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Spectral Data of Conformable Sturm-Liouville Direct Problems

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Abstract

In this study, we investigate spectral structure of conformable Sturm-Liouville problems and with this end, we obtain representation of solutions under different initial conditions and asymptotic formulas for eigenfunctions, eigenvalues, norming constants and normalized eigenfunctions. Consequently, we prove the existence of infinitely many eigenvalues. Also, we compare the solutions with graphics with different orders, different eigenvalues, different potentials and so, we observe the behaviors of eigenfunctions. We give an application to the α-orthogonality of eigenfunctions and reality of eigenvalues for conformable Sturm-Liouville problems defined by [15] in the last section.

Keywords: Sturm-Liouville; conformable derivative; asymptotic formula; spectral data

1 Introduction

Conformable derivative idea was firstly introduced by Khalil et al. [17]. This new derivative involves a shift as $\varepsilon t^{1-\alpha}$ in its limit definition differently from classical derivative definition. In the beginning, this derivative was used to be called as "conformable fractional" because of having fractional α power in the shift but afterwards, Jarad et al. [13] introduced a real fractional version of conformable derivative in Riemann-Liouville and Caputo sense and so, it deserves anymore the name of "fractional conformable derivative". The most important advantage of conformable derivative has similar properties with the ordinary derivative like the derivative of the product and quotient of two functions, and also it enables variation of order between $0 < \alpha \leq 1$, when $\alpha = 1$, it corresponds to the ordinary derivative. For this reason, many scientists showed great interest. Some of those are [11, 12, 18, 20, 21]. Conformable derivative of f of order $0 < \alpha \leq 1$ is defined by,

$$T_a^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}.$$

Another conformable derivative is proportional $\alpha-$derivative, is also known as Katugampola derivative, was introduced by Katugampola [22] as an alternative to conformable derivative. It

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is based on the control theory. It has the same properties with conformable derivative which we mentioned above but it differs in its limit definition. This definition involves a proportional ratio as $e^{\varepsilon t^{-\alpha}}$ instead of shifting. It was studied by Anderson and Ulness [23–25]. Proportional $\alpha-$derivative of f of order $0 < \alpha \leq 1$ is defined by,

$$T_a^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon}.$$

Sturm-Liouville problems are an important research area in mathematics, physics, engineering, electronics, vibrations, stability, hydrodynamics, elasticity, acoustics, electrodynamic etc. It is the form of one dimensional Schrödinger equation, which has a great importance in quantum mechanics. However, many applications of mathematical physics need investigation of eigenvalues and eigenfunctions of Sturm–Liouville problems. There are too much studies about Sturm-Liouville eigenvalue problems throughout its long history (see, e.g. [1–10, 14, 16]). Major results for the classical Sturm-LİouvilIe problem are included the asymptotic behavior of eigenvalues, eigenfunctions, norming constants, and this kind of approach is called as direct problem. In this study, we give asymptotic behavior of eigenfunctions, eigenvalues, norming constants and normalized eigenfunctions for conformable Sturm-Liouville problems and our this approachment will give rise to a lot of open problems.

Sturm-Liouville problems and its fundamental spectral properties involving confromable derivatives, which is our background study, were studied by [15,23,24]. Fractional Sturm-Liouville problems in Riemann-Liouville and Caputo sense were studied by [26–30].

In this study, we investigate the spectral structure of conformable Sturm-Liouville problems and with this end, we obtain representation of solutions under different initial conditions, asymptotic formulas for eigenfunctions, eigenvalues, norming constants and normalized eigenfunctions. Consequently, we prove the existence of infinitely many eigenvalues. Also, we compare the solutions with graphics with different orders, different eigenvalues, different potentials and so, we observe the behaviors of eigenfunctions. We give an application to the $\alpha-$orthogonality of eigenfunctions and reality of eigenvalues for conformable Sturm-Liouville problems defined by [15] in the last section.

2. Preliminaries

We give some necessary notations, definitions and lemmas related to confromable calculus theory. For more details about this field, see [17–20, 23–25].

Definition 2.1. [17,19] Let $f : (0,\infty) \to \mathbb{R}$ and $t > 0$. Then the conformable derivative of f of order $0 < \alpha \leq 1$ is defined by,

$$T_a^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon (t-a)^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{1}$$

Theorem 2.2. [17] If a function $f : [0,\infty) \to \mathbb{R}$ is $\alpha-$differentiable at $a > 0, \alpha \in (0,1]$, then f is continuous at a.

Theorem 2.3. [17,19] Let $\alpha \in (0,1]$ and f, g be $\alpha-$differentiable at a point $t > 0$. Then,

(i) $T_a^\alpha(C) = 0$, for all constant functions, $f(t) = C$;

(ii) $T_a^\alpha(t^n) = nt^{n-\alpha}$ for all $n \in \mathbb{R}$;

(iii) $T_a^\alpha[af + bg] = aT_a^\alpha[f] + bT_a^\alpha[g]$ for all $a, b \in \mathbb{R}$;

(iv) $T_a^\alpha[fg] = fT_a^\alpha[g] + gT_a^\alpha[f]$;

(v) $T_a^\alpha\left[\frac{f}{g}\right] = \frac{gT_a^\alpha[f] - fT_a^\alpha[g]}{g^2}$;

(vi) $T_a^\alpha[fog](t) = f'(g(t))T_a^\alpha g(t)$, for f differentiable at $g(t)$.

If, in addition, f is differentiable, then $T_a^\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

Theorem 2.4. [17] Let $a, n \in \mathbb{R}$ and $\alpha \in (0,1]$. Then we have the following results.

