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— Theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Closure Operators — *Definitions, Essential Properties, and Commutativity*

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Abstract. In a generalized topological space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, ordinary interior and ordinary closure operators $\mathrm{int}_{\mathfrak{g}}, \mathrm{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, are defined in terms of ordinary sets. In order to let these operators be as general and unified a manner as possible, and so to prove as many generalized forms of some of the most important theorems in generalized topological spaces as possible, thereby attaining desirable and interesting results, the present authors have defined the notions of generalized interior and generalized closure operators $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in terms of a new class of generalized sets which they studied earlier and studied their essential properties and commutativity. The outstanding result to which the study has led to is: $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathrm{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathrm{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. The elements supporting this fact are reported therein as a source of inspiration for more generalized operations.

Key words and phrases. *Generalized topological space, generalized sets, generalized interior operator, generalized closure operator, commutativity*

Contents

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1. Introduction

Just as the concepts of $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-interior[1] operators in $\mathscr{T}$-spaces (ordinary and generalized interior operators in ordinary topological spaces) and $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-closure operators in $\mathscr{T}$-spaces (ordinary and generalized closure operators in ordinary topological spaces) are essential operators in the study of $\mathfrak{T}$-sets in $\mathscr{T}$-spaces (arbitrary sets in ordinary topological spaces) [CJK04, Cs6, Cs5, Cs8, Cs7, GS17, JN19, Kal13, Lev70, Lev63, Lev61, MG16], so are the concepts of $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators in $\mathscr{T}_{\mathfrak{g}}$-spaces (ordinary and generalized interior operators in generalized topological spaces) and $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces (ordinary and generalized closure operators in generalized topological spaces) essential operators in the study of $\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathscr{T}_{\mathfrak{g}}$-spaces (arbitrary sets in generalized topological spaces) [DB11, GS14, Min10, Min05, Mus17].

Intuitively, $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-interior operators, respectively, in a $\mathscr{T}$-space can be characterized as one-valued maps $\mathrm{int}, \mathfrak{g}\text{-}\mathrm{Int} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ from the power set $\mathscr{P}(\Omega)$ of Ω into itself, assigning to each $\mathfrak{T}$-set in the $\mathscr{T}$-space the $\cup$-operation (union operation) of all $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-open subsets of the $\mathfrak{T}$-set [And96, Dix84, Nj5, Wil70]. When the role of $\cup$-operation and $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-open subsets, respectively, are given to $\cap$-operation (intersection operation) and $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-closed supersets of the $\mathfrak{T}$-set, the dual notions, called $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-closure operators in the $\mathscr{T}$-space follow [AON09, Cs8, Dix84, DM99, Kur22, Wil70], which can likewise be characterized as one-valued maps $\mathrm{cl}, \mathfrak{g}\text{-}\mathrm{Cl} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. Finally, when $(\mathscr{T}, \mathfrak{T}, \mathfrak{g}\text{-}\mathfrak{T}) \longmapsto (\mathscr{T}_{\mathfrak{g}}, \mathfrak{T}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}})$, the notions of $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in a $\mathscr{T}_{\mathfrak{g}}$-space follow [Cam19, Min11a, Pan11, SKK15, TC13], which can in a similar manner be characterized as one-valued maps of the types $\mathrm{int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathrm{cl}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively.

Thus, in a $\mathscr{T}$-space, $\mathrm{int}, \mathfrak{g}\text{-}\mathrm{Int} : \mathscr{S} \longmapsto \mathrm{int}(\mathscr{S}), \mathfrak{g}\text{-}\mathrm{Int}(\mathscr{S})$ describe two types of collections of points interior in $\mathscr{S}$ and, $\mathrm{cl}, \mathfrak{g}\text{-}\mathrm{Cl} : \mathscr{S} \longmapsto \mathrm{cl}(\mathscr{S}), \mathfrak{g}\text{-}\mathrm{Cl}(\mathscr{S})$ describe another two types of collections of points but close to $\mathscr{S}$. Similarly, in a $\mathscr{T}_{\mathfrak{g}}$-space, $\mathrm{int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ describe two types of collections of points interior in $\mathscr{S}_{\mathfrak{g}}$ and, $\mathrm{cl}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ describe another two types of collections of points but close to $\mathscr{S}_{\mathfrak{g}}$. Of all such operators int, cl, $\mathfrak{g}$-Int, $\mathfrak{g}\text{-}\mathrm{Cl} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in $\mathscr{T}$-spaces and $\mathrm{int}_{\mathfrak{g}}, \mathrm{cl}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in $\mathscr{T}_{\mathfrak{g}}$-spaces, $\mathrm{int}, \mathrm{cl} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are the oldest and $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are the newest. Hence, the studies of operators of these kinds have evolved from the studies of ordinary operators in ordinary topological spaces to the studies of generalized operators in generalized topological spaces.

In the literature of $\mathscr{T}_{\mathfrak{g}}$-spaces on $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators, some new types of one-valued maps $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ have been defined and investigated by Mathematicians.

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[1]Notes to the reader: The structures $\mathfrak{T} = (\Omega, \mathscr{T})$ and $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, respectively, are called ordinary and generalized topological spaces (briefly, $\mathscr{T}$-space and $\mathscr{T}_{\mathfrak{g}}$-space). The symbols $\mathscr{T}$ and $\mathscr{T}_{\mathfrak{g}}$, respectively, are called ordinary topology and generalized topology (briefly, topology and $\mathfrak{g}$-topology). Subsets of $\mathfrak{T}$ and $\mathfrak{T}_{\mathfrak{g}}$, respectively, are called $\mathfrak{T}$-sets and $\mathfrak{T}_{\mathfrak{g}}$-sets; subsets of $\mathscr{T}$ and $\mathscr{T}_{\mathfrak{g}}$, respectively, are called $\mathscr{T}$-open and $\mathscr{T}_{\mathfrak{g}}$-open sets, and their complements are called $\mathscr{T}$-closed and $\mathscr{T}_{\mathfrak{g}}$-closed sets. Generalizations of $\mathfrak{T}$-sets, $\mathscr{T}$-open and $\mathscr{T}$-closed sets in $\mathscr{T}$, respectively, are called $\mathfrak{g}$-$\mathfrak{T}$-sets, $\mathfrak{g}$-$\mathscr{T}$-open and $\mathfrak{g}$-$\mathscr{T}$-closed sets; generalizations of $\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets in $\mathscr{T}_{\mathfrak{g}}$, respectively, are called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-closed sets. By a Λ-operator is meant an operator using Λ-sets to characterize its argument, where $\Lambda \in \{\mathscr{T}, \mathfrak{T}, \mathfrak{g}\text{-}\mathscr{T}, \mathfrak{g}\text{-}\mathfrak{T}\} \cup \{\mathscr{T}_{\mathfrak{g}}, \mathfrak{T}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}\}$.

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In one paper, [Min09] has introduced $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators based on θ-sets in $\mathscr{T}_{\mathfrak{g}}$-spaces characterized by i_θ, $c_\theta : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively; the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators were used to study some properties of $\theta(g, g')$-continuity in $\mathscr{T}_{\mathfrak{g}}$-spaces. In one subsequent paper, [Min11b] has introduced another types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces characterized by $i_{\theta(\nu_1,\nu_2)}$, $c_{\theta(\nu_1,\nu_2)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively; the $\theta(\nu_1, \nu_2)$-operators were used to study other properties on mixed weak continuity on $\mathscr{T}_{\mathfrak{g}}$-spaces. In another subsequent paper, [Min11a] has made use of such θ, $\theta(\nu_1, \nu_2)$-interior and θ, $\theta(\nu_1, \nu_2)$-closure operators to study the notions of mixed θ-continuity on $\mathscr{T}_{\mathfrak{g}}$-spaces. In the work of [CYWW13], the authors have introduced and then used other $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}$-spaces called λ-interior and λ-closure operators and characterized by i_λ, $c_\lambda : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, where $\lambda \in \{\alpha, \beta, \sigma, \pi\}$.

In studying the properties of $\tilde{\mu}$-open sets in $\mathscr{T}_{\mathfrak{g}}$-spaces, [SKK15] have also used these $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets to define new $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators called $\tilde{\mu}$-interior and $\tilde{\mu}$-closure operators and characterized by $i_{\tilde{\mu}}$, $c_{\tilde{\mu}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and studied some of their properties. Thereafter, in studying a new family of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets called $\mathrm{g}_{\mathfrak{u}}$-semi closed sets in $\mathscr{T}_{\mathfrak{g}}$-spaces, [SJ16] have introduced new $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators called g-semi interior and g-semi closure operators and characterized by si_{g}, $\mathrm{sc}_{\mathrm{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively. In the paper of [Boo18], the author gave the definitions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators called $\delta(\mu)$-interior and $\delta(\mu)$-closure operators and characterized by $i_{\delta(\mu)}$, $c_{\delta(\mu)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and utilized them to study the properties of $\zeta_{\delta(\mu)}$ and $(\zeta, \delta(\mu))$-closed sets in strong in $\mathscr{T}_{\mathfrak{g}}$-spaces. Later on, in extending the notion of μ-$\hat{\beta}g$-closed set introduced by [KN12] in $\mathscr{T}$-spaces to $\mathscr{T}_{\mathfrak{g}}$-spaces and then studying their properties, [Cam19] has also investigated the related $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators called, μ-$\hat{\beta}g$-interior and μ-$\hat{\beta}g$-closure operators and characterized by $\hat{\beta}gi_\mu$, $\hat{\beta}gc_\mu : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively. Relative to the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators introduced by [Cs2, Cs5], the author found that the image of a $\mathfrak{T}_{\mathfrak{g}}$-set under $\hat{\beta}gi_\mu : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is a superset of that under $i_\mu : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and, the image of the $\mathfrak{T}_{\mathfrak{g}}$-set under $\hat{\beta}gc_\mu : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is a subset of its image under $c_\mu : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

In this paper titled *Theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Closure Operators* and subtitled *Definitions, Essential Properties, and Commutativity*, the authors attempt to add, in as unique and unified a way as possible, a further contribution to the field with these two research objectives in mind:

- I. To present the definitions and the essential properties of a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.
- II. To discuss the commutativity of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators of this class.

The rest of this paper is structured as thus: In Sect. 2, preliminary notions are described in Sect. 2.1 (Appx. A contains pre-preliminary notions extracted from the preliminary section of our first work titled *Theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Sets*) and the main results of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces are reported in Sect. 3: results associated with essential properties are given in Sect. 3.1 and those associated with the notion of commutativity are given in Sect. 3.2. In Sect. 4, the establishment of the various relationships between these $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators are discussed in Sects 4.1. To support the work, a nice application of

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the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in a $\mathscr{T}_{\mathfrak{g}}$-space is presented in Sect. 4.2. Finally, Sect. 4.3 provides concluding remarks and future directions of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

2. Theory

2.1. Preliminaries. Foreign terms used here are extracted from the preliminary section of our first work titled *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Sets* and are presented in Appx. A.

The discussion commences by defining the notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators of category ν in $\mathscr{T}_{\mathfrak{g}}$-spaces.

Definition 2.1 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Interior, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Closure Operators). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space, let $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}] \stackrel{\mathrm{def}}{=} \{\mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] : \mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}\}$ be the family of all $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open subsets of $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ relative to the class $\mathfrak{g}$-ν-$\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open sets, and let $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}] \stackrel{\mathrm{def}}{=} \{\mathscr{K}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] : \mathscr{K}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g}}\}$ be the family of all $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed supersets of $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ relative to the class $\mathfrak{g}$-ν-$\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed sets. Then, the one-valued maps of the types

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \stackrel{\mathrm{def}}{=} \{\mathscr{S}_{\mathfrak{g},\mu} \subseteq \Omega : \mu \in I^{*}_{\infty}\} \\ \mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \stackrel{\mathrm{def}}{=} \{\mathscr{S}_{\mathfrak{g},\mu} \subseteq \Omega : \mu \in I^{*}_{\infty}\} \\ \mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \end{aligned} \tag{2.2}$$

on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ are called, respectively, a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operator of category ν" and a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operator of category ν." The classes $\mathfrak{g}\text{-}\mathrm{I}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\mathrm{def}}{=} \{\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu} : \nu \in I^{0}_{3}\}$ and $\mathfrak{g}\text{-}\mathrm{C}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\mathrm{def}}{=} \{\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu} : \nu \in I^{0}_{3}\}$, respectively, are called the classes of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators.

Remark 2.2. According to their definitions, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is the *dual* of $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and conversely. For, the definition of the first rests on such concepts as $\cup$, $\subseteq$, $\mathscr{O}_{\mathfrak{g},1}$, $\mathscr{O}_{\mathfrak{g},2}$, ... whereas the second, on $\cap$, $\supseteq$, $\mathscr{K}_{\mathfrak{g},1}$, $\mathscr{K}_{\mathfrak{g},2}$, ..., which are dual concepts to $\cup$, $\subseteq$, $\mathscr{O}_{\mathfrak{g},1}$, $\mathscr{O}_{\mathfrak{g},2}$, ..., respectively.

It is interesting to view $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ as the components of some so-called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-vector operator.

Definition 2.3 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Vector Operator). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then, an operator of the type

$$\begin{aligned} \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) &\longmapsto \big(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu}(\mathscr{R}_{\mathfrak{g}}), \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})\big) \end{aligned} \tag{2.3}$$

on $\mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-vector operator of category ν." Then, $\mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\mathrm{def}}{=} \{\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} = (\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu}) : \nu \in I^{0}_{3}\}$ is called the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-vector operators.

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REMARK 2.4. Observing that, for every $\nu \in I_3^*$, the first and second components of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-vector operator $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} = \left(\mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}, \mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu}\right)$ are based on $\mathfrak{g}$-ν-O$\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathfrak{g}$-ν-K$\left[\mathfrak{T}_{\mathfrak{g}}\right]$, respectively, it follows that:

- I. $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} = \mathbf{ic}_{\mathfrak{g}} \stackrel{\text{def}}{=} \left(\text{int}_{\mathfrak{g}}, \text{cl}_{\mathfrak{g}}\right)$ if based on $\text{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\text{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$;
- II. $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} = \mathfrak{g}\text{-}\mathbf{Ic}_{\nu} \stackrel{\text{def}}{=} \left(\mathfrak{g}\text{-Int}_{\nu}, \mathfrak{g}\text{-Cl}_{\nu}\right)$ if based on $\mathfrak{g}$-ν-O$\left[\mathfrak{T}\right]$ and $\mathfrak{g}$-ν-K$\left[\mathfrak{T}\right]$;
- III. $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} = \mathbf{ic} \stackrel{\text{def}}{=} \left(\text{int}, \text{cl}\right)$ if based on $\text{O}\left[\mathfrak{T}\right]$ and $\text{K}\left[\mathfrak{T}\right]$.

In this way, $\mathbf{ic}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathbf{Ic}_{\nu}$, $\mathbf{ic} : \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$ are called a $\mathfrak{T}_{\mathfrak{g}}$-vector operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, a $\mathfrak{g}$-$\mathfrak{T}$-vector operator of category ν in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$ and a $\mathfrak{T}$-vector operator in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$, respectively. Accordingly,

$$
\begin{aligned}
(2.4)\ \mathfrak{g}\text{-IC}\left[\mathfrak{T}\right] &\stackrel{\text{def}}{=} \left\{\mathfrak{g}\text{-}\mathbf{Ic}_{\nu} = \left(\mathfrak{g}\text{-Int}_{\nu}, \mathfrak{g}\text{-Cl}_{\nu}\right) :\ \nu \in I_3^0\right\} \\
&\subseteq \left\{\mathfrak{g}\text{-Int}_{\nu} :\ \nu \in I_3^0\right\} \times \left\{\mathfrak{g}\text{-Cl}_{\nu} :\ \nu \in I_3^0\right\} \stackrel{\text{def}}{=} \mathfrak{g}\text{-I}\left[\mathfrak{T}\right] \times \mathfrak{g}\text{-C}\left[\mathfrak{T}\right].
\end{aligned}
$$

Then, $\mathfrak{g}$-IC$\left[\mathfrak{T}\right]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}$-vector operators in the $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$; $\mathfrak{g}$-I$\left[\mathfrak{T}\right]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}$-interior operators while $\mathfrak{g}$-C$\left[\mathfrak{T}\right]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}$-closure operators in the $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$.

DEFINITION 2.5 (Complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Operator). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then, the one-valued map

$$
(2.5)\qquad
\begin{aligned}
\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\
\mathscr{S}_{\mathfrak{g}} &\longmapsto \complement_{\mathscr{R}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g}}\right),
\end{aligned}
$$

where $\complement_{\mathscr{R}_{\mathfrak{g}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ denotes the relative complement of its operand with respect to $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, is called a "natural complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator" on $\mathscr{P}(\Omega)$.

For clarity, the notation $\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}$ is employed whenever $\mathscr{R}_{\mathfrak{g}} = \Omega$ or $\mathscr{R}_{\mathfrak{g}}$ is understood from the context. When $\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is with respect to $\mathscr{R}_{\mathfrak{g}} \in \text{S}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}\left[\mathfrak{T}\right]$ and $\mathscr{R}_{\mathfrak{g}} \in \text{S}\left[\mathfrak{T}\right]$, the terms natural complement $\mathfrak{T}_{\mathfrak{g}}$-operator, natural complement $\mathfrak{g}$-$\mathfrak{T}$-operator and natural complement $\mathfrak{T}$-operator are employed and these terms stand for $\text{Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}$, $\mathfrak{g}\text{-Op}_{\mathscr{R}_{\mathfrak{g}}}$, $\text{Op}_{\mathscr{R}_{\mathfrak{g}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively.

DEFINITION 2.6 (Symmetric Difference $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Operator). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then, the one-valued map

$$
(2.6)\qquad
\begin{aligned}
\mathfrak{g}\text{-Sd}_{\mathfrak{g}} : \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\
\left(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\right) &\longmapsto \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g}}\right) \cup \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{S}_{\mathfrak{g}}}\left(\mathscr{R}_{\mathfrak{g}}\right)
\end{aligned}
$$

is called the "symmetric difference $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator" on $\mathscr{P}(\Omega)$.

