TRACE OPERATOR INEQUALITY FOR SUPERQUADRATIC FUNCTIONS

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ABSTRACT. In this work, some operator trace inequalities are proved. An extension of Klein's inequality for all Hermitian matrices is proved. A non-commutative version (or Hansen-Pedersen version) of the Jensen trace inequality is provided as well. A generalization of the result for any positive Hilbert space operators acts on a positive unital linear map is established.

1. INTRODUCTION

Let $\mathscr{B}(\mathscr{H})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathscr{H}; \langle\cdot,\cdot\rangle)$ with the identity operator $1_{\mathscr{H}}$ in $\mathscr{B}(\mathscr{H})$. When $\mathscr{H} = \mathbb{C}^n$, we identify $\mathscr{B}(\mathscr{H})$ with the algebra $\mathfrak{M}_{n\times n}$ of n-by-n complex matrices. Then, $\mathfrak{M}^+_{n\times n}$ is just the cone of n-by-n positive semidefinite matrices.

Let $A \in \mathfrak{M}_{n\times n}(\mathbb{C})$, the trace of a square matrix equals the sum of the eigenvalues counted with multiplicities. Moreover, the trace of a Hermitian matrix is real. If A is a linear operator represented by a square matrix with real or complex entries and if $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of A, then $\operatorname{Tr}(A) = \sum_j \lambda_j$. This follows from the fact that A is always similar to its Jordan form, an upper triangular matrix having $\lambda_1, \cdots, \lambda_n$ on the main diagonal.

The inner product

$$\langle A, B\rangle = \operatorname{Tr}(A^*B)$$

which is defined on the space of all complex (or real) $m \times n$ matrices, is called the Frobenius norm, which satisfies submultiplicative property as matrix norm.

If A and B are real positive semi-definite matrices of the same size, using the Cauchy–Schwarz inequality, we have

$$0 \le \operatorname{Tr}^2(AB) \le \operatorname{Tr}\left(A^2\right)\operatorname{Tr}\left(B^2\right) \le \operatorname{Tr}^2(A)\operatorname{Tr}^2(B).$$

The concept of trace of a matrix is generalized to the trace class of compact operators on Hilbert spaces, and the analog of the Frobenius norm is called the Hilbert–Schmidt norm, which is can be defined as [19]:

$$\langle A, B\rangle = \operatorname{Tr}(A^*B) = \sum_i \langle Ae_i, Be_i\rangle$$

over all orthonormal basis of $\mathscr{H}$, $\{e_i : i \in I\}$.

A Hilbert–Schmidt operator, is a bounded operator A on a Hilbert space $\mathscr{H}$ with finite Hilbert–Schmidt norm

$$\|A\|^2_{\mathrm{HS}} = \operatorname{Tr}(A^*A) = \sum_{i\in I} \|Ae_i\|^2,$$

where $\|\cdot\|$ is the norm of $\mathscr{H}$ and Tr is the trace of a nonnegative selfadjoint operator. This definition is independent of the choice of the basis, and therefore

$$\|A\|^2_{\mathrm{HS}} = \sum_{i,j} |\langle e_i, Ae_j\rangle|^2 = \|A\|_2^2.$$

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A bounded linear operator A over a separable Hilbert space $\mathscr{H}$ is said to be in the trace class if for some (and hence all) orthonormal bases $\{e_i\}_i$ of $\mathscr{H}$, the sum of positive terms

$$\|A\|_1 = \operatorname{Tr}|A| := \sum_i \langle |A|\, e_i, e_i\rangle = \sum_i \left\langle \left(A^*A\right)^{1/2} e_i, e_i \right\rangle,$$

is finite. In this case, the trace of A, which is given by the sum

$$\operatorname{Tr}(A) = \sum_i \langle Ae_i, e_i\rangle,$$

is absolutely convergent and is independent of the choice of the orthonormal basis. When $\mathscr{H}$ is finite-dimensional, every operator is trace class and this definition of trace of A coincides with the definition of the trace of a matrix.