(i) $T_a^\alpha(1) = 0$,

(ii) $T_a^\alpha(e^{ax}) = ax^{1-\alpha}e^{ax}$,

(iii) $T_a^\alpha(\sin ax) = ax^{1-\alpha}\cos ax$,

(iv) $T_a^\alpha(\cos ax) = -ax^{1-\alpha}\sin ax$,

(v) $T_a^\alpha\left(\frac{1}{\alpha}t^a\right) = 1$.

It is easy to see from part (vi) of Theorem 2.3.

Theorem 2.5. [17] Let $\alpha \in (0,1]$ and $t > 0$. Then,

(i) $T_a^\alpha\left(\sin\frac{1}{\alpha}t^\alpha\right) = \cos\frac{1}{\alpha}t^\alpha$,

(ii) $T_a^\alpha\left(\cos\frac{1}{\alpha}t^\alpha\right) = -\sin\frac{1}{\alpha}t^\alpha$,

(iii) $T_a^\alpha\left(e^{\frac{1}{\alpha}t^\alpha}\right) = e^{\frac{1}{\alpha}t^\alpha}$.

Theorem 2.6. [17] Let $\alpha \in (0,1)$ and $t > 0$. Then, conformable integral is defined as following

$$(I_a^\alpha f)(x) = \int_a^x f(t)\,d_\alpha(t) = \int_a^x f(t)(t-a)^{\alpha-1}\,d(t).$$

Lemma 2.7. Let the conformable differential operator T_a^α be given as in (1), where $0 < \alpha \le 1$, $t \ge 0$ and assume the functions f and g are $\alpha-$differentiable as needed. Then

(i) $T_a^\alpha \ln t = t^{-\alpha}$ for $t > 0$;

(ii) $T_a^\alpha\left[\int\limits_a^t f(l,s)\,d_\alpha s\right] = f(l,l) + \int\limits_a^t T_a^\alpha\left[f(l,s)\right]d_\alpha s$, where $d_\alpha s \equiv s^{\alpha-1}ds$;

(iii) $\int\limits_a^b fT_a^\alpha[g]\,d_\alpha t = fg|_a^b - \int\limits_a^b gT_a^\alpha[f]\,d_\alpha t.$

Definition 2.7. If $x, y : I \to \mathbb{R}$ are $\alpha-$differentiable on I, then the conformable Wronskian of x and y is given by

$$W_\alpha(x,y)(t) = \det\begin{pmatrix} x(t) & y(t) \\ T_a^\alpha x(t) & T_a^\alpha y(t) \end{pmatrix}, \text{ for } t \in I.$$

Theorem 2.8. Let $a, b, c \in \mathbb{R}$ be constants and $\alpha \in (0,1]$. Then conformable homogeneous differential equation with constant coefficient

$$aT_a^\alpha T_a^\alpha y(t) + bT_a^\alpha y(t) + cy(t) = 0,\ t \in [0,\infty), \tag{2}$$

has the associated auxiliary equation

$$a\lambda^2 + b\lambda + c = 0,$$

and the general solution of (2) is given by one of the following for constants $c_1, c_2 \in \mathbb{R}$:

(i) $y(t) = c_1 e^{\lambda_1 t^\alpha/\alpha} + c_2 e^{\lambda_2 t^\alpha/\alpha}$, where $\lambda_1, \lambda_2 \in \mathbb{R}$ are distinct roots of (2);

(ii) $y(t) = c_1 e^{\lambda t^\alpha/\alpha} + c_2 t^\alpha e^{\lambda t^\alpha/\alpha}$, where λ is a repeated root of (2);

(iii) $y(t) = e^{\varsigma t^\alpha/\alpha}\left(c_1\cos\left(\frac{\beta}{\alpha}t^\alpha\right) + c_2\sin\left(\frac{\beta}{\alpha}t^\alpha\right)\right)$, where $\lambda = \varsigma \pm i\beta$ is a complex root of (2).

Lemma 2.6. Assume that f is continuous and $0 < \alpha \leq 1$. Then we have

$$\begin{aligned} T_a^\alpha I_a^\alpha f(t) &= f(t), \\ I_a^\alpha T_a^\alpha f(t) &= f(t) - f(a). \end{aligned}$$

3. Main Results

The most important advantage of conformable derivative has similar properties with the ordinary derivative like the derivative of the product and quotient of two functions, and also it enables variation of order between $0 < \alpha \leq 1$, when $\alpha = 1$, it corresponds to ordinary derivative. Levitan and Sargsjan [16] analyzed spectral theory of Sturm-Liouville problems in classical case. Differently from [16], we investigate spectral properties of Sturm-Liouville problems by using conformable derivative.

Conformable Sturm-Liouville Problems-Representation of Solutions and Asymptotic Formulas

Now, we consider conformable Sturm-Liouville problem

$$-T_0^{\alpha}T_0^{\alpha}y(x)+q(x)y(x)=\lambda y(x),0<\alpha\leq 1,x\in[0,\pi] \tag{3}$$

where T_0^{α} is conformable derivative operator, $q(x)$ is a real-valued and continuous function on $[0,\pi]$, $y(x)$ is $2\alpha-$continuously differentiable on $[0,\pi]$, $T_0^{\alpha}T_0^{\alpha}y(x)$ is continuous on $[0,\pi]$ and $y\in C^{2\alpha}[0,\pi]$.

We use the following boundary conditions throughout our study,

$$T_0^{\alpha}y(0)-hy(0) = 0, \tag{4}$$
$$T_0^{\alpha}y(\pi)+Hy(\pi) = 0, \tag{5}$$

where $-h=\cot\gamma$, $H=\cot\beta$. Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$, satisfy the initial conditions (4) and (5) respectively, are the solutions of (3)

$$\varphi(0,\lambda)=1,\ T_0^{\alpha}\varphi(0,\lambda)=h, \tag{6}$$

and

$$\psi(0,\lambda)=0,\ T_0^{\alpha}\psi(0,\lambda)=1. \tag{7}$$

Note that h and H are not equal to ∞ for the problem (3)−(6), the cases h or H equal to ∞ wil be investigated for the problem (3)−(7) in the sequel.