When the definition of $\mathfrak{g}\text{-Sd}_{\mathfrak{g}} : \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is based on $\text{Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}$, $\mathfrak{g}\text{-Op}_{\mathscr{R}_{\mathfrak{g}}}$, $\text{Op}_{\mathscr{R}_{\mathfrak{g}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ instead of $\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, the concepts of symmetric difference $\mathfrak{T}_{\mathfrak{g}}$-operator $\text{Sd}_{\mathfrak{g}} : \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, symmetric difference $\mathfrak{g}$-$\mathfrak{T}$-operator $\mathfrak{g}\text{-Sd} : \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and symmetric difference $\mathfrak{T}$-operator $\text{Sd} : \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, present themselves.

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The components of $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ are said to commute in a $\mathscr{T}_{\mathfrak{g}}$-space if and only if, for some $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, or equivalently, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Thus, the definition follows.

Definition 2.7 ($\mathfrak{g}$-ν-$\mathfrak{P}_{\mathfrak{g}}$-Property). A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is said to have $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property of category ν in $\mathfrak{T}_{\mathfrak{g}}$ if and only if it belongs to:

$$\mathfrak{g}\text{-}\nu\text{-P}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{g}} : \ \mathfrak{g}\text{-Int}_{\mathfrak{g},\nu} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})\big\}, \tag{2.7}$$

called the class of all $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property of category ν in $\mathfrak{T}_{\mathfrak{g}}$.

The following classes:

$$\begin{aligned} \text{P}[\mathfrak{T}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{g}} : \ \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big\}, \\ \mathfrak{g}\text{-}\nu\text{-P}[\mathfrak{T}] &\stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{g}} : \ \mathfrak{g}\text{-Int}_{\nu} \circ \mathfrak{g}\text{-Cl}_{\nu}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\nu} \circ \mathfrak{g}\text{-Int}_{\nu}(\mathscr{S}_{\mathfrak{g}})\big\}, \\ \text{P}[\mathfrak{T}] &\stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{g}} : \ \text{int} \circ \text{cl}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \text{cl} \circ \text{int}(\mathscr{S}_{\mathfrak{g}})\big\}, \end{aligned} \tag{2.8}$$

respectively, stand for the class of all $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$, the class of all $\mathfrak{T}$-sets having $\mathfrak{g}$-$\mathfrak{P}$-property of category ν in $\mathfrak{T}$ and the class of all $\mathfrak{T}$-sets having $\mathfrak{P}$-property in $\mathfrak{T}$. Thus, by $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$ is meant a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$ and by $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}] \stackrel{\text{def}}{=} \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-P}[\mathfrak{T}]$, a $\mathfrak{T}$-set having $\mathfrak{g}$-$\mathfrak{P}$-property in $\mathfrak{T}$. The notion of $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property of category ν may well be defined as thus.

Definition 2.8 ($\mathfrak{g}$-ν-$\mathfrak{Q}_{\mathfrak{g}}$-Property). A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is said to have $\mathfrak{g}$-ν-$\mathfrak{Q}_{\mathfrak{g}}$-property of category ν in $\mathfrak{T}_{\mathfrak{g}}$ if and only if it belongs to:

$$\mathfrak{g}\text{-}\nu\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{g}} : \ \mathfrak{g}\text{-Int}_{\mathfrak{g},\nu} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset\big\}, \tag{2.9}$$

called the class of all $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$.

In an analogous manner, the following classes:

$$\begin{aligned} \text{Nd}[\mathfrak{T}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{g}} : \ \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset\big\}, \\ \mathfrak{g}\text{-}\nu\text{-Nd}[\mathfrak{T}] &\stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{g}} : \ \mathfrak{g}\text{-Int}_{\nu} \circ \mathfrak{g}\text{-Cl}_{\nu} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset\big\}, \\ \text{Nd}[\mathfrak{T}] &\stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{g}} : \ \text{int} \circ \text{cl} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset\big\}, \end{aligned} \tag{2.10}$$

respectively, stand for the class of all $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$, the class of all $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{Q}$-property of category ν in $\mathfrak{T}$ and the class of all $\mathfrak{T}$-sets having $\mathfrak{Q}$-property in $\mathfrak{T}$. Hence, by $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is meant a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$ and by $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}] \stackrel{\text{def}}{=} \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-Nd}[\mathfrak{T}]$, a $\mathfrak{T}$-set having $\mathfrak{g}$-$\mathfrak{Q}$-property in $\mathfrak{T}$.

In the the following sections, the main results of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators are presented.

3. Main Results

Using the foregoing definitions, some essential properties as well as the commutativity of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces are presented below.

3.1. Essential Properties. The discussion begins by giving some of the basic consequences resulting from the foregoing definition.

Lemma 3.1. *If* $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} : \nu \in I_{\sigma}^{*}\}$ *be a collection of* $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$*-sets of a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\bigcap_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}\big] = \bigcap_{\nu\in I_{\sigma}^{*}} \mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]$,
- II. $\mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\big[\bigcup_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}\big] = \bigcup_{\nu\in I_{\sigma}^{*}} \mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]$.

Proof. Let $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} : \nu \in I_{\sigma}^{*}\}$ be a collection of $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then by virtue of $\mathfrak{T}_{\mathfrak{g}}$-set-theoretic $(\cap,\cup)$-operation, it results that

$$\begin{aligned}
\mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\textstyle\bigcap_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}\big] &= \big\{\mathscr{O}_{\mathfrak{g}} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] : \mathscr{O}_{\mathfrak{g}} \subseteq \textstyle\bigcap_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}\big\} \\
&= \big\{\mathscr{O}_{\mathfrak{g}} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] : \textstyle\bigwedge_{\nu\in I_{\sigma}^{*}} (\mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g},\nu})\big\} \\
&= \bigcap_{\nu\in I_{\sigma}^{*}} \big\{\mathscr{O}_{\mathfrak{g}} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] : \mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g},\nu}\big\} = \bigcap_{\nu\in I_{\sigma}^{*}} \mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]; \\
\mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\big[\textstyle\bigcup_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}\big] &= \big\{\mathscr{K}_{\mathfrak{g}} \in \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] : \mathscr{K}_{\mathfrak{g}} \supseteq \textstyle\bigcup_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}\big\} \\
&= \big\{\mathscr{K}_{\mathfrak{g}} \in \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] : \textstyle\bigvee_{\nu\in I_{\sigma}^{*}} (\mathscr{K}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g},\nu})\big\} \\
&= \bigcup_{\nu\in I_{\sigma}^{*}} \big\{\mathscr{K}_{\mathfrak{g}} \in \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] : \mathscr{K}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g},\nu}\big\} = \bigcup_{\nu\in I_{\sigma}^{*}} \mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}].
\end{aligned}$$

The proof of the lemma is complete. Q.E.D.

For any $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$, the relations $\mathscr{O}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})$ and $\mathscr{K}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})$ hold, or alternatively, $\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently,

$$\big(\mathscr{O}_{\mathfrak{g}} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\big) \wedge \big(\mathscr{K}_{\mathfrak{g}} \in \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathscr{K}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]\big).$$

As a consequence of the above lemma, the corollary follows.

Corollary 3.2. *If* $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} : \nu \in I_{\sigma}^{*}\}$ *be a collection of* $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$*-sets of a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\bigcap_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}\big] = \bigcap_{\nu\in I_{\sigma}^{*}} \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]$,
- II. $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\big[\bigcup_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}\big] = \bigcup_{\nu\in I_{\sigma}^{*}} \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]$.

Remark 3.3. It is easily seen that the relations $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\bigcap_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} = \emptyset\big] = \{\emptyset\}$ and $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\big[\bigcup_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}\big] = \{\Omega\}$ hold. On the other hand, $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\mathscr{S}_{\mathfrak{g}} = \Omega\big] = \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\big[\mathscr{S}_{\mathfrak{g}} = \emptyset\big] = \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$.

Proposition 3.4. *Let* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *be a* $\mathfrak{T}_{\mathfrak{g}}$*-set and, let* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$*, respectively, be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-interior and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-closure operators in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. Then, the necessary and sufficient conditions for* $(\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \times \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ *to hold in* $\mathfrak{T}_{\mathfrak{g}}$ *are:*

- I. $\xi \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \big(\exists \mathscr{O}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\mathscr{O}_{\mathfrak{g},\xi} \subseteq \mathscr{S}_{\mathfrak{g}}\big]$,
- II. $\zeta \in \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \big(\forall \mathscr{O}_{\mathfrak{g},\zeta} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\mathscr{O}_{\mathfrak{g},\zeta} \cap \mathscr{S}_{\mathfrak{g}} \neq \emptyset\big]$.

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PROOF. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set and, let $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose

$$(\xi,\zeta) \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \times \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \Bigg) \times \Bigg(\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \Bigg).$$

Then, since the relations

$$\begin{aligned} \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} &\longleftrightarrow \big\{\xi : \big(\exists \mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]\big)\big[\xi \in \mathscr{O}_{\mathfrak{g}}\big]\big\}, \\ \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} &\longleftrightarrow \big\{\zeta : \big(\forall \mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]\big)\big[\zeta \in \mathscr{K}_{\mathfrak{g}}\big]\big\} \end{aligned}$$

hold and $\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \supseteq \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}] \times \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]$, and, on the other hand, the relation $\xi \in \mathscr{O}_{\mathfrak{g},\xi} \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{K}_{\mathfrak{g},\xi}$ also holds for any $(\xi, \mathscr{O}_{\mathfrak{g},\xi}, \mathscr{K}_{\mathfrak{g},\xi}) \in \mathscr{S}_{\mathfrak{g}} \times \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}] \times \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]$, it follows that

$$\begin{aligned} \xi \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \big(\exists \mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]\big)\big[\xi \in \mathscr{O}_{\mathfrak{g}}\big] \\ &\longleftrightarrow \big(\exists \mathscr{O}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\mathscr{O}_{\mathfrak{g},\xi} \subseteq \mathscr{S}_{\mathfrak{g}}\big]; \\ \zeta \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \big(\forall \mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]\big)\big[\zeta \in \mathscr{K}_{\mathfrak{g}}\big] \\ &\longleftrightarrow \big(\forall \mathscr{O}_{\mathfrak{g},\zeta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\mathscr{O}_{\mathfrak{g},\zeta} \cap \mathscr{S}_{\mathfrak{g}} \neq \emptyset\big]. \end{aligned}$$

Hence, $\xi \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ is equivalent to $\big(\exists \mathscr{O}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\mathscr{O}_{\mathfrak{g},\xi} \subseteq \mathscr{S}_{\mathfrak{g}}\big]$ and $\zeta \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ is equivalent to $\big(\forall \mathscr{O}_{\mathfrak{g},\zeta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\mathscr{O}_{\mathfrak{g},\zeta} \cap \mathscr{S}_{\mathfrak{g}} \neq \emptyset\big]$. The proof of the proposition is complete. Q.E.D.

In a $\mathscr{T}_{\mathfrak{g}}$-space, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior of finite intersection and the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure of finite union equal intersections of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interiors and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closures, respectively, as proved in the following theorem.

THEOREM 3.5. *If* $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} : \nu \in I^{*}_{\sigma}\}$ *be a collection of* $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$*-sets of a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *then:*

- I. $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \bigcap_{\nu \in I^{*}_{\sigma}} \mathscr{S}_{\mathfrak{g},\nu} \longmapsto \bigcap_{\nu \in I^{*}_{\sigma}} \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}) \quad \forall\, \mathfrak{g}\text{-Int}_{\mathfrak{g}} \in \mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g}}]$,
- II. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \bigcup_{\nu \in I^{*}_{\sigma}} \mathscr{S}_{\mathfrak{g},\nu} \longmapsto \bigcup_{\nu \in I^{*}_{\sigma}} \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}) \quad \forall\, \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g}}]$.

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PROOF. Let $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} : \nu \in I_{\sigma}^{*}\}$ be a collection of $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then for any $\left(\mathfrak{g}\text{-Int}_{\mathfrak{g}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-I}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-C}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \bigcap_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\bigcap_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}\right]} \mathscr{O}_{\mathfrak{g}} \\
&= \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \bigcap_{\nu \in I_{\sigma}^{*}} \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]} \mathscr{O}_{\mathfrak{g}} \\
&= \bigcap_{\nu \in I_{\sigma}^{*}} \left(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]} \mathscr{O}_{\mathfrak{g}} \right) = \bigcap_{\nu \in I_{\sigma}^{*}} \mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},\nu}\right); \\
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \bigcup_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} &\longmapsto \bigcup_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\bigcup_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}\right]} \mathscr{K}_{\mathfrak{g}} \\
&= \bigcup_{\mathscr{K}_{\mathfrak{g}} \in \bigcup_{\nu \in I_{\sigma}^{*}} \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]} \mathscr{K}_{\mathfrak{g}} \\
&= \bigcup_{\nu \in I_{\sigma}^{*}} \left(\bigcup_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]} \mathscr{K}_{\mathfrak{g}} \right) = \bigcup_{\nu \in I_{\sigma}^{*}} \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},\nu}\right).
\end{aligned}$$

The proof of the theorem is complete. Q.E.D.

THEOREM 3.6. *If $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:*

$$(3.1)\qquad \left(\forall\, \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right)\left[\left(\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \subseteq \mathscr{S}_{\mathfrak{g}}\right) \wedge \left(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathscr{S}_{\mathfrak{g}}\right)\right].$$

PROOF. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set and $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, by virtue of the definition of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, it results that,

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \\
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}},
\end{aligned}$$

respectively. But, for every $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}] \times \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]$, the relation $(\mathscr{O}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subseteq (\mathscr{S}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}})$ holds. Hence, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}}$. This completes the proof of the theorem. Q.E.D.

A consequence of the above theorem is the following corollary.

COROLLARY 3.7. *If $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:*

$$(3.2)\qquad \left(\forall\, \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right)\left[\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right].$$

REMARK 3.8. Employing the terminology of [Lev63], any $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ which satisfies the relation $\mathscr{O}_{\mathfrak{g}} = \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})$ for some $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set $\mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ may well be termed a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-semi-open set."

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PROPOSITION 3.9. *If* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *be a strong* $\mathscr{T}_{\mathfrak{g}}$*-space, then:*

$$(\forall\, \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}])\big[\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} : (\Omega, \emptyset) \longmapsto (\Omega, \emptyset)\big]. \tag{3.3}$$

PROOF. Let $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, $(\Omega, \emptyset) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and, therefore, Ω is the biggest $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subset contained in itself and, $\emptyset$ is the smallest $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed superset containing itself. Consequently,

$$\begin{aligned}
\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} : (\Omega, \emptyset) &\longmapsto \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\Omega]} \mathscr{O}_{\mathfrak{g}}, \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\emptyset]} \mathscr{K}_{\mathfrak{g}} \Bigg) \\
&= \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \{\Omega\} \cup \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\Omega]} \mathscr{O}_{\mathfrak{g}}, \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \{\emptyset\} \cup \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\emptyset]} \mathscr{K}_{\mathfrak{g}} \Bigg) = (\Omega, \emptyset).
\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} : (\Omega, \emptyset) \longmapsto (\Omega, \emptyset)$. The proof of the proposition is complete. Q.E.D.

In a $\mathscr{T}_{\mathfrak{g}}$-space, the components of $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ are both idempotent $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators, as demonstrated in the following proposition.

PROPOSITION 3.10. *If* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *be any* $\mathfrak{T}_{\mathfrak{g}}$*-set in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall\, \mathfrak{g}\text{-Int}_{\mathfrak{g}} \in \mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g}}]$,
- II. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall\, \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g}}]$.

PROOF. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set and let $(\mathfrak{g}\text{-Int}_{\mathfrak{g}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}}) \in \mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g}}]$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}} :\ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}}; \\
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} :\ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}}.
\end{aligned}$$

But, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and consequently,

$$\begin{aligned}
\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}} &= \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}; \\
\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}} &= \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}}.
\end{aligned}$$

Hence, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. This completes the proof of the proposition. Q.E.D.

PROPOSITION 3.11. *If* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *be any* $\mathfrak{T}_{\mathfrak{g}}$*-set in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall (\mathfrak{g}\text{-Int}_{\mathfrak{g}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}}) \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$,
- II. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall (\mathfrak{g}\text{-Int}_{\mathfrak{g}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}}) \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$.

PROOF. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set and let $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator in a $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, the first and second components of $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} : \mathscr{P}(\Omega) \times$

$\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ operated on $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}}$ gives

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}}: \ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \ &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}} \\
&= \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \left(\mathscr{O}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right) \\
&= \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \left(\mathscr{O}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right) = \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \\
\mathfrak{g}\text{-Cl}_{\mathfrak{g}}: \ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \ &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}} \\
&= \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \left(\mathscr{K}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right) \\
&= \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \left(\mathscr{K}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}\right) = \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}},
\end{aligned}$$

respectively. Hence, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is complete. Q.E.D.

In a $\mathscr{T}_{\mathfrak{g}}$-space, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure of subset are subsets of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure, respectively, as shown in the theorem below.

THEOREM 3.12. *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *then, for every* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ *such that* $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$*:*

$$\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}). \tag{3.4}$$

PROOF. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Suppose $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ such that $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$ be an arbitrary pair of $\mathfrak{T}_{\mathfrak{g}}$-sets. Then, since for any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}_{\mathfrak{g}}(\Omega)$, $(\mathscr{O}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subseteq (\mathscr{S}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}})$ for every $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}] \times \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]$, it follows by virtue of the relation $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$ that $(\mathscr{O}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \subseteq (\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subseteq (\mathscr{S}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}})$ for any $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}] \times \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]$. Consequently, it results on the one hand that

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}}: \mathscr{R}_{\mathfrak{g}} \ &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} = \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]} \left(\mathscr{O}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right) \\
&\subseteq \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \left(\mathscr{O}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right) = \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} = \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),
\end{aligned}$$

and on the other hand,

$$\begin{aligned}\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\mathscr{R}_{\mathfrak{g}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} = \bigcap_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]} (\mathscr{K}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}}) \\ &\subseteq \bigcap_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} (\mathscr{K}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

These show that the images of $\mathscr{R}_{\mathfrak{g}}$ under $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, respectively, are subsets of $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Hence, $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}})\subseteq \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})$. The proof of the theorem is complete. Q.E.D.