A function $f : J \to \mathbb{R}$ is called convex iff

$$f\left(t\alpha + (1-t)\beta\right) \le tf(\alpha) + (1-t) f(\beta), \tag{1.1}$$

for all points $\alpha, \beta \in J$ and all $t \in [0,1]$. If $-f$ is convex then we say that f is concave. Moreover, if f is both convex and concave, then f is said to be affine.

Geometrically, for two point $(x, f(x))$ and $(y, f(y))$ on the graph of f are on or below the chord joining the endpoints for all $x, y \in I$, $x < y$. In symbols, we write

$$f(t) \le \frac{f(y) - f(x)}{y - x}(t - x) + f(x)$$

for any $x \le t \le y$ and $x, y \in J$.

Equivalently, given a function $f : J \to \mathbb{R}$, we say that f admits a support line at $x \in J$ if there exists a $\lambda \in \mathbb{R}$ such that

$$f(t) \ge f(x) + \lambda(t - x) \tag{1.2}$$

for all $t \in J$.

The set of all such λ is called the subdifferential of f at x, and it's denoted by ∂f. Indeed, the subdifferential gives us the slopes of the supporting lines for the graph of f. So that if f is convex then $\partial f(x) \ne \emptyset$ at all interior points of its domain.

From this point of view Abramovich et al. [2] extend the above idea for what they called superquadratic functions. Namely, a function $f : [0,\infty) \to \mathbb{R}$ is called superquadratic provided that for all $x \ge 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$f(t) \ge f(x) + C_x(t - x) + f(|t - x|) \tag{1.3}$$

for all $t \ge 0$. We say that f is subquadratic if $-f$ is superquadratic. Thus, for a superquadratic function we require that f lie above its tangent line plus a translation of f itself. If f is differentiable and satisfies $f(0) = f'(0) = 0$, then one sees easily that the C_x appearing in the definition is necessarily $f'(x)$, (see [1]).

Prima facie, superquadratic function looks to be stronger than convex function itself but if f takes negative values then it may be considered as a weaker function. Therefore, if f is superquadratic and non-negative, then f is convex and increasing [2] (see also [4]).

Moreover, the following result holds for superquadratic function.

Lemma 1. [2] *Let f be superquadratic function. Then*

(1) $f(0) \le 0$
(2) If f is differentiable and $f(0) = f'(0) = 0$, then $C_x = f'(x)$ for all $x \ge 0$.
(3) If $f(x) \ge 0$ for all $x \ge 0$, then f is convex and $f(0) = f'(0) = 0$.

The next result gives a sufficient condition when convexity (concavity) implies super(sub)quaradicity.

Lemma 2. [2] *If f' is convex (concave) and $f(0) = f'(0) = 0$, then is super(sub)quadratic. The converse of is not true.*

Remark 1. *In general, non-negative subquadratic functions does not imply concavity. In other words, there exists a subquadratic function which is convex. For example, $f(x) = x^p$, $x \ge 0$ and $1 \le p \le 2$ is subquadratic and convex.*

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Among others, Abramovich et al. [2] proved that the inequality

$$f\left(\int\varphi d\mu\right)\le\int f\left(\varphi\left(s\right)\right)-f\left(\left|\varphi\left(s\right)-\int\varphi d\mu\right|\right)d\mu\left(s\right) \tag{1.4}$$

holds for all probability measures μ and all nonnegative, μ-integrable functions φ if and only if f is superquadratic. For more details, recent result and generalization the reader may refer to [4],[5] and [8]. Related operator inequalities can be found in [6], [7],[10] and [14]–[16].

2. Jensen's Trace Inequality

Let $f:\mathbb{R}\to\mathbb{R}$ be continuous, and let n be any integer. It's well known that if $t\mapsto f\left(t\right)$ is convex (monotone increasing), then the trace function $A\mapsto\operatorname{Tr}\left(f\left(A\right)\right)$ is convex (monotone increasing) [13] (see also [12]).