Theorem 3.1. Let $\lambda=s^2$. Then, we have the representation of solutions for the problems (3)−(6) and (3)−(7) as follows respectively

$$\varphi(x,\lambda)=\cos s\frac{x^{\alpha}}{\alpha}+\frac{h}{s}\sin s\frac{x^{\alpha}}{\alpha}+\frac{1}{s}\int_0^x \sin\left[s\left(\frac{x^{\alpha}-\tau^{\alpha}}{\alpha}\right)\right]q(\tau)\varphi(\tau,\lambda)d\tau, \tag{8}$$

$$\psi(x,\lambda)=\frac{1}{s}\sin s\frac{x^{\alpha}}{\alpha}+\frac{1}{s}\int_0^x \sin\left[s\left(\frac{x^{\alpha}-\tau^{\alpha}}{\alpha}\right)\right]q(\tau)\varphi(\tau,\lambda)d\tau. \tag{9}$$

Proof. Proof of theorem can be easily seen by the variation of parameters method for conformable differential equations given in [18], thus we have the representation of solution as follows,

$$\begin{aligned} y(x,\lambda) &= c_1y_1(x)+c_2y_2(x)+\frac{y_1(x)}{W}\int_0^x q(\tau)y_2(\tau)y(\tau)d_{\alpha}\tau \\ &-\frac{y_2(x)}{W}\int_0^x q(\tau)y_1(\tau)y(\tau)d_{\alpha}\tau \end{aligned}$$

It follows from initial condition (6),

$$\varphi(x,\lambda)=\cos\frac{sx^{a}}{\alpha}+\frac{h}{s}\sin\frac{sx^{\alpha}}{\alpha}-\frac{1}{s}\int_0^x \left[\sin s\left(\frac{x^{\alpha}-\tau^{\alpha}}{\alpha}\right)\right]q(\tau)\varphi(\tau,\lambda)d_{\alpha}\tau.$$

(9) can be proved similarly by using initial condition (7).

Theorem 3.2. Let $s = \sigma + it$. Then there exists $s_0 > 0$ such that, for $|s| > s_0$, the asymptotic formulas for eigenfunctions

$$\varphi(x,\lambda) = O\left(e^{|t|x^{\alpha}/\alpha}\right),\ \psi = O\left(|s|^{-1}e^{|t|x^{\alpha}/\alpha}\right) \tag{10}$$

are valid; and more precisely,

$$\varphi(x,\lambda) = \cos s\frac{x^{\alpha}}{\alpha} + O\left(\frac{e^{|t|x^{\alpha}/\alpha}}{|s|}\right), \tag{11}$$

$$\psi(x,\lambda) = \frac{1}{s}\sin s\frac{x^{\alpha}}{a} + O\left(\frac{e^{|t|x^{\alpha}/\alpha}}{|s|^2}\right). \tag{12}$$

Proof. Let $\varphi(x,\lambda) = e^{|t|x}F(x)$. Then, we get from (8),

$$F(x) = \left\{\cos\frac{sx^{\alpha}}{\alpha} + \frac{h}{s}\sin\frac{sx^{\alpha}}{\alpha}\right\}e^{-|t|\frac{x^{\alpha}}{\alpha}} - \frac{1}{s}\int_0^x \sin\left[s\left(\frac{x^{\alpha}-\tau^{\alpha}}{\alpha}\right)\right]e^{-|t|\frac{x^{\alpha}-\tau^{\alpha}}{\alpha}}q(\tau)F(\tau)\,d_{\alpha}\tau.$$

Put $\mu = \max\limits_{0\le x\le\pi}|F(x)|$. Then we arrive

$$\mu \le 1 + \frac{|h|}{|s|} + \frac{\mu}{|s|}\int_0^{\pi}|q(\tau)|\,d_{\alpha}\tau,$$

and therefore

$$\mu \le \frac{1+\frac{|h|}{|s|}}{1-\frac{1}{|s|}\int_0^{\pi}|q(\tau)|\,d_{\alpha}\tau}$$

on condition that the denominator is positive for

$$|s| > \int_0^{\pi}|q(\tau)|\,d_{\alpha}\tau.$$

(10) is proved for $\varphi(x,\lambda)$. It can be proved similarly for $\psi(x,\lambda)$.

Now, we obtain asymptotic formulas for eigenvalues. Consequently, this proves the existence of infinitely many eigenvalues.

Theorem 3.3. We suppose that $h \neq \infty$ and $H \neq \infty$. Asymptotic formula for the eigenvalues corresponding eigenfunction $\varphi(x,\lambda)$ is as follows

$$s_n = \alpha n\pi^{1-\alpha} + \frac{c}{n} + O\left(\frac{1}{n^2}\right),$$

where

$$c = \frac{1}{\pi}\left(\frac{1}{2}\int_0^\pi q(\tau)\,d_\alpha\tau + H + h\right).$$

Proof. The function $\varphi(x,\lambda)$ satisfies the condition (6) for any λ. Therefore, we find the eigenvalues by substituting $\varphi(x,\lambda)$ in the initial condition (5). $\alpha-$differentiating (8) with respect to x, we obtain

$$T^\alpha\varphi(x,\lambda) = -s\sin\frac{sx^\alpha}{\alpha} + h\cos\frac{sx^\alpha}{\alpha} - \int_0^x \cos\left[s\left(\frac{x^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right]q(\tau)\varphi(\tau,\lambda)\,d_\alpha\tau$$