Theorem 3.13. *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}\in\mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *and* $\mathbf{ic}_{\mathfrak{g}}\in\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{int}_{\mathfrak{g}}$, $\mathrm{cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$*, then:*

$$(3.5)\quad (\forall\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}})\big[\big(\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\subseteq\big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\big].$$

Proof. Let $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Suppose $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}\in\mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathbf{ic}_{\mathfrak{g}}\in\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be an arbitrary $\mathfrak{T}_{\mathfrak{g}}$-set. Then,

$$\begin{aligned}\mathrm{int}_{\mathfrak{g}}:\mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{O}_{\mathfrak{g}}\subseteq\bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{O}_{\mathfrak{g}}=\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}});\\ \mathrm{cl}_{\mathfrak{g}}:\mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}}\supseteq\bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}}=\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Therefore, it follows that the images of $\mathscr{S}_{\mathfrak{g}}$ under $\mathrm{int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, respectively, are subsets of $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Hence, $\big(\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\subseteq\big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$. The proof of the theorem is complete. Q.E.D.

Proposition 3.14. *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}\in\mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *and* $\mathbf{ic}_{\mathfrak{g}}\in\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{int}_{\mathfrak{g}}$, $\mathrm{cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ *then, for any* $\mathfrak{T}_{\mathfrak{g}}$*-set* $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$*,*

$$(3.6)\ \big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\longrightarrow\big(\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big).$$

Proof.

Remark 3.15. If "$\mathfrak{g}\text{-Int}_{\mathfrak{g}}\succsim\mathrm{int}_{\mathfrak{g}}$" stands for "$\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$" and "$\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\precsim\mathrm{cl}_{\mathfrak{g}}$," for "$\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$," then the outstanding facts are: $\mathfrak{g}\text{-Int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathrm{int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ or, $\mathrm{int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-Int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$; $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathrm{cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ or, $\mathrm{cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$.

If $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}\in\mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathbf{ic}_{\mathfrak{g}}\in\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and, let $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be an arbitrary $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But since $\big(\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\subseteq\big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$ it follows that

$$\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).$$

Hence, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ implies $\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is complete. Q.E.D.

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PROPOSITION 3.16. *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{ic}_{\mathfrak{g}} \in \text{IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\text{int}_{\mathfrak{g}}$, $\text{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ *be any pair of* $\mathscr{T}_{\mathfrak{g}}$*-sets in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

$$(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \text{O}[\mathfrak{T}_{\mathfrak{g}}] \times \text{K}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) = \mathbf{ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}). \tag{3.7}$$

PROOF. Let $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathbf{ic}_{\mathfrak{g}} \in \text{IC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and, let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\text{S}[\mathfrak{T}_{\mathfrak{g}}] = \text{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \text{K}[\mathfrak{T}_{\mathfrak{g}}]$ and, $\text{O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \supseteq \text{K}[\mathfrak{T}_{\mathfrak{g}}]$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathbf{ic}_{\mathfrak{g}} : (\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) &\longmapsto \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \text{C}^{\text{sub}}_{\text{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \text{C}^{\text{sup}}_{\text{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \Bigg) \\
&= \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \text{C}^{\text{sub}}_{\text{O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \text{C}^{\text{sup}}_{\text{K}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \Bigg) \\
&= \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \text{C}^{\text{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \text{C}^{\text{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \Bigg) \\
&= \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}).
\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) = \mathbf{ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is complete. Q.E.D.

PROPOSITION 3.17. *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

$$\begin{aligned}
&(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)) \big[\big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \big) \\
&\qquad \wedge \big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \big) \big].
\end{aligned} \tag{3.8}$$

PROOF. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Suppose $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be an arbitrary $\mathfrak{T}_{\mathfrak{g}}$-set. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \text{C}^{\text{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}} \\
&\supseteq \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \text{C}^{\text{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} = \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}});\\
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \text{C}^{\text{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}} \\
&\subseteq \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \text{C}^{\text{sub}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).
\end{aligned}$$

Hence, the image of $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ under $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is a superset of $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and that of $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ under $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is a subset of $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is complete. Q.E.D.

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The theorem stated below and the corollary following it mark the end of this section.

THEOREM 3.18. *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

$$(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)) \left[\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\right]. \tag{3.9}$$

PROOF. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Suppose $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be an arbitrary $\mathfrak{T}_{\mathfrak{g}}$-set. Then, by virtue of the definition of $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}$, it results that,

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \\
&\subseteq \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) = \mathrm{op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}\Bigg); \\
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \\
&\supseteq \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}}) = \mathrm{op}_{\mathfrak{g}}\Bigg(\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}}\Bigg).
\end{aligned}$$

But since

$$\Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}},\Bigg) \in \mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}},$$

it follows, consequently, that $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. Hence, $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. This proves the theorem. Q.E.D.

An immediate consequence of the above theorem is the corollary stated below.

COROLLARY 3.19. *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\Omega]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *be any* $\mathfrak{T}_{\mathfrak{g}}$*-set in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then there exists* $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}}$ *such that:*

$$\left[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\right] \wedge \left[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\right]. \tag{3.10}$$

In view of THMS 3.2, 3.5 and PROPS 3.9, 3.10, it follows immediately that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}$: $\mathrm{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively possess similar properties analogous to the *Kuratowski closure Axioms* which can be grouped and stated in the form of a corollary.

COROLLARY 3.20. *Let* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-interior and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-closure operators in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. Then:*

- *For every* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$,
  - I. $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\Omega) = \Omega$,
  - II. $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}}$,
  - III. $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
  - IV. $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$.
- *For every* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$,

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– V. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\emptyset) = \emptyset$,
– VI. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{R}_{\mathfrak{g}}$,
– VII. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
– VIII. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$.

Some nice Mathematical vocabulary follow. In COR. 3.20, ITEMS I., II., III. and IV. state that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operator $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is Ω-*grounded*, *non-expansive*, *idempotent* and $\cap$-*additive*, respectively. ITEMS V., VI., VII. and VIII. state that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operator $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is $\emptyset$-*grounded*, *expansive*, *idempotent* and $\cup$-*additive*, respectively.

The axiomatic definitions of the concepts of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces follow.

DEFINITION 3.21 (Axiomatic Definition: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Interior Operator). A one-valued map of the type $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operator" on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ if and only if, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, it satisfies the following axioms:

- AX. I. $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}}$,
- AX. II. $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \longrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$.

Thus, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operator $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a non-expansive $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set-valued set map forming a generalization of the $\mathfrak{T}_{\mathfrak{g}}$-set-valued set map $\text{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, provided

$$\left[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}}\right] \wedge \left[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right] \tag{3.11}$$

holds for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$.

DEFINITION 3.22 (Axiomatic Definition: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Closure Operator). A one-valued map of the type $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operator" on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ if and only if, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, it satisfies the following axioms:

- AX. I. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{R}_{\mathfrak{g}}$,
- AX. II. $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \longrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$.

Hence, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operator $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is an expansive $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set-valued set map forming a generalization of the $\mathfrak{T}_{\mathfrak{g}}$-set-valued set map $\text{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, provided

$$\left[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{R}_{\mathfrak{g}}\right] \wedge \left[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right] \tag{3.12}$$

holds for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$. The discussion of the present section terminates here.

3.2. COMMUTATIVITY. It is the intent of the present section to give some characterizations on the commutativity of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators in $\mathscr{T}_{\mathfrak{g}}$-spaces, giving some characterizations of $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property and $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space.

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LEMMA 3.23. *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathfrak{g}\text{-Op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be the natural complement* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operator of its components in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

$$(3.13)\quad \begin{aligned} &\big(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\big)\big[\big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \\ &\qquad \wedge \big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\big]. \end{aligned}$$

PROOF. Let $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given and, let $\mathfrak{g}\text{-Op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be the natural complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator of its components in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, for a $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ taken arbitrarily, it follows that

$$\begin{aligned} \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}}\Bigg); \\ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}}\Bigg). \end{aligned}$$

Let $\{\mathscr{O}_{\mathfrak{g},\nu} : (\forall \nu \in I^*_{\infty})[\mathscr{O}_{\mathfrak{g},\nu} \subseteq \mathscr{S}_{\mathfrak{g}}]\}$ and $\{\mathscr{K}_{\mathfrak{g},\nu} : (\forall \nu \in I^*_{\infty})[\mathscr{K}_{\mathfrak{g},\nu} \supseteq \mathscr{S}_{\mathfrak{g}}]\}$ stand for $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$, respectively. Then,

$$\begin{aligned} \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}}\Bigg) &= \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\nu \in I^*_{\infty}} \big(\mathscr{O}_{\mathfrak{g},\nu} \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\Bigg) \\ &= \complement_{\Omega}\Bigg(\bigcup_{\nu \in I^*_{\infty}} \big(\mathscr{O}_{\mathfrak{g},\nu} \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\Bigg) \\ &= \bigcap_{\nu \in I^*_{\infty}} \big(\complement_{\Omega}(\mathscr{O}_{\mathfrak{g},\nu}) \supseteq \complement_{\Omega}(\complement_{\Omega}(\mathscr{S}_{\mathfrak{g}}))\big) \\ &= \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}}; \\ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}}\Bigg) &= \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\nu \in I^*_{\infty}} \big(\mathscr{O}_{\mathfrak{g},\nu} \subseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\Bigg) \\ &= \complement_{\Omega}\Bigg(\bigcap_{\nu \in I^*_{\infty}} \big(\mathscr{K}_{\mathfrak{g},\nu} \supseteq \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\Bigg) \\ &= \bigcup_{\nu \in I^*_{\infty}} \big(\complement_{\Omega}(\mathscr{K}_{\mathfrak{g},\nu}) \subseteq \complement_{\Omega}(\complement_{\Omega}(\mathscr{S}_{\mathfrak{g}}))\big) \\ &= \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}}. \end{aligned}$$

Since $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ is arbitrary, it follows that, for every $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, the relations

$$\begin{aligned} \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \\ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \end{aligned}$$

hold. The proof of the lemma is complete. Q.E.D.

A necessary and sufficient condition for a $\mathfrak{T}_{\mathfrak{g}}$-sets to have $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space is contained in the following theorem.

THEOREM 3.24. *A $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is said to have $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$ if and only if:*

$$\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]. \tag{3.14}$$

PROOF. *Necessity.* Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}}: \quad & \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
= \quad & \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
= \quad & \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
= \quad & \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
= \quad & \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
= \quad & \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
= \quad & \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)
\end{aligned}$$

Thus, it follows that

$$\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\big) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\big),$$

and hence, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. The condition of the theorem is, therefore, necessary.

Sufficiency. Conversely, suppose $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. Set $\mathscr{R}_{\mathfrak{g}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)$. Then,

$$\mathscr{S}_{\mathfrak{g}} \quad \longleftrightarrow \quad \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \quad \longleftrightarrow \quad \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right).$$

But $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and it in turn implies $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. Hence, it follows that $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ implies $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. The condition of the theorem is, therefore, sufficient. Q.E.D.

Two if-then conditions for a $\mathfrak{T}_{\mathfrak{g}}$-set to have $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space are given in the following proposition in terms of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and closure operators.

PROPOSITION 3.25. *If $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:*

- I. $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$,
- II. $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$.

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PROOF. I. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) &= \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)
\end{aligned}$$

Hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$. The proof of ITEM I. of the proposition is complete.

II. Suppose $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) &= \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)
\end{aligned}$$

Hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$. The proof of ITEM II. of the proposition is complete. Q.E.D.

THEOREM 3.26. *If* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *be a* $\mathfrak{T}_{\mathfrak{g}}$*-set of a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *such that* $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ *or* $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ *in* $\mathfrak{T}_{\mathfrak{g}}$*, then* $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$.

PROOF. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ such that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ or $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}$. Then:

CASE I. Suppose $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}$. Then, for every $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$, it follows that $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset$. But $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and consequently, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset$. Since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, it follows, furthermore, that $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset$. Therefore, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \emptyset = \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and, hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$.

CASE II. Suppose $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}$. Then, by virtue of the above case, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$ and by virtue of the fact that $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$ is equivalent to $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$, it results that $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$. The proof of the theorem is complete. Q.E.D.

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THEOREM 3.27. *Let $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g},\Gamma}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$, where $\mathscr{T}_{\mathfrak{g},\Gamma} : \mathscr{P}(\Gamma) \longmapsto \mathscr{T}_{\mathfrak{g},\Gamma} = \{\mathscr{O}_{\mathfrak{g}} \cap \Gamma : \ \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\Omega}\}$. Then:*

- I. $\Gamma \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ *implies* $\mathfrak{g}\text{-Int}_{\mathfrak{g},\Gamma}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$,
- II. $\Gamma \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ *implies* $\mathfrak{g}\text{-Cl}_{\mathfrak{g},\Gamma}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$.

PROOF. Let $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g},\Gamma}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and let $(\mathfrak{g}\text{-Int}_{\mathfrak{g},\Lambda}, \mathfrak{g}\text{-Cl}_{\mathfrak{g},\Lambda}) \in \mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g},\Lambda}] \times \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Lambda}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathfrak{g}\text{-Int}_{\mathfrak{g},\Lambda}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g},\Lambda} : \mathscr{P}(\Lambda) \longrightarrow \mathscr{P}(\Lambda)$, respectively, where $\Lambda \in \{\Omega, \Gamma\}$. Then:

I. Suppose $\Gamma \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ in $\mathfrak{T}_{\mathfrak{g},\Omega}$. Then,

$$\begin{aligned} \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega} : \mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \\ &= \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}]}[\Gamma \cap \mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \\ &\subseteq \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}]}[\Gamma]} \mathscr{O}_{\mathfrak{g}} = \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\Gamma) = \Gamma. \end{aligned}$$

Thus, $\Gamma \cap \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$. On the other hand,

$$\begin{aligned} \mathfrak{g}\text{-Int}_{\mathfrak{g},\Gamma} : \mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Gamma}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \\ &\longleftrightarrow \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Gamma}]}[\mathscr{S}_{\mathfrak{g}}]} (\mathscr{O}_{\mathfrak{g}} \cap \Gamma) \\ &\longleftrightarrow \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}]}[\mathscr{S}_{\mathfrak{g}}]} (\mathscr{O}_{\mathfrak{g}} \cap \Gamma) \\ &\longleftrightarrow \Gamma \cap \left(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \right) = \Gamma \cap \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}}). \end{aligned}$$

But $\Gamma \cap \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$ and hence, $\mathfrak{g}\text{-Int}_{\mathfrak{g},\Gamma}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$.

II. Suppose $\Gamma \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ in $\mathfrak{T}_{\mathfrak{g},\Omega}$. Then,

$$\begin{aligned} \mathfrak{g}\text{-Cl}_{\mathfrak{g},\Omega} : \mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \\ &\subseteq \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]}[\Gamma]} \mathscr{K}_{\mathfrak{g}} = \mathfrak{g}\text{-Cl}_{\mathfrak{g},\Omega}(\Gamma) = \Gamma. \end{aligned}$$

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Consequently, $\Gamma\cap\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$. On the other hand,

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Gamma}:\mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g},\Gamma}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}}\\ &\longleftrightarrow \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g},\Gamma}]}[\mathscr{S}_{\mathfrak{g}}]}(\mathscr{K}_{\mathfrak{g}}\cap\Gamma)\\ &\longleftrightarrow \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g},\Omega}]}[\mathscr{S}_{\mathfrak{g}}]}(\mathscr{K}_{\mathfrak{g}}\cap\Gamma)\\ &\longleftrightarrow \Gamma\cap\Bigg(\bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g},\Omega}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}}\Bigg)=\Gamma\cap\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

But $\Gamma\cap\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$ and hence, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Gamma}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$. The proof of the theorem is complete. Q.E.D.

