise, \end{cases}$$

$$[\Phi(f^+)](\varepsilon)(y) = \begin{cases} sup_{x\in\Phi_u{}^{-1}(y)}[sup_{e\in\Phi_p{}^{-1}\cap A} f^+(e)](x), & if \Phi_p^{-1}(\varepsilon)\cap A \neq \Phi and \Phi_u^{-1}(y) \neq \Phi \\ 0, & otherwise. \end{cases}$$

2. *Let $\Phi : \mathcal{IVFS}(X_1, E_1) \to \mathcal{IVFS}(X_2, E_2)$ be an $IVFS$-map from X_1 to X_2. The lower inverse image and the upper inverse image of $IVFS$ g_B under Φ denoting by $\Phi^{-1}(g_B)$, is an $IVFS$ over X_1, respectively, that for all $x \in X_1$ and $e \in E_1$ it is defined as below:*

$$[\Phi^{-1}(g^-)](e)(x) = \begin{cases} g^-_{\Phi_{p(e)}}\Phi_u(x), & if\ \Phi_p(e) \in B \\ 0, & otherwise, \end{cases}$$

$$[\Phi^{-1}(g^+)](e)(x) = \begin{cases} g^+_{\Phi_{p(e)}}\Phi_u(x), & if\ \Phi_p(e) \in B \\ 0 & otherwise. \end{cases}$$

Proposition 2.1. *Let $\Phi : \mathcal{IVFS}(X, E) \to \mathcal{IVFS}(Y, F)$ be an $IVFS$-mapping between X and X, and Let $\{f_{iA}\}_{i\in J} \subset \mathcal{IVFS}(X, E)$ and $\{g_{iB}\}_{i\in J} \subset \mathcal{IVFS}(Y, F)$ be two families of $IVFS$ sets over X and Y, respectively, where $A \subseteq E$ and $B \subseteq F$, then the following properties hold.*

1. *$[\Phi(f_{jA})]^c \tilde{\leq} \Phi(f_{jA})^c$ for each $j \in J$.*
2. *$[\Phi^{-1}(g_{jB})]^c = \Phi^{-1}(g_{jB})^c$ for each $j \in J$.*
3. *If $g_{iB} \tilde{\leq} g_{jB}$, then $\Phi^{-1}(g_{iB}) \tilde{\leq} \Phi^{-1}(g_{jB})$ for each $i, j \in J$.*
4. *If $f_{iA} \tilde{\leq} f_{jA}$, then $\Phi(f_{iA}) \tilde{\leq} \Phi(f_{jA})$ for each $i, j \in J$.*
5. *$\Phi(\Phi^{-1}(g_{jB})) \leq g_{jB}$ for each $j \in J$.*
6. *$(f_{jA}) \leq \Phi(\Phi^{-1}(F_{jA}))$ for each $j \in J$.*

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7. $\Phi[\tilde{\vee}_{j\in J} f_{jA}] = \tilde{\vee}_{j\in J}\Phi(f_{jA})$ *and* $\Phi^{-1}[\tilde{\vee}_{j\in J} g_{jB}] = \tilde{\vee}_{j\in J}\Phi^{-1}(g_{jB})$.

8. $\Phi[\tilde{\wedge}_{j\in J} f_{jA}] = \tilde{\wedge}_{j\in J}\Phi(f_{jA})$ *and* $\Phi^{-1}[\tilde{\wedge}_{j\in J} g_{jB}] = \tilde{\wedge}_{j\in J}\Phi^{-1}(g_{jB})$.

Proof. We only prove part(7). The other parts follow the similar technique. For any $k \in F, y \in Y$, and $a \in A$ Then

$$\begin{aligned}
\Phi[\tilde{\vee}_{j\in J} f_{jA}](k)(y) &= sup_{x\in\Phi_u^{-}(y)}(sup_{z\in\Phi_p^{-1}(k)}(\tilde{\vee}_{j\in J}) f_{jA})(z)(x) \\
&= sup_{x\in\Phi_u^{-1}(y)}(sup_{z\in\Phi_p^{-1}(k)}(\max_{j\in J}([f_{ja}^{-}, f_{ja}^{+}])))(k)(y) \\
&= sup_{x\in\Phi_u^{-1}(y)}(\max_{j\in J}(sup_{z\in\Phi_p^{-1}(k)}[f_{ja}^{-}(k), f_{ja}^{+}(k)]))(y) \\
&= \max_{j\in J}(sup_{x\in\Phi_u^{-1}(y)}(sup_{z\in\Phi_p^{-1}(k)}[f_{ja}^{-}(k))(y), f_{ja}^{+}(k)]))(y)] \\
&= \max_{j\in J}(sup_{x\in\Phi_u^{-}(y)}(sup_{z\in\Phi_p^{-1}(k)} f_{jA}(k)(y))) \\
&= \max_{j\in J}\Phi(f_{jA})(k)(y) \\
&= \tilde{\vee}_{j\in J}\Phi(f_{jA})(k)(y).
\end{aligned}$$

Now we prove that $\Phi^{-1}[\tilde{\vee}_{j\in J} g_{jB}] = \tilde{\vee}_{j\in J}\Phi^{-1}(g_{jB})$. For any $e \in E, x \in X$ and $b \in B$

$$\begin{aligned}
\Phi^{-1}[\tilde{\vee}_{j\in J} g_{jB}](e)(x) &= (\tilde{\vee}_{j\in J}) g_{jB}(\Phi_p(e))(\Phi_u(x)) \\
&= [\max_{j\in J} g_{jb}^{-}, \max_{j\in J} g_{jb}^{+}](\Phi_p(e))(\Phi_u(x)) \\
&= [[\max_{j\in J} g_{jb}^{-}(\Phi_p(e))(\Phi_u(x)), \max_{j\in J} g_{jb}^{+}(\Phi_p(e))(\Phi_u(x))] \\
&= [\max_{j\in J}\Phi_u^{-1}(g_{jb}^{-})(e)(x), \max_{j\in J}\Phi_u^{-1}(g_{jb}^{+})(e)(x)] \\
&= \max_{j\in J}[\Phi_u^{-1}(g_{jb}^{-})(e)(x), \Phi_u^{-1}(g_{jb}^{+})(e)(x)] \\
&= \max_{j\in J}\Phi_u^{-1}(g_{jB})(e)(x) \\
&= \tilde{\vee}_{j\in J}\Phi_u^{-1}(g_{jB})(e)(x).
\end{aligned}$$

□

3 Interval-valued fuzzy soft topological spaces

The interval-valued fuzzy topology $IVFT$ was discussed by Mondal and Samanta [17]. In this section, we recall their definition and then present different neighborhood structures in the interval-valued fuzzy soft topology ($IVFST$).

Definition 3.1. *Let X be a non-empty set and let τ be a collection of interval valued fuzzy soft set over X with the following properties:*

(i) $\emptyset_E$, X_E *belong to* τ,

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(ii) *If* f_{1E}, f_{2E} *are* $IVFS$ *sets belong to* τ.

As the ordinary topologies, the indiscrete $IVFST$ *over* X *contains only* $\emptyset_E$ *and* X_E*, while the discrete* $IVFST$ *over* X *contains all* $IVFS$ *sets. Every member of* τ *is called an interval-valued fuzzy soft open set (*$IVFS$*-open) in* X*. The complement of an* $IVFS$*-open set is said an* $IVFS$*-closed set. Then* $f_{1E}\tilde{\wedge}f_{2E}$ *belong to* τ.

(iii) *If the collection of* $IVFS$ *sets* $\{f_{jE}|j \in J\}$ *where* J *is an index set, belong to* τ *then* $\tilde{\vee}_{j\in J}f_{jE}$ *belong to* τ,

then τ *is called interval-valued fuzzy soft topology over* X *and the triplet* (X, E, τ) *is called the interval-valued fuzzy soft topological space* $(IVFST)$.

Remark 3.1. *If* $f_e^-(x) = f_e^+(x) = a \in [0,1]$*. Then we put* $[f_e^-(x), f_e^+(x)] = [a,a] = a$.

Example 3.1. *Let* $X = [0,1]$ *and* E *be any subset of* X*. Consider* $IVFS$ *set* f_E *over* X *by the mapping*

$$f : E \to \mathcal{IVF}([0,1])$$

Such that for any $e \in E, x \in X$

$$\tilde{f}_e(x) = \begin{cases} 1 & 0 \le x \le e \\ 0 & e < x \le 1. \end{cases}$$

The collection $\tau = \{\Phi_E, X_E, f_E\}$ *is an* $IVFST$ *over* X.

1. *Clearly* $X_E, \emptyset_E \in \tau$.

2. *Let* $\{f_{jE}\}_{j\in J}$ *is a sub-family of* τ *where for any* $j \in J$ *if* $x \in X$ *such that for all* $e \in E$

$$f_{je}(x) = \begin{cases} 1 & 0 \le x \le e \\ 0 & e < x \le 1. \end{cases}$$

Since

$$\vee_j f_{je}(x) = \begin{cases} 1 & 0 \le x \le e \\ 0 & e < x \le 1 \end{cases}$$

Then $\tilde{\vee}_j f_{jE} \in \tau$.

