 doi:10.20944/preprints201912.0134.v1

Article

Dunkl generalization of Phillips operators and approximation in weighted spaces

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Abstract: Purpose of this article is to introduce a modification of Phillips operators on the interval $\left[\frac{1}{2},\infty\right)$ via Dunkl generalization. This type of modification enables a better error estimation on the interval $\left[\frac{1}{2},\infty\right)$ rather than the classical Dunkl Phillips operators on $[0,\infty)$. We discuss the convergence results and obtain the degrees of approximations. Furthermore, we calculate the rate of convergence by means of modulus of continuity, Lipschitz type maximal functions, Peetre's K-functional and second order modulus of continuity.

Keywords: Szász operator; Dunkl analogue; generalization of exponential function; Korovkin type theorem; modulus of continuity; order of convergence.

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1. Preliminaries and Introduction

Szász operators [18] provide an extension to Bernstein operators [4] on the interval $[0,\infty)$. In the present years, several authors have studied the Dunkl type generalization of Szász operators (see [1,2,9–12,14–16,19]).

Sucu [17] introduced Dunkl analogue of Szász operators. That is, for $f \in C[0,\infty)$, $x \geq 0$, $v \geq 0$ and $n \in \mathbb{N}$,

$$\mathcal{S}_n^*(f;x) := \frac{1}{e_v(nx)} \sum_{\ell=0}^{\infty} \frac{(nx)^\ell}{\gamma_v(\ell)} f\left(\frac{\ell + 2v\theta_\ell}{n}\right), \tag{1}$$

where $\mathbb{N}$ is the set of all natural numbers and

$$e_v(x) = \sum_{\ell=0}^{\infty} \frac{x^\ell}{\gamma_v(\ell)}. \tag{2}$$

The coefficients γ_v are given as

$$\gamma_v(2\ell) = \frac{2^{2\ell}\ell!\Gamma\left(\ell+v+\frac{1}{2}\right)}{\Gamma\left(v+\frac{1}{2}\right)}, \quad \gamma_v(2\ell+1) = \frac{2^{2\ell+1}\ell!\Gamma\left(\ell+v+\frac{3}{2}\right)}{\Gamma\left(v+\frac{1}{2}\right)} \tag{3}$$

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with recursion

$$\frac{\gamma_v(\ell+1)}{(\ell+1+2v\theta_{\ell+1})} = \gamma_v(\ell) \tag{4}$$

where

$$\theta_\ell = \begin{cases} 0 & \text{if } \ell = 0, 2, 4, \cdots, \\ 1 & \text{if } \ell = 1, 3, 5, \cdots, . \end{cases} \tag{5}$$

Studies on Dunkl type generalizations [13] and previous studies of Szász type operators [6,7] demonstrate an error estimation to the operators which allow us much faster approximation to the function which is being approximated. In this paper, we modify the Phillips operators given by [13] via Dunkl generalization. Our main idea is to approximate these operators by well known Korovkin's and weighted Korovkin's theorems. We estimate the degrees of approximations and calculate the rate of convergence by means of modulus of continuity, Lipschitz type maximal functions, Peetre's K-functional and second order modulus of continuity.

2. New operators and their moments

Let $\{\chi_n(x)\}$ be a sequence of nonnegative continuous functions on $[0,\infty)$ as

$$\chi_n(x) = \left(x - \frac{1}{2n}\right)_+, \quad n \in \mathbb{N}, \tag{6}$$

where

$$\kappa_+ = \begin{cases} \kappa & \text{if } \kappa \geq 0, \\ 0 & \text{if } \kappa < 0. \end{cases} \tag{7}$$

Moreover, suppose

$$\mathcal{J}_{n,v}(x) = \frac{e_v(-n\chi_n(x))}{e_v(n\chi_n(x))} \tag{8}$$

For $f \in C_\zeta(\mathbb{R}^+) = \{f \in C[0,\infty) : f(t) = O(t^\zeta), \zeta > n, \quad n \in \mathbb{N}\}$, we define

$$\mathcal{P}_{n,v}(f;x) = \frac{n^2}{e_v(n\chi_n(x))} \sum_{\ell=0}^{\infty} \frac{(n\chi_n(x))^\ell}{\gamma_v(\ell)} \int_0^\infty \frac{e^{-nt} n^{\ell+2v\theta_\ell-1} t^{\ell+2v\theta_\ell}}{\gamma_v(\ell)} f(t)\mathrm{d}t, \; v \geq 0$$

where $e_v(x)$, γ_v and θ_ℓ are defined as in [17] by (2), (3) and (5), respectively.

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Lemma 1. *Let $e_\ell = t^{\ell-1}$, $\ell = 1,2,3,4,5$ and $\mathcal{J}_{n,v}(x)$ defined by (8). Then for $x \geq 0$, $\mathcal{P}_{n,v}(e_1;x) = 1$ and for any $x \geq \frac{1}{2n}$ we have*

$$\begin{aligned}
(1) \quad \mathcal{P}_{n,v}(e_2;x) &= x + \frac{1}{2n},\\
(2) \quad \mathcal{P}_{n,v}(e_3;x) &= x^2 + \frac{1}{n}\left(3 + 2v\mathcal{J}_{n,v}(x)\right)x - \frac{1}{4n^2}\mathcal{J}_{n,v}(x),\\
(3) \quad \mathcal{P}_{n,v}(e_4;x) &= x^3 + \frac{1}{2n}\left(15 - 4v\mathcal{J}_{n,v}(x)\right)x^2\\
&+ \frac{1}{4n^2}\left(39 + 16v + 72v\mathcal{J}_{n,v}(x)\right)x\\
&- \frac{1}{8n^3}\left(7 + 16v + 68v\mathcal{J}_{n,v}(x)\right),\\
(4) \quad \mathcal{P}_{n,v}(e_5;x) &= x^4 + \frac{1}{n}\left(14 + 4v\mathcal{J}_{n,v}(x)\right)x^3\\
&+ \frac{1}{2n^2}\left(99 - 68v\mathcal{J}_{n,v}(x) + 8v^2\right)x^2\\
&+ \frac{1}{2n^3}\left(131 + 294v\mathcal{J}_{n,v}(x) + 96v^2 + 16v^3\mathcal{J}_{n,v}(x)\right)x\\
&+ \frac{1}{16n^4}\left(-367 + 808v\mathcal{J}_{n,v}(x) + 432v^2 + 64v^3\mathcal{J}_{n,v}(x)\right).
\end{aligned}$$