THEOREM 3.28. *Let* $\mathscr{Q}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open-closed set and let* $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\subseteq\mathfrak{T}_{\mathfrak{g}}\times\mathfrak{T}_{\mathfrak{g}}$ *be a pair of* $\mathfrak{T}_{\mathfrak{g}}$*-sets in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. *If* $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\subseteq(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}))$, *then:*

$$(3.15)\quad (\forall\,\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{I}[\mathfrak{T}_{\mathfrak{g}}])\Big[\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\textstyle\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\big)=\displaystyle\bigcup_{\sigma=\alpha,\beta}\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\sigma})\Big].$$

PROOF. Let $\mathscr{Q}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set, let $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\subseteq\mathfrak{T}_{\mathfrak{g}}\times\mathfrak{T}_{\mathfrak{g}}$ be a pair of $\mathfrak{T}_{\mathfrak{g}}$-sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ and, suppose $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\subseteq(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}))$. Then, for every $\mathscr{S}_{\mathfrak{g}}\in\{\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta}\}$,

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}:\mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{O}_{\mathfrak{g}}\\ &\subseteq \bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\alpha}\cup\mathscr{S}_{\mathfrak{g},\beta}]}\mathscr{O}_{\mathfrak{g}}=\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\textstyle\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\big).\end{aligned}$$

Consequently, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\big)\supseteq\bigcup_{\sigma=\alpha,\beta}\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\sigma})$. Set $\hat{\mathscr{S}}_{\mathfrak{g},\alpha}=\mathscr{S}_{\mathfrak{g},\alpha}\cap\mathscr{Q}_{\mathfrak{g}}$ and $\hat{\mathscr{S}}_{\mathfrak{g},\beta}=\mathscr{S}_{\mathfrak{g},\beta}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})$. Then, since $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\subseteq(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}))$, it follows that

$$\begin{aligned}
\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\right] &= \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\bigcup_{\sigma=\alpha,\beta}\hat{\mathscr{S}}_{\mathfrak{g},\sigma}\right] \\
&= \left\{\mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] : \ \mathscr{O}_{\mathfrak{g}} \subseteq \bigcup_{\sigma=\alpha,\beta} \hat{\mathscr{S}}_{\mathfrak{g},\sigma}\right\} \\
&= \left\{\mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] : \ \bigvee_{\sigma=\alpha,\beta} \left(\mathscr{O}_{\mathfrak{g}} \subseteq \hat{\mathscr{S}}_{\mathfrak{g},\sigma}\right)\right\} \\
&= \bigcup_{\sigma=\alpha,\beta} \left\{\mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] : \ \mathscr{O}_{\mathfrak{g}} \subseteq \hat{\mathscr{S}}_{\mathfrak{g},\sigma}\right\} \\
&= \bigcup_{\sigma=\alpha,\beta} \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\hat{\mathscr{S}}_{\mathfrak{g},\sigma}\right] = \bigcup_{\sigma=\alpha,\beta} \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g},\sigma}\right].
\end{aligned}$$

Therefore, $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\right] = \bigcup_{\sigma=\alpha,\beta}\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g},\sigma}\right]$, as a consequence of the condition $(\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta}) \subseteq \left(\mathscr{Q}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{Q}_{\mathfrak{g}}\right)\right)$. Taking this fact into account, it follows, moreover, that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g},\alpha}\cup\mathscr{S}_{\mathfrak{g},\beta}\right]} \mathscr{O}_{\mathfrak{g}} \\
&\subseteq \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \bigcup_{\sigma=\alpha,\beta}\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g},\sigma}\right]} \mathscr{O}_{\mathfrak{g}} \\
&\subseteq \bigcup_{\sigma=\alpha,\beta}\left(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g},\sigma}\right]} \mathscr{O}_{\mathfrak{g}}\right) = \bigcup_{\sigma=\alpha,\beta} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},\sigma}\right).
\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\left(\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\right) \subseteq \bigcup_{\sigma=\alpha,\beta}\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},\sigma}\right)$. The proof of the theorem is complete. Q.E.D.

Theorem 3.29. *Let* $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ *be a* $\mathscr{T}_{\mathfrak{g}}$*-subspace of a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$*, where* $\mathscr{T}_{\mathfrak{g},\Gamma} : \mathscr{P}(\Gamma) \longmapsto \mathscr{T}_{\mathfrak{g},\Gamma} = \left\{\mathscr{O}_{\mathfrak{g}} \cap \Gamma : \ \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\Omega}\right\}$*. If* $\Gamma \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right] \cap \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ *and* $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$*, then* $\mathscr{S}_{\mathfrak{g}} \cap \Gamma \in \mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g},\Gamma}\right]$.

Proof. Let $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ be a $\mathscr{T}_{\mathfrak{g}}$-subspace of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and, suppose $\Gamma \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right] \cap \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ and $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$. Then, since $\Gamma \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right] \cap \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ implies $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Gamma}\left(\mathscr{S}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Omega}\left(\mathscr{S}_{\mathfrak{g}}\right)$ and $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Gamma}\left(\mathscr{S}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}\left(\mathscr{S}_{\mathfrak{g}}\right)$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Gamma} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Gamma} : \mathscr{S}_{\mathfrak{g}} \cap \Gamma &\longmapsto \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Omega} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}\left(\mathscr{S}_{\mathfrak{g}} \cap \Gamma\right) \\
&\subseteq \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Omega} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}\left(\mathscr{S}_{\mathfrak{g}}\right).
\end{aligned}$$

Since $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$, it follows, moreover, that $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Omega} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset$. Consequently, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Gamma} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Gamma} : \mathscr{S}_{\mathfrak{g}} \cap \Gamma \longmapsto \emptyset$ and hence, $\mathscr{S}_{\mathfrak{g}} \cap \Gamma \in \mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g},\Gamma}\right]$. The proof of the theorem is complete. Q.E.D.

Theorem 3.30. *In order that a* $\mathfrak{T}_{\mathfrak{g}}$*-set* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *satisfies the condition* $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$*, it is necessary and sufficient that there exist a*

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$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property such that it be expressible as:

$$\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}}). \tag{3.16}$$

Proof. *Sufficiency.* Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and let there exist $\mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ such that the relation $\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})$ holds. Clearly, $(\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}}) \subseteq (\mathscr{Q}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}))$, implying

$$\begin{aligned} \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\left[(\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})\right] &= \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}\right] \\ &\cup \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}}\right]. \end{aligned}$$

Set $\mathscr{S}_{\mathfrak{g},(q,r)} = \mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}$ and $\mathscr{S}_{\mathfrak{g},(r,q)} = \mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}}$. Then, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(q,r)} \cup \mathscr{S}_{\mathfrak{g},(r,q)}\right) = \mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right)$. Since $\left(\mathscr{S}_{\mathfrak{g},(q,r)}, \mathscr{S}_{\mathfrak{g},(r,q)}\right) \subseteq \left(\mathscr{Q}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\right)$ and $\mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$, it follows that

$$\begin{aligned} \mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right) &= \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right), \\ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right) &= \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right), \\ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right) &= \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right), \\ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right) &= \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right). \end{aligned}$$

Consequently,

$$\begin{aligned} \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right]} \mathscr{O}_{\mathfrak{g}} \\ &= \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right) \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right)\right]} \mathscr{O}_{\mathfrak{g}} \\ &= \left(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right)\right]} \mathscr{O}_{\mathfrak{g}}\right) \\ &\cup \left(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right)\right]} \mathscr{O}_{\mathfrak{g}}\right) \\ &= \mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right) \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right)\right) \\ &= \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right) \\ &\cup \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right) \\ &= \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right) \\ &\cup \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right). \end{aligned}$$

Thus, it follows that

$$\begin{aligned} \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &= \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right) \\ &\cup \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right). \end{aligned}$$

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Similarly,

$$\begin{aligned}
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longmapsto & \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\sup}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}} \\
= & \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\sup}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)}) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)})\right]} \mathscr{K}_{\mathfrak{g}} \\
= & \left(\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\sup}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)})]} \mathscr{K}_{\mathfrak{g}} \right) \\
\cup & \left(\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\sup}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)})]} \mathscr{K}_{\mathfrak{g}} \right) \\
= & \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)}) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)})\right) \\
= & \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)}) \\
\cup & \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)}) \\
= & \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)}) \\
\cup & \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)}) .
\end{aligned}$$

Hence, it results that

$$\begin{aligned}
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = & \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)}) \\
\cup & \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)}) .
\end{aligned}$$

By virtue of the relation $(\mathscr{S}_{\mathfrak{g},(q,r)}, \mathscr{S}_{\mathfrak{g},(r,q)}) \subseteq (\mathscr{Q}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}))$, it is plain that $\mathscr{S}_{\mathfrak{g},(q,r)} = \mathscr{Q}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}} \cap \mathscr{R}_{\mathfrak{g}}$ and $\mathscr{S}_{\mathfrak{g},(r,q)} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) \cap \mathscr{R}_{\mathfrak{g}}$. Since $\mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$, it follows that $\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{R}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathscr{Q}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) \cap \mathscr{R}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})$. But $\mathscr{S}_{\mathfrak{g},(q,r)} = \complement_{\mathscr{Q}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}})$ and $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently, $\mathscr{R}_{\mathfrak{g}}$ has $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in $\mathscr{Q}_{\mathfrak{g}}$ and hence,

$$\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)}) = \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)}).$$

On the other hand, the statement that $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) \cap \mathscr{R}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})$ implies that $\mathscr{S}_{\mathfrak{g},(r,q)}$ has $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})$ and therefore,

$$\begin{aligned}
& \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)}) \\
& \qquad = \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)}).
\end{aligned}$$

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When all the foregoing set-theoretic expressions are taken into account, it results that

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &= \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right)\\ &\cup \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right)\\ &= \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right)\\ &\cup \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right)\\ &= \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Hence, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The condition of the theorem is, therefore, sufficient.

Necessity. Conversely, suppose that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$. Then, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Set $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathscr{Q}_{\mathfrak{g}} = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Then, $\mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$, meaning that $\mathscr{Q}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set in $\mathfrak{T}_{\mathfrak{g}}$. Set $\mathscr{S}_{\mathfrak{g},(s,q)} = \mathscr{S}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}}$ and $\mathscr{S}_{\mathfrak{g},(q,s)} = \mathscr{Q}_{\mathfrak{g}} - \mathscr{S}_{\mathfrak{g}}$. Then,

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(s,q)}\right) &\subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathscr{Q}_{\mathfrak{g}};\\ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(s,q)}\right) &\subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\right) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}).\end{aligned}$$

But $\mathscr{Q}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) = \emptyset$ and consequently, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g},(s,q)} \longmapsto \emptyset$, meaning that $\mathscr{Q}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathscr{S}_{\mathfrak{g}}$. On the other hand,

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(q,s)}\right) &\subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) = \mathscr{Q}_{\mathfrak{g}};\\ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(q,s)}\right) &\subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)\\ &= \mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\ &= \mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}).\end{aligned}$$

Since $\mathscr{Q}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) = \emptyset$ it follows, consequently, that $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g},(q,s)} \longmapsto \emptyset$, meaning that $\mathscr{S}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathscr{Q}_{\mathfrak{g}}$. Set $\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g},(q,s)} \cup \mathscr{S}_{\mathfrak{g},(s,q)}$. Then,

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} &\longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(q,s)}\cup\mathscr{S}_{\mathfrak{g},(s,q)}\right)\\ &= \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(q,s)}\right)\cup\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},(s,q)}\right)\\ &= \emptyset\cup\emptyset = \emptyset,\end{aligned}$$

implying that $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$. Having evidenced the existence of a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property, it only remains to show that $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is expressible as $\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}})\cup(\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})$.

Observe that

$$\begin{aligned}
&\mathscr{S}_{\mathfrak{g},(q,r)} \cup \mathscr{S}_{\mathfrak{g},(r,q)} \\
&\quad = \{\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\} \cup \{\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\} \\
&\quad = \{\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}[(\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \cup (\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}))]\} \\
&\quad \cup \{[(\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \cup (\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}))] \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\} \\
&\quad = \{\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}))\} \\
&\quad \cup \{\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\} \\
&\quad = \{\mathscr{Q}_{\mathfrak{g}} \cap (\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) \cup \mathscr{S}_{\mathfrak{g}}) \cap (\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathscr{Q}_{\mathfrak{g}})\} \\
&\quad \cup \{\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\} \\
&\quad = \{(\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \cap (\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathscr{Q}_{\mathfrak{g}})\} \cup \{\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\} \\
&\quad = (\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \cup (\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})).
\end{aligned}$$

But since $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathscr{Q}_{\mathfrak{g}} = \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and the latter in turn implies $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}))$, it follows that $\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}$ and $\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) = \emptyset$. Consequently, $\mathscr{S}_{\mathfrak{g},(q,r)} \cup \mathscr{S}_{\mathfrak{g},(r,q)} = \mathscr{S}_{\mathfrak{g}}$. But, $\mathscr{S}_{\mathfrak{g},(q,r)} \cup \mathscr{S}_{\mathfrak{g},(r,q)} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})$ and hence, $\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})$. The condition of the theorem is, therefore, necessary. Q.E.D.

Observe that $\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}}) = \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{Q}_{\mathfrak{g}}) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}})$. Thus, an immediate consequence of the above theorem is the following corollary.

COROLLARY 3.31. *Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$ if and only if:*

$$(\exists \mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}])(\exists \mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}])[\mathscr{S}_{\mathfrak{g}} = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}})]. \tag{3.17}$$

PROPOSITION 3.32. *If $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property, then $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \neq \Omega$:*

$$\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \neq \Omega. \tag{3.18}$$

PROOF. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, it follows that $\Omega \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\Omega) = \Omega$. But, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \emptyset$. Thus, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \emptyset \neq \Omega = \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\Omega)$, implying $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \neq \Omega$. The proof of the proposition is complete. Q.E.D.

PROPOSITION 3.33. *If $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and $\mathfrak{T}_{\mathfrak{g}}$ be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, then:*

$$\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}] \longleftrightarrow (\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]) \vee (\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]). \tag{3.19}$$

PROOF. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and $\mathfrak{T}_{\mathfrak{g}}$ be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. Suppose $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$. Then, there exist a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property such that $\mathscr{S}_{\mathfrak{g}}$ be expressible as $\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})$. Since the strong $\mathscr{T}_{\mathfrak{g}}$-space

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$\mathfrak{T}_{\mathfrak{g}}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, the only $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set are the improper $\mathfrak{T}_{\mathfrak{g}}$-sets $\emptyset$, $\Omega \subset \mathfrak{T}_{\mathfrak{g}}$. Consequently,

$$\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}] \longleftrightarrow \big(\mathscr{Q}_{\mathfrak{g}} \in \{\emptyset, \Omega\}\big)\big[\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})\big].$$

CASE I. Suppose $\mathscr{Q}_{\mathfrak{g}} = \emptyset$. Then $\mathscr{S}_{\mathfrak{g}} = (\emptyset - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \emptyset)$. But $\emptyset - \mathscr{R}_{\mathfrak{g}} = \emptyset$ and $\mathscr{R}_{\mathfrak{g}} - \emptyset = \mathscr{R}_{\mathfrak{g}}$. Therefore, $\mathscr{S}_{\mathfrak{g}} = \emptyset \cup \mathscr{R}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}$. Thus, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$.

CASE II. Suppose $\mathscr{Q}_{\mathfrak{g}} = \Omega$. Then $\mathscr{S}_{\mathfrak{g}} = (\Omega - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \Omega)$. But $\Omega - \mathscr{R}_{\mathfrak{g}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ and $\mathscr{R}_{\mathfrak{g}} - \Omega = \emptyset$. Consequently, $\mathscr{S}_{\mathfrak{g}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \emptyset = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ and therefore, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathscr{R}_{\mathfrak{g}}$. Hence, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$. The proof of the proposition is complete. Q.E.D.

LEMMA 3.34. *If* $(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ *be a triple of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-sets and* $\mathfrak{g}\text{-Sd}_{\mathfrak{g}} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be the symmetric difference* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operator in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

- I. $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\mathscr{Q}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$,
- II. $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}), \mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}})) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$,
- III. $\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{R}_{\mathfrak{g}}, \mathscr{Q}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}})$.

PROOF. Let $(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ and, let $\mathfrak{g}\text{-Sd}_{\mathfrak{g}} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be the symmetric difference $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. The proof that $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\mathscr{Q}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ holds for any $(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ is first supplied. It is evident that

$$\begin{aligned} \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) &= \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{Q}_{\mathfrak{g}}) \\ &= \big(\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \cup \big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\big) \subseteq \mathscr{Q}_{\mathfrak{g}} \cup \mathscr{R}_{\mathfrak{g}}, \end{aligned}$$

implying $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{Q}_{\mathfrak{g}} \cup \mathscr{R}_{\mathfrak{g}}$. Since $\mathscr{Q}_{\mathfrak{g}} \cup \mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$, it follows that $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$. Items I., II. and III. are now proved.

I. Since the order of the operands under the $\cup$-operation does not change, it follows that

$$\begin{aligned} \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) &= \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{Q}_{\mathfrak{g}}) \\ &= \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{Q}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{Q}_{\mathfrak{g}}). \end{aligned}$$

Hence, $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{Q}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$.

II. For any $(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$, it is plain that $\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g}}) = \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Therefore,

$$\begin{aligned} \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}), \mathscr{S}_{\mathfrak{g}}) &= \{\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\} \\ &\cup \{\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}))\} \\ &= \{\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\} \\ &\cup \{\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\} \\ &\cup \{\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\} \\ &\cup \{\mathscr{S}_{\mathfrak{g}} \cap \mathscr{Q}_{\mathfrak{g}} \cap \mathscr{R}_{\mathfrak{g}}\}. \end{aligned}$$

If $\text{P}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, then

$$\begin{aligned} \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}), \mathscr{S}_{\mathfrak{g}}) &= \text{P}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \cup \text{P}(\mathscr{R}_{\mathfrak{g}}, \mathscr{Q}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \\ &\cup \text{P}(\mathscr{S}_{\mathfrak{g}}, \mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{S}_{\mathfrak{g}} \cap \mathscr{Q}_{\mathfrak{g}} \cap \mathscr{R}_{\mathfrak{g}}). \end{aligned}$$

Since $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}),\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}))$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})) &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}=\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}}))\\ &= \mathrm{P}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}})\cup\mathrm{P}(\mathscr{S}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}})\\ &\cup\ \mathrm{P}(\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\cup(\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

But by virtue of the associativity and distributive properties of the $\cap$, $\cup$-operations, the relations $\mathrm{P}(\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) = \mathrm{P}(\mathscr{Q}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}})$, $\mathrm{P}(\mathscr{R}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) = \mathrm{P}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}})$, $\mathrm{P}(\mathscr{S}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}) = \mathrm{P}(\mathscr{S}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}})$, and $\mathscr{S}_{\mathfrak{g}}\cap\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}} = \mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}$ hold. Thus, $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}),\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$.

III. Since the relation $\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g}}) = \mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds for any $(\mathscr{S}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$, it results that

$$\begin{aligned}\mathscr{Q}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) &= \mathscr{Q}_{\mathfrak{g}}\cap(\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{S}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}}))\\ &= (\mathscr{Q}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g}}))\cup(\mathscr{Q}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{S}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}}))\\ &= (\mathscr{Q}_{\mathfrak{g}}\cap(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})))\cup(\mathscr{Q}_{\mathfrak{g}}\cap(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})))\\ &= ((\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}))\cup((\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}))\\ &= \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}})\\ &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Hence, $\mathscr{Q}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$. The proof of the lemma is complete.