3. *Let* $f_E, g_E \in \tau$*, where*

$$f_e(x) = \begin{cases} 1 & 0 \le x \le e \\ 0 & e < x \le 1, \end{cases}$$

and

$$g_e(x) = \begin{cases} 1 & 0 \le x \le e \\ 0 & e < x \le 1. \end{cases}$$

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Since

$$f_e(x)\wedge g_e(x) = \begin{cases} 1 & 0 \leq x \leq e \\ 0 & e < x \leq 1. \end{cases}$$

Thus, $f_E \wedge g_E \in \tau$.

Example 3.2. [24] *Let* $\mathbb{R}$ *be the set of all real numbers with the usual topology* τ_u *where* $\tau_u = \langle\{(a,b), a, b \in \mathbb{R}\}\rangle$ *and* E *be a parameter set. Let* $U = (a,b) \subset \mathbb{R}$ *be an open interval in* $\mathbb{R}$*, we define* $IVFS$ $\tilde{U}_E$ *over* $\mathbb{R}$ *by the mapping*

$$\tilde{U} : E \to (Int[0,1])^{\mathbb{R}}$$

such that for all $x \in \mathbb{R}$

$$\tilde{U}_e(x) = \begin{cases} 1 & x \in (a,b) \\ 0 & x \notin (a,b). \end{cases}$$

The family $\{\tilde{U}_E : (a,b) \subset \mathbb{R}, \forall a, b \in \mathbb{R}\}$ *generates an* $IVFS$ *over* $\mathbb{R}$*, we denote it by* $\tau_u^{(IVFS)}$:

1. *Clearly* $\mathbb{R}_E, \emptyset_E \in \tau_u^{(IVFS)}$ *where for all* $e \in E$ *and* $k \in \mathbb{R}, \mathbb{R}_E(e)(k) = [1,1]$ *and* $\emptyset_e(k) = 0$

2. *Let* $\{\tilde{U}_{jE}\}_{j\in J}$ *is a sub-family of* $\tau_u^{(IVFS)}$ *where for any* $j \in J$ *if* $x \in (a_j, b_j)$ *and interval* (a_j, b_j) *in* $\mathbb{R}$ *such that for all* $e \in E$

$$\tilde{U}_{je}(x) = \begin{cases} 1 & x \in (a_j, b_j) \\ 0 & x \notin (a_j, b_j). \end{cases}$$

   Since $\tilde{\vee}_j \tilde{U}_{jE} = (\widetilde{\cup_j U_j}, E)$ *where* $\cup_j U_{jE} \in \tau_u$*. Then* $\tilde{\vee}_j \tilde{U}_{jE} \in \tau_u^{(IVFS)}$

3. *Let* $\tilde{U}_E, \tilde{V}_E \in \tau_u^{(IVFS)}$*. Then* $\tilde{U}_E \tilde{\wedge} \tilde{V}_E \in \tau_u^{(IVFS)}$ *since* $\tilde{U}_E \tilde{\wedge} \tilde{V}_E = (\widetilde{U \cap V}, E)$ *where* $U \cap V \in \tau_u$.

Definition 3.2. *Let interval* $[\lambda_e^-, \lambda_e^+] \subseteq [0,1]$ *for all* $e \in E$*. Then* $\tilde{x}_E$ *is called an interval-valued fuzzy soft point (in short* $IVFS$*-Point)with support* $x \in X$ *and e-lower value* λ_e^- *and e-upper value* λ_e^+ *if for each* $y \in X$

$$\tilde{x}(e)(y) = \begin{cases} [\lambda_e^-, \lambda_e^+] & y = x \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.3. *Let* $X = [0,1]$ *and* E *be any subset of* X*. Consider* $IVFS$*-point* $\tilde{x}_E$ *with support* x *and lower value* 0 *and upper value* 0.3*, we define* $IVFS$*-point* $\tilde{x}_E$ *by*

$$\tilde{x}(e)(c) = \begin{cases} [0, 0.3] & c = x \\ 0 & \text{otherwise.} \end{cases}$$

For any $e \in E$ *and* $c \in X$.

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Definition 3.3. *The IVFS-point $\tilde{x}_E$ belong to IVFS set f_E, denoting by $\tilde{x}_E \tilde{\in} f_E$, whenever for all $e \in E$ we have $\lambda_e^- \leq f_e^-(x)$ and $\lambda_e^+ \leq f_e^+(x)$.*

Theorem 3.1. *Let f_E be an IVFS set. f_E is the union of all its IVFS-points i.e $f_E = \tilde{\vee}_{\tilde{x}_E \tilde{\in} f_E} \tilde{x}_E$.*

Proof. Let $x \in X$ be a fixed point, $y \in X$ and $e \in E$. Take all $\tilde{x}_E \tilde{\in} f_E$ with different e-lower and e-upper values $\lambda_{je}^-, \lambda_{je}^+$ where $j \in J$ there exists $\lambda_{je}^- = f_e^-, \lambda_{je}^+ = f_e^+$

$$\begin{aligned} \tilde{\vee}_{\tilde{x}_E \in f_E} \tilde{x}_e(y) &= [sup\ \tilde{x}_e^-(y), sup\ \tilde{x}_e^+(y)] \\ &= [sup_{\lambda_{je}^- \tilde{\leq} f^-(x)} \lambda_{je}^-,\ sup_{\lambda_{je}^+ \tilde{\leq} f^+(x)} \lambda_{je}^+] \\ &= [f_e^-(x), f_e^+(x)] \end{aligned}$$

□

Proposition 3.1. *Let $\{f_{jE}\}_{j \in J}$ be a family of IVFS sets over X, where J is an index set and $\tilde{x}_E$ be an IVFS-point with support x and e-lower value λ_e^- and e-upper value λ_e^+. If $\tilde{x} \tilde{\in} \tilde{\wedge}_{j \in J} \{f_{jE}\}$, then $\tilde{x}_E \tilde{\in} \{f_{jE}\}$ for each $j \in J$.*

Proof. Let $\tilde{x}_E$ be an $IVFS$-point with support x and e-lower value λ_e^- and e-upper value λ_e^+ and let $\tilde{x} \tilde{\in} \tilde{\wedge}_{j \in J} \{f_{jE}\}$, then $\lambda_e^- \leq \wedge_{j \in J} \{f_{je}^-\}(x) \leq \{f_{je}^-\}(x)$ for each $e \in E, x \in X$ and $\lambda_e^+ \leq \wedge_{j \in J} \{f_{je}^+\}(x) \leq \{f_{je}^+\}(x)$ for each $e \in E, x \in X$. Then, $[\lambda_e^-, \lambda_e^+] \leq [\{f_{je}^-\}(x), \{f_{je}^+\}(x)]$, for each $e \in E, x \in X$. Hence $\tilde{x}_E \tilde{\in} \{f_{jE}\}_{j \in J}$. □

Remark 3.2. *If $\tilde{x}_E \tilde{\in} f_E \tilde{\vee} g_E$ dose not imply $\tilde{x}_E \tilde{\in} f_E$ or $\tilde{x}_E \tilde{\in} g_E$.*

This is shown in following example.

Example 3.4. *Let τ be an IVFST over X, where $\tau = \{\emptyset_E, X_E, f_E, g_E, f_E \tilde{\wedge} g_E\}$ and $\tilde{x}_E$ be absolute IVFS-point with support x and e-lower value λ_e^- and e-upper value λ_e^+. If f_E and g_E are two IVFS sets in X defined as below:*

$$f : E \to \mathcal{IVF}([0,1])$$

and

$$g : E \to \mathcal{IVF}([0,1])$$

Such that for any $e \in E, x \in X$

$$f_e(x) = \begin{cases} [1, 0.5] & 0 \leq x \leq e \\ 0 & e < x \leq 1, \end{cases}$$

and

$$g_e(x) = \begin{cases} [0.2, 1] & 0 \leq x \leq e \\ 0 & e < x \leq 1. \end{cases}$$

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Since

$$f_e(x)\vee g_e(x) = \begin{cases} 1 & \text{if } 0 \le x \le e \\ 0 & \text{if } e < x \le 1. \end{cases}$$

Then $\tilde{x}_E \tilde{\in} f_E \tilde{\vee} g_E$, *but* $\tilde{x}_E \tilde{\notin} f_E$ *and* $\tilde{x}_E \tilde{\notin} g_E$.