The next result shows that the trace function is real positive superquaratic function.

Lemma 3. *Given $f:[0,\infty)\to\mathbb{R}$ be a continuous function. If f is superquadratic, then the associated trace function, $A\mapsto\operatorname{Tr}\left(f\left(A\right)\right)$ is superquadratic function, for all Hermitian $n\times n$ matrices. Moreover, we have*

$$\operatorname{Tr}\left(f\left(\frac{A+B}{2}\right)\right)+\operatorname{Tr}\left(f\left(\frac{\left|A-B\right|}{2}\right)\right)\le\frac{\operatorname{Tr}\left(f\left(A\right)\right)+\operatorname{Tr}\left(f\left(B\right)\right)}{2} \tag{2.1}$$

for all Hermitian $n\times n$ matrices.

Proof. Let $\lambda_1,\cdots,\lambda_n$ and $\mu_1,\cdots,\mu_n$ be the eigenvalues of A and B; respectively. Then, for $0\le\alpha\le1$, we have

$$\begin{aligned}
&\operatorname{Tr}\left(f\left(\alpha A+\left(1-\alpha\right)B\right)\right)\\
&=\sum_j f\left(\alpha\lambda_j+\left(1-\alpha\right)\mu_j\right)\\
&\le\sum_j\left[\alpha f\left(\lambda_j\right)+\left(1-\alpha\right)f\left(\mu_j\right)-\alpha f\left(\left(1-\alpha\right)\left|\lambda_j-\mu_j\right|\right)-\left(1-\alpha\right)f\left(\alpha\left|\lambda_j-\mu_j\right|\right)\right]\\
&=\alpha\sum_j f\left(\lambda_j\right)+\left(1-\alpha\right)\sum_j f\left(\mu_j\right)-\alpha\sum_j f\left(\left(1-\alpha\right)\left|\lambda_j-\mu_j\right|\right)-\left(1-\alpha\right)\sum_j f\left(\alpha\left|\lambda_j-\mu_j\right|\right)\\
&=\alpha\operatorname{Tr}\left(f\left(A\right)\right)+\left(1-\alpha\right)\operatorname{Tr}\left(f\left(B\right)\right)-\alpha\operatorname{Tr}\left(f\left(\left(1-\alpha\right)\left|A-B\right|\right)\right)-\left(1-\alpha\right)\operatorname{Tr}\left(f\left(\alpha\left|A-B\right|\right)\right),
\end{aligned}$$

which proves that $\operatorname{Tr}\left(f\left(A\right)\right)$ is superquadratic function on the set of all positive definite Hermitian $n\times n$ matrices. The inequality holds by setting $\alpha=\frac{1}{2}$. ☐

Remark 2. *Another proof of Lemma 3 could be done using the Spectral Decomposition of A such that the real function f is defined on the spectrum of A.*

For all Hermitian $n\times n$ matrices A and B and all differentiable convex functions $f:\mathbb{R}\to\mathbb{R}$ with derivative f', the well known Klein inequality reads that [18]:

$$\operatorname{Tr}\left[f\left(A\right)-f\left(B\right)-\left(A-B\right)f'\left(B\right)\right]\ge0, \tag{2.2}$$

The equality holds if and only if $A=B$.

For the choice $f(t)=t\log t$ $(t>0)$, we obtain [17]:

$$S\left(A,B\right)=\operatorname{Tr}A\left(\log A-\log B\right)\ge\operatorname{Tr}\left(A-B\right)$$

for B strictly positive and A nonnegative. The left-hand side is called relative entropy. If A and B are density matrices, i.e., $\operatorname{Tr}\left(A\right)=\operatorname{Tr}\left(B\right)=1$, then $S\left(A,B\right)\ge0$. This is a classical application of the Klein inequality (cf. [18]).

