Peer-reviewed version available at *Symmetry* **2019**, *12*, 63; doi:10.3390/sym12010063

Article

Quantum Correction for Newton's Law of Motion

Timur Kamalov [1] 0000-0002-4349-4747

[1] Moscow Institute of Physics and Technology; t.kamalov@phystech.edu
* Correspondence: timkamalov@mail.ru; Tel.:+7-916-293-6637

Abstract: A description of the motion in Non-Inertial Reference Frames by means of the inclusion of high time derivatives is studied. It is noted that incompleteness of the description of physical reality is a problem of any theory: both quantum mechanics and classical physics. The "stability principle" is put forward. We also provide macroscopic examples of Non-Inertial Mechanics and verify the use of high-order derivatives as non-local hidden variables based the equivalence principle when acceleration is equal to the gravitational field. Acceleration in this case is a function of the higher derivatives with respect to time. The definition of Dark Metrics for matter and energy is presented to replace the standard notions of Dark Matter and Dark Energy.

Keywords: quantum correction, Non-Inertial Mechanics, Dark Metric

PACS: 03.65.Ud

1. Introduction

The problem of physics axiomatizing being one of Hilbert problems entails a search for unified axiomatic of both classical and quantum physics. In this paper the problem of incompleteness of quantum-mechanical description of the physical reality shall be replaced with the problem of incompleteness of classical physics. The implementation of the search for a unified axiomatic of classical and quantum physics is suggested through complementing classical physics as well. This is related to the fact that quantum physics is much richer in variables than classical one, and complementing classical physics with hidden variables is more reasonable than doing it with quantum physics, which has been practice for the last 100 years by numerous authors in the effort to sew together classical and quantum physics.

2. Why Newton's Law of Motion is Second-Order Derivatives Equation?

Since 1935 the contradiction of classical and quantum mechanics and the search for a satisfactory quantum axiomatic were one of the important problems. But the search for this quantum axiomatic was unsuccessful. A more successful solution in providing the consistency of classical and quantum physics would be a unified axiomatic. According to Gödel's theorem, in any theory there are provisions that cannot be proved in the framework of this theory. It can also add that no theory is complete. The axioms of any theory have not been proved, but guessed, so any system of axioms can be replaced by another. The main idea of Newton's laws in Prisipia postulate a description dynamics of mechanical systems with second-order differential equations. Are there any cases of describing reality using higher order differential equations? The answer is yes, but this is not Newtonian mechanics. In fact, it is difficult to find an inertial reference frame, since there always exist random weak external fields and forces, but we can assume that an inertial reference frame exists theoretically. One possible example of

Peer-reviewed version available at Symmetry 2019, 12, 63; doi:10.3390/sym12010063

non-Newtonian mechanics is quantum mechanics. A non-inertial reference frames is needed in order to add one of the most important properties of micro-objects of quantum mechanics - non-locality. Non-locality in quantum mechanics has problems with a description and contradicts one of the basic physical postulates of the maximum speed of light. The use of non-inertial reference frames eliminates these difficulties. In this case, the role of non-local hidden variables is played by acceleration and its higher derivatives with respect to time. In a quivering frame of reference, the oscillations of two classical particles will correlate, since the acceleration and its higher time derivatives will not depend on their coordinates. The description of mechanical systems by non-inertial mechanics is performed using high-order derivatives differential equations.

Definition. Non-Inertial Reference Frame is a Reference Frame with the generalised transformation of coordinates and time as

$$Q = \langle q(t) \rangle = \int_{-\tau}^{+\tau} \psi^* q \psi dt$$

$$T = t$$

Here τ is a time interval for averaging and

$$\psi = \psi_0 e^{i\frac{\Delta p \Delta q + \Delta E \Delta t}{\hbar}} = \psi_0 e^{i\frac{\frac{\Delta p}{\Delta t} - F}{\frac{\hbar}{\Delta q \Delta t}}} = \psi_0 e^{i\frac{f_0}{f_Q}} \tag{1}$$

is wave function with the inertial force f_0 depend of high-order derivatives coordinates on time and f_Q corresponds of inertial forces and constant force.