Theorem 3.2. *Let* $\tilde{x}_E$ *be an IVFS-point with support* x *and e-lower value* λ_e^- *and e-upper value* λ_e^+ *and* f_E *and* g_E *be an IVFS sets. If* $\tilde{x}_E \tilde{\in} f_E \tilde{\vee} g_E$, *then there exists IVFS-point* $\tilde{x}_{1E} \tilde{\in} f_E$ *and IVFS-point* $\tilde{x}_{2E} \tilde{\in} g_E$ *such that* $\tilde{x}_E = \tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E}$

Proof. Let $\tilde{x}_E \tilde{\in} f_E \tilde{\vee} g_E$, then, $\lambda_e^- \le f_e^-(x) \vee g_e^-(x)$, and $\lambda_e^+ \le f_e^+(x) \vee g_e^+(x)$, for each $e \in E, x \in X$. Let choose

$E_1 = \{e \in E | \lambda_e^- \le f_e^-(x), \lambda_e^+ \le f_e^+(x) : x \in X\}$ $E_2 = \{e \in E | \lambda_e^- \le g_E^-(x), \lambda_e^+ \le g_E^+(x) : x \in X\}$ and

$$\tilde{x}_1(e)(y) = \begin{cases} [\lambda_e^-, \lambda_e^+] & \text{if } y = x_1, \ e \in E_1 \\ 0, & \text{otherwise.} \end{cases}$$

$$\tilde{x}_2(e)(y) = \begin{cases} [\lambda_e^-, \lambda_e^+], & \text{if } y = x_2, e \in E_2 \\ 0, & \text{otherwise.} \end{cases}$$

Since $x_{1e}^- \le f_{1e}^-(x)$, and $x_{1e}^+ \le f_{1e}^+(x)$ for each $e \in E_1, x \in X$, that implies $\tilde{x}_{1E} \tilde{\in} f_{1E}$ and also $x_{2e}^- \le f_{2e}^-(x)$, and $x_{2e}^+ \le f_{2e}^+(x)$ for each $e \in E_2, x \in X$, that implies $\tilde{x}_{2E} \tilde{\in} f_{2E}$. Consequently, $E_1 \tilde{\vee} E_2 = E$ and $\tilde{x}_E = \tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E}$. □

Definition 3.4. *Let* (X, E, τ) *be an IVFST space and* $\tilde{x}_E$ *be an IVFS-point with support* x, *e-lower value* λ_e^- *and e-upper value* λ_e^+. *The IVFS set* g_E *is called interval-valued fuzzy soft neighbourhood (IVFSN) of IVFS-point* $\tilde{x}_E$ *if there exists the IVFS-open set* f_E *in* X *such that* $\tilde{x}_E \tilde{\in} f_E \tilde{<} g_E$. *So the IVFS-open set* f_E *is an IVFSN of the IVFS-point* $\tilde{x}_E$ *if* $\forall e \in E, x \in X$ *such that* $\lambda_e^- < f_e^-(x)$ *and* $\lambda_e^+ < f_e^+(x)$.

Definition 3.5. *Let* (X, E, τ) *be an IVFST space and* $\tilde{x}_E$ *be an IVFS-point with support* x, *e-lower value* λ_e^- *and e-upper value* λ_e^+ *and* $\tilde{x}_E^\star$ *be an IVFS-point with support* $x^\star$, *e-lower value* ε_e^- *and e-upper value* ε_e^+. $\tilde{x}_E^\star$ *is called to compatible with* λ_e^-, λ_e^+, *if* $\tilde{x}_E^\star$ *provides that* $0 \le \varepsilon_e^- \le \lambda_e^-$ *and* $0 \le \varepsilon_e^+ \le \lambda_e^+$ *for each* $e \in E$.

Proposition 3.2.

1. *If* f_E *is an IVFSN of the IVFS-point* $\tilde{x}_E$ *and* $f_E \tilde{\le} h_E$, *then* h_E *is also an IVFSN of* $\tilde{x}_E$.

2. *If* f_E *and* g_E *are two IVFSN of the IVFS-point* $\tilde{x}_E$, *then* $f_E \tilde{\wedge} g_E$ *is also IVFSN of* $\tilde{x}_E$.

3. *If* f_E *is an IVFSN of the IVFS-point* $\tilde{x}_E^\star$ *with support* $x^\star$, *e-lower value* $\lambda_e^- - \varepsilon_e^-$ *and e-upper value* $\lambda_e^+ - \varepsilon_e^+$, *for all* ε_e^- *compatible with* λ_e^- *and* ε_e^+ *compatible with* λ_e^+. *Then* f_E *is an IVFSN of the IVFS-point* $\tilde{x}_E$.

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4. *If f_E is an $IVFSN$ of the $IVFS$-point $\tilde{x}_{1E}$ and g_E is an $IVFSN$ of the $IVFS$-point $\tilde{x}_{2E}$ then $f_E \tilde{\vee} g_E$ is also an $IVFSN$ of $\tilde{x}_{1E}$ and $\tilde{x}_{2E}$.*

5. *If f_E is an $IVFSN$ of the $IVFS$-point $\tilde{x}_E$, then there exists $IVFSN$ g_E of $\tilde{x}_E$ such that $g_E \tilde{\leq} f_E$ and g_E is $IVFSN$ of $IVFS$-point $\tilde{y}$ with support y and e-lower value γ_e^- and e-upper value γ_e^+, for all $\tilde{y}_E \tilde{\in} g_E$.*

Proof. 1. Let f_E be an $IVFSN$ of the $IVFS$-point $\tilde{x}$. Then there exists the $IVFS$-open set g_E in X such that $\tilde{x}_E \tilde{\in} g_E \tilde{\leq} f_E$. Since $f_E \tilde{\leq} h_E$, then $\tilde{x}_E \tilde{\in} g_E \tilde{\leq} f_E \tilde{\leq} h_E$, therefore h_E is an $IVFSN$ of $\tilde{x}_E$.

2. Let f_E and g_E are two $IVFSN$ of the $IVFS$-point $\tilde{x}_E$. Then there exists two $IVFS$-open sets h_E, k_E in X such that $\tilde{x}_E \tilde{\in} h_E \tilde{\leq} f_E$ and $\tilde{x}_E \tilde{\in} k_E \tilde{\leq} g_E$, so $\tilde{x}_E \tilde{\in} h_E \tilde{\wedge} k_E \tilde{\leq} f_E \tilde{\wedge} g_E$. Since $h_E \tilde{\wedge} k_E$ is $IVFS$-open set, then $g_E \tilde{\wedge} f_E$ is an $IVFSN$ of $\tilde{x}_E$.

3. Let f_E be an $IVFSN$ of the $IVFS$-point $\tilde{x}^\star_E$ with support $x^\star$ and e-lower value $\lambda_e^- - \varepsilon_e^-$ and e-upper value $\lambda_e^+ - \varepsilon_e^+$, for all ε_e^- compatible with λ_e^- and ε_e^+ compatible with λ_e^+. Then, there exists $IVFS$-open set $g_E^{x^\star}$ such that $\tilde{x}^\star_E \tilde{\in} g_E^{x^\star} \tilde{\leq} f_E$. Let $g_E = \tilde{\vee}_{x^\star} g_E^{x^\star}$, then g_E is $IVFS$-open in X and $g_E \tilde{\leq} f_E$. By the Theorem 3.2 and since for all $e \in E$, then $\tilde{\vee} \tilde{x}^\star_E = \tilde{x}_E \tilde{\leq} \tilde{\vee}_{x^\star} g_E^{x^\star} = g_E \tilde{\leq} f_E$. Hence, $\tilde{x}_E \tilde{\in} g_E \tilde{\leq} f_E$, i.e f_E is $IVFSN$ of $\tilde{x}_E$.

4. Let f_E be an $IVFSN$ of the $IVFS$-point $\tilde{x}_{1E}$ with support x_1 and e-lower value λ_{1e}^- and e-upper value λ_{1e}^+ and g_E be an $IVFSN$ of the $IVFS$-point $\tilde{x}_{2E}$ with support x_2 and e-lower value λ_{2e}^- and e-upper value λ_{2e}^+. Then there exists $IVFS$-open sets h_{1E}, h_{2E} such that $\tilde{x}_{1E} \tilde{\in} h_{1E} \tilde{\leq} f_E$ and $\tilde{x}_{2E} \tilde{\in} h_{2E} \tilde{\leq} f_E$, respectively, Since $\tilde{x}_{1E} \tilde{\in} h_{1E}$, then $\lambda_{1e}^- \leq h_{1e}^-(x), \lambda_{1e}^+ \leq h_{1e}^+(x)$ for each $e \in E$ and $x \in X$, Since $\tilde{x}_{2E} \tilde{\in} h_{2E}$, then $\lambda_{2e}^- \leq h_{2e}^-(x), \lambda_{2e}^+ \leq h_{2e}^+(x)$ for each $e \in E$ and $x \in X$. Then, we have

$max\{[\lambda_{1e}^-, \lambda_{1e}^+], [\lambda_{2e}^-, \lambda_{2e}^+]\} \leq max\{[h_{1e}^-(x), h_{1e}^+(x)], [h_{2e}^-(x), h_{2e}^+(x)]\}$ for each $e \in E, x \in X$. So $\tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E} \tilde{\in} h_{1E} \tilde{\vee} h_{2E}$ and $h_{1E} \tilde{\vee} h_{2E} \in \tau$ and $h_{1E} \tilde{\vee} h_{2E} \tilde{\leq} f_E \tilde{\vee} g_E$. Consequently, $f_E \tilde{\vee} g_E$ is $IVFSN$ of $x_{1E} \tilde{\vee} x_{2E}$.

5. Let f_E be an $IVFSN$ of the $IVFS$-point $\tilde{x}_E$, with support x and e-lower value λ_e^- and e-upper value λ_e^+. Then there exists $IVFS$-open set g_E such that $\tilde{x}_E \tilde{\in} g_E \tilde{\leq} f_E$. Since g_E $IVFS$-open set, g_E is a neighborhood of its points, i.e g_E is $IVFSN$ of $IVFS$-point $\tilde{y}_E$ with support y and e-lower value γ_e^- and e-upper value γ_e^+, for all $e \in E$. Also, g_E is $IVFSN$ of $IVFS$-point $\tilde{x}_E$ since $\tilde{x}_E \tilde{\in} g_E$. Therefore, there exists g_E is $IVFSN$ of $\tilde{x}_E$ such that $g_E \tilde{\leq} f_E$ and g_E is $IVFSN$ of $\tilde{y}_E$, Since f_E is $IVFSN$ of $\tilde{x}_E$.