Let R,F be two Hermitian $n\times n$ matrices such that $\operatorname{Tr}\left(e^R\right)=1$. Define $g=\operatorname{Tr}\left(Fe^R\right)$, then we obtain the Peierl–Bogoliubov inequality [18]:

$$\operatorname{Tr}\left(e^Fe^R\right)\ge\operatorname{Tr}\left(e^{F+R}\right)\ge\operatorname{Tr}\left(e^g\right).$$

The proof of this inequality follows from Klein inequality. Take $f\left(t\right)=e^x$, $A=R+F$ and $B=R+gI$.

Employing the concept of superquadricity, one may has the following refinement of Klein's inequality (2.2).

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Theorem 1. *For all positive-definite Hermitian $n\times n$ matrices A and B, and all differentiable superquadratic functions $f:[0,\infty)\to\mathbb{R}$, satisfy that $f(0)=f'(0)=0$, the inequality*

$$\operatorname{Tr}\left[f\left(A\right)-f\left(B\right)-\left(A-B\right)f'\left(B\right)-f\left(\left|A-B\right|\right)\right]\ge 0, \tag{2.3}$$

holds. The equality holds if and only if $A=B$.

Proof. Let f be superquadratic on $[0,\infty)$. Since f is differentiable and satisfies $f(0)=f'(0)=0$, then one sees that the C_x in (1.3) is necessarily $f'(x)$ ([1]). Therefore, we have

$$f\left(t\right)\ge f\left(x\right)+f'\left(x\right)\left(t-x\right)+f\left(\left|t-x\right|\right)$$

for all $t,x\ge 0$. Therefore, if λ_j and μ_j $(1\le j\le n)$ are the eigenvalues of positive-definite Hermitian $n\times n$ matrices A and B, respectively; thus we have

$$f\left(\lambda_j\right)\ge f\left(\mu_j\right)+f'\left(\mu_j\right)\left(\lambda_j-\mu_j\right)+f\left(\left|\lambda_j-\mu_j\right|\right).$$

Taking the sum over j, we get

$$\sum_j f\left(\lambda_j\right)\ge \sum_j f\left(\mu_j\right)+\sum_j f'\left(\mu_j\right)\left(\lambda_j-\mu_j\right)+\sum_j f\left(\left|\lambda_j-\mu_j\right|\right),$$

which is equivalent to write

$$\operatorname{Tr}\left(f\left(A\right)\right)\ge \operatorname{Tr}\left(f\left(B\right)\right)+\operatorname{Tr}\left(\left(A-B\right)f'\left(B\right)\right)+\operatorname{Tr}\left(f\left(\left|A-B\right|\right)\right),$$

and this is exactly (2.2). □

For the choice of the superquadratic function $f(t)=t^2\log(t)$ $(t>0)$, one could introduce a new refinement of the relative entropy defined above; which it has important applications in statistical mechanics. Similarly, one could refine the Peierl–Bogoliubov inequality using (2.3). For more details see [17], [18] and the references therein. We left the details to the interested reader.

A generalization of Lemma 3 could be stated as follows:

Theorem 2. *Let $w_k\ge 0$ $(k=1,\cdots,n)$ be positive scalars such that $W_n=\sum\limits_{j=1}^n w_k$. Let f be a real-valued continuous function defined on an interval I and let m and n be natural numbers. If f is superquadratic function (in ordinary sense), then the inequality*

$$\begin{aligned}\operatorname{Tr}\left(f\left(\frac{1}{W_n}\sum_{k=1}^n w_kA_k\right)\right)\\ \le \frac{1}{W_n}\sum_{k=1}^n w_k\operatorname{Tr}\left(f\left(A_k\right)\right)-\frac{1}{W_n}\sum_{k=1}^n w_k\operatorname{Tr}\left(f\left(\left|A_k-\frac{1}{W_n}\sum_{l=1}^n w_l\operatorname{Tr}\left(A_l\right)\right|\right)\right)\end{aligned} \tag{2.4}$$

holds for every n-tuple $(A_1,\cdots,A_n)$ of positive $m\times m$ matrices with spectra contained in I.