THEOREM 3.35. *If $\mathscr{S}_{\mathfrak{g},1}, \mathscr{S}_{\mathfrak{g},2}, \ldots, \mathscr{S}_{\mathfrak{g},\sigma} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$ are $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$, then $\bigcap_{\nu\in I_{\sigma}^{*}}\mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$.*

PROOF. Let $\mathscr{S}_{\mathfrak{g},1}, \mathscr{S}_{\mathfrak{g},2}, \ldots, \mathscr{S}_{\mathfrak{g},\sigma} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$ be $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$. Then, since $\mathscr{S}_{\mathfrak{g},1}, \mathscr{S}_{\mathfrak{g},2}, \ldots, \mathscr{S}_{\mathfrak{g},\sigma} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$, there exist $\sigma \geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed sets $\mathscr{Q}_{\mathfrak{g},1}, \mathscr{Q}_{\mathfrak{g},2}, \ldots, \mathscr{Q}_{\mathfrak{g},\sigma} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g},1}, \mathscr{R}_{\mathfrak{g},2}, \ldots, \mathscr{R}_{\mathfrak{g},\sigma} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property such that

$$\begin{aligned}\mathscr{S}_{\mathfrak{g},1} &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g},1},\mathscr{R}_{\mathfrak{g},1}),\\ \mathscr{S}_{\mathfrak{g},2} &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g},2},\mathscr{R}_{\mathfrak{g},2}),\ \ldots,\ \mathscr{S}_{\mathfrak{g},\sigma} = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g},\sigma},\mathscr{R}_{\mathfrak{g},\sigma}).\end{aligned}$$

For an arbitrary pair $(\nu,\mu) \in I_{\sigma}^{*}\times I_{\sigma}^{*}$, set $\mathscr{Q}_{\mathfrak{g},(\nu,\mu)} = \mathscr{Q}_{\mathfrak{g},\nu}\cap\mathscr{Q}_{\mathfrak{g},\mu}$, $\mathscr{W}_{\mathfrak{g},(\nu,\mu)} = \mathscr{Q}_{\mathfrak{g},\nu}\cap\mathscr{R}_{\mathfrak{g},\mu}$, and $\mathscr{R}_{\mathfrak{g},(\nu,\mu)} = \mathscr{R}_{\mathfrak{g},\nu}\cap\mathscr{R}_{\mathfrak{g},\mu}$. Then,

$$\begin{aligned}\mathscr{S}_{\mathfrak{g},\nu}\cap\mathscr{S}_{\mathfrak{g},\mu} &= \mathscr{S}_{\mathfrak{g},\nu}\cap\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g},\mu},\mathscr{R}_{\mathfrak{g},\mu})\\ &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}\cap\mathscr{Q}_{\mathfrak{g},\mu},\mathscr{S}_{\mathfrak{g},\nu}\cap\mathscr{R}_{\mathfrak{g},\mu})\\ &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}[\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g},\nu},\mathscr{R}_{\mathfrak{g},\nu})\cap\mathscr{Q}_{\mathfrak{g},\mu},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g},\nu},\mathscr{R}_{\mathfrak{g},\nu})\cap\mathscr{R}_{\mathfrak{g},\mu}]\\ &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}[\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g},(\nu,\mu)},\mathscr{W}_{\mathfrak{g},(\mu,\nu)}),\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g},(\nu,\mu)},\mathscr{R}_{\mathfrak{g},(\nu,\mu)})]\\ &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\{\mathscr{Q}_{\mathfrak{g},(\nu,\mu)},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}[\mathscr{W}_{\mathfrak{g},(\mu,\nu)},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g},(\nu,\mu)},\mathscr{R}_{\mathfrak{g},(\nu,\mu)})]\}.\end{aligned}$$

But, $\mathscr{R}_{\mathfrak{g},\nu}, \mathscr{R}_{\mathfrak{g},\mu} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathscr{R}_{\mathfrak{g},(\nu,\mu)} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$, $(\mathscr{Q}_{\mathfrak{g},\nu},\mathscr{R}_{\mathfrak{g},\mu}) \in (\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}])\times\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathscr{W}_{\mathfrak{g},(\nu,\mu)} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ and, $\mathscr{Q}_{\mathfrak{g},\nu}, \mathscr{Q}_{\mathfrak{g},\mu} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathscr{Q}_{\mathfrak{g},(\nu,\mu)} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. Thus, $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g},(\nu,\mu)},\mathscr{R}_{\mathfrak{g},(\nu,\mu)}) \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$, implying $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}[\mathscr{W}_{\mathfrak{g},(\mu,\nu)},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g},(\nu,\mu)},\mathscr{R}_{\mathfrak{g},(\nu,\mu)})] = \hat{\mathscr{R}}_{\mathfrak{g},(\nu,\mu)} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$. Therefore, $\mathscr{S}_{\mathfrak{g},\nu}\cap\mathscr{S}_{\mathfrak{g},\mu} = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g},(\nu,\mu)},\hat{\mathscr{R}}_{\mathfrak{g},(\nu,\mu)})$, where $\mathscr{Q}_{\mathfrak{g},(\nu,\mu)} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap$

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$\mathfrak{g}$-K$[\mathfrak{T}_{\mathfrak{g}}]$ and $\hat{\mathscr{R}}_{\mathfrak{g},(\nu,\mu)} \in \mathfrak{g}$-Nd$[\mathfrak{T}_{\mathfrak{g}}]$, and consequently, $\mathscr{S}_{\mathfrak{g},\nu} \cap \mathscr{S}_{\mathfrak{g},\mu} \in \mathfrak{g}$-P$[\mathfrak{T}_{\mathfrak{g}}]$ for any $(\nu,\mu) \in I_{\sigma}^{*} \times I_{\sigma}^{*}$. Hence, $\bigcap_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}$-P$\big[\mathfrak{T}_{\mathfrak{g}}\big]$. The proof of the theorem is complete. Q.E.D.

Proposition 3.36. *If $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} :\ \nu \in I_{\sigma}^{*}\}$ be a collection of $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets each of which having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then $\bigcup_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}$ has also $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$:*

$$\bigwedge\nolimits_{\nu\in I_{\sigma}^{*}} \big(\mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]\big) \;\longrightarrow\; \bigcup_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]\,. \tag{3.20}$$

Proof. Let $\mathscr{S}_{\mathfrak{g},1}, \mathscr{S}_{\mathfrak{g},2}, \ldots, \mathscr{S}_{\mathfrak{g},\sigma} \in \mathfrak{g}$-P$[\mathfrak{T}_{\mathfrak{g}}]$ be $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\mathscr{S}_{\mathfrak{g}} = \mathfrak{g}$-Op$_{\mathfrak{g}} \circ \mathfrak{g}$-Op$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ for any $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, it follows that $\mathscr{S}_{\mathfrak{g},\nu} \cup \mathscr{S}_{\mathfrak{g},\mu} = \mathfrak{g}$-Op$_{\mathfrak{g}} \circ \mathfrak{g}$-Op$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu} \cup \mathscr{S}_{\mathfrak{g},\mu}) =$ $\mathfrak{g}$-Op$_{\mathfrak{g}}\big(\mathfrak{g}$-Op$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}) \cap \mathfrak{g}$-Op$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu})\big)$ for any arbitrary pair $(\nu,\mu) \in I_{\sigma}^{*} \times I_{\sigma}^{*}$. But, $\mathfrak{g}$-Op$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu})$, $\mathfrak{g}$-Op$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu}) \in \mathfrak{g}$-P$[\mathfrak{T}_{\mathfrak{g}}]$ and therefore, $\mathfrak{g}$-Op$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}) \cap \mathfrak{g}$-Op$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu}) \in$ $\mathfrak{g}$-P$[\mathfrak{T}_{\mathfrak{g}}]$. Set $\mathfrak{g}$-Op$_{\mathfrak{g}}\big(\hat{\mathscr{S}}_{\mathfrak{g}}\big) = \mathfrak{g}$-Op$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}) \cap \mathfrak{g}$-Op$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu})$. Then, since $\mathfrak{g}$-Op$_{\mathfrak{g}}\big(\hat{\mathscr{S}}_{\mathfrak{g}}\big) \in$ $\mathfrak{g}$-P$[\mathfrak{T}_{\mathfrak{g}}]$ is equivalent to $\mathfrak{g}$-Op$_{\mathfrak{g}} \circ \mathfrak{g}$-Op$_{\mathfrak{g}}\big(\hat{\mathscr{S}}_{\mathfrak{g}}\big) \in \mathfrak{g}$-P$[\mathfrak{T}_{\mathfrak{g}}]$ and, the relation $\mathscr{S}_{\mathfrak{g},\nu} \cup$ $\mathscr{S}_{\mathfrak{g},\mu} = \mathfrak{g}$-Op$_{\mathfrak{g}} \circ \mathfrak{g}$-Op$_{\mathfrak{g}}\big(\hat{\mathscr{S}}_{\mathfrak{g}}\big)$ holds, it follows that $\mathscr{S}_{\mathfrak{g},\nu} \cup \mathscr{S}_{\mathfrak{g},\mu} \in \mathfrak{g}$-P$[\mathfrak{T}_{\mathfrak{g}}]$. The proof of the proposition is complete. Q.E.D.

Theorem 3.37. *Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. If $\mathscr{S}_{\mathfrak{g}}$ has $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$, then it has also $\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$:*

$$\big(\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}\big)\big[\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}] \;\longrightarrow\; \mathscr{S}_{\mathfrak{g}} \in \text{P}\,[\mathfrak{T}_{\mathfrak{g}}]\big]. \tag{3.21}$$

Proof. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}$-P$[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} =$ $(\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, it satisfies the relation $\mathfrak{g}$-Int$_{\mathfrak{g}} \circ \mathfrak{g}$-Cl$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}$-Cl$_{\mathfrak{g}} \circ \mathfrak{g}$-Int$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Since $\big(\text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \subseteq \big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$, it follows that

$$\begin{aligned}
&\text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \text{int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\\
&\text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\text{int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &= \text{int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\
&= \text{int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \text{cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),
\end{aligned}$$

implying $\text{cl}_{\mathfrak{g}} \circ \mathfrak{g}$-Int$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, $\text{cl}_{\mathfrak{g}} \circ \mathfrak{g}$-Int$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) =$ $\text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\text{int}_{\mathfrak{g}} \circ \mathfrak{g}$-Cl$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \text{int}_{\mathfrak{g}} \circ \mathfrak{g}$-Cl$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Consequently, it results that $\text{int}_{\mathfrak{g}} \circ \mathfrak{g}$-Cl$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ which, in turn, implies $\text{cl}_{\mathfrak{g}} \circ \mathfrak{g}$-Int$_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Therefore, $\text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, meaning that $\mathscr{S}_{\mathfrak{g}}$ has also $\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$. Hence, $\mathscr{S}_{\mathfrak{g}} \in$ P$[\mathfrak{T}_{\mathfrak{g}}]$. The proof of the theorem is complete. Q.E.D.

Proposition 3.38. *If $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} :\ \nu \in I_{\sigma}^{*}\}$ be a collection of $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then $\bigcup_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}$ has also $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$:*

$$\bigwedge\nolimits_{\nu\in I_{\sigma}^{*}} \big(\mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Nd}\,[\mathfrak{T}_{\mathfrak{g}}]\big) \;\longrightarrow\; \bigcup_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Nd}\,[\mathfrak{T}_{\mathfrak{g}}]\,. \tag{3.22}$$

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PROOF. Let $\{\mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}] : \nu \in I_{\sigma}^{*}\}$ be a collection of $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose $\bigwedge_{\nu \in I_{\sigma}^{*}}(\mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}])$ implies $\bigcup_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is an untrue logical statement. Then, $\bigwedge_{\nu \in I_{\sigma}^{*}}(\mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}])$ is true and $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \bigcup_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \longmapsto \emptyset$ is untrue. Thus, to prove the proposition, it suffices to prove that $\bigcup_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \notin \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is a contradiction. For arbitrary $(\nu, \mu(\nu)) \in I_{\sigma}^{*} \times I_{\sigma(\nu)}^{*}$ such that $I_{\sigma(\nu)}^{*} = I_{\sigma}^{*} \setminus \{\nu\}$, set $\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))} = \mathscr{S}_{\mathfrak{g},\nu} \cup \mathscr{S}_{\mathfrak{g},\mu(\nu)}$, where $\{\mathscr{S}_{\mathfrak{g},\nu}, \mathscr{S}_{\mathfrak{g},\mu(\nu)}\} \subset \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$. Since $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}) \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)})$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) &\cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)}) \\ &\subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)}) \\ &= \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}).\end{aligned}$$

Thus, for arbitrary $(\nu, \mu(\nu)) \in I_{\sigma}^{*} \times I_{\sigma(\nu)}^{*}$ such that $I_{\sigma(\nu)}^{*} = I_{\sigma}^{*} \setminus \{\nu\}$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left[\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)})\right] \\ \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}) = \emptyset.\end{aligned}$$

Since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, it results that

$$\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)}) = \emptyset,$$

and therefore, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)})$. On the other hand, since $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, it follows that

$$\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)}) = \emptyset,$$

Thus, $\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ holds for arbitrary $(\nu, \mu(\nu)) \in I_{\sigma}^{*} \times I_{\sigma(\nu)}^{*}$ such that $I_{\sigma(\nu)}^{*} = I_{\sigma}^{*} \setminus \{\nu\}$ and hence, $\bigcup_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$. The relation $\bigcup_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \notin \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is therefore a contradiction. The proof of the proposition is complete. Q.E.D.

THEOREM 3.39. *Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. If $\mathscr{S}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$, then it has also $\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$:*

$$(\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}})\left[\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}] \longleftarrow \mathscr{S}_{\mathfrak{g}} \in \text{Nd}[\mathfrak{T}_{\mathfrak{g}}]\right]. \tag{3.23}$$

PROOF. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose $\mathscr{S}_{\mathfrak{g}} \in \text{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is an untrue logical statement. Then, $\mathscr{S}_{\mathfrak{g}} \in \text{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is true and $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset$ is untrue. Thus, to prove the theorem, it suffices to prove that $\mathscr{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is a contradiction. Since $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it follows that $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Consequently,

$$\text{int}_{\mathfrak{g}}\left[\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right] \subseteq \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).$$

Since $\mathscr{S}_{\mathfrak{g}} \in \text{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, it follows that $\text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset$ and therefore, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \emptyset$. Since $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it results that

$$\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \emptyset,$$

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implying $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset$. Hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}\,[\mathfrak{T}_{\mathfrak{g}}]$. The relation $\mathscr{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-Nd}\,[\mathfrak{T}_{\mathfrak{g}}]$ is therefore a contradiction. The proof of the theorem is complete. Q.E.D.

The important remark given below ends the present section.

REMARK 3.40. In a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, the converse of the following statements with respect to some $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ are in general untrue:

- I. $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}\,(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]$,
- II. $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\,(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]$,
- III. $\big(\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}\,[\mathfrak{T}_{\mathfrak{g}}]\big) \vee \big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}\,(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}\,[\mathfrak{T}_{\mathfrak{g}}]\big) \longrightarrow \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]$.

Because, in the event that $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}}) = \big(\mathbb{R}, \mathscr{T}_{\mathfrak{g},\mathbb{R}}\big) = \mathfrak{T}_{\mathfrak{g},\mathbb{R}}$ and $\mathscr{S}_{\mathfrak{g}} = \mathbb{Q}$ ($\mathbb{Q}$ and $\mathbb{R}$, respectively, denote the sets of rational and real numbers, where $\mathbb{R} \supset \mathbb{Q}$), the converse of ITEMS I.,II. and III., reading

- IV. $\mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big] \longleftarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}\,(\mathbb{Q}) \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$,
- V. $\mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big] \longleftarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\,(\mathbb{Q}) \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$,
- VI. $\big(\mathbb{Q} \in \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]\big) \vee \big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}\,(\mathbb{Q}) \in \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]\big) \longleftarrow \mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$,

respectively, are all untrue. In fact, every $\mathscr{T}_{\mathfrak{g}}$-open set $\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\mathbb{R}}$ contains both points $\xi \in \mathbb{Q}$ and $\zeta \in \mathbb{R} \setminus \mathbb{Q}$. Consequently, there are no $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior points of $\mathbb{Q}$. Therefore, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\,(\mathbb{Q}) = \emptyset$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\,(\mathbb{Q}) = \mathbb{R}$ and thus, $\mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big] \ni \mathbb{R} = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\,(\mathbb{R}) = \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\,(\mathbb{Q}) \neq \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\,(\mathbb{Q}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\,(\emptyset) = \emptyset \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$; $\big(\mathbb{Q}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}\,(\mathbb{Q})\big) \notin \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big] \times \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$. In ITEMS IV., V. and VI., the consequents $\mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$, $\mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$ and $\big(\mathbb{Q} \in \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]\big) \vee \big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}\,(\mathbb{Q}) \in \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]\big)$ are all untrue and on the other hand, their antecedents $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\,(\mathbb{Q}) \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\,(\mathbb{Q}) \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$ and $\mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$ are all true. Consequently, ITEMS IV., V. and VI. are all untrue statements and hence, the converse of ITEMS I., II. and III. are untrue statements. In addition, since $\big(\mathbb{Q}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}\,(\mathbb{Q})\big) \notin \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big] \times \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$ it follows that, for some $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, the condition $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\,(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}\,[\mathfrak{T}_{\mathfrak{g}}]$ can be satisfied without the condition $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}\,[\mathfrak{T}_{\mathfrak{g}}]$ being satisfied, though $\mathscr{O}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\,(\mathscr{S}_{\mathfrak{g}}) \neq \emptyset$ for every $\mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}\,[\mathfrak{T}_{\mathfrak{g}}]$ is a consequence of $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}\,[\mathfrak{T}_{\mathfrak{g}}]$.

The categorical classifications of $\mathfrak{g}$-$\mathfrak{T}$-interior and $\mathfrak{g}$-$\mathfrak{T}$-closure operators, $\mathfrak{T}$-sets having $\mathfrak{g}$-$\mathfrak{P}$-property and $\mathfrak{T}$-sets having $\mathfrak{g}$-$\mathfrak{Q}$-property in the $\mathscr{T}$-space $\mathfrak{T}$ and, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators, $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property and $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ are discussed and diagrammed on this basis in the next sections.