□

Definition 3.6. *Let (X, E, τ) be an $IVFST$ space and f_E be an $IVFS$ set. The $IVFS$-closure of f_E denoted by Clf_E is intersection of all $IVFS$-closed super sets of f_E. Clearly, Clf_E is the smallest $IVFS$-closed set over X which contains f_E.*

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Example 3.5. [24] *Consider* $IVFST$ τ_u^{IVFS} *over* $\mathbb{R}$ *as introduced in Example* 3.2 *if* $\tilde{H}_E$ *is an* $IVFS$ *over* $\mathbb{R}$ *related of the open interval* $H = (a, b) \subset \mathbb{R}$ *by mapping*

$$\tilde{H} : E \to (Int[0,1])^{\mathbb{R}}$$

$$\tilde{H}_e(x) = \begin{cases} 1 & x \in (a,b) \\ 0 & x \notin (a,b), \end{cases}$$

where $e \in E$ *and* $x \in \mathbb{R}$*. Then closure of* $\tilde{H}_E$ *defined as*

$$Cl\tilde{H} : E \to (Int[0,1])^{\mathbb{R}}$$

$$\tilde{H}_e(x) = \begin{cases} 1 & x \in [a,b] \\ 0 & x \notin [a,b]. \end{cases}$$

Remark 3.3. *By replacing* $\tilde{x}_E$ *for* f_E*. The* $IVFS$*-closure of* $\tilde{x}_E$ *denoted by* $Cl\tilde{x}_E$ *is intersection of all* $IVFS$*-closed super sets of* $\tilde{x}_E$*.*

Proposition 3.3. *Let* (X, E, τ) *be an* $IVFST$ *space and* f_E *and* g_E *be two* $IVFSS$ *over* X*. Then*

1. $Cl\emptyset_E = \emptyset_E$ *and* $Cl\tilde{X}_E = \tilde{X}_E$.
2. $f_E \tilde{\leq} Clf_E$*, and* Clf_E *is the smallest* $IVFS$*-closed set containing the* $IVFS f_E$*.*
3. $Cl(Clf_E) = Clf_E$.
4. *if* $f_E \tilde{\leq} g_E$*, then* $(Clf_E) \tilde{\leq} Clg_E$*.*
5. f_E *is an*$IVFS$*-closed set if and only if* $f_E = Clf_E$*.*
6. $Cl(f_E \tilde{\vee} g_E) = Clf_E \tilde{\vee} Clg_E$.
7. $Cl(f_E \tilde{\wedge} g_E) \tilde{\leq} Clf_E \tilde{\wedge} Clg_E$.

Proof. We only prove part(6). The similar technique is used to show the other parts.

Since $f_E \tilde{\leq} f_E \tilde{\vee} g_E$ and $g_E \tilde{\leq} f_E \tilde{\vee} g_E$, by part(4) we have $Clf_E \tilde{\leq} Cl(f_E \tilde{\vee} g_E)$ and $Clg \tilde{\leq} Cl(f_E \tilde{\vee} g_E)$. Thus $Clf_E \tilde{\vee} Clg_E \tilde{\leq} Cl(f_E \tilde{\vee} g_E)$.

Conversely, we have $f_E \tilde{\leq} Clf_E$ and $g_E \tilde{\leq} Clg_E$, by part(2).Hence, $f_E \tilde{\vee} g_E \tilde{\leq} Clf_E \tilde{\vee} Clg_E$ where $Clf_E \tilde{\vee} Clg_E$ is an $IVFS$-closed set. Thus, $Cl(f_E \tilde{\vee} g_E) \tilde{\leq} Clf_E \tilde{\vee} Clg_E$.

So $Cl(f_E \tilde{\vee} g_E) = Clf_E \tilde{\vee} Clg_E$. □

Definition 3.7. *Let* (X_1, E_1, τ_1) *and* (X_2, E_2, τ_2) *be two* $IVFSTS$ *and*

$$\Phi : (X_1, E_1, \tau_1) \to (X_2, E_2, \tau_2)$$

be an $IVFS$ *map. Ten* Φ *is called an*

1. *interval-valued fuzzy soft continuous* $(IVFSC)$ *map if and only if for each* $g_{E_2} \in \tau_2$*, we have* $\Phi^{-1}(g_{E_2}) \in \tau_1$*.*

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2. *interval-valued fuzzy soft open* ($IVFSO$) *map if and only if for each* $f_E \in \tau_1$, *we have* $\Phi(f_{E_1}) \in \tau_2$.

Theorem 3.3. *Let* (X_1, E_1, τ_1) *and* (X_2, E_2, τ_2) *be two* $IVFST$ *and* Φ *be an* $IVFS$-*mapping from* X_1 *to* X_2, *then the following statements are equivalent:*

1. Φ *is* $IVFC$.

2. *for each* $IVFS$-*point* $\tilde{x}_E$ *on* X_1 *the inverse of every neighbourhood of* $\Phi(\tilde{x}_E)$ *under* Φ *is neighbourhood of* $\tilde{x}_E$.

3. *for each* $IVFS$-*point* $\tilde{x}_E$ *on* X_1 *and each neighbourhood* g_E *of* $\Phi(\tilde{x}_E)$, *there exists a neighbourhood* f_E *of* $\tilde{x}_E$ *such that* $\Phi(f_E)\tilde{\leq}g_E$.

Proof.
(1) $\Rightarrow$ (2) Let g_E be an $IVFSN$ of $\Phi(\tilde{x}_E)$ in τ_2. Then there exists $IVFS$-open set f_E in τ_2 such that $\Phi(\tilde{x}_E)\tilde{\in}f_E\tilde{\leq}g_E$, since Φ is $IVFSC$, $\Phi^{-1}(f_E)$ is an $IVFS$-open in τ_1 and we have $\tilde{x}_E\tilde{\in}\Phi^{-1}(f_E)\tilde{\leq}\Phi^{-1}(g_E)$.

(2) $\Rightarrow$ (3) Let g_E be an $IVFSN$ of $\Phi(\tilde{x}_E)$. By hypothesis $\Phi^{-1}(g_E)$ is an $IVFSN$ of $\tilde{x}_E$. Consider the $f_E = \Phi^{-1}(g_E)$ is an $IVFSN$ of $\tilde{x}_E$. Therefore, we have $\Phi(f_E) = \Phi(\Phi^{-1}(g_E))\tilde{\leq}g_E$.

(3) $\Rightarrow$ (1) Let g_E be an $IVFS$-open set in τ_2. We must show that $\Phi^{-1}(g_E)$ is an $IVFS$-open set in τ_1. Now let $\tilde{x}_E\tilde{\in}\Phi^{-1}(g_E)$. Then $\Phi(\tilde{x}_E)\tilde{\in}g_E$ and since g_E is $IVFS$-open set in τ_2, we get g_E is an $IVFSN$ $\Phi(\tilde{x}_E)$ in τ_2. By hypothesis there exists $IVFS$-open set f_E is $IVFSN$ of $\tilde{x}_E$ such that $\Phi(f_E)\tilde{\leq}g_E$, then $f_E\tilde{\leq}\Phi^{-1}[\Phi(f_E)]\tilde{\leq}\Phi^{-1}(g_E)$ for f_E is an $IVFSN$ of $\tilde{x}_E$. Form here, $f_E\tilde{\leq}\Phi^{-1}(g_E)$, for f_E is an $IVFSN$ of $\tilde{x}_E$. Hence,$\Phi^{-1}(g_E)\tilde{\in}\tau_1$.

□

4 Quasi coincident neighbourhood structure of interval-valued fuzzy soft topological spaces

In this section, we present quasi coincident neighborhood structure in the interval-valued fuzzy soft topology ($IVFST$) and its properties.

Definition 4.1. *The* $IVFS$-*point* $\tilde{x}_E$ *is called a soft quasi-coincident whit* $IVFS$ f_E, *denoting by* $\tilde{x}_E\tilde{q}f_E$, *if and only if there exists* $e \in E$ *such that* $\lambda_e^- + f_e^-(x) > 1$ *and* $\lambda_e^+ + f_e^+(x) > 1$. *If* f_E *is not soft quasi-coincident whit* f_E, *we write* $f_E\neg\tilde{q}g_E$.

Definition 4.2. *The* $IVFS$-*set* f_E *is called a soft quasi-coincident whit* $IVFS$ g_E, *denoting by* $f_E\tilde{q}g_E$, *if and only if there exists* $e \in E$ *such that* $f_e^-(x) + g_e^-(x) > 1$ *and* $f_e^+(x) + g_e^+(x) > 1$.

Proposition 4.1. $\tilde{x}_E$ *be an* $IVFS$-*point with support* x *and e-lower value* λ_e^- *and e-upper value* λ_e^+ *and* f_E, g_E *two* $IVFS$ *sets :*

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(i) $f_E \tilde{\leq} g_E \Leftrightarrow f_E \neg \tilde{q} g^c_E$.

(ii) $\tilde{x}_E \tilde{\in} f_E \Leftrightarrow \tilde{x}_E \neg \tilde{q} f^c_E$.