Non-Inertial Reference Frames are a method for describing the influence of random fields on both the particle to be described and the observer. The transition from a Non-Inertial reference frame to an Inertial one causes a free particle to oscillate randomly, correlating with vibrations of other free particles. Transformations of Non-Inertial Reference Frames differ from Galileo-Lorentz transformations by residual terms in the Taylor expansion. Then free particles in Inertial Reference Frames will be described with uncertainty in coordinates and momentum, time and energy, equal to the rest of the terms of the Taylor expansion. If the transformation of a Non-Inertial Reference Frame to another described by the Taylor expansion contains a remainder term with index N, then we can say that this free particle conserves its time derivative of the N-th order. Such a free particle is described by N derivatives and preserves this state until interactions with other bodies (forces) perturb this state. If such a particle interacts with other bodies (a force acts on it), then the dynamics of such a particle is described by differential equations of the (N + 1) -th order. In other words, the influence of force adds one more derivative to the description of particle dynamics. Considering a particle in an Inertial Reference Frame instead of Non-Inertial ones, one shall either introduce inertial forces, that is, change from higher-derivative description to the description without higher derivatives, but with inertia forces, or take into account remainder terms of Taylor expansion.

Modern physics (both classical and quantum) is the physics of Inertial Reference Frames. The case of a Non-Inertial Reference Frame usually comes down to the introduction of inertia forces into the Inertial Reference Frame. The use of inertia forces makes it possible to reduce the problems of the dynamics of a physical system in a Non-Inertial Reference Frame to the tasks in an Inertial Reference Frame by artificially introducing inertial forces or applying the Alembert principle. At the same time, an Inertial Reference Frame does not exist in nature, since any reference frame is always influenced by infinitesimal disturbing fields or forces. In this study, we propose to consider only non-inertial reference frames as real. Since the introduction of the d'Alembert principle to the present, the reality of the forces of inertia has been a debatable issue. We believe that the question of the reality of inertia forces can be reduced to the question of the reality of Inertial Reference Frames. How can one describe physical systems in non-inertial reference systems without introducing inertia forces? The reference frame is called Inertial, if Newton's laws hold. They postulate description of physical systems by second-order differential equations. The rejection of higher order derivatives of coordinates is associated with the problem of inertia forces in Inertial Reference Frames. Thus, in order to answer the question posed above, we must consider a more general case of higher order differential equations and expand classical physics with a description using higher order derivatives. The transition from an Inertial Reference Frame to a Non-Inertial one without introducing inertia forces means a transition from a description of physical systems of second-order differential equations to their description using higher-order differential equations. The rejection of the use of higher order derivatives with respect to coordinates in classical Newtonian physics does not mean that they do not exist. They exist in certain cases. But this is not Newtonian physics.

In the most general case, a transformation from the Non-Inertial Reference Frame to another one can be expressed as:

$$Q = q(\tau, \dot{q}, \ddot{q}, \dddot{q}, ..., q^{(n)})$$
$$T = t$$

Conversion of coordinates of a point particle between two Non-Inertial Reference Frames provided τ is a time interval for averaging, shall be expressed as

$$Q = q(t) + \dot{q}(t)\tau + \Delta q(t),$$
$$\Delta q(t) = \sum_{k=2}^{n} (-1)^k \tfrac{1}{k!} \tau^k q^{(n)}(t)$$

same holds for momentum

$$P = p(t) + \dot{p}(t)\tau + \Delta p(t),$$
$$\Delta p(t) = \sum_{k=2}^{n} (-1)^k \tfrac{1}{k!} \tau^k p^{(n)}(t)$$

$$\langle P \rangle = \frac{1}{2}[p(t+\tau) + p(t-\tau)] \tag{2}$$

Here, $\Delta q(t), \Delta p(t)$ are remainder terms of the Taylor expansion. The remainder terms $\Delta q(t), \Delta p(t)$ in Non-Inertial Reference Frame may be interpreted as uncertainties of coordinate and momentum of a point particle in this reference system. In quantum mechanics, uncertainties of coordinate and momentum of a micro particle obey to the rule

$$\Delta q(t) \Delta p(t) \geq \hbar/2 \tag{3}$$

In Non-Inertial Physics can be introduced an General Uncertainty Relation, as there always exist random small fields and forces influencing either the very system to be described or an observer, that is

$$[\sum_{k=2}^{n} (-1)^k \tfrac{1}{k!} \tau^k x^{(k)}(t)][\sum_{k=2}^{n} (-1)^k \tfrac{1}{k!} \tau^k p^{(k)}(t)] \geq H/2$$

inertial one.