Remark 1. *For any $0 \leq x \leq \frac{1}{2n}$, we have $\mathcal{P}_{n,v}(e_2;x) = \frac{1}{n}$; $\mathcal{P}_{n,v}(e_3;x) = \frac{2}{n^2}$; $\mathcal{P}_{n,v}(e_4;x) = \frac{6}{n^3}$; $\mathcal{P}_{n,v}(e_5;x) = \frac{24}{n^4}$.*

Here we also introduce the Stancu type generalization to the operators defined by (9). Thus, for each $f \in C_\zeta(\mathbb{R}^+)$ the modified version of the operators (9) is defined as

$$\mathcal{S}^*_{n,v}(f;x) = \frac{n^2}{e_v(n\chi_n(x))}\sum_{\ell=0}^{\infty}\frac{(n\chi_n(x))^\ell}{\gamma_v(\ell)}\int_0^\infty \frac{e^{-nt}n^{\ell+2v\theta_\ell-1}t^{\ell+2v\theta_\ell}}{\gamma_v(\ell)} f\left(\frac{nt+\alpha}{n+\beta}\right)\mathrm{d}t$$

where $0 \leq \alpha \leq \beta$. Note that if we take $\alpha = \beta = 0$ in (9), then the operators $\mathcal{S}^*_{n,v}$ reduce to operators defined by (9) and if take $\chi_n(x) = x$ in $\mathcal{P}_{n,v}$, then we get the operators defined studied in [13].

Lemma 2. *For $x \geq 0$, $\mathcal{S}^*_{n,v}(e_1;x) = 1$ and for $x \geq \frac{1}{2n}$, we have*

$$\begin{aligned}
1^\circ \quad \mathcal{S}^*_{n,v}(e_2;x) &= \frac{n}{n+\beta}\mathcal{P}_{n,v}(e_2;x) + \frac{\alpha}{n+\beta},\\
2^\circ \quad \mathcal{S}^*_{n,v}(e_3;x) &= \frac{n^2}{(n+\beta)^2}\mathcal{P}_{n,v}(e_3;x) + \frac{2\alpha n}{(n+\beta)^2}\mathcal{P}_{n,v}(e_2;x) + \frac{\alpha^2}{(n+\beta)^2},\\
3^\circ \quad \mathcal{S}^*_{n,v}(e_4;x) &= \frac{n^3}{(n+\beta)^3}\mathcal{P}_{n,v}(e_4;x) + \frac{3\alpha n^2}{(n+\beta)^3}\mathcal{P}_{n,v}(e_3;x) + \frac{3\alpha^2 n}{(n+\beta)^3}\mathcal{P}_{n,v}(e_2;x) + \frac{\alpha^3}{(n+\beta)^3},\\
4^\circ \quad \mathcal{S}^*_{n,v}(e_5;x) &= \frac{n^4}{(n+\beta)^4}\mathcal{P}_{n,v}(e_5;x) + \frac{4\alpha n^3}{(n+\beta)^4}\mathcal{P}_{n,v}(e_4;x) + \frac{6\alpha^2 n^2}{(n+\beta)^4}\mathcal{P}_{n,v}(e_3;x)\\
&+ \frac{4\alpha^3 n}{(n+\beta)^4}\mathcal{P}_{n,v}(e_2;x) + \frac{\alpha^4}{(n+\beta)^4},
\end{aligned}$$

Lemma 3. *For $0 \leq x \leq \frac{1}{2n}$, we have*

$$
\begin{aligned}
(1)^{\circ\circ} \quad \mathcal{S}^{*}_{n,v}(e_2;x) &= \frac{\alpha+1}{n+\beta}, \\
(2)^{\circ\circ} \quad \mathcal{S}^{*}_{n,v}(e_3;x) &= \frac{2+\alpha+\alpha^2}{(n+\beta)^2}, \\
(3)^{\circ\circ} \quad \mathcal{S}^{*}_{n,v}(e_4;x) &= \frac{6+6\alpha+3\alpha^2+\alpha^3}{(n+\beta)^3}, \\
(4)^{\circ\circ} \quad \mathcal{S}^{*}_{n,v}(e_5;x) &= \frac{24+4\alpha+12\alpha^2+4\alpha^3+\alpha^4}{(n+\beta)^4}.
\end{aligned}
$$

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Lemma 4. *Suppose* $\eta_j = (e_2 - x)^j$ *for* $j = 1, 2, 3, 4$, *where* e_2 *is defined in Lemma 2. Then, for* $x \geq \frac{1}{2n}$ *we have*