Proof. We just prove part(1). The similar technique is used to show the part (2). For two $IVFS$ sets f_E, g_E we have:

$$\begin{aligned} f_E \tilde{\leq} g_E &\Leftrightarrow \forall e \in E : [f^-_e(x), f^+_e(x)] \leq [g^-_e(x), g^+_e(x)], \forall x \in X \\ &\Leftrightarrow \forall e \in E : f^-_e(x) \leq g^-_e(x) \text{ and } f^+_e(x) \leq g^+_e(x), \forall x \in X \\ &\Leftrightarrow \forall e \in E : f^-_e(x) + 1 - g^-_e(x) \leq 1 \text{ and } f^+_e(x) + 1 - g^+_e(x) \leq 1, \forall x \in X \\ &\Leftrightarrow \forall e \in E : f^-_e(x) + {g^-}^c_e(x) \leq 1 \text{ and } f^+_e(x) + {g^+}^c_e(x) \leq 1, \forall x \in X \\ &\Leftrightarrow f_E \neg \tilde{q} g^c_E. \end{aligned}$$

□

Proposition 4.2. *Let $\{f_{jE} : j \in J\}$ is a family of $IVFS$ sets over X and $\tilde{x}_E$, be an $IVFS$-point with support x and e-lower value λ^-_e and e-upper value λ^+_e. If $\tilde{x}_E \tilde{q}(\tilde{\wedge} f_{jE})$, then $\tilde{x}_E \tilde{q} f_{jE}$ for each $j \in J$.*

Proof. Let $\tilde{x}_E \tilde{q}(\tilde{\wedge} f_{jE})$. Then $\lambda^-_e \tilde{q}(\tilde{\wedge}_j f^-_{je})(x)$ and $\lambda^+_e \tilde{q}(\tilde{\wedge}_j f^+_{je})(x)$ for $e \in E$ and $x \in X$. This implies that $\lambda^-_e > 1 - \wedge_j (f^-_{je})(x)$ and $\lambda^+_e > 1 - \wedge_j (f^+_{je})(x)$, $x \in X$. Since $\wedge_j f^-_{je}(x) \leq f^-_{je}(x)$ and $\wedge_j f^+_{je}(x) \leq f^+_{je}(x)$, then $\lambda^-_e > 1 - \wedge_j (f^-_{je})(x) > 1 - f^-_{je}(x)$ for each $e \in E, x \in X$ and $\lambda^+_e > 1 - \wedge_j (f^+_{je})(x) > 1 - f^+_{je}(x)$ for each $e \in E, x \in X$. Hence $\lambda^-_e > 1 - f^-_{je}(x)$ and $\lambda^+_e > 1 - f^+_{je}(x)$. So, $[\lambda^-_e, \lambda^+_e] > [1, 1] - [f^-_{je}(x), f^+_{je}(x)]$, implies that $\tilde{x}_E > 1 - f^-_{jE}$ and $\tilde{x}_E \tilde{q} f_{jE}$ for each $j \in J$. □

Remark 4.1. *$\tilde{x}_E \tilde{q}(f_E \vee g_E)$ does not imply $\tilde{x}_E \tilde{q} f_E$ or $\tilde{x}_E \tilde{q} g_E$. This is shown in the following example.*

Example 4.1. *Let consider Example 3.5 in this example $\tilde{x}_E \tilde{q}(f_E \tilde{\vee} g_E)$ but $\tilde{x}_E \neg \tilde{q} f_E$ and $\tilde{x}_E \neg \tilde{q} g_E$.*

Theorem 4.1. *Let $\tilde{x}_E$ be an$IVFS$-point $\tilde{x}_E$ with support x and e-lower value λ^-_e and e-upper value λ^+_e and f_E, g_E are $IVFS$-sets over X. If $\tilde{x}_E \tilde{q}(f_E \vee g_E)$, then there exists $\tilde{x}_{1E} \tilde{q} f_E$ and $\tilde{x}_{2E} \tilde{q} g_E$ such that $\tilde{x}_E = \tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E}$.*

proof Analogously with Theorem 3.2.

Definition 4.3. *Let (X, E, τ) be an $IVFSTS$ and $\tilde{x}_E$ be an $IVFS$-point with support x, e-lower values λ^-_e and e-upper values λ^+_e. The $IVFS$ set g_E is called a quasi soft neighbourhood ($QIVFSN$) of $IVFS$-point $\tilde{x}_E$ if there exists the $IVFS$-open set f_E in X such that $\tilde{x}_E \tilde{q} f_E \tilde{\leq} g_E$. Thus the $IVFS$-open set f_E is a $QIVFSN$ of the $IVFS$-point $\tilde{x}_E$ if and only if $\exists e \in E, x \in X$ such that $\lambda^-_e + f^-_e(x) > 1$ and $\lambda^+_e + f^+_e(x) > 1$.*

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Remark 4.2. *A quasi-coincident soft neighbourhood of $IVFS$-point generally does not contain the point itself. It is shown by the following:*

Example 4.2. *Let $X = [0, 1]$ and E be any subset of X. Consider two $IVFS$ sets f_E, g_E over X by the mapping $f : E \to \mathcal{IVF}([0, 1])$ and $f : E \to \mathcal{IVF}([0, 1])$ Such that for any $e \in E, x \in X$*

$$\tilde{f}_e(x) = \begin{cases} [0.4, 0.5] & 0 \leq x \leq e \\ 0 & e < x \leq 1, \end{cases}$$

and

$$\tilde{g}_e(x) = \begin{cases} [0.6, 0.7] & 0 \leq x \leq e \\ 0 & e < x \leq 1, \end{cases}$$

and $\tilde{x}_E$ be any $IVFS$-point defined by

$$\tilde{x}_e(c) = \begin{cases} [0.4, 0.5] & c = x \\ 0 & c \neq x. \end{cases}$$

Let $\tau = \{\emptyset_E, X_E, f_E, g_E\}$. Then clearly τ an $IVFST$ over X. Since $f_E \tilde{\leq} g_E$ and $\tilde{x}\tilde{q}f_E$. Then g_E is $QIVFSN$ of $\tilde{x}_E$. But $\tilde{x}_E \notin g_E$.

Proposition 4.3. *(1) If $f_E \tilde{\leq} g_E$ and f_E is $QINVSN$ of $\tilde{x}_E$, then g_E is also $QINVSN$ of $\tilde{x}_E$.*

(2) If f_E, g_E are $QINVSN$ of $\tilde{x}_E$, then $f_E \tilde{\wedge} g_E$ is also $QINVSN$ of $\tilde{x}_E$.

(3) If f_E is $QINVSN$ of $\tilde{x}_{1E}$ and g_E is $QINVSN$ of $\tilde{x}_{2E}$, then $f_E \tilde{\vee} g_E$ is also $QINVSN$ of $\tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E}$.

(4) If f_E is $QINVSN$ of $\tilde{x}_E$. Then there exists g_E is $QINVSN$ of $\tilde{x}_E$, such that $g_E \tilde{\leq} f_E$ and g_E is $QINVSN$ of y_E, $\forall y_E \tilde{q} g_E$.

Proof. (1),(2) are straightforward.

(3) Let f_E is $QINVSN$ of $\tilde{x}_{1E}$ and g_E is $QINVSN$ of $\tilde{x}_{2E}$, then there exists $IVFS$-open set h_{1E} in X such that $\tilde{x}_{1E}\tilde{q}h_{1E} \tilde{\leq} f_E$ and g_E is $QINVSN$ of $\tilde{x}_{2E}$, then there exists $IVFS$-open set h_{2E} in X such that $\tilde{x}_{2E}\tilde{q}h_{2E} \tilde{\leq} g_E$. Since $\tilde{x}_{1E}\tilde{q}h_{1E}$, then for each $e \in E, x \in X, \lambda^-_{1e} + h^-_{1e} > 1$, $\lambda^+_{1e} + h^+_{1e} > 1$ and this implies that $\lambda^-_{1e} > 1 - h^-_{1e}, \lambda^+_{1e} > 1 - h^+_{1e}$ for each $e \in E$ and since $\tilde{x}_{2E}\tilde{q}h_{2E}$, then for each $e \in E, \lambda^-_{2e} + h^-_{2e} > 1$, $\lambda^+_{2e} + h^+_{2e} > 1$ and this implies that $\lambda^-_{2e} > 1 - h^-_{2e}, \lambda^+_{2e} > 1 - h^+_{2e}$ for each $e \in E, x \in X$. From here, $max(\lambda^-_{1e}, \lambda^-_{2e}) > max(1 - h^-_{1e}(x)), (1 - h^-_{2e}(x)), max(\lambda^+_{1e}, \lambda^+_{2e}) > max(1 - h^+_{1e}(x)), (1 - h^+_{2e}(x))$. Hence, $\tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E} \tilde{q}(h_{1E} \tilde{\vee} h_{2E}) \tilde{\leq} f_E \tilde{\vee} g_E$. Consequently, $f_E \tilde{\vee} g_E$ is $QINVSN$ of $\tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E}$.

(4) Let f_E is $QINVSN$ of $\tilde{x}_E$, then there exists g_E is $QINVSN$ of $\tilde{x}_E$ such that $\tilde{x}_E \tilde{q} g_E \tilde{\leq} f_E$. Consider the $g_E = h_E$. Indeed, since $\tilde{x}_E \tilde{q} h_E$ and h_E is $IVFS$-open set, then h_E is $QINVSN$ of $\tilde{x}_E$, we obtain h_E is $QINVSN$ of $\tilde{y}_E$.