The supremum of the difference of the action function in Non-Inertial Reference Frames (with higher time derivatives of the generalized coordinate) from the classical mechanics action functions (without higher derivatives) is: In this case, higher derivatives are non-local additional variables and disclose the sense of the classical analogue H of the Planck's constant. The H constant defines the supremum of the influence of random fields onto the physical system and the observer. We shall analyse this case in terms of Non-Inertial Reference Frame. In this case, H defines the supremum of the difference between a Non-Inertial Reference Frame and an inertial

$$\sup \left| S(q, \dot{q}, \ddot{q}, \dddot{q}, ..., q^{(n)}, ...) - S(q, \dot{q}) \right| = H. \tag{4}$$

Peer-reviewed version available at Symmetry 2019, 12, 63; doi:10.3390/sym12010063

Action functions in higher-derivative of Non-Inertial Reference Frame describe physical systems dynamics and differ from the action function neglecting random fields, which are accounted for via Non-Inertial Reference Frame. In our case, the classical space is featured by infinite number of variables, same as Hilbert one. In the search for a unified axiomatic the classical constant H shall coincide with the quantum one, i.e. the Planck constant $\hbar$. In this approach, the estimate of the Planck constant may be determined by higher derivatives, playing the role of non-local hidden variables.

In this case the state of quantum object can be describe

$$|\psi(t)\rangle = \left|q, \dot{q}, \ddot{q}, \dddot{q}, ..., q^{(n)}\right\rangle = |Q(t)\rangle \,.$$

The transfer object from point 1 to point 2 is

$$\langle Q_1, t_1 | Q_2, t_2 \rangle = \int_{t_1}^{t_2} DQ \exp(\tfrac{i}{\hbar} L(Q)) dt.$$

And introduced function can be represent

$$A_{Q(R)} = \left\langle q(t), \dot{q}(t), \ddot{q}(t), ..., q^{(n)}(t) | [i\hbar \tfrac{\partial}{\partial t} - H] | q(t), \dot{q}(t), \ddot{q}(t), ..., q^{(n)}(t) \right\rangle dt$$

3. Stability Principal

Classical mechanics describe a stable trajectories and Non-Inertial Mechanics to add instability random trajectories with high-order derivatives variables. The stability condition in calculations of mechanical trajectories is put forward in publications of N.G. Chetayev [2]. According to him, "stability is probably an essentially general phenomenon that has to manifest itself in principal laws of Nature." In his opinion, stability is not a mere casualness, but rather, is a consequence of system being affected by persistent infinitesimal perturbations, which, no matter how small, affect the state of a mechanical system. The condition of stability usually used in Mechanics can be extended to other areas of Physics. In this case the condition of stability can be named the stability principle. The stability principle is a generalization of basic fundamental physical laws, such as the least action principle, Newton's laws, Euler-Lagrange equations, Schrödinger equation, et al. We draw your attention for our definition stability condition which we extend to the other areas of Physics. Let us define a stable state of a physical system through the stability principle.

Stability principle: *The state A of a physical system is considered stable if it returns to the initial state after finished the action of external factors and the variance of the variable with itself is zero* $Var(A) = \sigma_A = 0$.

We consider Non-Inertial Reference Frames due the influence of the background of random fields and waves because the variance of the action function for unstable trajectory with itself can be represented as $Var(S_r) = \sigma_{Sr} = H$ and the complex variance can be define in the form

$$Var(S_c) = Var(S(q, \dot{q})) + iVar(S_r(q, \dot{q}, \ddot{q}, \dddot{q}, ..., q^{(n)}, ...) = \sigma_C = \sigma_S + i\sigma_{Sr}.$$

where for the classical stable trajectory the variance is $\sigma_S = 0$ and for the unstable trajectory $\sigma_{Sr} = H$.

4. Quantum Correlations and Illusion of Superluminal Interaction

The discussing the non-locality of entangled states quantum correlations for observers Alice and Bob, we may notice the following. The emerging illusion of transfer from A to B, or interaction of entangled quantum objects in A and B follows from experimentally observed correlation of their states. So it would be correct to negate not only faster-than-light interaction or transfer, but the very fact of any interaction or transfer. Existence of quantum correlations and non-locality of micro-object quantum states may be describe by non-inertial nature of Non-Inertial Reference. In other words, existence of quantum non-locality and quantum correlations means an illusion rather than realness of any transfer or faster-than-light interaction of these objects.

Peer-reviewed version available at *Symmetry* **2019**, *12*, 63; doi:10.3390/sym12010063

Let us perform an imaginary experiment of the classical analogue of teleportation of quantum polarization states of bi-photons.