In particular, we have

$$\operatorname{Tr}\left(f\left(\frac{1}{n}\sum_{k=1}^n A_k\right)\right)\le \frac{1}{n}\sum_{k=1}^n\operatorname{Tr}\left(f\left(A_k\right)\right)-\frac{1}{n}\sum_{k=1}^n\operatorname{Tr}\left(f\left(\left|A_k-\frac{1}{n}\sum_{l=1}^n\operatorname{Tr}\left(A_l\right)\right|\right)\right) \tag{2.5}$$

Proof. Since f is superquadratic then the inequality

$$f\left(\frac{1}{W_n}\sum_{k=1}^n w_ka_k\right)\le \sum_{k=1}^n\frac{w_k}{W_n}f\left(a_k\right)-\sum_{k=1}^n\frac{w_k}{W_n}f\left(\left|a_k-\sum_{l=1}^n\frac{w_l}{W_n}a_l\right|\right), \tag{2.6}$$

holds for every finite positive sequence of real numbers a_k $(k=1,\cdots,n)$ and every positive scalars w_k such that $W_n=\sum\limits_{j=1}^n w_k$, (see [2]).

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Now, let $\lambda_1^{(j)}, \cdots, \lambda_m^{(j)}$ be the eigenvalues of A_j $(1 \le j \le n)$; respectively. Then, on utilizing the above inequality we have

$$\begin{aligned}
\operatorname{Tr}\left(f\left(\frac{1}{W_n}\sum_{k=1}^{n} w_k A_k\right)\right) &= f\left(\frac{1}{W_n}\sum_{k=1}^{n} w_k \left(\sum_{j=1}^{m}\lambda_k^{(j)}\right)\right) \\
&\le \sum_{k=1}^{n}\frac{w_k}{W_n} f\left(\sum_{j=1}^{m}\lambda_k^{(j)}\right) - \sum_{k=1}^{n}\frac{w_k}{W_n} f\left(\left|\sum_{j=1}^{m}\lambda_k^{(j)} - \sum_{l=1}^{n}\frac{w_l}{W_n}\sum_{j=1}^{m}\lambda_l^{(j)}\right|\right) \\
&= \operatorname{Tr}\left(\sum_{k=1}^{n}\frac{w_k}{W_n} f(A_k)\right) - \operatorname{Tr}\left(\sum_{k=1}^{n}\frac{w_k}{W_n} f\left(\left|A_k - \operatorname{Tr}\left(\sum_{l=1}^{n}\frac{w_l}{W_n}A_l\right)\right|\right)\right) \\
&= \frac{1}{W_n}\sum_{k=1}^{n} w_k \operatorname{Tr}(f(A_k)) - \frac{1}{W_n}\sum_{k=1}^{n} w_k \operatorname{Tr}\left(f\left(\left|A_k - \frac{1}{W_n}\sum_{l=1}^{n} w_l \operatorname{Tr}(A_l)\right|\right)\right)
\end{aligned}$$

which gives the required result. The particular case follows by setting $w_k = 1$ for all $k = 1, \cdots, n$ so that $W_n = n$. □

Corollary 1. *Let $w_k \ge 0$ $(k = 1, \cdots, n)$ be positive scalars such that $W_n = \sum_{j=1}^{n} w_k$. For every n-tuple $(A_1, \cdots, A_n)$ of positive $m \times m$ matrices with spectra contained in $[0, \infty)$. Then the inequality*

$$\operatorname{Tr}\left(\left(\frac{1}{W_n}\sum_{k=1}^{n} w_k A_k\right)^p\right) \le \frac{1}{W_n}\sum_{k=1}^{n} w_k \operatorname{Tr}(A_k^p) - \frac{1}{W_n}\sum_{k=1}^{n} w_k \operatorname{Tr}\left(\left|A_k - \frac{1}{W_n}\sum_{l=1}^{n} w_l \operatorname{Tr}(A_l)\right|^p\right) \tag{2.7}$$