$$
\begin{aligned}
1^* \quad \mathcal{S}^*_{n,v}(\eta_1; x) &= \left(\frac{n}{n+\beta} - 1\right) x + \frac{1+2\alpha}{2(n+\beta)}, \\
2^* \quad \mathcal{S}^*_{n,v}(\eta_2; x) &= \left[\frac{n^2}{(n+\beta)^2} - \frac{2n}{n+\beta} + 1\right] x^2 \\
&+ \left[\frac{n}{(n+\beta)^2}\left(3 + 2v\mathcal{J}_{n,v}(x)\right) + \frac{2\alpha n}{(n+\beta)^2} - \frac{2\alpha+1}{n+\beta}\right] x \\
&+ \frac{\alpha+\alpha^2}{(n+\beta)^2} - \frac{1}{4(n+\beta)^2} v\mathcal{J}_{n,v}(x), \\
3^* \quad \mathcal{S}^*_{n,v}(\eta_4; x) &= \left[\frac{n^4}{(n+\beta)^4} - \frac{4n^3}{(n+\beta)^3} + \frac{6n^2}{(n+\beta)^2} - \frac{4n}{n+\beta} + 1\right] x^4 \\
&+ \left[\frac{n^3}{(n+\beta)^4}\left(14 + 4\alpha + 4v\mathcal{J}_{n,v}(x)\right)\right. \\
&- \frac{2n^2}{(n+\beta)^3}\left(15 + 6\alpha - 4v\frac{e_v(-n\chi(x))}{e_v(n\chi(x))}\right) \\
&+ \left.\frac{6n}{(n+\beta)^2}\left(3 + 2\alpha + 2v\mathcal{J}_{n,v}(x)\right) - \frac{2+4\alpha}{n+\beta}\right] x^3 \\
&+ \left[\frac{n^2}{2(n+\beta)^4}\left(99 + 60\alpha + 8v^2 + 4(17-4\alpha)4v\mathcal{J}_{n,v}(x)\right)\right. \\
&- \frac{n}{(n+\beta)^3}\left(39 + 16v + 12(3+\alpha)(2+\alpha)v\mathcal{J}_{n,v}(x)\right) \\
&+ \left.\frac{1}{2(n+\beta)^2}\left(12\alpha + 12\alpha^2 + 12\alpha^2 n^2 - 3v\mathcal{J}_{n,v}(x)\right)\right] x^2 \\
&+ \left[\frac{n}{2(n+\beta)^4}\left(131 + 78\alpha + 16\alpha^3 + 31\alpha v + 96v^2\right.\right. \\
&+ \left.(294 + 144\alpha + 16v^2)v\mathcal{J}_{n,v}(x)\right) \\
&+ \frac{1}{2(n+\beta)^3}\left(7 + 16v - 12\alpha^2 - 8\alpha^3 + (68+6\alpha)v\mathcal{J}_{n,v}(x)\right) \\
&+ \left.\frac{6\alpha^2 n}{(n+\beta)^2}\left(3 + 2v\mathcal{J}_{n,v}(x)\right)\right] x \\
&+ \frac{1}{16(n+\beta)^4}\Big(-367 + 432v^2 + 64v^3\mathcal{J}_{n,v}(x) - 56\alpha - 128\alpha v \\
&+ 16\alpha^2(2\alpha+1) + (808 - 544\alpha)v\mathcal{J}_{n,v}(x)\Big) - \frac{3\alpha^2}{2(n+\beta)^2} v\mathcal{J}_{n,v}(x).
\end{aligned}
$$

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Lemma 5. *Suppose $\eta_j = (e_2 - x)^j$ for $j = 1,2,3,4$, where e_2 defined in Lemma 2. Then, for any $0 \leq x \leq \frac{1}{2n}$ we have*

$$\begin{aligned}
(1)^{**} \quad \mathcal{S}^*_{n,v}(\eta_1; x) &= \frac{\alpha+1}{n+\beta} - x, \\
(2)^{**} \quad \mathcal{S}^*_{n,v}(\eta_2; x) &= x^2 - \frac{2(\alpha+1)}{(n+\beta)}x + \frac{2+\alpha+\alpha^2}{(n+\beta)^2}, \\
(3)^{**} \quad \mathcal{S}^*_{n,v}(\eta_4; x) &= x^4 - \frac{4(\alpha+1)}{(n+\beta)}x^3 + \frac{6(2+\alpha+\alpha^2)}{(n+\beta)^2}x^2 \\
&- \frac{4(6+6\alpha+3\alpha^2+\alpha^3)}{(n+\beta)^3}x + \frac{24+24\alpha+12\alpha^2+4\alpha^3+\alpha^4}{(n+\beta)^4}.
\end{aligned}$$

3. Korovkin type approximation

In the present section the results related to uniform convergence of the operators defined by (9) are given via well-known Korovkin's and weighted Korovkin's type theorems.

Let $\mathbb{R}^+ = [0,\infty)$ and $C_B(\mathbb{R}^+)$ denote the linear normed space with the norm

$$\| f \|_{C_{B(\mathbb{R}^+)}} = \sup_{x\geq 0} | f(x) | .$$

Let

$$\mathcal{H} := \{f : \lim_{x\to\infty} \frac{f(x)}{1+x^2} \text{ exists, } x \in [0,\infty)\}.$$

Theorem 1. *Let the function $f \in C[0,\infty) \cap \mathcal{H}$ and the operators $\mathcal{S}^*_{n,v}(\cdot\,;\cdot)$ defined by (9). Then*

$$\lim_{n\to\infty} \mathcal{S}^*_{n,v}(f;x) = f(x)$$

uniformly on U, where U is any compact subset of $[0,\infty)$.