For this purpose, let us consider the classical analogue of teleportation of bi-photon polarization states quantum entanglement. A classical analogue of this situation may be considered on the example of newspapers with news printed, say, in the city O and sent to cities A and B.

If a reader in the city A reads the news, then coincidence of his/her information with that in B may be described with a non-zero correlation factor. This is so because the news information in A and B shall correlate with a non-zero factor.

Let us emphasize that the complete match of the news information could only occur provided readers A and B read newspapers with the same title and of the same date.

If the newspapers are different but both of the same date, then the correlation factor will not be unity, but at the same time, it will not be zero. To achieve complete match of the news information with the correlation factor unity, the reader A shall advise to the reader B both the title of the newspaper and its date.

To provide teleportation of bi-photon quantum states from A to B we may consider a primary photon, which, with the aid of a non-linear crystal (e.g. BBO), is split in the point O into two photons with vertical H and horizontal V polarizations. Photon B may be compared with photon C, entangled with photon D. Therefore, in points A and D measurements of polarizations of the photons shall always coincide.

Let us repeat the proof of the Bell's theorem incorporating influences of any random fields, waves, or forces onto both particles A and B and the observers. We consider here Non-Inertial Reference Frame. We may consider that in the Inertial Reference Frame these particles are influenced with random inertia forces, which, due to the equivalence principle, can be described by a random metrics.

5. Quantum Correction to Second Newton's Law

Ostrogradsky formalism [1] uses Lagrange function is

$$L = L(q, \dot{q}, \ddot{q}, ..., q^{(n)}, ...)$$

but not

$$L = L(q, \dot{q})$$

Euler-Lagrange equation in this case is follow from least action principal [3-6]

$$\delta S = \delta \int L(q, \dot{q}, \ddot{q}, \dddot{q}, ..., q^{(n)})dt = \int \sum_{n=0}^{N} (-1)^n \frac{d^n}{dt^n} (\frac{\partial L}{\partial q^{(n)}}) \delta q^{(n)} dt = 0.$$

or

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} - \frac{d^3}{dt^3}\frac{\partial L}{\partial \dddot{q}} + ... + (-1)^n \frac{d^n}{dt^n}\frac{\partial L}{\partial q^{(n)}} + ... = 0$$

This equation can be write in the form of corrected Newton Second Law of Motion in Non-Inertial Reference Frames

$$F - ma + f_0 = 0$$

Here

$$f_0 = mw = w(t) + \dot{w}(t)\tau + \sum_{k=2}^{n} (-1)^k \frac{1}{k!} \tau^k w^{(n)}(t)$$

is a random inertial force (1) which can be represent by Taylor expansion with high-order derivatives coordinates on time

$$F - ma + \tau m\dot{a} - \frac{1}{2}\tau^2 m a^{(2)} + ... + \frac{1}{n!}(-1)^n \tau^n m a^{(n)} + ... = 0$$

In the Inertial Reference Frame $w = 0$.

Peer-reviewed version available at *Symmetry* **2019**, *12*, 63; doi:10.3390/sym12010063

6. Dark Metric for Matter and Energy

On the one hand, the force F is expressed using the infinite Taylor expansion. On the other hand, the gravitational force F_g can also be represented as a series, as follows from the principle of equivalence. If this series is replaced by exponential [7], then we can write a metric,

$$ds^2 = \exp(-\tfrac{r_0}{r})dt^2 - \exp(\tfrac{r_0}{r})dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

which we call the Dark Metric, where $r_0 = 2GM$.

The Dark Metric is the asymptotic of the Schwarzschild metric for $r_0 < r$ [8,9] and can be applied for the description of the phantom Dark Matter and Dark Energy. The Dark Metric can be obtained from the Standard metric too

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2.$$

Conditions $A(r)B(r) = 1$ and $\lim A(r) = B(r) = 1$ for $r \to \infty$ must be satisfied. The Dark Metric satisfies these conditions. Please note that gravitational forces are presented as a series with changing signs.