Hence, we obtain from the initial condition (5),

$$\begin{aligned}&-s\sin\frac{s\pi^\alpha}{\alpha} + h\cos\frac{s\pi^\alpha}{\alpha} - \int_0^\pi \cos\left[s\left(\frac{\pi^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right]q(\tau)\varphi(\tau,\lambda)\,d_\alpha\tau \\ &+H\left\{-s\sin\frac{sx^\alpha}{\alpha} + h\cos\frac{sx^\alpha}{\alpha} - \int_0^x \cos\left[s\left(\frac{x^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right]q(\tau)\varphi(\tau,\lambda)\,d_\alpha\tau\right\} = 0.\end{aligned} \tag{13}$$

Since eigenvalues λ are real, we have $\operatorname{Im} s = 0$ and thus, for positive eigenvalues λ, (11) can be written as follows,

$$\varphi(x,\lambda) = \cos s\frac{x^\alpha}{\alpha} + O\left(\frac{1}{s}\right).$$

From here, substituting (11) in (13) we have

$$\begin{aligned}&-s\sin\frac{s\pi^\alpha}{\alpha} + h\cos\frac{s\pi^\alpha}{\alpha} - \int_0^\pi \cos\left[s\left(\frac{\pi^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right]q(\tau)\varphi(\tau,\lambda)\,d_\alpha\tau \\ &+H\cos\frac{s\pi^\alpha}{\alpha} - \frac{Hh}{s}\sin\frac{s\pi^\alpha}{\alpha} - \frac{H}{2s}\sin\frac{s\pi^\alpha}{\alpha}\int_0^\pi q(\tau)\,d_\alpha\tau + O\left(\frac{1}{s^2}\right) = 0.\end{aligned}$$

We assume that $q(x)$ has a bounded $\alpha-$derivative. From here

$$-s\sin\frac{s\pi^\alpha}{\alpha} + h\cos\frac{s\pi^\alpha}{\alpha} - \frac{1}{2}\cos s\frac{\pi^\alpha}{\alpha}\int_0^\pi q(\tau)\,d_\alpha\tau + H\cos\frac{s\pi^\alpha}{\alpha} + O\left(\frac{1}{s}\right) = 0$$

and hence, it can be easily seen from the method of classical eigenvalue calculation

$$s_n = \alpha n\pi^{1-\alpha} + \frac{1}{2n\pi}\int_0^\pi q(\tau)\,d_\alpha\tau - \frac{(H+h)}{s} + O\left(\frac{1}{(\alpha n)^2}\right)$$

Therefore

$$s_n = \alpha n\pi^{1-\alpha} + \frac{c}{n} + O\left(\frac{1}{(\alpha n)^2}\right), \tag{14}$$

where

$$c = \frac{1}{\pi}\left(\frac{1}{2}\int_0^{\pi} q(\tau)\,d_\alpha\tau + H + h\right).$$

Asymptotic formula (11) can be written as follows by inserting value s_n

$$\varphi_n(x,\lambda) = \cos(n\pi^{1-\alpha}x^\alpha) + \frac{\beta(x)}{n}\sin(n\pi^{1-\alpha}x^\alpha) + O\left(\frac{1}{(\alpha n)^2}\right),$$

where $\beta(x) = -c\dfrac{x^\alpha}{\alpha} + \dfrac{h}{\alpha\pi^{1-\alpha}} + \dfrac{1}{2\alpha\pi^{1-\alpha}}\displaystyle\int_0^x q(\tau)\,d_\alpha\tau.$

Theorem 3.4. Asymptotic formula for the norming constants is as follows,

$$\alpha_n = \sqrt{\frac{\pi^\alpha}{2\alpha} + O\left(\frac{1}{(\alpha n)^2}\right)}.$$

Proof. Let's consider the following integral

$$\begin{aligned}
\alpha_n^2 &= \int_0^{\pi} \varphi_n^2(x)\,d_\alpha x \\
&= \int_0^{\pi} \cos^2 n\frac{x^\alpha}{\alpha}d_\alpha x + \frac{1}{n}\int_0^{\pi}\beta(x)\sin\left(2n\frac{x^\alpha}{\alpha}\right)d_\alpha x + O\left(\frac{1}{n^2}\right) \\
&= \int_0^{\pi} \cos^2 s\frac{x^\alpha}{\alpha}d_\alpha x + O\left(\frac{1}{(\alpha n)^2}\right) \\
&= \frac{x^\alpha}{2\alpha}\Big|_0^{\pi} + O\left(\frac{1}{(\alpha n)^2}\right) \\
&= \frac{\pi^\alpha}{2\alpha} + O\left(\frac{1}{(\alpha n)^2}\right).
\end{aligned}$$

Hence, we have the norming constants $\alpha_n = \sqrt{\dfrac{\pi^\alpha}{2\alpha} + O\left(\dfrac{1}{(\alpha n)^2}\right)}$.

Theorem 3.5. Asymptotic formula for the normalized eigenfunction is as follows,

$$\frac{\varphi_n(x)}{\alpha_n} = \sqrt{\frac{2\alpha}{\pi^\alpha}}\left(\cos(n\pi^{1-\alpha}x^\alpha)\alpha + \frac{\beta(x)}{n}\sin(n\pi^{1-\alpha}x^\alpha)\right) + O\left(\frac{1}{(\alpha n)^2}\right).$$