Proof of Theorem 1. We apply the well-known Korovkin's theorem to prove the uniform convergence of operators $\mathcal{S}^*_{n,v}(\cdot\,;\cdot)$. Therefore, for $\ell = 1,2,3$, we see $\lim_{n\to\infty} \mathcal{S}^*_{n,v}(e_\ell;x) = x^{\ell-1}$ uniformly. Therefore, we have

$$\lim_{n\to\infty} \mathcal{S}^*_{n,v}(e_1;x) = 1; \quad \lim_{n\to\infty} \mathcal{S}^*_{n,v}(e_2;x) = x; \quad \lim_{n\to\infty} \mathcal{S}^*_{n,v}(e_3;x) = x^2.$$

Hence proved. □

We recall the weighted spaces defined by Gadžiev [8]. We write $B_\sigma(\mathbb{R}^+)$ for the set of all functions such that

$$| f(x) | \leq m_f \sigma(x)$$

with $||f||_\sigma = \sup_{x\geq 0} \frac{|f(x)|}{\sigma(x)}$, where m_f is a constant depends on f, and $x \to \phi(x)$ is a continuous and strictly increasing function such as $\sigma(x) = 1 + \phi^2(x)$ with $\lim_{x\to\infty} \sigma(x) = \infty$. Let $C_\sigma(\mathbb{R}^+) = B_\sigma(\mathbb{R}^+) \cap C(\mathbb{R}^+)$. Note that [8] the sequence of positive linear operators $\{L_n\}_{n\geq 1}$ maps $C_\sigma(\mathbb{R}^+)$ into $B_\sigma(\mathbb{R}^+)$ if and only if

$$|L_n(\sigma;x)| \leq K\sigma(x)$$

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with $\sigma(x) = 1 + \phi^2(x)$, $x \in \mathbb{R}^+$ and K is a positive constant. Let $C^0_\sigma(\mathbb{R}^+)$ be a subset of $C_\sigma(\mathbb{R}^+)$ such that

$$\lim_{x\to\infty} \frac{f(x)}{\sigma(x)} = k_f < \infty.$$

Theorem 2. *Let $\mathcal{S}^*_{n,v}$ be the sequence of positive linear operators acting from $C_\sigma(\mathbb{R}^+)$ into $B_\sigma(\mathbb{R}^+)$ such that*

$$\lim_{n\to\infty} \parallel \mathcal{S}^*_{n,v}(\varphi^k(t);x) - \varphi^k(x) \parallel_\sigma = 0 \quad k = 0,1,2.$$

Then for all $f \in C^0_\sigma(\mathbb{R}^+)$, we have

$$\lim_{n\to\infty} \parallel \mathcal{S}^*_{n,v}(f(t);x) - f(x) \parallel_\sigma = 0.$$

Proof of Theorem 2. Consider $\varphi(x) = x$, $\sigma(x) = 1 + x^2$ and

$$\left|\left|\mathcal{S}^*_{n,v}\left(e_\ell;x\right) - x^{\ell-1}\right|\right|_\sigma = \sup_{x\geq 0} \frac{\left|\mathcal{S}^*_{n,v}(e_\ell;x) - x^{\ell-1}\right|}{1+x^2}.$$

Then by Korovkin's theorem, it is easily proved that $\lim_{n\to\infty} \left|\left|\mathcal{S}^*_{n,v}\left(e_\ell;x\right) - x^{\ell-1}\right|\right|_\sigma = 0$, for $\ell = 1,2,3$. Hence for any $f \in C^0_\sigma(\mathbb{R}^+)$, we get

$$\lim_{n\to\infty} \left|\left|\mathcal{S}^*_{n,v}\left(f(t);x\right) - f(x)\right|\right|_\sigma = 0.$$

□

Theorem 3. *Let $\mathcal{S}^*_{n,v}(\,\cdot\,;\,\cdot\,)$ be the operators defined by (9). Then for every $f \in C^0_\sigma(\mathbb{R}^+)$, we have*

$$\lim_{n\to\infty} \parallel \mathcal{S}^*_{n,v}(f;x) - f \parallel_\sigma = 0.$$

Proof of Theorem 3. We prove this theorem in the light of 2. Take $f(t) = e_\ell$ defined by Lemma 2. Then, for any $f(t) \in C^0_\sigma(\mathbb{R}^+)$, $\mathcal{S}^*_{n,v}(e_\ell;x) \to x^{\ell-1}$ $(\ell = 1,2,3)$ uniformly as $n \to \infty$. For $\ell = 1$, by applying Lemma 2, we get $\mathcal{S}^*_{n,v}(e_1;x) = 1$, so that

$$\lim_{n\to\infty} \left|\left|\mathcal{S}^*_{n,v}\left(e_1;x\right) - 1\right|\right|_\sigma = 0. \tag{9}$$

Take $\ell = 2$ and $x \geq \frac{1}{2n}$, we get

$$\begin{aligned}
&\left|\left|\mathcal{S}^*_{n,v}\left(e_2;x\right) - x\right|\right|_\sigma \\
&= \sup_{x\geq 0} \frac{\left|\mathcal{S}^*_{n,v}(e_2;x) - x\right|}{1+x^2} \\
&= \sup_{x\geq 0} \frac{\left|\frac{n}{n+\beta}\mathcal{P}_{n,v}(e_2;x) - x + \frac{\alpha}{n+\beta}\right|}{1+x^2} \\
&\leq \left(\frac{n}{n+\beta} - 1\right) \sup_{x\geq 0} \frac{x}{1+x^2} + \frac{1+2\alpha}{2(n+\beta)} \sup_{x\geq 0} \frac{1}{1+x^2}.
\end{aligned}$$

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In case of $0 \leq x \leq \frac{1}{2n}$, we get

$$\begin{aligned}
&\left|\left|\mathcal{S}^*_{n,v}\left(e_2;x\right)-x\right|\right|_\sigma \\
&= \max_{0\leq x\leq \frac{1}{2n}} \frac{\left|\mathcal{S}^*_{n,v}(e_2;x)-x\right|}{1+x^2} \\
&\leq \max_{0\leq x\leq \frac{1}{2n}} \left|\mathcal{S}^*_{n,v}(e_2;x)-x\right| \\
&\leq \max_{0\leq x\leq \frac{1}{2n}} \left|\frac{\alpha+1}{n+\beta}-x\right| \\
&= \frac{1}{n+\beta}\max\left\{\alpha+1,\left|\alpha-\frac{\beta}{2n}\right|\right\}.
\end{aligned}$$