4. DISCUSSION

4.1. CATEGORICAL CLASSIFICATIONS. Having adopted a categorical approach in the classifications of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators, $\mathfrak{T}_{\mathfrak{g}}$-sets with $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property, and $\mathfrak{T}_{\mathfrak{g}}$-sets with $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property, the twofold purposes here are to establish the various relationships between the classes of $\mathfrak{g}$-$\mathfrak{T}$-interior operators in the $\mathscr{T}$-space $\mathfrak{T}$ and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, the classes of $\mathfrak{g}$-$\mathfrak{T}$-closure operators in the $\mathscr{T}$-space $\mathfrak{T}$ and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, the classes of $\mathfrak{T}$-sets with $\mathfrak{g}$-$\mathfrak{P}$-property in the $\mathscr{T}$-space $\mathfrak{T}$ and $\mathfrak{T}_{\mathfrak{g}}$-sets with $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ and, the classes of $\mathfrak{T}$-sets with $\mathfrak{g}$-$\mathfrak{Q}$-property in the

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$\mathscr{T}$-space $\mathfrak{T}$ and $\mathfrak{T}_{\mathfrak{g}}$-sets with $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, and to illustrate them through diagrams.

In a $\mathscr{T}$-space $\mathfrak{T}$, for every $\mathscr{O}_{\mathfrak{g}} \in \mathrm{O}\big[\mathfrak{T}\big]$, the relation $\mathrm{op}_0(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathrm{op}_1(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathrm{op}_3(\mathscr{O}_{\mathfrak{g}}) \supseteq \mathrm{op}_2(\mathscr{O}_{\mathfrak{g}})$ holds implying, for any $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{T}$, $\mathfrak{g}\text{-}\mathrm{Int}_0(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Int}_1(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Int}_3(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Int}_2(\mathscr{S}_{\mathfrak{g}})$. Likewise, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, for every $\mathscr{O}_{\mathfrak{g}} \in \mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, the relation $\mathrm{op}_{\mathfrak{g},0}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},1}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},3}(\mathscr{O}_{\mathfrak{g}}) \supseteq \mathrm{op}_{\mathfrak{g},2}(\mathscr{O}_{\mathfrak{g}})$ holds implying, for any $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}})$. But, for every $\nu \in I_3^0$, it results that $\mathscr{O}_{\mathfrak{g}} \subseteq \mathrm{op}_{\nu}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}})$ implying, for any $(\nu, \mathscr{S}_{\mathfrak{g}}) \in I_3^0 \times \mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Int}_{\nu}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$. Consequently, this diagram, which is to be read horizontally, from left to right and vertically, from top to bottom, follows:

$$
\begin{array}{ccccccc}
\mathscr{O}_{\mathfrak{g}} & = & \mathscr{O}_{\mathfrak{g}} & = & \mathscr{O}_{\mathfrak{g}} & = & \mathscr{O}_{\mathfrak{g}} \\
\cap\!\!| & & \cap\!\!| & & \cap\!\!| & & \cap\!\!| \\
\mathrm{op}_0(\mathscr{O}_{\mathfrak{g}}) & \subseteq & \mathrm{op}_1(\mathscr{O}_{\mathfrak{g}}) & \subseteq & \mathrm{op}_3(\mathscr{O}_{\mathfrak{g}}) & \supseteq & \mathrm{op}_2(\mathscr{O}_{\mathfrak{g}}) \\
\cap\!\!| & & \cap\!\!| & & \cap\!\!| & & \cap\!\!| \\
\mathrm{op}_{\mathfrak{g},0}(\mathscr{O}_{\mathfrak{g}}) & \subseteq & \mathrm{op}_{\mathfrak{g},1}(\mathscr{O}_{\mathfrak{g}}) & \subseteq & \mathrm{op}_{\mathfrak{g},3}(\mathscr{O}_{\mathfrak{g}}) & \supseteq & \mathrm{op}_{\mathfrak{g},2}(\mathscr{O}_{\mathfrak{g}}).
\end{array}
\tag{4.1}
$$

In FIG. 1, we present the relationships between the elements of the collections $\big\{\mathfrak{g}\text{-}\mathrm{Int}_{\nu}(\mathscr{S}_{\mathfrak{g}}) : \ \nu \in I_3^0\big\}$ in the $\mathscr{T}$-space $\mathfrak{T}$ and $\big\{\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) : \ \nu \in I_3^0\big\}$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$; FIG. 1 may well be called a $\big(\mathfrak{g}\text{-}\mathrm{Int}, \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big)$*-valued diagram.*

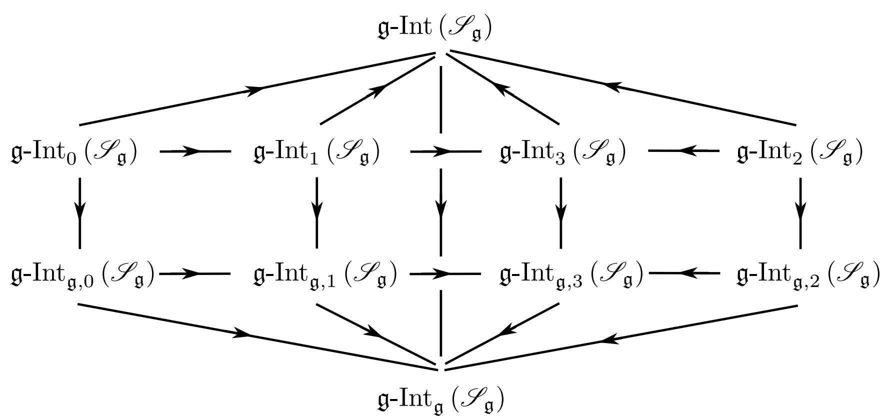

FIGURE 1. Relationships: $\mathfrak{g}$-$\mathfrak{T}$-interior operators in $\mathscr{T}$-spaces and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

On the other hand, for every $\mathscr{K}_{\mathfrak{g}} \in \mathrm{K}\big[\mathfrak{T}\big]$, the relation $\neg\,\mathrm{op}_0(\mathscr{K}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_1(\mathscr{K}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_3(\mathscr{K}_{\mathfrak{g}}) \subseteq \mathrm{op}_2(\mathscr{K}_{\mathfrak{g}})$ holds implying, for any $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{T}$, $\mathfrak{g}\text{-}\mathrm{Cl}_0(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cl}_1(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cl}_3(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Cl}_2(\mathscr{S}_{\mathfrak{g}})$. Likewise, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, for every $\mathscr{K}_{\mathfrak{g}} \in \mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, the relation $\neg\,\mathrm{op}_{\mathfrak{g},0}(\mathscr{K}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g},1}(\mathscr{K}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g},3}(\mathscr{K}_{\mathfrak{g}}) \subseteq \neg\,\mathrm{op}_{\mathfrak{g},2}(\mathscr{K}_{\mathfrak{g}})$ holds implying, for any $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}})$. But, for every $\nu \in I_3^0$, it results that $\mathscr{K}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\nu}(\mathscr{K}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g}})$ implying, for any $(\nu, \mathscr{S}_{\mathfrak{g}}) \in I_3^0 \times \mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\nu}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$. Consequently, this diagram, which is to be read horizontally, from left to right and vertically, from top to bottom,

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follows:

$$
\begin{array}{ccccccc}
\mathscr{K}_{\mathfrak{g}} & = & \mathscr{K}_{\mathfrak{g}} & = & \mathscr{K}_{\mathfrak{g}} & = & \mathscr{K}_{\mathfrak{g}} \\
\text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} \\
\neg\,\mathrm{op}_0(\mathscr{K}_{\mathfrak{g}}) & \supseteq & \neg\,\mathrm{op}_1(\mathscr{K}_{\mathfrak{g}}) & \supseteq & \neg\,\mathrm{op}_3(\mathscr{K}_{\mathfrak{g}}) & \subseteq & \neg\,\mathrm{op}_2(\mathscr{K}_{\mathfrak{g}}) \\
\text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} \\
\neg\,\mathrm{op}_{\mathfrak{g},0}(\mathscr{K}_{\mathfrak{g}}) & \supseteq & \neg\,\mathrm{op}_{\mathfrak{g},1}(\mathscr{K}_{\mathfrak{g}}) & \supseteq & \neg\,\mathrm{op}_{\mathfrak{g},3}(\mathscr{K}_{\mathfrak{g}}) & \subseteq & \neg\,\mathrm{op}_{\mathfrak{g},2}(\mathscr{K}_{\mathfrak{g}}).
\end{array}
\tag{4.2}
$$

In FIG. 2, we present the relationships between the elements of the collections $\{\mathfrak{g}\text{-}\mathrm{Cl}_{\nu}(\mathscr{S}_{\mathfrak{g}}) : \nu \in I_3^0\}$ in the $\mathscr{T}$-space $\mathfrak{T}$ and $\{\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) : \nu \in I_3^0\}$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$; FIG. 2 may well be called a $(\mathfrak{g}\text{-}\mathrm{Cl}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}})$-*valued diagram.*

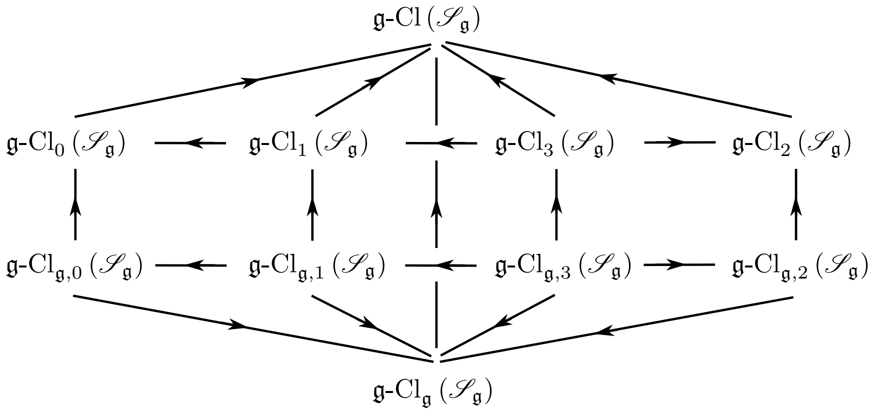

FIGURE 2. Relationships: $\mathfrak{g}$-$\mathfrak{T}$-closure operators in $\mathscr{T}$-spaces and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

Since $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\bigvee_{\nu\in I_3^0}(\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}])$, it follows that, $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}} \longleftarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{P}_{\mathfrak{g}}$ in $\mathfrak{T}_{\mathfrak{g}}$ for every $\nu \in I_3^0$; likewise, $\mathfrak{g}\text{-}\mathfrak{P} \longleftarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{P}$ in $\mathfrak{T}$ for every $\nu \in I_3^0$, since $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}]$ implies $\bigvee_{\nu\in I_3^0}(\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{P}[\mathfrak{T}])$. Therefore, $\mathfrak{g}\text{-}0\text{-}\mathfrak{P}_{\mathfrak{g}} \longrightarrow \mathfrak{g}\text{-}1\text{-}\mathfrak{P}_{\mathfrak{g}} \longrightarrow \mathfrak{g}\text{-}3\text{-}\mathfrak{P}_{\mathfrak{g}} \longleftarrow \mathfrak{g}\text{-}2\text{-}\mathfrak{P}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-}0\text{-}\mathfrak{P} \longrightarrow \mathfrak{g}\text{-}1\text{-}\mathfrak{P} \longrightarrow \mathfrak{g}\text{-}3\text{-}\mathfrak{P} \longleftarrow \mathfrak{g}\text{-}2\text{-}\mathfrak{P}$. Finally, $\mathfrak{g}\text{-}\mathfrak{P} \longleftarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{P} \longrightarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{P}_{\mathfrak{g}} \longrightarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{P}_{\mathfrak{g}}$ for every $\nu \in I_3^0$. Altogether, EQ. (4.3) present itself which may well be called $(\mathfrak{g}\text{-}\mathfrak{P}, \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}})$-*properties diagram.*

$$
\begin{array}{ccccccc}
\mathfrak{g}\text{-}\mathfrak{P} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathfrak{g}\text{-}0\text{-}\mathfrak{P} & \longrightarrow & \mathfrak{g}\text{-}1\text{-}\mathfrak{P} & \longrightarrow & \mathfrak{g}\text{-}3\text{-}\mathfrak{P} & \longleftarrow & \mathfrak{g}\text{-}2\text{-}\mathfrak{P} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}\text{-}0\text{-}\mathfrak{P}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-}1\text{-}\mathfrak{P}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-}3\text{-}\mathfrak{P}_{\mathfrak{g}} & \longleftarrow & \mathfrak{g}\text{-}2\text{-}\mathfrak{P}_{\mathfrak{g}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}
\end{array}
\tag{4.3}
$$

In terms of the classes of the collections $\{\mathfrak{g}\text{-}\nu\text{-}\mathrm{P}[\mathfrak{T}] : \nu \in I_3^*\}$ and $\{\mathfrak{g}\text{-}\nu\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}] : \nu \in I_3^*\}$, FIG. 3 present itself which may well be called $(\mathfrak{g}\text{-}\mathfrak{P}, \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}})$-*classes diagram.*

Since $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\bigvee_{\nu\in I_3^0}(\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}])$, it follows that, $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}} \longleftarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{Q}_{\mathfrak{g}}$ in $\mathfrak{T}_{\mathfrak{g}}$ for every $\nu \in I_3^0$; likewise, $\mathfrak{g}\text{-}\mathfrak{Q} \longleftarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{Q}$ in $\mathfrak{T}$ for every $\nu \in I_3^0$, since

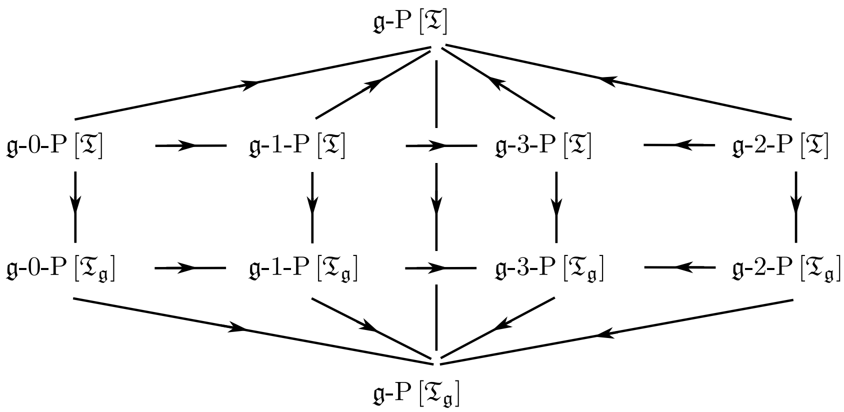

FIGURE 3. Relationships: $(\mathfrak{g}\text{-}\mathfrak{P}, \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}})$-*classes diagram* in $\mathscr{T}_{\mathfrak{g}}$-spaces.

$\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}]$ implies $\bigvee_{\nu \in I_3^0}(\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}])$. Therefore, $\mathfrak{g}\text{-0-}\mathfrak{Q}_{\mathfrak{g}} \longrightarrow \mathfrak{g}\text{-1-}\mathfrak{Q}_{\mathfrak{g}} \longrightarrow \mathfrak{g}\text{-3-}\mathfrak{Q}_{\mathfrak{g}} \longleftarrow \mathfrak{g}\text{-2-}\mathfrak{Q}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-0-}\mathfrak{Q} \longrightarrow \mathfrak{g}\text{-1-}\mathfrak{Q} \longrightarrow \mathfrak{g}\text{-3-}\mathfrak{Q} \longleftarrow \mathfrak{g}\text{-2-}\mathfrak{Q}$. Finally, $\mathfrak{g}\text{-}\mathfrak{Q} \longleftarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{Q} \longrightarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{Q}_{\mathfrak{g}} \longrightarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{Q}_{\mathfrak{g}}$ for every $\nu \in I_3^0$. Altogether, EQ. (4.4) present itself which may well be called $(\mathfrak{g}\text{-}\mathfrak{Q}, \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}})$-*properties diagram*.

$$
\begin{array}{ccccccc}
\mathfrak{g}\text{-}\mathfrak{Q} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathfrak{g}\text{-0-}\mathfrak{Q} & \longrightarrow & \mathfrak{g}\text{-1-}\mathfrak{Q} & \longrightarrow & \mathfrak{g}\text{-3-}\mathfrak{Q} & \longleftarrow & \mathfrak{g}\text{-2-}\mathfrak{Q} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}\text{-0-}\mathfrak{Q}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-1-}\mathfrak{Q}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-3-}\mathfrak{Q}_{\mathfrak{g}} & \longleftarrow & \mathfrak{g}\text{-2-}\mathfrak{Q}_{\mathfrak{g}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}
\end{array}
\tag{4.4}
$$

In terms of the classes of the collections $\{\mathfrak{g}\text{-}\nu\text{-Nd}[\mathfrak{T}] : \nu \in I_3^*\}$ and $\{\mathfrak{g}\text{-}\nu\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}] : \nu \in I_3^*\}$, FIG. 4 present itself which may well be called $(\mathfrak{g}\text{-}\mathfrak{Q}, \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}})$-*classes diagram*.

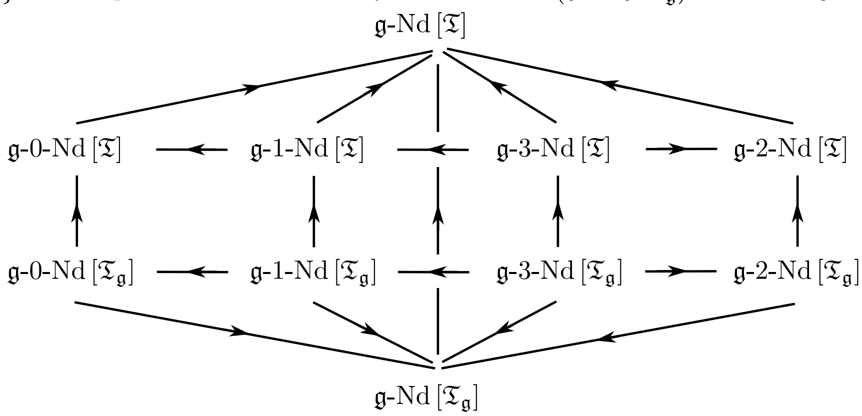

FIGURE 4. Relationships: $(\mathfrak{g}\text{-}\mathfrak{Q}, \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}})$-*classes diagram* in $\mathscr{T}_{\mathfrak{g}}$-spaces.