Now, we analyze the case $h = \infty$, $H \neq \infty$ (the case $h \neq \infty$, $H = \infty$ can be analyzed by substituting to $t = \pi - x$). We assume that the condition (4) is in the form of

$$y(0) = 0. \tag{15}$$

Theorem 3.6. We suppose that $h=\infty$, $H\neq\infty$. Asymptotic formula for the eigenvalues corresponding eigenfunction $\psi(x,\lambda)$ is as follows

$$s_n=\frac{\alpha\pi}{\pi^{\alpha}}\left(n+\frac{1}{2}\right)-\alpha\pi\frac{H_1}{\left(n+\frac{1}{2}\right)}+O\left(\frac{1}{(\alpha n)^2}\right),$$

where

$$H_1=H+\frac{1}{2}\int_0^{\pi}q(\tau)\,d_{\alpha}\tau.$$

Proof. The function $\psi(x,\lambda)$ satisfies condition (15) from (4). Therefore, for this case we can settle the eigenvalues by replacing the function $\psi(x,\lambda)$ in the initial condition (5). $\alpha-$differentiating (9) with respect to x, we obtain

$$T^{\alpha}\psi(x,\lambda)=\cos\frac{sx^{\alpha}}{\alpha}-\int_0^{x}\cos\left[s\left(\frac{x^{\alpha}}{\alpha}-\frac{\tau^{\alpha}}{\alpha}\right)\right]q(\tau)\psi(\tau,\lambda)\,d_{\alpha}\tau$$

Hence, we obtain from the the initial condition (5),

$$\cos\frac{s\pi^{\alpha}}{\alpha}-\int_0^{\pi}\cos\left[s\left(\frac{\pi^{\alpha}}{\alpha}-\frac{\tau^{\alpha}}{\alpha}\right)\right]q(\tau)\psi(\tau,\lambda)\,d_{\alpha}\tau \tag{16}$$
$$+H\left\{\frac{1}{s}\sin\frac{s\pi^{\alpha}}{\alpha}-\frac{1}{s}\int_0^{\pi}\sin\left[s\left(\frac{\pi^{\alpha}}{\alpha}-\frac{\tau^{\alpha}}{\alpha}\right)\right]q(\tau)\psi(\tau,\lambda)\,d_{\alpha}\tau\right\}\;=\;0,$$

substituting (12) in (16), we get

$$\cos\frac{s\pi^{\alpha}}{\alpha}-\frac{1}{s}\int_0^{\pi}\cos\left[s\left(\frac{\pi^{\alpha}}{\alpha}-\frac{\tau^{\alpha}}{\alpha}\right)\right]q(\tau)\sin\frac{s\tau^{\alpha}}{\alpha}d_{\alpha}\tau+\frac{H}{s}\sin\frac{s\pi^{\alpha}}{\alpha}+O\left(\frac{1}{s^2}\right)=0,$$

we assume that $q(x)$ has a bounded $\alpha-$derivative, so

$$\int_0^{\pi}q(\tau)\cos\left[s\left(\frac{\pi^{\alpha}}{\alpha}-\frac{\tau^{\alpha}}{\alpha}\right)\right]\sin\frac{s\tau^{\alpha}}{\alpha}d_{\alpha}\tau$$
$$=\;\cos\frac{s\pi^{\alpha}}{\alpha}\int_0^{\pi}q(\tau)\cos s\frac{\tau^{\alpha}}{\alpha}\sin\frac{s\tau^{\alpha}}{\alpha}d_{\alpha}\tau$$
$$+\sin\frac{s\pi^{\alpha}}{\alpha}\int_0^{\pi}q(\tau)\sin^2\frac{s\tau^{\alpha}}{\alpha}d_{\alpha}\tau$$
$$=\;\frac{1}{2}\sin\frac{s\pi^{\alpha}}{\alpha}\int_0^{\pi}q(\tau)\,d_{\alpha}\tau+O\left(\frac{1}{s}\right).$$

Hence

$$\cos\frac{s\pi^{\alpha}}{\alpha}-\frac{1}{s}\left\{\frac{1}{2}\sin\frac{s\pi^{\alpha}}{\alpha}\int_{0}^{\pi}q(\tau)\,d_{\alpha}\tau+O\left(\frac{1}{s}\right)\right\}$$
$$+\frac{H}{s}\sin\frac{s\pi^{\alpha}}{\alpha}+O\left(\frac{1}{s^{2}}\right)=0,$$

and

$$\cos\frac{s\pi^{\alpha}}{\alpha}+\frac{1}{s}\sin\frac{s\pi^{\alpha}}{\alpha}\left\{H+\frac{1}{2}\int_{0}^{\pi}q(\tau)\,d_{\alpha}\tau\right\}+O\left(\frac{1}{s^{2}}\right)$$
$$=\cos\frac{s\pi^{\alpha}}{\alpha}+\frac{H_{1}}{s}\sin\frac{s\pi^{\alpha}}{\alpha}+O\left(\frac{1}{s^{2}}\right)=0.$$

It can be easily seen from the method of classical eigenvalue calculation

$$s_{n}=\frac{\alpha\pi}{\pi^{\alpha}}\left(n+\frac{1}{2}-\frac{H_{1}}{s}\right)+O\left(\frac{1}{s^{2}}\right)$$

Therefore

$$s_{n}=\frac{\alpha\pi}{\pi^{\alpha}}\left(n+\frac{1}{2}\right)+\frac{H_{1}}{n+\frac{1}{2}}+O\left(\frac{1}{n^{2}}\right),$$

where

$$H_{1}=H+\frac{1}{2}\int_{0}^{\pi}q(\tau)\,d_{\alpha}\tau.$$

Asymptotic formula (12) can be written as follows by inserting value s_n

$$s_{n}=\frac{\alpha\pi}{\pi^{\alpha}}\left(n+\frac{1}{2}\right)-\alpha\pi\frac{H_{1}}{\left(n+\frac{1}{2}\right)}+O\left(\frac{1}{(\alpha n)^{2}}\right),$$

Theorem 3.7. Asymptotic formula for the norming constants is as follows,

$$\alpha_{n}=\frac{1}{\alpha\pi^{1-\alpha}}\sqrt{\frac{\pi^{\alpha}}{2\alpha}}\left(\frac{1}{n+\frac{1}{2}}\right)+O\left(\frac{1}{\alpha n}\right).$$

Proof. To find asymptotic formula for the norming constants, let's consider the following integral

$$\alpha_{n}^{2}=\int_{0}^{\pi}\psi_{n}^{2}(x)\,d_{\alpha}x$$
$$=\left(\frac{1}{\alpha\pi^{1-\alpha}}\right)^{2}\frac{\pi^{\alpha}}{2\alpha}\left(\frac{1}{n+\frac{1}{2}}\right)+O\left(\frac{1}{(\alpha n)^{2}}\right).$$

Hence, we have the norming constants $\alpha_n = \frac{1}{\alpha\pi^{1-\alpha}}\sqrt{\frac{\pi^\alpha}{2\alpha}}\left(\frac{1}{n+\frac{1}{2}}\right)+O\left(\frac{1}{\alpha n}\right)$.