Then

$$\lim_{n\to\infty}\left|\left|\mathcal{S}^*_{n,v}\left(e_2;x\right)-x\right|\right|_\sigma = 0. \tag{10}$$

In similar way if take $\ell = 3$ and $x \geq \frac{1}{2n}$, we get

$$\begin{aligned}
&\left|\left|\mathcal{S}^*_{n,v}\left(e_3;x\right)-x^2\right|\right|_\sigma \\
&= \sup_{x\geq 0}\frac{\left|\mathcal{S}^*_{n,v}(e_3;x)-x^2\right|}{1+x^2} \\
&= \sup_{x\geq 0}\frac{\left|\frac{n^2}{(n+\beta)^2}\mathcal{P}_{n,v}(e_3;x)+\frac{2\alpha n}{(n+\beta)^2}\mathcal{P}_{n,v}(e_2;x)+\frac{\alpha^2}{(n+\beta)^2}-x^2\right|}{1+x^2} \\
&\leq \left(\frac{n^2}{(n+\beta)^2}-1\right)\sup_{x\geq 0}\frac{x^2}{1+x^2}+\frac{n}{(n+\beta)^2}\left(2\alpha+3+2v\mathcal{J}_{n,v}(x)\right)\sup_{x\geq 0}\frac{x}{1+x^2} \\
&+\frac{1}{4(n+\beta)^2}\left(4\alpha+4\alpha^2-\mathcal{J}_{n,v}(x)\right)\sup_{x\geq 0}\frac{1}{1+x^2},
\end{aligned}$$

In case of $0 \leq x \leq \frac{1}{2n}$, we get

$$\begin{aligned}
&\left|\left|\mathcal{S}^*_{n,v}\left(e_3;x\right)-x\right|\right|_\sigma \\
&= \max_{0\leq x\leq \frac{1}{2n}} \frac{\left|\mathcal{S}^*_{n,v}(e_2;x)-x\right|}{1+x^2} \\
&\leq \max_{0\leq x\leq \frac{1}{2n}} \left|\mathcal{S}^*_{n,v}(e_2;x)-x\right| \\
&\leq \max_{0\leq x\leq \frac{1}{2n}} \left|\frac{2+\alpha+\alpha^2}{(n+\beta)^2}-x^2\right| \\
&= \frac{2+\alpha+\alpha^2}{(n+\beta)^2}.
\end{aligned}$$

$$\lim_{n\to\infty}\left|\left|\mathcal{S}^*_{n,v}\left(e_3;x\right)-x^2\right|\right|_\sigma = 0. \tag{11}$$

This proves the theorem. □

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4. Order of approximation of $\mathcal{S}^*_{n,v}(\,\cdot\,;\,\cdot\,)$

We denote the set of all uniformly continuous functions by $\tilde{C}[0,\infty)$. Let $\tilde{\omega}(f;\tilde{\delta})$ denote the modulus of continuity of $f \in \tilde{C}[0,\infty)$, i.e.

$$\tilde{\omega}(f;\tilde{\delta}) = \sup_{|x_1-x_2|\le\delta} \mid f(x_1) - f(x_2) \mid; \quad x_1, x_2 \in [0,\infty),\ \tilde{\delta} > 0. \tag{12}$$

which satisfies that $\lim_{\tilde{\delta}\to 0+} \tilde{\omega}(f;\tilde{\delta}) = 0$, and

$$\mid f(x_1) - f(x_2) \mid \le \left(\frac{\mid x_1 - x_2 \mid}{\tilde{\delta}} + 1 \right) \tilde{\omega}(f;\tilde{\delta}). \tag{13}$$

In the light of Lemma 4 and 5 we use the notation

$$\sqrt{\mathcal{S}^*_{n,v}\left(\eta_2;x\right)} = \tilde{\delta}_{n,v}(x), \tag{14}$$

where

$$\left(\tilde{\delta}_{n,v}(x)\right)^2 = \begin{cases} x^2 - \frac{2(\alpha+1)}{(n+\beta)}x + \frac{2+\alpha+\alpha^2}{(n+\beta)^2}; & \text{if} \quad 0 \le x \le \frac{1}{2n}, \\ \left[\frac{n^2}{(n+\beta)^2} - \frac{2n}{n+\beta} + 1\right] x^2 & \\ + \left[\frac{n}{(n+\beta)^2}\left(3 + 2v\mathcal{J}_{n,v}(x)\right) + \frac{2\alpha n}{(n+\beta)^2} - \frac{2\alpha+1}{n+\beta}\right] x & \\ + \frac{\alpha+\alpha^2}{(n+\beta)^2} - \frac{1}{4(n+\beta)^2} v\mathcal{J}_{n,v}(x); & \text{if} \quad x \ge \frac{1}{2n} \end{cases} \tag{15}$$

and

$$\mathcal{J}_{n,v}(x) = \mathcal{J}^*_{n,v}(n\chi_n(x)) = \begin{cases} 1; & \text{if} \quad 0 \le x \le \frac{1}{2n}, \\ \mathcal{J}^*_{n,v}\left(nx - \frac{1}{2}\right); & \text{if} \quad x \ge \frac{1}{2n}. \end{cases} \tag{16}$$

Theorem 4. *For any $f \in \tilde{C}[0,\infty)$,*

$$\mid \mathcal{S}^*_{n,v}(f;x) - f(x) \mid \le 2\tilde{\omega}\left(f;\tilde{\delta}_{n,v}(x)\right),$$

where $\tilde{\delta}_{n,v}(x)$ is defined by (15).