7. Macro-examples of Non-Inertial Mechanics

The behaviour of macroscopic mechanical systems in non-inertial reference frames can be described by higher-order differential equations. Here we consider the case when the contribution of higher derivatives is small compared to lower ones. Therefore, at this stage, we restrict ourselves to only the third derivatives of the coordinates with respect to time. There are many examples of the description of mechanical systems in non-inertial reference frames [3-6] due to the influence of the backgrounds of random fields and waves. Theoretical descriptions of such cases do not always fully describe the physical reality of the processes occurring in this process. Such cases include Kapica's pendulum, the movement of bulk materials upwards, against the action of gravity, Chalomey's pendulum, and others [10]. For describing vibrating mechanical systems, the principle of least action is traditionally used to obtain critical states of mechanical systems. All such cases are described by second-order differential equations. In this case, the direction of the resultant force remains uncertain. This is the main disadvantage of this method of description. Using the extended Newton's second law [3]

$$F - ma + \tau m\dot{a} - \frac{1}{2}\tau^2 m a^{(2)} + ... + \frac{1}{n!}(-1)^n \tau^n m a^{(n)} + ... = 0 \tag{5}$$

where $\tau = 1/\omega$ is the averaging time during the transition from the micro world to the macro, is inverse to the average cyclic frequency, we obtain the direction of the resultant force that coincides with the direction of the motion. In [9], the behaviour of such systems is described by introducing experimental vibration forces. The introduction of vibration forces in these cases, in our opinion, is not justified and is introduced axiomatically.

Here we will use a third order differential equation. This allows, first, to get the correct direction of the resultant force. Secondly, it explains its occurrence and does not contradict the already known descriptions.

Comparing the two descriptions: the differential equations of the second order and the third order can be argued the consistency of these two descriptions. Indeed, in mathematics, there is a method of transition from higher-order differential equations to lower ones by changing variables. In our case, from a third-order differential equation, we can go to two equations of order not higher than the second.

For example, consider the description of the Kapica pendulum using the differential equation (6), limiting ourselves to the third order of the derivative of the coordinate with respect to time

$$F - ma + \tau m\dot{a} = 0 \tag{6}$$

Peer-reviewed version available at Symmetry 2019, 12, 63; doi:10.3390/sym12010063

Or

$$F = ma - mj\tau$$

where $j = \dot{a} = \frac{d^3q}{dt^3}$ is third-order derivative coordinate q on the time named Jerk and $\tau = 1/\omega$ is the averaging time during the transition from the micro-world to the macro, the opposite of the average cyclic frequency.

Using the substitution, we get

$$F + V = ma \tag{7}$$

where the vibration force V is equal to

$$V = mA\omega^2 \sin \omega t \tag{8}$$

Thus, we have shown that equation (6) can be replaced by two equations (7) and (8). In this case, the description with high-order derivatives of mechanical systems is more complete than the description with second-order derivatives [11-22].

8. Verifications of High-Order Derivatives as Non-Local Hidden Variables

The role of High-Order Derivatives as Hidden Variables can be verified by using the Equivalence Principal when acceleration is equal to gravitational field. Then the correlation factor for entangled photons polarization measurements may be presented as

$$|M| = |\langle AB \rangle| = \left| \left\langle (\lambda^i A^k g_{ik})(\lambda^m A^n g_{mn}) \right\rangle \right| \tag{9}$$

Here, the random variables distribution function may be considered uniform, with the photon polarization varying from 0 to π:

$$\frac{1}{\pi} \int_0^\pi \rho(\phi) d\phi = 1.$$

According to the definition,

$$\cos\phi = \frac{\lambda^i A^k g_{ik}}{\sqrt{\lambda^i \lambda_i}\sqrt{A^k A_k}},$$

$$\cos(\phi + \theta) = \frac{\lambda^n B^n g_{mn}}{\sqrt{\lambda^m \lambda_m}\sqrt{B^m B_n}}.$$

Hence, the correlation factor is

$$|M| = \left| \frac{1}{\pi} \int_0^\pi \rho(\phi) \cos\phi cos(\phi + \theta) d\phi + \frac{1}{\pi} \int_\pi^{2\pi} \rho(\phi) \cos\phi cos(\phi + \theta) d\phi \right| = |\cos\theta| .$$

The Bell's observable in our case differs from that calculated by Bell and does not contradict to experimental data. Bell's inequality are not violated in either classical or quantum cases of accounting for random fields, forces and waves.

9. Conclusion

Contemporary physics, both Classical and Quantum, requires a notion of inertial reference frames. However, how to find a physical inertial frame in reality where there always exist random weak forces? We suggest a description of the motion in Non-Inertial Reference Frames by means of inclusion of higher time derivatives. They may play a role of non-local hidden variables in a more general description can be named Non-Inertial Mechanics complementing both classical and quantum mechanics.