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□

Theorem 4.2. *In the $IVFST(X, E, \tau)$, the $IVFS$-point $\tilde{x}_E$ belongs to Clf_E if and only if each $QIVFS$ of $\tilde{x}_E$ is soft quasi-coincident with f_E.*

Proof. Let $IVFS$-point $\tilde{x}_E$ with support x, e-lower value λ_e^- and e-upper value λ_e^+ belongs to $Clf_E, i.e, \tilde{x}_E \tilde{\in} Clf_E$. For any $IVFS$-closed g_E which containing f_E, $\tilde{x}_E \tilde{\in} g_E$ which implies that $\lambda_e^- \leq g_e^-(x)$ and $\lambda_e^+ \leq g_e^+(x)$, for all $x \in X, e \in E$. Consider h_E be an $QIVFN$ of the $IVFS$-point $\tilde{x}_E$ and $h_E \neg\tilde{q} f_E$. Then for any $e \in E$ and $x \in X$, $h_e^-(x) + f_e^-(x) \leq 1, h_e^+(x) + f_e^+(x) \leq 1$ and so $f_E \tilde{\leq} h_E^c$. Since h_E is $QIVFSN$ of the $IVFS$-point $\tilde{x}_E$, by $\tilde{x}_E$ dose not belong to h_E^c. Therefore, we have that $\tilde{x}_E$ dose not belong to Clf_E. This is a contradiction.

Conversely, let any $QIVFSN$ of the $IVFS$-point $\tilde{x}_E$ be soft quasi-coincident with f_E. Consider $\tilde{x}_E$ dose not belong to $Clf_E, i.e, \tilde{x}_E \notin Clf_E$. Then there exists an $IVFS$-closed set g_E which is containing f_E such that $\tilde{x}_E$ dose not belong to g_E. we have $\tilde{x}_E \tilde{q} g_E^c$. Then g_E^c is an $QIVFSN$ of the $IVFS$-point $\tilde{x}_E$ and $f_E \neg\tilde{q} g_E^c$. This is a contradiction with the hypothesis. □

5 IVFS quasi-separation axioms

In this section we develop the separation axioms to $IVFST$, so-called $IVFSQ$-separation axioms($IVFSq$-T_i axioms)for $i = 0, 1, 2, 3, 4$ and consider some properties of them.

Definition 5.1. *Let (X, E, τ) be an $IVFST$ space. Let $\tilde{x}_E$ and $\tilde{y}_E$ are $IVFS$-points over X where*

$$\tilde{x}(e)(z) = \begin{cases} [\lambda_e^-, \lambda_e^+] & z = x \\ 0 & otherwis \end{cases}$$

$$\tilde{y}(e)(z) = \begin{cases} [\gamma_e^-, \gamma_e^+] & z = y \\ 0 & otherwis. \end{cases}$$

$\tilde{x}_E$ and $\tilde{y}_E$ are said distinct if and only if $\tilde{x}_E \tilde{\wedge} \tilde{y}_E = \emptyset_E$, which meaning $x \neq y$.

Definition 5.2. *Let (X, E, τ) be an $IVFST$ space. The $IVFS$-point $\tilde{x}_E$ is called a crisp $IVFS$-point $x_E^{[1,1]}$ if $\lambda_e^- = \lambda_e^+ = 1$ for all $e \in E$.*

Definition 5.3. *Let (X, E, τ) be an $IVFST$ space and $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$-points. If there exists $IVFS$ open sets f_E and g_E such that:*

(a.) when $\tilde{x}_E$ and $\tilde{y}_E$ be two distinct $IVFS$-points with different supports x and y and e-lower values and e-upper values λ_e^-, λ_e^+ and γ_e^-, γ_e^+, respectively, and f_E is $IVFSN$ of the $IVFS$-point $\tilde{x}_E$ and $\tilde{y}_E \neg\tilde{q} f_E$ or g_E is $IVFSN$ of the $IVFS$-point $\tilde{y}_E$ and $\tilde{x}_E \neg\tilde{q} g_E$.

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(b.) *when $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$-points with the same supports $x = y$ and e-value $\lambda_e^- < \gamma_e^-$ and e-value $\lambda_e^+ < \gamma_e^+$ and f_E is a $QIVFSN$ of the $IVFS$-point $\tilde{y}_E$ such that $\tilde{x}_E \neg \tilde{q} f_E$.*

Then (X, E, τ) be interval-valued fuzzy soft quasi-T_0 space ($IVFSq$-T_0 space).

Example 5.1. *Consider $IVFS$ set defined in Example 3.1. and $\tilde{x}_E, \tilde{y}_E$ be any two distinct $IVFS$-point in X defined by*

$$\tilde{x}(e)(z) = \begin{cases} 1 & z = x \\ 0 & z \neq x. \end{cases}$$

and

$$\tilde{y}(e)(z) = \begin{cases} 0 & \text{if } z = y \\ 1 & \text{if } z \neq y \end{cases}$$

f_E is $IVFSN$ of $\tilde{x}_E$ and $\tilde{y}_E \neg \tilde{q} f_E$. Thus X is $IVFSq$-T_0 space.

Theorem 5.1. *(X, E, τ) is an $IVFSq$-T_0 space if and only if for every two $IVFS$-points $\tilde{x}_E$, $\tilde{y}_E$ and $\tilde{x}_E \notin Cl\tilde{y}_E$ or $\tilde{y}_E \notin Cl\tilde{x}_E$.*

Proof. Let (X, E, τ) is an $IVFS$q-T_0 space and $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$-points in X. First consider that $\tilde{x}_E$ and $\tilde{y}_E$ be two distinct $IVFS$-points with different supports x and y and e-lower values and e-upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+, respectively, a crisp $IVFS$-point $\tilde{x}_E^{[1,1]}$ has an $IVFSN$ f_E such that $\tilde{y}_E \neg \tilde{q} f_E$ or a crisp $IVFS$-point $\tilde{y}_E^{[1,1]}$ has an $IVFSN$ g_E such that $\tilde{x}_E \neg \tilde{q} f_E$. Consider the crisp $IVFS$-point $\tilde{x}_E^{[1,1]}$ has an $IVFSN$ f_E such that $\tilde{y}_E \neg \tilde{q} f_E$. Moreover, f_E is an $QINFSN$ of $\tilde{x}_E$ and $\tilde{y}_E \neg \tilde{q} f_E$. Hence $\tilde{x_E} \notin Cl\tilde{y_E}$. Next we consider the case $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$-points with the same supports $x = y$ and e-lower value $\lambda_e^- < \gamma_e^-$ and e-upper value $\lambda_e^+ < \gamma_e^+$, then $\tilde{y}_E$ has a $QIVFSN$ which is not quasi-coincident with $\tilde{x}_E$ and so by Theorem 4.1 $\tilde{x_E} \notin Cl\tilde{y_E}$.

Conversely, let $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$-points in X. Consider the without loss of generality, that $\tilde{x_E} \notin Cl\tilde{y_E}$. First consider that $\tilde{x}_E$ and $\tilde{y}_E$ be two distinct $IVFS$-points with different supports x and y and e- lower values and e-upper values λ_e^-, γ_e^- and λ_e^+, γ_e^-, respectively, since $\tilde{x}_E \notin Cl\tilde{y}_E$ for any $e \in E$, $f_e^-(y) = f_e^+(y) = 0$ and $f_e^-(x) = f_e^+(x) = 1$. Then $Cl(\tilde{y}_E)^c$ is an $IVFSN$ of $\tilde{x}_E$ such that $Cl(\tilde{y}_E)^c \neg \tilde{q} \tilde{y}_E$. Next, let when $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$-points with the same supports $x = y$ and we must have e- lower value $\lambda_e^- > \gamma_e^-$ and e-upper value $\lambda_e^+ > \gamma_e^+$ and then $\tilde{x}_E$ has $QIVFSN$ which is not quasi-coincident with $\tilde{y}_E$. □

Definition 5.4. *Let (X, E, τ) be an $IVFST$ and $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$-points, if there exists $IVFS$ open sets f_E and g_E such that:*

(a.) *when $\tilde{x}_E$ and $\tilde{y}_E$ be two distinct $IVFS$-points with different supports x and y and e-lower values and e-upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+, respectively and f_E is $IVFSN$ of the $IVFS$-point $\tilde{x}_E$ and $\tilde{y}_E \neg \tilde{q} f_E$ and g_E is $IVFSN$ of the $IVFS$-point $\tilde{y}_E$ and $\tilde{x}_E \neg \tilde{q} g_E$.*

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(b.) *when $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$-points with the same supports $x = y$ and e-value $\lambda_e^- < \gamma_e^-$ and e-value $\lambda_e^+ < \gamma_e^+$ and f_E is a $QIVFSN$ of the $IVFS$-point $\tilde{y}_E$ such that $\tilde{x}_E \neg \tilde{q} f_E$.*

Then (X, E, τ) be interval-valued fuzzy soft quasi-T_1 space ($IVFS$q-T_1 space).

Theorem 5.2. *(X, E, τ) is an $IVFS$q-T_1 space if and only if for any $IVFS$-point $\tilde{x_E}$ in X is an $IVFS$-closed set.*

Proof. Suppose that for each $IVFS$-point $\tilde{x_E}$ in X is an $IVFS$-closed set, i.e, $g_E = \tilde{x}_E^c$. Then g_E is an $IVFS$-open set. Let x_E and y_E two $IVFS$-point such that: First consider that $\tilde{x}_E$ and $\tilde{y}_E$ be two distinct $IVFS$-points with different supports x and y and e-lower values and e-upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+, respectively, g_E is an $IVFS$-open set such that g_E is $IVFSN$ of $IVFS$-point $\tilde{y}_E$ and $\tilde{x}_E \neg \tilde{q} g_E$. Similarly, $f_E = \tilde{y}_E^c$ is $IVFS$-open set and f_E is $IVFSN$ of the $IVFS$-point $\tilde{x}_E$ and $\tilde{y}_E \neg \tilde{q} f_E$. Next, we consider the case $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$-points with the same supports $x = y$ and e-value $\lambda_e^- < \gamma_e^-$ and e-value $\lambda_e^+ < \gamma_e^+$, then $\tilde{y}_E$ has a $QIVFSN$ g_E which is not quasi-coincident with $\tilde{x}_E$. Thus X is an $IVFS$q-T_1 space.