Theorem 3.8. Asymptotic formula for the normalized eigenfunction is as follows,

$$\frac{\psi_n(x)}{\alpha_n} = \sqrt{\frac{2\alpha}{\pi^\alpha}}\frac{1}{\alpha\pi^{1-\alpha}}\sin\left[\alpha\pi^{1-\alpha}\left(n+\frac{1}{2}\right)\frac{x^\alpha}{\alpha}\right]+O\left(\frac{1}{\alpha n}\right).$$

Finally, let's analyze the case $h=\infty$ and $H=\infty$. This requires that the boundary conditions (4)−(5) have the form

$$y(0) = y(\pi) = 0.$$

Theorem 3.9. We suppose that $h=\infty$, $H=\infty$. Asymptotic formula for the eigenvalues corresponding eigenfunction $\psi(x,\lambda)$ is as follows

$$s_n = n\alpha\pi^{1-\alpha} + \frac{c}{n\pi} + O\left(\frac{1}{(\alpha n)^2}\right),$$

where

$$c = \frac{1}{2}\int_0^\pi q(\tau)\,d_\alpha\tau.$$

Proof. Eigenvalues are the roots of

$$\psi(\pi) = 0.$$

Thus,

$$\begin{aligned}\psi(\pi,\lambda) &= \sin\frac{s\pi^\alpha}{\alpha}\left(1-\int_0^\pi \cos s\frac{\tau^\alpha}{\alpha}q(\tau)\psi(\tau,\lambda)\,d_\alpha\tau\right) \\ &\quad -\cos\frac{s\pi^\alpha}{\alpha}\int_0^\pi \sin s\frac{\tau^\alpha}{\alpha}q(\tau)\psi(\tau,\lambda)\,d_\alpha\tau \;=\; 0,\end{aligned} \tag{17}$$

substituting (12) in (17), we get

$$\begin{aligned}&\sin\frac{s\pi^\alpha}{\alpha} - \sin\frac{s\pi^\alpha}{\alpha}\frac{1}{s}\int_0^\pi \cos s\frac{\tau^\alpha}{\alpha}\sin\frac{s\tau^\alpha}{\alpha}q(\tau)\,d_\alpha\tau \\ &-\cos\frac{s\pi^\alpha}{\alpha}\int_0^\pi \frac{1}{s}\sin^2\frac{s\tau^\alpha}{\alpha}q(\tau)\,d_\alpha\tau + O\left(\frac{1}{s^2}\right) \;=\; 0,\end{aligned}$$

we assume that $q(x)$ has a bounded α−derivative, so

$$\sin\frac{s\pi^\alpha}{\alpha} - \cos\frac{s\pi^\alpha}{\alpha}\frac{1}{2s}\int_0^\pi q(\tau)\,d_\alpha\tau + O\left(\frac{1}{s^2}\right) = 0.$$

From here,

$$s_n = \frac{\alpha}{\pi^\alpha}\left(n + \frac{1}{2} - \frac{H_1}{s}\right) + O\left(\frac{1}{s^2}\right),$$

Therefore,

$$s_n = n\alpha\pi^{1-\alpha} + \frac{c}{n\pi} + O\left(\frac{1}{n^2}\right),$$

where

$$c = \frac{1}{2}\int_0^\pi q(\tau)\, d_\alpha \tau.$$

Asymptotic formula (12) can be written as follows by inserting value

$$s_n = n\alpha\pi^{1-\alpha} + \frac{c}{n\pi} + O\left(\frac{1}{(\alpha n)^2}\right),$$

Theorem 3.10. Asymptotic formula for the norming constants is as follows,

$$\alpha_n = \frac{1}{\alpha n\pi^{1-\alpha}}\sqrt{\frac{\pi^\alpha}{2\alpha}} + O\left(\frac{1}{(\alpha n)^2}\right).$$

Proof. Norming constants can be found as follows

$$\begin{aligned}\alpha_n^2 &= \int_0^\pi \psi_n^2(x)\, d_\alpha x \\ &= \frac{\pi^{\alpha-1}}{2\alpha n} + O\left(\frac{1}{n^2}\right).\end{aligned}$$

Hence, we have the norming constants $\alpha_n = \frac{1}{\alpha n\pi^{1-\alpha}}\sqrt{\frac{\pi^\alpha}{2\alpha}} + O\left(\frac{1}{(\alpha n)^2}\right)$.

Theorem 3.11. Asymptotic formula for the normalized eigenfunction is as follows,

$$\frac{\psi_n(x)}{\alpha_n} = \sqrt{\frac{2\alpha}{\pi^{1-\alpha}}}\sin\left(n\pi^{1-\alpha}x^\alpha\right) + O\left(\frac{1}{n}\right).$$

4. Visual Results

In this section, the eigenfunctions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ of the problems (3)−(6) and (3)−(7) are compared for different cases visually. Assume that $h = 1$ for all figures, and $q(x) = 0$ for all figures except for $Fig4$ and $Fig8$.