Since $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathscr{S}_{\mathfrak{g}} \in \text{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ imply $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$, $\mathscr{S}_{\mathfrak{g}} \in \text{P}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$, respectively, in $\mathfrak{T}_{\mathfrak{g}}$, it follows that $\mathfrak{Q}_{\mathfrak{g}} \longrightarrow$

$\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}} \longrightarrow \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}} \longrightarrow \mathfrak{P}_{\mathfrak{g}}$ in $\mathfrak{T}_{\mathfrak{g}}$; likewise, $\mathfrak{Q} \longrightarrow \mathfrak{g}\text{-}\mathfrak{Q} \longrightarrow \mathfrak{g}\text{-}\mathfrak{P} \longrightarrow \mathfrak{P}$ in $\mathfrak{T}$, since $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}]$, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}]$ and $\mathscr{S}_{\mathfrak{g}} \in \mathrm{Nd}[\mathfrak{T}]$ imply $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}]$, $\mathscr{S}_{\mathfrak{g}} \in \mathrm{P}[\mathfrak{T}]$ and $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}]$, respectively, in $\mathfrak{T}$. Finally, $\mathscr{S}_{\mathfrak{g}} \in \mathrm{Nd}[\mathfrak{T}]$ and $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}]$ imply $\mathscr{S}_{\mathfrak{g}} \in \mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$, respectively, and, $\mathscr{S}_{\mathfrak{g}} \in \mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]$ imply $\mathscr{S}_{\mathfrak{g}} \in \mathrm{P}[\mathfrak{T}]$ and $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}]$, respectively. Altogether, EQ. (4.5) present itself which may well be called $(\mathfrak{P}, \mathfrak{g}\text{-}\mathfrak{P}; \mathfrak{Q}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathfrak{Q})$*-properties diagram.*

$$\begin{array}{ccccccc} \mathfrak{Q} & \longrightarrow & \mathfrak{g}\text{-}\mathfrak{Q} & \longrightarrow & \mathfrak{g}\text{-}\mathfrak{P} & \longrightarrow & \mathfrak{P} \\ \downarrow & & \downarrow & & \uparrow & & \uparrow \\ \mathfrak{Q}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}} & \longrightarrow & \mathfrak{P}_{\mathfrak{g}} \end{array} \tag{4.5}$$

In terms of the classes of the collection $\{\mathrm{Nd}[\mathfrak{T}], \mathrm{P}[\mathfrak{T}], \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}], \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}]\}$ and the classes of the collection $\{\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}], \mathrm{P}[\mathfrak{T}_{\mathfrak{g}}], \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}], \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]\}$, FIG. 5 present itself which may well be called $(\mathfrak{P}, \mathfrak{g}\text{-}\mathfrak{P}; \mathfrak{Q}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathfrak{Q})$*-classes diagram.*

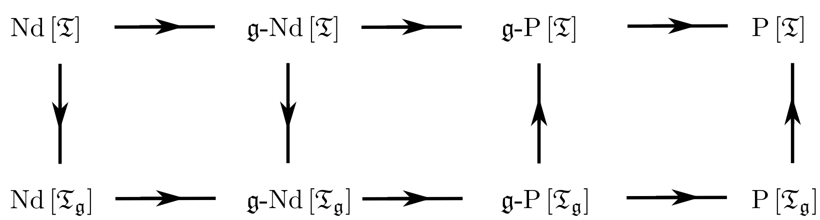

FIGURE 5. Relationships: $(\mathfrak{P}, \mathfrak{g}\text{-}\mathfrak{P}; \mathfrak{Q}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathfrak{Q})$*-classes diagram* in $\mathscr{T}_{\mathfrak{g}}$-spaces.

As in the works of other authors [CJS05, Don97, JJLL08, TC16], the manner we have positioned the arrows in the $(\mathfrak{g}\text{-}\mathrm{Int}, \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}})$, $(\mathfrak{g}\text{-}\mathrm{Cl}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}})$*-valued diagrams* (FIGS 1, 2), the $(\mathfrak{g}\text{-}\mathfrak{P}, \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}})$, $(\mathfrak{g}\text{-}\mathfrak{Q}, \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}})$, $(\mathfrak{P}, \mathfrak{g}\text{-}\mathfrak{P}; \mathfrak{Q}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathfrak{Q})$*-classes diagrams* (FIGS 3, 4, 5), and the $(\mathfrak{g}\text{-}\mathfrak{P}, \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}})$, $(\mathfrak{g}\text{-}\mathfrak{Q}, \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}})$, $(\mathfrak{P}, \mathfrak{g}\text{-}\mathfrak{P}; \mathfrak{Q}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathfrak{Q})$*-property diagrams* (EQS (4.3), (4.4), (4.5)) is solely to stress that, in general, the implications in FIGS 1–5 and EQS (4.3)–(4.5) are irreversible.

At this stage, a nice application is worth considering, and is presented in the following section.

4.2. A NICE APPLICATION. Focusing on essential concepts from the standpoint of the theory of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closure operators in an attempt to shed lights on the essential properties established in the earlier sections, we shall now present a nice application. Let $\Omega = \{\xi_{\nu} : \nu \in I_5^*\}$ denotes the underlying set and consider the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, where Ω is topologized by the choice:

$$\begin{aligned} \mathscr{T}_{\mathfrak{g}}(\Omega) &= \{\emptyset,\ \{\xi_1\},\ \{\xi_1, \xi_3, \xi_5\},\ \Omega\} \\ &= \{\mathscr{O}_{\mathfrak{g},1},\ \mathscr{O}_{\mathfrak{g},2},\ \mathscr{O}_{\mathfrak{g},3},\ \mathscr{O}_{\mathfrak{g},4}\}; \end{aligned} \tag{4.6}$$

$$\begin{aligned} \neg\mathscr{T}_{\mathfrak{g}}(\Omega) &= \{\Omega,\ \{\xi_2, \xi_3, \xi_4, \xi_5\},\ \{\xi_2, \xi_4\}, \emptyset\} \\ &= \{\mathscr{K}_{\mathfrak{g},1},\ \mathscr{K}_{\mathfrak{g},2},\ \mathscr{K}_{\mathfrak{g},3},\ \mathscr{K}_{\mathfrak{g},4}\}. \end{aligned} \tag{4.7}$$

Evidently, the set-valued set maps $\mathscr{T}_{\mathfrak{g}}, \neg\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\{\xi_{\nu} : \nu \in I_5^*\})$ establish the classes of $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets, respectively. Since conditions $\mathscr{T}_{\mathfrak{g}}(\emptyset) = \emptyset$, $\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) \subseteq \mathscr{O}_{\mathfrak{g},\nu}$ for every $\nu \in I_4^*$, $\mathscr{T}_{\mathfrak{g}}(\Omega) = \Omega$, and $\mathscr{T}_{\mathfrak{g}}\big(\bigcup_{\nu \in I_4^*} \mathscr{O}_{\mathfrak{g},\nu}\big) =$

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$\bigcup_{\nu\in I_4^*}\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ are satisfied, it is clear that the one-valued map $\mathscr{T}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}\big(\{\xi_\nu:\ \nu\in I_5^*\}\big)$ is a strong $\mathfrak{g}$-topology and hence, $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space. On the other hand, because the additional condition $\mathscr{T}_{\mathfrak{g}}\big(\bigcap_{\nu\in I_4^*}\mathscr{O}_{\mathfrak{g},\nu}\big)=\bigcap_{\nu\in I_4^*}\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ is satisfied, $\mathscr{T}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}\big(\{\xi_\nu:\ \nu\in I_5^*\}\big)$ is also a topology and thus, $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ is a $\mathscr{T}$-space $\mathfrak{T}=(\Omega,\mathscr{T})$. Moreover, it is easily checked that $\mathscr{O}_{\mathfrak{g},\mu}\in\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}\big]$ for every $(\nu,\mu)\in I_3^0\times I_4^*$. Thus, the $\mathscr{T}_{\mathfrak{g}}$-open sets forming the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}\big(\{\xi_\nu:\ \nu\in I_5^*\}\big)$ of the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ are $\mathfrak{g}$-$\mathfrak{T}$-open sets relative to the $\mathscr{T}$-space $\mathfrak{T}=(\Omega,\mathscr{T})$.

Clearly, the cardinality $\mathrm{card}\big(\mathscr{P}(\Omega)\big)=2^{\mathrm{card}(\Omega)}$ is very large. For convenience of notation, express $\mathscr{P}(\Omega)$ in set-builder notation as a collection indexed by the Cartesian product $I^*_{\mathrm{card}(\mathscr{P}(\Omega))}\times I^0_{\mathrm{card}(\Omega)}$:

$$(4.8)\qquad \mathscr{P}(\Omega)\;=\;\big\{\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\in\mathscr{P}(\Omega):\ (\nu,\mu)\in I^*_{\mathrm{card}(\mathscr{P}(\Omega))}\times I^0_{\mathrm{card}(\Omega)}\big\},$$

where $\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\in\mathscr{P}(\Omega)$ denotes a $\mathfrak{T}_{\mathfrak{g}}$-set labeled $\nu\in I^*_{\mathrm{card}(\mathscr{P}(\Omega))}$ and containing $\mu\in I^0_{\mathrm{card}(\Omega)}$ elements. Below is established the indexing by the Cartesian product $I^*_{\mathrm{card}(\mathscr{P}(\Omega))}\times I^0_{\mathrm{card}(\Omega)}$ by the choice: $\mathscr{S}_{\mathfrak{g},(1,0)}=\emptyset$, …, $\mathscr{S}_{\mathfrak{g},(\nu,\mu)}=\{\xi_1,\xi_2,\ldots,\xi_\mu\}$, …, $\mathscr{S}_{\mathfrak{g},(32,5)}=\Omega$.

For $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$ such that $\mathrm{card}(\mathscr{S}_{\mathfrak{g}})\in\{0,5\}$, let $\mathscr{S}_{\mathfrak{g},(1,0)}=\emptyset$ and $\mathscr{S}_{\mathfrak{g},(32,5)}=\Omega$. For $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$ such that $\mathrm{card}(\mathscr{S}_{\mathfrak{g}})\in\{1,4\}$, let $\mathscr{S}_{\mathfrak{g},(2,1)}=\{\xi_1\}$, $\mathscr{S}_{\mathfrak{g},(3,1)}=\{\xi_2\}$, $\mathscr{S}_{\mathfrak{g},(4,1)}=\{\xi_3\}$, $\mathscr{S}_{\mathfrak{g},(5,1)}=\{\xi_4\}$, and $\mathscr{S}_{\mathfrak{g},(6,1)}=\{\xi_5\}$; $\mathscr{S}_{\mathfrak{g},(27,4)}=\{\xi_1,\xi_2,\xi_3,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(28,4)}=\{\xi_2,\xi_3,\xi_4,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(29,4)}=\{\xi_1,\xi_3,\xi_4,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(30,4)}=\{\xi_1,\xi_2,\xi_3,\xi_5\}$, and $\mathscr{S}_{\mathfrak{g},(31,4)}=\{\xi_1,\xi_2,\xi_4,\xi_5\}$. For $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$ such that $\mathrm{card}(\mathscr{S}_{\mathfrak{g}})\in\{2,3\}$, let $\mathscr{S}_{\mathfrak{g},(7,2)}=\{\xi_1,\xi_2\}$, $\mathscr{S}_{\mathfrak{g},(8,2)}=\{\xi_1,\xi_3\}$, $\mathscr{S}_{\mathfrak{g},(9,2)}=\{\xi_1,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(10,2)}=\{\xi_1,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(11,2)}=\{\xi_2,\xi_3\}$, $\mathscr{S}_{\mathfrak{g},(12,2)}=\{\xi_2,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(13,2)}=\{\xi_2,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(14,2)}=\{\xi_3,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(15,2)}=\{\xi_3,\xi_5\}$, and $\mathscr{S}_{\mathfrak{g},(16,2)}=\{\xi_4,\xi_5\}$; $\mathscr{S}_{\mathfrak{g},(17,3)}=\{\xi_1,\xi_2,\xi_3\}$, $\mathscr{S}_{\mathfrak{g},(18,3)}=\{\xi_1,\xi_3,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(19,3)}=\{\xi_1,\xi_4,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(20,3)}=\{\xi_1,\xi_2,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(21,3)}=\{\xi_1,\xi_2,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(22,3)}=\{\xi_1,\xi_3,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(23,3)}=\{\xi_2,\xi_3,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(24,3)}=\{\xi_2,\xi_3,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(25,3)}=\{\xi_3,\xi_4,\xi_5\}$, and $\mathscr{S}_{\mathfrak{g},(26,3)}=\{\xi_2,\xi_4,\xi_5\}$.

A first series of calculations shows that, for every $(\nu,\mu)\in I^*_{\mathrm{card}(\mathscr{P}(\Omega))}\times I^0_{\mathrm{card}(\Omega)}$,

$$(4.9)\quad \begin{aligned}\mathrm{int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big)\subseteq\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big)&=\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\\&=\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big)\subseteq\mathrm{cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big).\end{aligned}$$

That for every $(\nu,\mu)\in I^*_{\mathrm{card}(\mathscr{P}(\Omega))}\times I^0_{\mathrm{card}(\Omega)}$, the relation

$$(4.10)\quad \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big)=\mathscr{S}_{\mathfrak{g},(\nu,\mu)}=\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big)$$

holds is evidently an immediate consequence of the above relation. Introduce $J_{28}^*=I_1^*\cup(I_7^*\setminus I_2^*)\cup(I_{16}^*\setminus I_{10}^*)\cup(I_{26}^*\setminus I_{22}^*)\cup(I_{28}^*\setminus I_{27}^*)$. Then, a second series of calculations shows that, for every $(\nu,\mu)\in J_{28}^*\times I_4^0$ and every $(\delta,\eta)\in\big(I^*_{\mathrm{card}(\mathscr{P}(\Omega))}\setminus J_{28}^*\big)\times I^0_{\mathrm{card}(\Omega)}$,

$$(4.11)\quad \begin{aligned}\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big)&=\emptyset=\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big);\\ \mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\delta,\eta)}\big)&=\Omega=\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\delta,\eta)}\big).\end{aligned}$$

On inspecting each of Eqs (4.9)–(4.11), some interesting features can be remarked and thus, some interesting conclusions can be drawn.

Having ordered the $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators $\mathrm{int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, respectively, by setting $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\succsim\mathrm{int}_{\mathfrak{g}}$ if and only if $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and

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the $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathrm{cl}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, by setting $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \precsim \mathrm{cl}_{\mathfrak{g}}$ if and only if $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, where $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ is arbitrary, Eq. (4.9), then, is but a result validating the following outstanding facts: $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathrm{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathrm{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathrm{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathrm{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

From Eq. (4.10), it is thus evident that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, do *commute* in which case, it is no error to consider the following interpretation: $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is both *coarser and finer* (or, *smaller and larger*, *weaker and stronger*) than $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. Consequently, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]$ for any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. Furthermore, it is easily checked from Eq. (4.10) that, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]$ is untrue if and only if $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is true and $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]$ is untrue.

From Eq. (4.11), both $\mathscr{S}_{\mathfrak{g},(\nu,\mu)} \in \mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\nu,\mu) \in J^{*}_{28} \times I^{0}_{4}$ and $\mathscr{S}_{\mathfrak{g},(\delta,\eta)} \in \mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\delta,\eta) \in \big(I^{*}_{\mathrm{card}(\mathscr{P}(\Omega))} \setminus J^{*}_{28}\big) \times I^{0}_{\mathrm{card}(\Omega)}$ are easily checked. Moreover, it results from Eqs (4.10), (4.11) that, $\mathscr{S}_{\mathfrak{g},(\nu,\mu)} \in \mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is true and $\mathscr{S}_{\mathfrak{g},(\nu,\mu)} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is untrue for every $(\nu,\mu) \in \big(J^{*}_{28} \setminus I^{*}_{1}\big) \times I^{0}_{4}$. This confirms the statement that, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}] \longleftarrow \mathscr{S}_{\mathfrak{g}} \in \mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is untrue if and only if $\mathscr{S}_{\mathfrak{g}} \in \mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is true and $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is untrue. Observing that, for every $(\nu,\mu) \in J^{*}_{28} \times I^{0}_{4}$ and every $(\delta,\eta) \in \big(I^{*}_{\mathrm{card}(\mathscr{P}(\Omega))} \setminus J^{*}_{28}\big) \times I^{0}_{\mathrm{card}(\Omega)}$, the relations

$$\begin{aligned}
\emptyset = \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big) &\subseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big) \\
&= \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big) \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big) = \emptyset, \\
\mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\delta,\eta)}\big) = \Omega &\supseteq \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\delta,\eta)}\big) \\
&= \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\delta,\eta)}\big) \subseteq \Omega = \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\delta,\eta)}\big),
\end{aligned}$$

respectively, hold, of which the first relation is the dual of the second, and conversely, it follows that the logical statement $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathscr{S}_{\mathfrak{g}} \in \mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]$ is satisfied for any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

If the discussions of this nice application be explore a step further, other interesting conclusions can be drawn. The next section provides concluding remarks and future directions of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators discussed in the preceding sections.