Conversely, Let (X, E, τ) be an $IVFS$q-T_1 space. Suppose that any $IVFS$-point $\tilde{x}_E$ is not $IVFS$-closet set in X, i.e, $f_E \doteq \tilde{x}_E^c$. Then $\tilde{f}_E \neq Cl\tilde{f}_E$ and there exists $\tilde{y}_E \tilde{\in} Cl\tilde{f}_E$ such that $\tilde{x}_E \neq \tilde{y}_E$. First consider that $\tilde{x}_E$ and $\tilde{y}_E$ be two distinct $IVFS$-points with different supports x and y and e-lower values and e-upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+, respectively, suppose that e-lower value $\lambda_e^- \leq 0.5$ and e-upper value $\lambda_e^+ \leq 0.5$. Since $\tilde{y}_E \tilde{\in} Cl f_E$, by Theorem 4.1 for any f_E is $QIVFSN$ of $\tilde{y}_E$ and $\tilde{x}_E \tilde{q} f_E$. Then there exists $IVFS$-open set h_E such that $\tilde{y} \tilde{q} h_E \tilde{\leq} f_E$. Hence $h_e^-(y) + \gamma_e^- > 1$ Next,let $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$-points with the same supports $x = y$ and e-value $\lambda_e^- < \gamma_e^-$ and e-value $\lambda_e^+ < \gamma_e^+$, since $y_E \tilde{\in} Cl x_E$, by Theorem 4.1 for each f_E is $QIVFSN$ of $IVFS$-point $\tilde{y}_E$, $\tilde{x}_E \tilde{q} f_E$. This is contradiction.

□

Definition 5.5. *Let (X, E, τ) be an $IVFST$ and $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$-points, if there exists $IVFS$ open sets f_E and g_E such that:*

(a.) *when $\tilde{x}_E$ and $\tilde{y}_E$ be two distinct $IVFS$-points with different supports x and y and e-lower values and e-upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+, respectively, f_E is $IVFSN$ of the $IVFS$-point $\tilde{x}_E$, g_E is $IVFSN$ of the $IVFS$-point $\tilde{y}_E$, such that $f_E \neg \tilde{q} g_E$.*

(b.) *when $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$-points with the same supports $x = y$ and e-value $\lambda_e^- < \gamma_e^-$ and e-value $\lambda_e^+ < \gamma_e^+$, f_E is a $IVFSN$ of the $IVFS$-point $\tilde{x}_E$, g_E is a $QIVFSN$ of the $IVFS$-point $\tilde{y}_E$.*

Then (X, E, τ) be interval-valued fuzzy soft quasi-T_2 space ($IVFS$ q-T_2 space).

Example 5.2. *Suppose that $X = [0, 1]$ and E be any proper$(E \subset X)$ Consider $IVFS$ sets f_E and g_E over X defined as below: $f : E \to \mathcal{IVF}([0, 1])$ and $g : E \to$*

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$\mathcal{IVF}([0,1])$, such that for any $e \in E, x \in X$

$$f(e)(x) = \begin{cases} 1 & 0 \leq x \leq e \\ 0 & e < x \leq 1, \end{cases}$$

and

$$g(e)(x) = \begin{cases} 0 & 0 \leq x \leq e \\ 1 & e \leq x \leq 1. \end{cases}$$

Let $\tau = \{\emptyset_E, X_E, f_E, g_E\}$. Then clearly τ is an $IVFST$ over X. Therefore for any two absolute distinct $IVFS$-points $\tilde{x}_E, \tilde{y}_E$ in X defined by

$$\tilde{x}(e)(z) = \begin{cases} 1 & z = x \\ 0 & z \neq x \end{cases}$$

and

$$\tilde{y}(e)(z) = \begin{cases} 0 & \text{if } z = y \\ 1 & \text{if } z \neq y \end{cases}$$

f_E is $IVFSN$ of the $\tilde{x}_E$ and g_E is $IVFSN$ of $\tilde{y}_E$, such that $f_E \neg \tilde{q} g_E$. Then X is $IVFS$ q-T_2 space

Theorem 5.3. *The $IVFST(X, E, \tau)$ is an $IVFS$q-T_2 space if and only if for any $x \in X$, we have*

$$\tilde{x}_E = \tilde{\bigwedge}\{Clf_E : f_E \in IVFSN \text{ of } \tilde{x}_E\}.$$

Proof. Let (X, E, τ) be a crisp $IVFS$q-T_2 space and $\tilde{x}_E$ be $IVFS$-point with support x, e-lower value λ_e^- and e-upper value γ_e^+. For any y_E be a crisp $IVFS$-point with support y, e-lower value γ_e^- and e-upper value λ_e^+. If $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$-points with different supports x and y and e-lower values and e-upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+, respectively, then there exist two $IVFS$-open sets f_E and g_E containing $IVFS$-points $\tilde{y}_E$ and $\tilde{x}_E$ respectively, such that $f_E \neg \tilde{q} g_E$. Then g_E is $IVFSN$ of $IVFS$-point $\tilde{x}_E$ and f_E is $QIVFSN$ of $\tilde{y}_E$ such that $f_E \neg \tilde{q} g_E$. Hence $\tilde{y}_E \notin Clg_E$. If $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$-points with the same supports $x = y$, then $\gamma_e^- > \lambda_e^-$and $\gamma_e^- > \lambda_e^+$ and hence there are $QIVFSN$ f_E of $IVFS$-point $\tilde{y}_E$ and $IVFSN$ g_E such that $f_E \neg \tilde{q} g_E$. Then $\tilde{y}_E \notin Clg_E$.

Conversely, let $\tilde{x}_E$ and $\tilde{y}_E$ be two distinct $IVFS$-points with different supports x and y and e-lower values and e-upper values λ_e^-, λ_e^+ and γ_e^-, γ_e^+ ,respectively. Since $\tilde{x}_E = \tilde{\bigwedge}\{Clf_E : f_E \in IVFSN$of $\tilde{x}_E\}$, then $\tilde{\bigwedge}\{Cl([f_e^-, f_e^+])(y) : f_E \in IVFSN$ of$\tilde{x}_E\} = 0$ then, $\tilde{y}_E \neg \tilde{q} \tilde{\bigwedge}\{Clf_E : f_E \in IVFSN\ mboxof \tilde{x}_E\}$. Therefore, there exists f_E is $IVFSN$ of $\tilde{x}$ and $\tilde{y}_E \neg \tilde{q} Clf_E$. Take two τ-$IVFS$-open sets f_E and $(Clf_E)^c$. So f_E is $IVFSN$ of $IVFS$-point $\tilde{x}_E$ and $(Clf_E)^c$ is $IVFSN$ of $IVFS$-point $\tilde{y}_E$, and $f_E \neg \tilde{q}(Clf_E)^c$.

□

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Definition 5.6. *Let (X, E, τ) be an $IVFST$. If for any $IVFS$-point $\tilde{x}_E$ with support x, e-lower values λ_e^- and e-upper values λ_e^+ and any $IVFS$-closed set f_E in X such that $\tilde{x}_E \neg\tilde{q} f_E$, there exists two $IVFS$-open sets h_E and k_E such that $\tilde{x}_E \tilde{\in} h_E$ and $f_E \tilde{\leq} k_E, h_E \neg\tilde{q} k_E$. Then (X, E, τ) is called interval-valued fuzzy soft quasi regular space($IVFS$ q-regular space).*

(X, E, τ) is called an interval-valued fuzzy soft quasi T_3 space, if it is $IVFS$ q-regular space and $IVFS$ q-T_1 space.

Theorem 5.4. *The $IVFST(X, E, \tau)$ is an $IVFS$ q-T_3 space if and only if for any $IVFSN$ g_E of $IVFS$-point $\tilde{x}_E$ there exists an $IVFS$-open set f_E in X such that $\tilde{x}_E \tilde{\in} f_E \tilde{\leq} cl f_E \tilde{\leq} g_E$.*

Proof. Let g_E be an $IVFS$ set in X and $\tilde{x}_E$ be $IVFS$-point with support x, e-lower value λ_e^- and e-upper value λ_e^+ such that $\tilde{x}_E \tilde{\in} g_E$, then clearly, g_E^c is $IVFS$-closed set. Since X is an $IVFS$ q-T_3 space, there exists two $IVFS$-open sets f_E, h_E such that $\tilde{x}_E \tilde{\in} f_E, g_E^c \tilde{\leq} h_E, h_E$ and $f_E \neg\tilde{q} h_E$. So, $f_E^c \tilde{\leq} h_E^c$. Then $Cl f_E \tilde{\leq} h_E^c$ implies $Cl f_E \tilde{\leq} g_E$. Hance $\tilde{x}_E \tilde{\in} f_E \tilde{\leq} Cl f_E \tilde{\leq} g_E$.