holds for all $p \ge 2$. In particular, we have

$$\operatorname{Tr}\left(\left(\frac{1}{n}\sum_{k=1}^{n} A_k\right)^p\right) \le \frac{1}{n}\sum_{k=1}^{n} \operatorname{Tr}(A_k^p) - \frac{1}{n}\sum_{k=1}^{n} \operatorname{Tr}\left(\left|A_k - \frac{1}{W_n}\sum_{l=1}^{n} \operatorname{Tr}(A_l)\right|^p\right) \tag{2.8}$$

In 2003, Hansen & Pedersen [11], proved the following non-commutative trace version of Jensen's inequality

$$\operatorname{Tr}\left(f\left(\sum_{k=1}^{n} C_k^* A_k C_k\right)\right) \le \operatorname{Tr}\left(\sum_{k=1}^{n} C_k^* f(A_k) C_k\right)$$

for every n-tuple $(A_1, \cdots, A_n)$ of positive $m \times m$ matrices with spectra contained in I and every n-tuple $(C_1, \cdots, C_n)$ of $m \times m$ matrices with $\sum_{k=1}^{n} C_k^* C_k = 1$, where f is assumed to convex on I.

Using the concept of supequdratic functions, one could give the following refinement of Hansen–Pedersen trace inequality.

Theorem 3. *Let f be a real-valued continuous function defined on an interval I and let m and n be natural numbers. If f is superquadratic function (in ordinary sense), then the inequality*

$$\begin{aligned}
&\operatorname{Tr}\left(f\left(\sum_{k=1}^{n} C_k^* A_k C_k\right)\right) \\
&\le \operatorname{Tr}\left(\sum_{k=1}^{n} C_k^* f(A_k) C_k\right) - \operatorname{Tr}\left(\sum_{k=1}^{n} C_k^* f\left(\left|A_k - \operatorname{Tr}\left(\sum_{j=1}^{n} C_j^* A_j C_j\right)\right|\right) C_k\right)
\end{aligned} \tag{2.9}$$

holds for every n-tuple $(A_1, \cdots, A_n)$ of positive $m \times m$ matrices with spectra contained in I and every n-tuple $(C_1, \cdots, C_n)$ of $m \times m$ matrices with $\sum_{k=1}^{n} C_k^* C_k = 1$. Conversely, if the inequality (2.9) is satisfied for some n and m, where $n > 1$, then f is superquadratic function. If f is subquadratic, then the inequality (2.9) is reversed.

Proof. Our proof is motivated by [11]. Let $A_k = \sum_{\mathrm{sp}(A_k)} \lambda E_k(\lambda)$ denote the spectral resolution of A_k for $1 \le k \le n$. Thus, $E_k(\lambda)$ is the spectral projection of A_k on the eigenspace corresponding to λ; if λ is an eigenvalue for A_k, otherwise $E_k(\lambda) = 0$. For each unit vector ξ in $\mathbb{C}^m$. Define the probability measure

$$\mu_\xi(S) = \left\langle \sum_{k=1}^n C_k^* E_k(S) C_k \xi, \xi \right\rangle = \sum_{k=1}^n \langle E_k(S) C_k \xi, C_k \xi \rangle$$

for any (Borel) set S in $\mathbb{R}$. Note that if $y = \sum_{k=1}^n C_k^* A_k C_k$, then

$$\begin{aligned} \langle y\xi, \xi \rangle &= \left\langle \sum_{k=1}^n C_k^* A_k C_k \xi, \xi \right\rangle \\ &= \left\langle \sum_{k=1}^n \sum_{\mathrm{sp}(x_k)} \lambda E_k(\lambda) C_k \xi, C_k \xi \right\rangle \\ &= \int \lambda d\mu_\xi(\lambda). \end{aligned}$$