Proof of Theorem 4. Using (12) and (13), we get

$$\begin{aligned} \mathcal{S}^*_{n,v}(f;x) - f(x) &= \mathcal{S}^*_{n,v}(f;x) - f(x)\mathcal{S}^*_{n,v}(e_1;x) \\ &= \mathcal{S}^*_{n,v}\left(f(t) - f(x);x\right) \\ &\le \mathcal{S}^*_{n,v}\left(|f(t) - f(x)|\,;x\right) \end{aligned}$$

Since $\mathcal{S}^*_{n,v}(e_1;x) = 1$, by (13) we get

$$\begin{aligned} \left|\mathcal{S}^*_{n,v}(f;x) - f(x)\right| &\le \mathcal{S}^*_{n,v}\left(1 + \frac{|\eta_1|}{\tilde{\delta}};x\right)\tilde{\omega}(f;\tilde{\delta}) \\ &= \left(1 + \frac{1}{\tilde{\delta}}\mathcal{S}^*_{n,v}(|\eta_1|\,;x)\right)\tilde{\omega}(f;\tilde{\delta}). \end{aligned}$$

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From the Cauchy-Schwarz inequality we conclude that

$$\begin{aligned}\mathcal{S}_{n,v}^{*}\left(|\eta_1|\,;x\right) &\leq \mathcal{S}_{n,v}^{*}\left(e_1;x\right)^{\frac{1}{2}}\mathcal{S}_{n,v}^{*}\left(\eta_2;x\right)^{\frac{1}{2}}\\ &= \mathcal{S}_{n,v}^{*}\left(\eta_2;x\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore,

$$\left|\mathcal{S}_{n,v}^{*}(f;x)-f(x)\right| \leq \left(1+\frac{1}{\tilde{\delta}}\mathcal{S}_{n,v}^{*}\left(\eta_2;x\right)^{\frac{1}{2}}\right)\tilde{\omega}(f;\tilde{\delta}).$$

Choose $\tilde{\delta}=\tilde{\delta}_{n,v}(x)=\sqrt{\mathcal{S}_{n,v}^{*}\left(\eta_2;x\right)}$, then we get the result. □

Here we use the usual class of Lipschitz functions and obtain the rate of convergence of the sequence of positive linear operators $\mathcal{S}_{n,v}^{*}(\,\cdot\,;\,\cdot\,)$ (9). For $\mathcal{L}>0,\quad 0<\varrho\leq 1$ and for the continuous functions f on $[0,\infty)$, the class of Lipschitz functions $Lip_{\mathcal{L},\varrho}(f)$ is

$$Lip_{\mathcal{L},\varrho}(f)=\{f:\mid f(\varsigma_1)-f(\varsigma_2)\mid\leq\mathcal{L}\mid\varsigma_1-\varsigma_2\mid^{\varrho};\ \mathcal{L}>0,0<\varrho\leq 1\quad(\varsigma_1,\varsigma_2\in[0,\infty))\} \tag{17}$$

Theorem 5. *For any $f\in Lip_{\mathcal{L},\varrho}$, we have*

$$\mid\mathcal{S}_{n,v}^{*}(f;x)-f(x)\mid\leq\mathcal{L}\left(\tilde{\delta}_{n,v}(x)\right)^{\varrho}$$

where $\tilde{\delta}_{n,v}(x)$ is defined by (15).

Proof of Theorem 5. By Hölder inequality and (17), we get

$$\begin{aligned}\mid\mathcal{S}_{n,v}^{*}(f;x)-f(x)\mid &\leq \mid\mathcal{S}_{n,v}^{*}(f(t)-f(x);x)\mid\\ &\leq \mathcal{S}_{n,v}^{*}\left(\mid f(t)-f(x)\mid;x\right)\\ &\leq \mathcal{L}\mathcal{S}_{n,v}^{*}\left(\mid\eta_1\mid^{\varrho};x\right)\\ &\leq \mathcal{L}\left(\mathcal{S}_{n,v}^{*}(e_1;x)\right)^{\frac{2-\varrho}{2}}\left(\mathcal{S}_{n,v}^{*}(\mid\eta_1\mid^{2};x)\right)^{\frac{\varrho}{2}}\\ &= \mathcal{L}\left(\mathcal{S}_{n,v}^{*}(\eta_2;x)\right)^{\frac{\varrho}{2}}.\end{aligned}$$

□

The space of all that continuous and bounded functions on $\mathbb{R}^{+}$ is denoted by $C_B(\mathbb{R}^{+})$ and

$$C_B^2(\mathbb{R}^{+})=\{\psi\in C_B(\mathbb{R}^{+}):\psi',\psi''\in C_B(\mathbb{R}^{+})\}. \tag{18}$$

The norm on $C_B^2(\mathbb{R}^{+})$ is given by

$$\parallel\psi\parallel_{C_B^2(\mathbb{R}^{+})}=\parallel\psi''\parallel_{C_B(\mathbb{R}^{+})}+\parallel\psi'\parallel_{C_B(\mathbb{R}^{+})}+\parallel\psi\parallel_{C_B(\mathbb{R}^{+})}, \tag{19}$$

where the norm for $C_B(\mathbb{R}^{+})$ is

$$\parallel\psi\parallel_{C_B(\mathbb{R}^{+})}=\sup_{x\geq 0}\mid\psi(x)\mid. \tag{20}$$

Theorem 6. *Let $\psi\in C_B^2(\mathbb{R}^{+})$. Then*

$$\mid\mathcal{S}_{n,v}^{*}(\psi;x)-\psi(x)\mid\leq\mu_{n,v}(x)||\psi||_{C_B^2(\mathbb{R}^{+})},$$

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where $\mu_{n,v}(x) = \tilde{\delta}_{n,v}(x) + \frac{(\tilde{\delta}_{n,v}(x))^2}{2}$.