Acknowledgments: Author have to thank an unknown referee for asking me to do so for improve the article.

Peer-reviewed version available at *Symmetry* **2019**, *12*, 63; doi:10.3390/sym12010063

References

1. Ostrogradsky, M.V. Memoires sur les equations differentielles relatives au probleme des isoperimetres. *Mem. Acad. St. Petersburg* **1850**, 6 (4), 385.
2. Chetayev, N.G. On stable trajectories in dynamics. *Mathematics-1, Proceedings of Kazan State University* **1931**, 91, 4, 3 - 8.
3. Kamalov, T.F. Classical and quantum-mechanical axioms with the higher time derivative formalism. *J. of Physics Conf. Ser.* **2013**, 442, 012051.
4. Kamalov, T.F. Quantum Extension for Newton's Law of Motion, *J. of Physics Conf. Ser.* **2019,** 1251(1),012022.
5. Kamalov, T.F. Axiomatizing of Mechanics. *Quantum Computers and Computing* **2011**,11, 52-57.
6. Kamalov, T.F. Instability States and Ostrogradsky Formalism. *J. of Physics Conf. Ser.* **2018,** 1051(1) 012033.
7. Kamalov, T.F. Simulation the nuclear interaction. In: *Particle Physics on the Eve of LHC*; ed. A. I Studenikin; World Scientific: Singapore, 439-442, 2010**.**
8. Donoghue J.F. General relativity as an effective field theory: The leading quantum corrections, *Physical Review D* **1994,** 50 (6), 3874.
9. Ward, B.F.L. Quantum Corrections to Newton's Laws, *Mod.Phys.Lett.* **2002**, A17, 2371-2382.
10. Blekhman, I.I. *Vibrational Mechanics;* World Scientific: Singapore,New Jersey, Hong Cong, 2000.
11. El-Nabulsi, R.A. Non-Standard Non-Local-in-Time Lagrangians in Classical Mechanics, *Qual. Theory Dyn. Syst.* **2014,** 13, 149–160.
12. Musielak, Z.E.; Fry, J.L.; Kanan, G.W. New fundamental dynamical equation for higher derivative quantum field theories, *Adv. Stud. in Theor. Physics* **2015**, 9(5), 213-232 .
13. El-Nabulsi, R.A. Fourth-Order Ginzburg-Landau differential equation a la Fisher-Kolmogorov and quantum aspects of superconductivity, *Physica C: Superconductivity and its Applications* **2019,** 567,1353545.
14. Musielak, Z.E. General conditions for the existence of non-standard Lagrangians for dissipative dynamical systems, *Chaos Solitons Fractals* **2009,** 42 (15), 2645-2652.
15. El-Nabulsi, R.A. Jerk in Planetary Systems and Rotational Dynamics, Nonlocal Motion Relative to Earth and Nonlocal Fluid Dynamics in Rotating Earth Frame, *Earth, Moon and Planets* **2018**, 122(1-2), 15-41.
16. El-Nabulsi, R.A. Quantization of Non-standard Hamiltonians and the Riemann Zeros, *Qual. Theory Dyn. Syst.* **2019**, 18(1), 69-84.
17. Suykens, J.A.K. Extending Newtons law from nonlocal-in-time kinetic energy, *Phys. Lett.* **2009**, A373, 1201. 1211.
18. El-Nabulsi, R.A. The Hamilton-Jacobi Analysis of Powers of Singular Lagrangians: A Connection Between the Modified Schredinger and the Navier-Stokes Equations, *Qual. Theory Dyn. Syst.* **2018**, 17(3), 583-608.
19. El-Nabulsi, R.A. Time-nonlocal kinetic equations, jerk and hyperjerk in plasmas and solar physics, *Adv. Space Res.* **2018**, 61(12), 2914-2931.
20. El-Nabulsi, R.A. Complex Backward Forward Derivative Operator in Non-local-In-Time Lagrangians Mechanics, *Qual. Theory Dyn. Syst.* **2017**, 16(2), 223-234.
21. Musielak, Z.E. Standard and non-standard Lagrangians for dissipative dynamical systems with variable coefficients, *J. Phys. A* **2008**, 41, 055205-055222.
22. El-Nabulsi, R.A. Nonlinear dynamics with non-standard Lagrangians, *Qual. Theory Dyn. Syst.* **2013**, 13, 273-291.