If a unit vector ξ is an eigenvector for y, then the corresponding eigenvalue is $\langle y\xi, \xi \rangle$, and ξ is also an eigenvector for $f(y)$ with corresponding eigenvalue $\langle f(y)\xi, \xi \rangle = f(\langle y\xi, \xi \rangle)$. In this case we have

$$\begin{aligned} \left\langle f\left(\sum_{k=1}^n C_k^* A_k C_k\right) \xi, \xi \right\rangle &= \langle f(y)\xi, \xi \rangle \\ &= f(\langle y\xi, \xi \rangle) \\ &= f\left(\int \lambda d\mu_\xi(\lambda)\right) \\ &\le \int \left[f(\lambda) - f\left(\left|\lambda - \int \lambda d\mu_\xi(\lambda)\right|\right)\right] d\mu_\xi(\lambda) \qquad \text{(by 1.4)} \\ &= \sum_{k=1}^n \left\langle \sum_{sp(x_k)} f(\lambda) - f\left(\left|\lambda - \int \lambda d\mu_\xi(\lambda)\right|\right) E_k(\lambda) C_k \xi, C_k \xi \right\rangle \\ &= \sum_{k=1}^n \langle [C_k^* f(A_k) C_k - C_k^* f(|A_k - \langle y\xi, \xi \rangle|) C_k] \xi, \xi \rangle \\ &= \sum_{k=1}^n \left\langle \left[C_k^* f(A_k) C_k - C_k^* f\left(\left|A_k - \left\langle \sum_{j=1}^n C_j^* A_j C_j \xi, \xi \right\rangle\right|\right) C_k \right] \xi, \xi \right\rangle. \end{aligned}$$

Summing over an orthonormal basis of eigenvectors for y we get the desired result in (2.9). □

Corollary 2. *Let f be a real-valued continuous function defined on an interval I and let m and n be natural numbers. If f is superquadratic function (in the ordinary sense), then the inequality*

$$\mathrm{Tr}(f(C^* A C)) \le \mathrm{Tr}(C^* f(A) C) - \mathrm{Tr}(C^* f(|A - \mathrm{Tr}(C^* A C)|) C), \tag{2.10}$$

holds for every positive $m \times m$ matrix A with spectrum contained in I and every $m \times m$ matrix C with $C^* C = 1$. If f is subquadratic, then the inequality (2.10) is reversed. Furthermore, we have

$$\mathrm{Tr}((C^* A C)^p) \le \mathrm{Tr}(C^* A^p C) - \mathrm{Tr}(C^* |A - \mathrm{Tr}(C^* A C)|^p C)$$

for every $p \ge 2$, and

$$\mathrm{Tr}((C^* A C)^p) \ge \mathrm{Tr}(C^* A^p C) - \mathrm{Tr}(C^* |A - \mathrm{Tr}(C^* A C)|^p C)$$

for every $p \in (0, 2]$.

Proof. The result follows by setting $n = 1$ in Theorem 3. The second inequality follows by applying the superquadratic function $f(t) = t^p$, $p \ge 2$. Similarly, the last inequality follows by applying the subquardtic function $f(t) = t^p$, $p \in (0, 2]$. □

The inequality (2.10) could be extended for general positive Hilbert space operators mapped under positive unital linear map, as follows:

Theorem 4. *Let f be a real-valued continuous function defined on $[0,\infty)$. Let $A \in \mathscr{B}(\mathscr{H})$ be a positive operator and $\Phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$ be a positive unital linear map. If f is superquadratic, then we have*

$$\operatorname{Tr}\left(\Phi\left(f\left(A\right)\right)\right) \geq \operatorname{Tr}\left(f\left(\Phi\left(A\right)\right)\right) + \operatorname{Tr}\left(\Phi\left(f\left(\left|A - \operatorname{Tr}\left(\Phi\left(A\right)\right)\right|\right)\right)\right). \tag{2.11}$$

In particular case, for the choice $\Phi(A) = C^*AC$, where $C \in \mathscr{B}(\mathscr{H})$ such that $C^*C = 1_{\mathscr{H}}$, we get the inequality (2.10).