Proof of Theorem 6. By Taylor series expansion for $\psi \in C_B^2(\mathbb{R}^+)$ we obtain

$$\psi(t) = \psi(x) + \psi'(x)\eta_1 + \psi''(\varphi)\,\frac{\eta_2}{2} \quad \text{where } \eta_1, \eta_2 \text{ are given by (4), (5)}$$

$$|\psi(t) - \psi(x)| \leq \mathcal{U}_1 \mid \eta_1 \mid + \frac{1}{2}\mathcal{U}_2\eta_2,$$

where

$$\mathcal{U}_1 = \sup_{x\geq 0} \left|\psi'(x)\right| = ||\psi'||_{C_B(\mathbb{R}^+)} \leq ||\psi||_{C_B^2(\mathbb{R}^+)},$$

$$\mathcal{U}_2 = \sup_{x\geq 0} \left|\psi''(x)\right| = ||\psi''||_{C_B(\mathbb{R}^+)} \leq ||\psi||_{C_B^2(\mathbb{R}^+)}.$$

Therefore,

$$|\psi(t) - \psi(x)| \leq \left(\mid \eta_1 \mid + \frac{1}{2}\eta_2\right) ||\psi||_{C_B^2(\mathbb{R}^+)}.$$

By using linearity of $\mathcal{S}^*_{n,v}$ we get

$$\left|\mathcal{S}^*_{n,v}(\psi, x) - \psi(x)\right| \leq \left(\mathcal{S}^*_{n,v}(\mid \eta_1 \mid; x) + \frac{1}{2}\mathcal{S}^*_{n,v}(\eta_2; x)\right) ||\psi||_{C_B^2(\mathbb{R}^+)}.$$

So that

$$\left|\mathcal{S}^*_{n,v}(\psi, x) - \psi(x)\right| = \left|\mathcal{S}^*_{n,v}(\psi(t) - \psi(x); x)\right| \leq \mathcal{S}^*_{n,v}\left(|\psi(t) - \psi(x)|\,; x\right)$$

From Cauchy-Schwarz inequality,

$$\mathcal{S}^*_{n,v}(\mid \eta_1 \mid; x) \leq \left(\mathcal{S}^*_{n,v}\left(\eta_2; x\right)\right)^{\frac{1}{2}}.$$

Thus, we have

$$\left|\mathcal{S}^*_{n,v}(\psi, x) - \psi(x)\right| \leq \left(\tilde{\delta}_{n,v}(x) + \frac{(\tilde{\delta}_{n,v}(x))^2}{2}\right) ||\psi||_{C_B^2(\mathbb{R}^+)}.$$

□

5. Peetre's K-functional and Direct theorem of $\mathcal{S}^*_{n,v}(\cdot\,;\cdot)$

The Peetre's K-functional is an influence of potential research work on approximation process has given by J. Peetre in 1968. The Peetre was enable to to investigate the interpolation spaces between two Banach spaces and an interactions to the real interpolation on K-functional. For any $f \in C_B(\mathbb{R}^+)$, the Peetre's, well-known K-functional property defined as:

$$\mathcal{K}_2(f, \breve{\delta}) = \inf\left\{\left(\| f - \psi \|_{C_B(\mathbb{R}^+)} + \breve{\delta} \| \psi'' \|_{C_B^2(\mathbb{R}^+)}\right) : \psi \in \mathcal{W}^2\right\}, \tag{21}$$

where

$$\mathcal{W}^2 = \{\psi | \quad \psi, \psi' \quad \text{and} \quad \psi'' \in C_B(\mathbb{R}^+)\}. \tag{22}$$

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For any $\breve{\delta} > 0$ and a positive constant $\mathcal{C}$ one has $\mathcal{K}_2(f;\breve{\delta}) \leq \mathcal{C}\omega_2(f;\breve{\delta}^{\frac{1}{2}})$, where

$$\omega_2(f;\breve{\delta}^{\frac{1}{2}}) = \sup_{0<h<\breve{\delta}^{\frac{1}{2}}} \sup_{t\geq 0} \mid f(t+2h) - 2f(t+h) + f(t) \mid . \tag{23}$$

Theorem 7. *Let $f \in C_B(\mathbb{R}^+)$. Then there exists a positive constant $\mathcal{D}$ such as*

$$\begin{aligned} &\left|\mathcal{S}^*_{n,v}(f;x) - f(x)\right| \\ &\leq 2\mathcal{D}\left\{\omega_2\left(f;\sqrt{\frac{\lambda_{n,v}(x)}{2}}\right) + \min\left(1;\frac{\lambda_{n,v}(x)}{2}\right)||f||_{C_B(\mathbb{R}^+)}\right\}, \end{aligned}$$

where $\lambda_{n,v}(x)$ is given by 6 and $\omega_2\left(f;\frac{\lambda_{n,v}(x)}{2}\right)$ is given by (15).

Proof of Theorem 7. Take $\psi \in C^2_B(\mathbb{R}^+)$. Thus we get

$$\begin{aligned} \left|\mathcal{S}^*_{n,v}(f;x) - f(x)\right| &\leq \left|\mathcal{S}^*_{n,v}(f-\psi;x)\right| + \left|\mathcal{S}^*_{n,v}(\psi;x) - \psi(x)\right| + |f(x) - \psi(x)| \\ &\leq 2||f-\psi||_{C_B(\mathbb{R}^+)} + \lambda_{n,v}(x)||\psi||_{C^2_B(\mathbb{R}^+)} \\ &= 2\left(||f-\psi||_{C_B(\mathbb{R}^+)} + \frac{\lambda_{n,v}(x)}{2}||\psi||_{C^2_B(\mathbb{R}^+)}\right). \end{aligned}$$

By taking the infimum over all $\psi \in C^2_B(\mathbb{R}^+)$ and by using (21), we get

$$\left|\mathcal{S}^*_{n,v}(f;x) - f(x)\right| \leq 2K_2\left(f;\frac{\lambda_{n,v}(x)}{2}\right).$$