Conversely, let $\tilde{x}_E$ be an $IVFS$-point with different support x and e-lower value λ_e^- and e-upper value λ_e^+ and let g_E be an $IVFS$-closed set such that $\tilde{x}_E \neg\tilde{q} g_E$. Then g_E^c is an $IVFS$-open set containing the $IVFS$-point $\tilde{x}_E$, i.e, $\tilde{x}_E \tilde{\in} g_E^c$. Therefore, there exists an $IVFS$-open set f_E containing $\tilde{x}_E$ such that $\tilde{x}_E \tilde{\in} f_E \tilde{\leq} Cl f_E \tilde{\leq} g_E$ $g_E \tilde{\leq} (Cl f_E)^c$. Then clearly, $(Cl f_E)^c$ is an $IVFS$-open set containing g_E and $f_E \neg\tilde{q} (Cl f_E)^c$. Hance X is $IVFS$ q-T_3 space. □

Definition 5.7. *Let (X, E, τ) be an $IVFST$.If for any two $IVFS$-closed sets f_E and g_E such that $f_E \neg\tilde{q} g_E$, there exists two $IVFS$-open sets h_E and k_E such that $f_E \tilde{\leq} h_E$ and $g_E \tilde{\leq} k_E$. Then (X, E, τ) is called interval-valued fuzzy soft quasi normal space($IVFS$ q-normal space).*

(X, E, τ) is called an interval-valued fuzzy soft quasi T_4 space, if it is $IVFS$ q-normal space and $IVFS$q-T_1 space.

Theorem 5.5. *The $IVFST(X, E, \tau)$ is an $IVFS$ q-T_4 space if and only if for any $IVFS$-closed set f_E and of $IVFS$-open set containing f_E, there exists an $IVFS$-open set h_E in X such that $f_E \tilde{\leq} h_E \tilde{\leq} cl h_E \tilde{\leq} g_E$.*

Proof. Let f_E be an $IVFS$-closed set in X and g_E be an $IVFS$-open set in X containing f_E, i.e, $f_E \tilde{\leq} g_E$. So, g_E^c is an $IVFS$-closed set such that $f_E \neg\tilde{q} g_E^c$. Since X is an $IVFS$ q-T_4 space, there exists two $IVFS$-open sets h_E, k_E such that $f_E \tilde{\leq} h_E, g_E^c \tilde{\leq} k_E$, and $h_E \neg\tilde{q} k_E$. Thus, $h_E \tilde{\leq} k_E^c$, but $Cl h_E \tilde{\leq} Cl k_E^c = k_E$. Also $g_E^c \tilde{\leq} k_E$ implies $k^c \tilde{\leq} g_E$. $IVFS$-closed set over X. So $Cl h_E \tilde{\leq} k_E^c$. Hence we have $f_E \tilde{\leq} h_E \tilde{\leq} Cl h_E \tilde{\leq} g_E$.

Conversely, let $\tilde{f}_E$ and g_E be any $IVFS$-closed sets such that $f_E \neg\tilde{q} g_E$. So $f_E \tilde{\leq} g_E^c$. There exists an $IVFS$-open set h_E such that $f_E \tilde{\leq} h_E \tilde{\leq} Cl h_E \tilde{\leq} g_E$. Thus there are two $IVFS$-open sets h_E and $(Cl h_E)^c$ such that $f_E \tilde{\leq} h_E$, $g_E \tilde{\leq} (Cl h_E)^c$. This shows that X is an $IVFS$ q-T_4 space. □

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Theorem 5.6. *If $\Phi : (X_1, E_1, \tau_1) \to (X_2, E_2, \tau_2)$ is a $IVFSC$ and $IVFSO$ map where $\Phi_u X_1 \to X_2$ and $\Phi_p E_1 \to E_2$ are two ordinary bijections, then X_1 is an $IVFS$q-T_i space if and only if X_2 is an $IVFS$q-T_i space for $i = 0, 1, 2, 3, 4$.*

Proof. We just prove when $i = 2$. The other parts similarly.

Suppose that we have two $IVFS$-points $\tilde{k}_{E_2}$ and $\tilde{s}_{E_2}$ with different supports k and s and e-lowers value and e-upper values λ_e^-, λ_e^+ and γ_e^-, γ_e^+, respectively. for any $e \in E_2$. The inverse lower and upper image of $IVFS$-point $\tilde{k}_{E_2}$ under the $IVFSO$ map Φ is an $IVFS$-point in X_1 with different support $\Phi^{-1}(k)$ as below:

$$\Phi^{-1}(\tilde{k}^-)(e)(x) = \tilde{k}^-(\Phi_p(e))(\Phi_u(x)) \text{ and } \Phi^{-1}(\tilde{k}^+)(e)(x) = \tilde{k}^+(\Phi_p(e))(\Phi_u(x)).$$

And also the inverse lower and upper image of $IVFS$-point $\tilde{s}_{E_2}$ under the $IVFSO$ map Φ is an $IVFS$-point in X_1 with different support $\Phi^{-1}(s)$ as below:

$$\Phi^{-1}(\tilde{s}^-)(e)(x) = \tilde{s}^-(\Phi_p(e))(\Phi_u(x)) \text{ and } \Phi^{-1}(\tilde{s}^+)(e)(x) = \tilde{s}^+(\Phi_p(e))(\Phi_u(x)).$$

Since (X_1, E_1, τ_1) is an $IVFS$q-T_2 space, there exist two $IVFS$-open sets f_E and g_E in X_1 such that $\Phi^{-1}(\tilde{k}_{E_2})\tilde{\in} f_E$, $\Phi^{-1}(\tilde{s}_{E_2})\tilde{\in} g_E$, and $f_E \neg\tilde{q} g_E$. So $\tilde{k}_{E_2}\tilde{\in} f_E$ and $\tilde{s}_{E_2}\tilde{\in} g_E$, while $\Phi(f_E)\neg\tilde{q}\Phi(g_E)$. Then (X_2, E_2, τ_2) is an $IVFS$q-T_2 space.

Conversely, Suppose that we have two $IVFS$-points $\tilde{x}_E$ and $\tilde{y}_E$ with different supports $x, y \in X_1$ and e-lower value and e-upper value λ_e^-, λ_e^+ and γ_e^-, γ_e^+, respectively. The lower and upper image of an $IVFS$-point $\tilde{x}_E$ under the $IVFSC$ map Φ is an $IVFS$-point in X_2 with different support $\Phi_u(x)$ as below:

$$\begin{aligned}\Phi(\tilde{x}^-)(\varepsilon)(k) &= sup_{z\in\Phi^{-1}(k)}[sup_{e\in\Phi_p^{-1}(\varepsilon)}(\tilde{x}^-)(e)](z)\\ &= \begin{cases} \lambda_e^- & \text{if } k = \Phi_u(x)\\ 0 & \text{otherwis,}\end{cases}\end{aligned}$$

$$\begin{aligned}\Phi(\tilde{x}^+)(\varepsilon)(k) &= sup_{z\in\Phi^{-1}(k)}[sup_{e\in\Phi_p^{-1}(\varepsilon)}(\tilde{x}^+)(e)](z)\\ &= \begin{cases} \lambda_e^+ & \text{if } k = \Phi_u(x)\\ 0 & \text{otherwis,}\end{cases}\end{aligned}$$

and The lower and upper image of an $IVFS$-point $\tilde{y}_E$ under the $IVFSC$ map Φ is an $IVFS$-point in X_2 with different support $\Phi_u(y)$ as below:

$$\begin{aligned}\Phi(\tilde{y}^-)(\varepsilon)(k) &= sup_{z\in\Phi^{-1}(k)}[sup_{e\in\Phi_p^{-1}(\varepsilon)}(\tilde{y}^-)(e)](z)\\ &= \begin{cases} \gamma_e^- & \text{if } k = \Phi_u(y)\\ 0 & \text{otherwis,}\end{cases}\end{aligned}$$

$$\begin{aligned}\Phi(\tilde{y}^+)(\varepsilon)(k) &= sup_{z\in\Phi^{-1}(k)}[sup_{e\in\Phi_p^{-1}(\varepsilon)}(\tilde{y}^+)(e)](z)\\ &= \begin{cases} \gamma_e^+ & \text{if } k = \Phi_u(y)\\ 0 & \text{otherwis.}\end{cases}\end{aligned}$$

Since (X_2, E_2, τ_2) is an $IVFS$q-T_2 space, there exist two $IVFS$-open sets f_{E_2} and g_{E_2} in X_2 such that $\Phi(\tilde{x})\tilde{\in} f_{E_2}$, $\Phi(\tilde{y})\tilde{\in} g_{E_2}$, and $f_{E_2}\neg\tilde{q} g_{E_2}$. Clearly, $\tilde{x}_E\tilde{\in}\Phi^{-1}(f_{E_2})$, $\tilde{y}_E\tilde{\in}\Phi^{-1}(g_{E_2})$ and $\Phi^{-1}(f_{E_2})\neg\tilde{q}\Phi^{-1}(g_{E_2})$. Then (X_1, E_1, τ_1) is an $IVFS$q-T_2 space. □

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6 Conclusion

In this paper, we have introduced a new definition of interval-valued fuzzy soft point and then consider some properties of it, and different types of neighbourhoods of $IVFS$-point were studied in interval-valued fuzzy soft topological space. The separation axioms of interval-valued fuzzy soft topological is presented and of its basic properties are also studied.

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Conflicts of interest

The authors declare they have no conflict of interest.

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