Proof. Let $A \in \mathscr{B}(\mathscr{H})$ be positive. Assume that $\mathcal{A}$ is the C^*-subalgebra of $\mathscr{B}(\mathscr{H})$ generated by A and $1_{\mathscr{H}}$. Without loss of generality, we may assume that Φ is defined on $\mathcal{A}$. Since every unital positive linear map on a commutative C^*-algebra is completely positive. It follows that Φ is completely positive. So there exists (by Stinespring's theorem [20]), some isometry $V : \mathscr{H} \to \mathscr{K}$; and a unital $*$-homomorphism ρ from $\mathcal{A}$ into the C^*-algebra $\mathscr{B}(\mathscr{H})$ such that $\Phi(A) = V^*\rho(A)V$. Clearly, $f(\rho(A)) = \rho(f(A))$, for all continuous function f.

Now, let $\{e_i : i \in I\}$ be a set of an orthonormal basis of a Hilbert space $\mathscr{H}$. On utilizing the continuous functional calculus for the operator $A \geq 0$ Thus,

$$\begin{aligned}
\operatorname{Tr}\left(f\left(\Phi\left(A\right)\right)\right) &= \operatorname{Tr}\left(f\left(V^*\rho\left(A\right)V\right)\right) \\
&\leq \operatorname{Tr}\left(V^* f\left(\rho\left(A\right)\right)V\right) - \operatorname{Tr}\left(V^* f\left(\left|\rho\left(A - V^*\rho\left(A\right)V\right)\right|\right)V\right) \qquad \text{(by (2.10))} \\
&= \operatorname{Tr}\left(V^*\rho\left(f\left(A\right)\right)V\right) - \operatorname{Tr}\left(V^*\rho\left(f\left(\left|A - \Phi\left(A\right)\right|\right)\right)V\right) \\
&= \operatorname{Tr}\left(\Phi\left(f\left(A\right)\right)\right) - \operatorname{Tr}\left(\Phi\left(f\left(\left|A - \Phi\left(A\right)\right|\right)\right)\right).
\end{aligned}$$

which proves the required inequality. □

Corollary 3. *Let $A \in \mathscr{B}(\mathscr{H})$ be a positive operator and $\Phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$ be a positive unital linear map. Then we have*

$$\operatorname{Tr}\left(\Phi\left(A^p\right)\right) \geq \operatorname{Tr}\left(\Phi^p\left(A\right)\right) + \operatorname{Tr}\left(\Phi\left(\left|A - \operatorname{Tr}\left(\Phi\left(A\right)\right)\right|^p\right)\right).$$

for all $p \geq 2$. The inequality is reversed for $p \in (0,2]$.

One can easily generalized (2.11) by using Theorem 4, as follows:

Corollary 4. *Let f be a real-valued continuous function defined on $[0,\infty)$. Let $A_j \in \mathscr{B}(\mathscr{H})$ $(j = 1,\cdots,n)$ be positive operators. Let $\Phi_j : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$ $(j = 1,\cdots,n)$ be a positive linear map, such that $\sum\limits_{j=1}^{n} \Phi_j\left(1_{\mathscr{H}}\right) = 1_{\mathscr{K}}$. If f is superquadratic function, then*

$$\operatorname{Tr}\left(\sum_{j=1}^{n} \Phi_j\left(f\left(A_j\right)\right)\right) \geq \operatorname{Tr}\left(f\left(\sum_{j=1}^{n} \Phi_j\left(A_j\right)\right)\right) + \operatorname{Tr}\left(\sum_{j=1}^{n} \Phi_j\left(f\left(\left|A_j - \operatorname{Tr}\left(\sum_{j=1}^{n} \Phi_j\left(A_j\right)\right)\right|\right)\right)\right).$$

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