Now, for an absolute constant $\mathcal{D} > 0$ in [5], we use the following relation:

$$K_2(f;\breve{\delta}) \leq \mathcal{D}\{\omega_2(f;\sqrt{\breve{\delta}}) + \min(1;\breve{\delta})||f||_{C_B(\mathbb{R}^+)}\}$$

where $\breve{\delta} = \frac{\lambda_{n,v}(x)}{2}$. This completes the proof. □

For an arbitrary $f \in C^0_\sigma(\mathbb{R}^+)$ the following weighted modulus of continuity was defined in [3]

$$\bar{\Omega}(f;\hat{\delta}) = \sup_{|h|\leq\hat{\delta},\quad x\geq 0} \frac{\mid f(x+h) - f(x) \mid}{(1+x^2)(1+h^2)}, \tag{24}$$

satisfying

$$\lim_{\hat{\delta}\to 0} \bar{\Omega}(f;\hat{\delta}) = 0, \tag{25}$$

and

$$\mid f(t) - f(x) \mid \leq 2\left(\frac{\mid t-x \mid}{\hat{\delta}} + 1\right)(1+\hat{\delta}^2)(1+x^2)\left((t-x)^2 + 1\right)\bar{\Omega}(f;\hat{\delta}). \tag{26}$$

Theorem 8. *For any $f \in C^0_\sigma(\mathbb{R}^+)$, we have*

$$\sup_{x\in[0,\mathcal{A}_{n,v})} \frac{\left|\mathcal{S}^*_{n,v}(f;x) - f(x)\right|}{1+x^2} \leq \mathcal{A}\Big(1 + O(\mathcal{A}_{n,v})\Big)\Omega\Big(f;O(\sqrt{\mathcal{A}_{n,v}})\Big),$$

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where $\mathcal{A}$ is a positive constant and for $x \geq \frac{1}{2n}$

$$\mathcal{A}_{n,v} = \max\left\{\frac{n^2}{(n+\beta)^2} - \frac{2n}{n+\beta} + 1, \frac{\alpha+\alpha^2}{(n+\beta)^2} - \frac{1}{4(n+\beta)^2} v\mathcal{J}_{n,v}(x), \frac{n}{(n+\beta)^2}(3+2v\mathcal{J}_{n,v}(x)) + \frac{2\alpha n}{(n+\beta)^2} - \frac{2\alpha+1}{n+\beta}\right\},$$

and for $0 \leq x \leq \frac{1}{2n}$,

$$\mathcal{A}_{n,v} = \max\left\{1, \frac{2(\alpha+1)}{(n+\beta)}, \frac{2+\alpha+\alpha^2}{(n+\beta)^2}\right\}.$$

Proof of Theorem 8. We prove it by using (24), (26) and Cauchy-Schwarz inequality. We have

$$\left|\mathcal{S}^*_{n,v}(f;x) - f(x)\right| \leq 2(1+\hat{\delta}^2)(1+x^2)\Omega(f;\hat{\delta})\left(1 + \mathcal{S}^*_{n,v}(\eta_2;x) + \mathcal{S}^*_{n,v}\left((1+\eta_2)\frac{|\eta_1|}{\hat{\delta}};x\right)\right). \tag{27}$$

From the Lemma 4, 5 we easily conclude that,

$$\mathcal{S}^*_{n,v}(\eta_2;x) \leq \mathcal{A}_1\, O\left(\mathcal{A}_{n,v}\right)(1+x+x^2) \leq \mathcal{A}_2(1+x+x^2)$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are positive constants, and for $x \geq \frac{1}{2n}$

$$\mathcal{A}_{n,v} = \max\left\{\frac{n^2}{(n+\beta)^2} - \frac{2n}{n+\beta} + 1, \frac{\alpha+\alpha^2}{(n+\beta)^2} - \frac{1}{4(n+\beta)^2} v\mathcal{J}_{n,v}(x), \frac{n}{(n+\beta)^2}(3+2v\mathcal{J}_{n,v}(x)) + \frac{2\alpha n}{(n+\beta)^2} - \frac{2\alpha+1}{n+\beta}\right\}, \tag{28}$$

and for $0 \leq x \leq \frac{1}{2n}$,

$$\mathcal{A}_{n,v} = \max\left\{1, \frac{2(\alpha+1)}{(n+\beta)}, \frac{2+\alpha+\alpha^2}{(n+\beta)^2}\right\}.$$

By apply the Cauchy-Schwarz inequality we easily see that

$$\mathcal{S}^*_{n,v}\left((1+\eta_2)\frac{|\eta_1|}{\delta};x\right) \leq 2\left(\mathcal{S}^*_{n,v}(1+\eta_4;x)\right)^{\frac{1}{2}}\left(\mathcal{S}^*_{n,v}\left(\frac{\eta_2}{\hat{\delta}^2};x\right)\right)^{\frac{1}{2}}. \tag{29}$$

Similarly for the constants $\mathcal{A}_3 > 0$ and $\mathcal{A}_4 > 0$, we have

$$\left(\mathcal{S}^*_{n,v}(1+\eta_4;x)\right)^{\frac{1}{2}} \leq \mathcal{A}_3\left(1+x^2+x^3+x^4\right)^{\frac{1}{2}}$$

$$\left(\mathcal{S}^{*}_{n,v}\left(\frac{\eta_2}{\hat{\delta}^2};x\right)\right)^{\frac{1}{2}} \leq \frac{1}{\hat{\delta}}\,\mathcal{A}_4\,O\Big(\mathcal{A}_{n,v}\Big)^{\frac{1}{2}}\Big(1+x+x^2\Big)^{\frac{1}{2}}$$

Finally, in view of (27), (28) and (29), and choosing $\hat{\delta} = O\left(\sqrt{\mathcal{A}_{n,v}}\right)$, and $\mathcal{A} = 2(1+\mathcal{A}_2+2\mathcal{A}_3\mathcal{A}_4)$, we easily led to the desire result. □

6. Conflicts of Interest

The author declares no conflict of interest.

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 doi:10.20944/preprints201912.0134.v1

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