Application

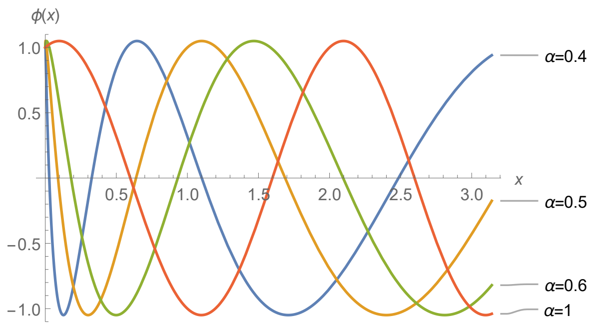

Figure 1: Comparions of eigenfunctions for $\varphi(x,\lambda)$ under different orders, $s=\pi$

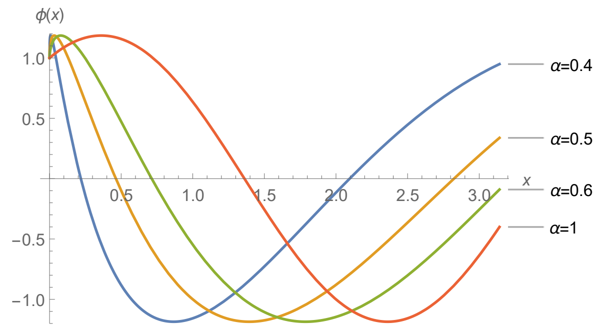

Figure 2: Comparions of eigenfunctions for $\varphi(x,\lambda)$ under different orders, $s=\frac{\pi}{2}$

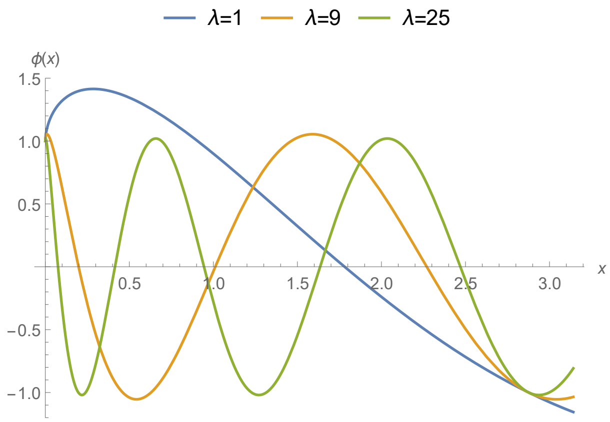

Figure 3: Comparions of eigenfunctions for $\varphi(x,\lambda)$ under different eigenvalues, $\alpha=0.6$

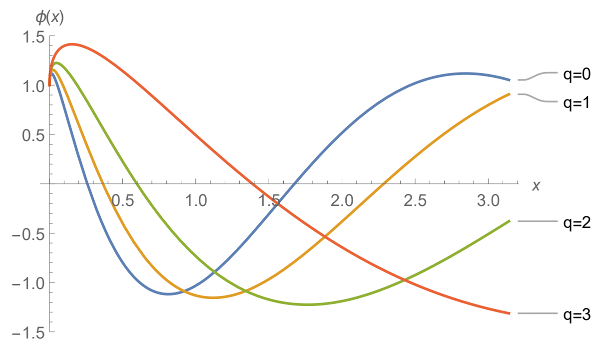

Figure 4: Comparions of eigenfunctions for $\varphi(x,\lambda)$ under different potentials,$s=4,\alpha=0.5$

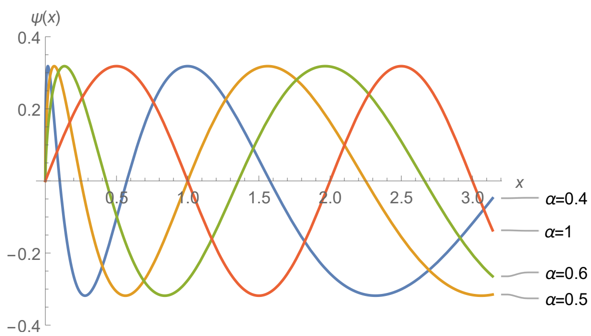

Figure 5: Comparions of eigenfunctions for $\psi(x,\lambda)$ under different orders,$s=\pi$

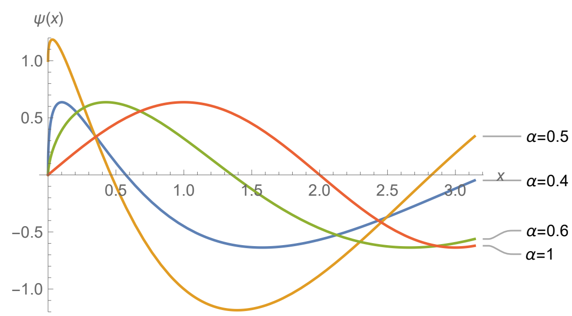

Figure 6: Comparions of eigenfunctions for $\psi(x,\lambda)$ under different orders,$s=\frac{\pi}{2}$

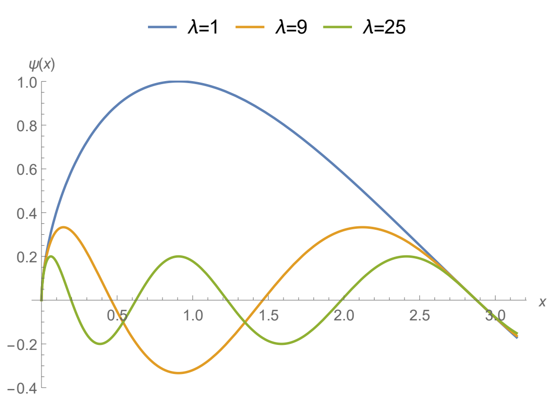

Figure 7: Comparions of eigenfunctions for $\psi(x,\lambda)$ under different eigenvalues, $\alpha=0.6$

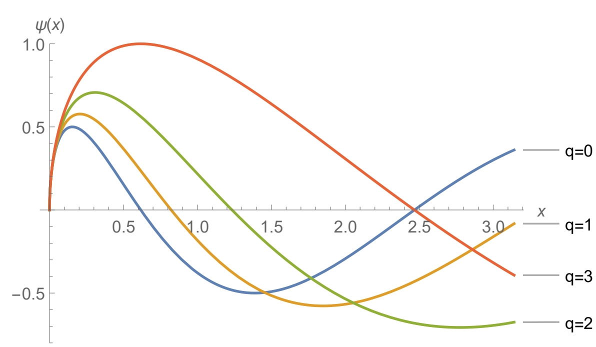

Figure 8: Comparions of eigenfunctions for $\psi(x,\lambda)$ under different potentials, $s=4$

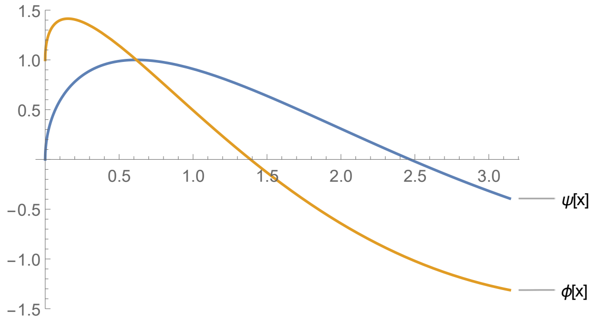

Figure 9: Comparions of eigenfunctions of $\varphi(x,\lambda)$ and $\psi(x,\lambda)$, $\alpha = 0.5, s = 1$

Let's give an application to the $\alpha-$orthogonality of eigenfunctions and reality of eigenvalues for conformable Sturm-Liouville problem defined by [15] under Dirichlet boundary conditions.

$$-T_0^\alpha T_0^\alpha y(x) = \lambda y(x),\ 0 < x < \ell, \ell \in \mathbb{R}^+, \tag{18}$$

$$y(0) = y(\ell) = 0. \tag{19}$$

Solution of the problem $(18-19)$ is as follows,

$$y_n(x) = \sin\left(\frac{n\pi x^\alpha}{\ell^\alpha}\right), \tag{20}$$

so $y_1 = \sin\left(\frac{n\pi}{\ell^\alpha}\right)$, $y_2 = \sin\left(\frac{n\pi 2^\alpha}{\ell^\alpha}\right)$, $y_3 = \sin\left(\frac{n\pi 3^\alpha}{\ell^\alpha}\right)$, ... and eigenvalues of the problem $(18-19)$ is found as

$$\lambda_n = \left(\frac{n\pi\alpha}{\ell^\alpha}\right)^2.$$

so $\lambda_1 = \left(\frac{\pi\alpha}{\ell^\alpha}\right)^2$, $\lambda_2 = \left(\frac{2\pi\alpha}{\ell^\alpha}\right)^2$, $\lambda_3 = \left(\frac{3\pi\alpha}{\ell^\alpha}\right)^2$, ... and $\alpha-$orthogonality of eigenfunctions y_n and y_m corresponding to distinct eigenvalues λ_n and λ_m can be seen as follows,

$$\langle y_n, y_m\rangle_\alpha = \int_0^\ell \sin\left(\frac{m\pi\tau^\alpha}{\ell^\alpha}\right)\sin\left(\frac{n\pi\tau^\alpha}{\ell^\alpha}\right) d_\alpha\tau = 0, m \neq n.$$

Remark.

All results obtained above are also valid for conformable Sturm-Liouville problems with proportional $\alpha-$derivative.

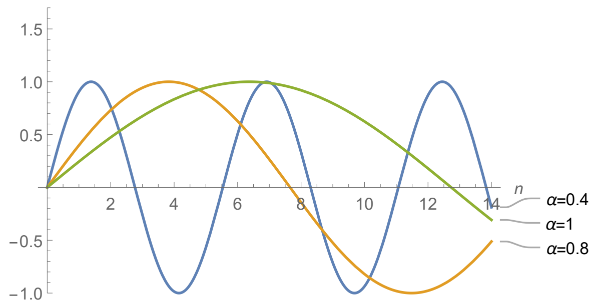

Figure 10: Comparions of eigenfunctions y_n for (20) under different orders, $x = \frac{\pi}{4}, l = 10$

5. Conclusion

In this study, we investigate spectral structure of conformable Sturm-Liouville problems and with this end, we obtain representation of solutions under different initial conditions and asymptotic formulas for eigenfunctions, eigenvalues, norming constants and normalized eigenfunctions. So, we prove the existence of infinitely many eigenvalues. Also, we compare the solutions with graphics with different orders, different eigenvalues, different potentials and so, we observe the behaviors of eigenfunctions.

We compare the eigenfunctions of the problem $(3)-(6)$ under different orders, different eigenvalues and different potential functions in $Fig1$, $Fig2$, $Fig3$, and $Fig4$. We compare the eigenfunctions of the problems $(3)-(7)$ under different orders, different eigenvalues and different potential functions in $Fig5$, $Fig6$, $Fig7$ and $Fig8$. We compare the eigenfunctions of the problems $(3)-(6)$ and $(3)-(7)$ with each other in $Fig9$.

The most important advantage of conformable derivative has similar properties with the ordinary derivative like the derivative of the product and quotient of two functions, and also it enables variation of order between $0 < \alpha \leq 1$, and when $\alpha = 1$, it corresponds to ordinary derivative. Levitan and Sargsjan [16] analyzed spectral theory of Sturm-Liouville problems in usual case. Differently from [16], we investigate spectral properties of Sturm-Liouville problems by using conformable derivative .

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