4.3. Concluding Remarks. In this paper, a new theory called *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Interior and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Closure Operators* has been developed. The definitions of the notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces were presented in as general and unified a manner as possible and, the essential properties and the commutativity of such $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators were discussed in such a way as to show that much of the fundamental structure of $\mathscr{T}_{\mathfrak{g}}$-spaces is better considered for $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ than for the $\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathrm{int}_{\mathfrak{g}}$, $\mathrm{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively. If "$\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \succsim \mathrm{int}_{\mathfrak{g}}$" stands for "$\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$" and "$\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \precsim \mathrm{cl}_{\mathfrak{g}}$," for "$\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$," then the outstanding facts are: $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathrm{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathrm{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow$

$\mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\text{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\text{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

In its own rights, the proposed theory has also several advantages. The very first advantage is that the theory offers very nice features for the passage from $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators to $\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{T}_{\mathfrak{g}}$-closure operators, $\mathfrak{g}$-$\mathfrak{T}$-interior and $\mathfrak{g}$-$\mathfrak{T}$-closure operators and $\mathfrak{T}$-interior and $\mathfrak{T}$-closure operators, respectively. Hence, the theory holds equally well when $(\Omega, \mathscr{T}_{\mathfrak{g}}) = (\Omega, \mathscr{T})$ and other features adapted on this ground, in which case it might be called *Theory of* $\mathfrak{g}$-$\mathfrak{T}$-*Interior and* $\mathfrak{g}$-$\mathfrak{T}$-*Closure Operators*.

In a $\mathscr{T}_{\mathfrak{g}}$-space the theoretical framework categorises such pairs of concepts as the pair $\left(\mathfrak{g}\text{-Int}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}), \mathfrak{g}\text{-Cl}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}})\right)$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, the pair $\left(\mathfrak{g}\text{-Int}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}), \mathfrak{g}\text{-Cl}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}})\right)$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-semi-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-semi-closed sets, the pair $\left(\mathfrak{g}\text{-Int}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}}), \mathfrak{g}\text{-Cl}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}})\right)$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-preopen and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-preclosed sets, and the pair $\left(\mathfrak{g}\text{-Int}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}), \mathfrak{g}\text{-Cl}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}})\right)$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-semi-preopen and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-semi-preclosed sets as pairs of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of categories 0, 1, 2, and 3, respectively, and theorises the concepts in a unified way. In a $\mathscr{T}$-space the theoretical framework categorises such pairs of concepts as the pair $\left(\mathfrak{g}\text{-Int}_{0}(\mathscr{S}), \mathfrak{g}\text{-Cl}_{0}(\mathscr{S})\right)$ of $\mathfrak{g}$-$\mathfrak{T}$-open and $\mathfrak{g}$-$\mathfrak{T}$-closed sets, the pair $\left(\mathfrak{g}\text{-Int}_{1}(\mathscr{S}), \mathfrak{g}\text{-Cl}_{1}(\mathscr{S})\right)$ of $\mathfrak{g}$-$\mathfrak{T}$-semi-open and $\mathfrak{g}$-$\mathfrak{T}$-semi-closed sets, the pair $\left(\mathfrak{g}\text{-Int}_{2}(\mathscr{S}), \mathfrak{g}\text{-Cl}_{2}(\mathscr{S})\right)$ of $\mathfrak{g}$-$\mathfrak{T}$-preopen and $\mathfrak{g}$-$\mathfrak{T}$-preclosed sets, and the pair $\left(\mathfrak{g}\text{-Int}_{3}(\mathscr{S}), \mathfrak{g}\text{-Cl}_{3}(\mathscr{S})\right)$ of $\mathfrak{g}$-$\mathfrak{T}$-semi-preopen and $\mathfrak{g}$-$\mathfrak{T}$-semi-preclosed sets as pairs of $\mathfrak{g}$-$\mathfrak{T}$-open and $\mathfrak{g}$-$\mathfrak{T}$-closed sets of categories 0, 1, 2, and 3, respectively, and theorises the concepts in a unified way.

Making the theorization of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators of mixed categories in $\mathscr{T}_{\mathfrak{g}}$-spaces a prime subject of inquiry is an interestingly promising avenue for future research. More precisely, for some pair $(\nu, \mu) \in I_3^0 \times I_3^0$ such that $\nu \neq \mu$, to develop the theory of $\mathfrak{g}$-(ν,μ)-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-(ν,μ)-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathfrak{g}\text{-Int}_{\mathfrak{g},\nu\mu}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu\mu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ respectively, in $\mathscr{T}_{\mathfrak{g}}$-spaces, where $\mathfrak{g}\text{-Int}_{\mathfrak{g},\nu\mu} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g},\nu\mu}(\mathscr{S}_{\mathfrak{g}})$ describes a type of collection of points interior in $\mathscr{S}_{\mathfrak{g}}$ and interiorness are characterized by $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets belonging to the class $\left\{\mathscr{O}_{\mathfrak{g}} = \mathscr{O}_{\mathfrak{g},\nu} \cup \mathscr{O}_{\mathfrak{g},\mu} :\ (\mathscr{O}_{\mathfrak{g},\nu}, \mathscr{O}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\mu\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right\}$; $\mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu\mu} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu\mu}(\mathscr{S}_{\mathfrak{g}})$ describes a type of collection of points close to $\mathscr{S}_{\mathfrak{g}}$ and closeness are characterized by $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets belonging to the class $\left\{\mathscr{K}_{\mathfrak{g}} = \mathscr{K}_{\mathfrak{g},\nu} \cap \mathscr{K}_{\mathfrak{g},\mu} :\ (\mathscr{K}_{\mathfrak{g},\nu}, \mathscr{K}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\mu\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right\}$. Such an interestingly promising theory is what the present authors thought would certainly be worth considering, and the discussion of this paper ends here.

Appendix A. Pre-preliminaries

In this pre-preliminaries section, the elements accompanying the foregoing preliminary section are given below. In actual fact, they are the elements extracted from the preliminaries section of two previous works of the authors entitled *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Sets* and *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Connectedness*. As in all the previous works of the authors (See, *Theories of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Sets*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Maps*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Connectedness*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Separation Axioms*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Compactness*), $\mathfrak{U}$ is the *universe* of discourse, fixed within the framework of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators and containing as elements all sets (Ω, Γ-sets; $\mathscr{T}$, $\mathfrak{g}$-$\mathscr{T}$, $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-sets; $\mathscr{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$, $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets) considered in this theory, and $I_n^0 \stackrel{\text{def}}{=} \left\{\nu \in \mathbb{N}^0 :\ \nu \leq n\right\}$; index sets I_∞^0, I_n^*, I_∞^* are

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defined similarly. A set $\Gamma \subset \mathfrak{U}$ is a subset of the set $\Omega \subset \mathfrak{U}$ and, for some $\mathscr{T}_{\mathfrak{g}}$-open set $\mathscr{O}_{\mathfrak{g}} \in \mathscr{T} \cup \mathfrak{g}\text{-}\mathscr{T} \cup \mathscr{T}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}$, these implications hold:

$$\mathscr{O}_{\mathfrak{g}} \in \mathscr{T} \Rightarrow \mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathscr{T} \Rightarrow \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}} \Rightarrow \mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}} \Rightarrow \mathscr{O}_{\mathfrak{g}} \subset \Omega \subset \mathfrak{U}. \tag{A.1}$$

In a natural way, a monotonic map $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ from the power set $\mathscr{P}(\Omega)$ of Ω into itself can be associated to a given mapping $\pi_{\mathfrak{g}} : \Omega \longrightarrow \Omega$, thereby inducing a $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}} \subset \mathscr{P}(\Omega)$ on the underlying set $\Omega \subset \mathfrak{U}$ [PC12]. When some further axioms [LR15] is specified for $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ with no separation axioms assumed unless otherwise stated, the notion of a $\mathscr{T}_{\mathfrak{g}}$-space follows.

Definition A.1 ($\mathscr{T}_{\mathfrak{g}}$-Space). Let $\Omega \subset \mathfrak{U}$ be a given set and let $\mathscr{P}(\Omega) \overset{\text{def}}{=} \{\mathscr{O}_{\mathfrak{g},\nu} \subseteq \Omega : \nu \in I_{\infty}^{*}\}$ be the family of all subsets $\mathscr{O}_{\mathfrak{g},1}, \mathscr{O}_{\mathfrak{g},2}, \ldots,$ of Ω. Then every one-valued map of the type $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ satisfying the following axioms:

- Ax. I. $\mathscr{T}_{\mathfrak{g}}(\emptyset) = \emptyset$,
- Ax. II. $\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathscr{O}_{\mathfrak{g}}$,
- Ax. III. $\mathscr{T}_{\mathfrak{g}}\big(\bigcup_{\nu \in I_{\infty}^{*}} \mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu \in I_{\infty}^{*}} \mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$,

is called a "$\mathfrak{g}$-topology on Ω," and the structure $\mathfrak{T}_{\mathfrak{g}} \overset{\text{def}}{=} (\Omega, \mathscr{T}_{\mathfrak{g}})$ is called a "$\mathscr{T}_{\mathfrak{g}}$-space."

In Def. A.1, by Ax. I., Ax. II. and Ax. III., respectively, are meant that the unary operation $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \to \mathscr{P}(\Omega)$ preserves nullary union, is contracting and preserves binary union. Any element $\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}} \overset{\text{def}}{=} \{\mathscr{O}_{\mathfrak{g}} : \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}}\}$ of the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ is called a $\mathscr{T}_{\mathfrak{g}}$-open set and its complement element $\complement_{\Omega}(\mathscr{O}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g}} \in \neg\mathscr{T}_{\mathfrak{g}} \overset{\text{def}}{=} \{\mathscr{K}_{\mathfrak{g}} : \complement(\mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g}}\}$, a $\mathscr{T}_{\mathfrak{g}}$-closed set; by convention, $\mathscr{T}_{\mathfrak{g}}$ and $\neg\mathscr{T}_{\mathfrak{g}}$, respectively, stand for the classes of all $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets relative to the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}}$. If there exists a $\nu \in I_{\infty}^{*}$ such that $\mathscr{O}_{\mathfrak{g},\nu} = \Omega$, then $\mathfrak{T}_{\mathfrak{g}}$ is called a strong $\mathscr{T}_{\mathfrak{g}}$-space [Cs5, PC12]. Moreover, if $\mathscr{T}_{\mathfrak{g}}\big(\bigcap_{\nu \in I_{n}^{*}} \mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcap_{\nu \in I_{n}^{*}} \mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ holds for any index set $I_{n}^{*} \subset I_{\infty}^{*}$ such that $n < \infty$, then $\mathfrak{T}_{\mathfrak{g}}$ is called a quasi $\mathscr{T}_{\mathfrak{g}}$-space [Cs8].

In the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, the operator $\operatorname{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ carrying each $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ into its interior $\operatorname{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \Omega - \operatorname{cl}_{\mathfrak{g}}(\Omega \setminus \mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}}$ is called a "$\mathfrak{T}_{\mathfrak{g}}$-interior operator;" the operator $\operatorname{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ carrying each $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ into its closure $\operatorname{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \Omega - \operatorname{int}_{\mathfrak{g}}(\Omega \setminus \mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}}$ is called a "$\mathfrak{T}_{\mathfrak{g}}$-closure operator." The classes $\mathrm{C}_{\mathscr{T}_{\mathfrak{g}}}^{\mathrm{sub}}[\mathscr{S}_{\mathfrak{g}}] \overset{\text{def}}{=} \{\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}} : \mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}\}$ and $\mathrm{C}_{\neg\mathscr{T}_{\mathfrak{g}}}^{\mathrm{sup}}[\mathscr{S}_{\mathfrak{g}}] \overset{\text{def}}{=} \{\mathscr{K}_{\mathfrak{g}} \in \neg\mathscr{T}_{\mathfrak{g}} : \mathscr{K}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g}}\}$, respectively, denote the classes of $\mathscr{T}_{\mathfrak{g}}$-open subsets and $\mathscr{T}_{\mathfrak{g}}$-closed supersets of the $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ relative to the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}}$. That $\mathrm{C}_{\mathscr{T}_{\mathfrak{g}}}^{\mathrm{sub}}[\mathscr{S}_{\mathfrak{g}}] \subseteq \mathscr{T}_{\mathfrak{g}}(\Omega)$ and $\neg\mathscr{T}_{\mathfrak{g}}(\Omega) \supseteq \mathrm{C}_{\neg\mathscr{T}_{\mathfrak{g}}}^{\mathrm{sup}}[\mathscr{S}_{\mathfrak{g}}]$ are true for the $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in question are clear from the context. To this end, the $\mathfrak{T}_{\mathfrak{g}}$-closure and the $\mathfrak{T}_{\mathfrak{g}}$-interior of a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space define themselves as

$$\operatorname{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{=} \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}_{\mathscr{T}_{\mathfrak{g}}}^{\mathrm{sub}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \quad \operatorname{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{=} \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}_{\neg\mathscr{T}_{\mathfrak{g}}}^{\mathrm{sup}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}}. \tag{A.2}$$

We note in passing that, $\operatorname{cl}_{\mathfrak{g}}(\cdot) \neq \operatorname{cl}(\cdot)$ and $\operatorname{int}_{\mathfrak{g}}(\cdot) \neq \operatorname{int}(\cdot)$, because the resulting sets obtained from the intersection of all $\mathscr{T}_{\mathfrak{g}}$-closed supersets and the union of all $\mathscr{T}_{\mathfrak{g}}$-open subsets, respectively, relative to the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}}$ are not necessarily equal to those which would be obtained from the intersection of all $\mathscr{T}$-closed supersets and the union of all $\mathscr{T}$-open subsets relative to the topology $\mathscr{T}$ [BKR13]. Throughout this work, by $\operatorname{cl}_{\mathfrak{g}} \circ \operatorname{int}_{\mathfrak{g}}(\cdot)$, $\operatorname{int}_{\mathfrak{g}} \circ \operatorname{cl}_{\mathfrak{g}}(\cdot)$, and $\operatorname{cl}_{\mathfrak{g}} \circ \operatorname{int}_{\mathfrak{g}} \circ \operatorname{cl}_{\mathfrak{g}}(\cdot)$, respectively, are meant $\operatorname{cl}_{\mathfrak{g}}(\operatorname{int}_{\mathfrak{g}}(\cdot))$, $\operatorname{int}_{\mathfrak{g}}(\operatorname{cl}_{\mathfrak{g}}(\cdot))$, and $\operatorname{cl}_{\mathfrak{g}}(\operatorname{int}_{\mathfrak{g}}(\operatorname{cl}_{\mathfrak{g}}(\cdot)))$; other composition operators are defined in a similar way. Also, the backslash $\Omega \setminus \mathscr{S}_{\mathfrak{g}}$ refers to the set-theoretic difference $\Omega - \mathscr{S}_{\mathfrak{g}}$. Finally, for convenience of notation, let $\mathscr{P}^{*}(\Omega) = \mathscr{P}(\Omega) \setminus \{\emptyset\}$, $\mathscr{T}_{\mathfrak{g}}^{*} = \mathscr{T}_{\mathfrak{g}} \setminus \{\emptyset\}$, and $\neg\mathscr{T}_{\mathfrak{g}}^{*} = \neg\mathscr{T}_{\mathfrak{g}} \setminus \{\emptyset\}$.

Definition A.2 ($\mathfrak{g}$-Operation). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then, a mapping $\mathrm{op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ is called a "$\mathfrak{g}$-operation" if and only if the following statements hold:

$$(\text{A.3})\quad \big(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}^{*}(\Omega)\big)\big(\exists (\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g}}^{*} \times \neg\mathscr{T}_{\mathfrak{g}}^{*}\big)\big[\big(\mathrm{op}_{\mathfrak{g}}(\emptyset) = \emptyset\big) \vee \big(\neg\,\mathrm{op}_{\mathfrak{g}}(\emptyset) = \emptyset\big) \\ \vee\big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\big)\big],$$

where $\neg\,\mathrm{op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is called the "complementary $\mathfrak{g}$-operation" on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ and, for all $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g},\nu}, \mathscr{S}_{\mathfrak{g},\mu} \in \mathscr{P}^{*}(\Omega)$, the following axioms are satisfied:

- Ax. i. $\big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\big)$,
- Ax. ii. $\big(\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g}} \circ \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}} \circ \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\big)$,
- Ax. iii. $\big(\mathscr{S}_{\mathfrak{g},\nu} \subseteq \mathscr{S}_{\mathfrak{g},\mu} \longrightarrow \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\mu})\big) \vee \big(\mathscr{S}_{\mathfrak{g},\mu} \subseteq \mathscr{S}_{\mathfrak{g},\nu} \longleftarrow \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\mu}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\nu})\big)$,
- Ax. iv. $\big(\mathrm{op}_{\mathfrak{g}}\big(\bigcup_{\sigma=\nu,\mu} \mathscr{S}_{\mathfrak{g},\sigma}\big) \subseteq \bigcup_{\sigma=\nu,\mu} \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\sigma})\big) \vee \big(\neg\,\mathrm{op}_{\mathfrak{g}}\big(\bigcup_{\sigma=\nu,\mu} \mathscr{S}_{\mathfrak{g},\sigma}\big) \supseteq \bigcup_{\sigma=\nu,\mu} \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\sigma})\big)$,

for some $\mathscr{T}_{\mathfrak{g}}$-open sets $\mathscr{O}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g},\nu}, \mathscr{O}_{\mathfrak{g},\mu} \in \mathscr{T}_{\mathfrak{g}}^{*}$ and $\mathscr{T}_{\mathfrak{g}}$-closed sets $\mathscr{K}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g},\nu}, \mathscr{K}_{\mathfrak{g},\mu} \in \neg\mathscr{T}_{\mathfrak{g}}$.

The formulation of Def. A.2 is based on the axioms of the Čech closure operator [Boo11] and the various axioms used by many mathematicians to define closure operators [MHD83].

Definition A.3 ($\mathbf{op}_{\mathfrak{g}}$-Elements). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then, the class $\mathscr{L}_{\mathfrak{g}}[\Omega] \overset{\text{def}}{=} \{\mathbf{op}_{\mathfrak{g},\nu} = (\mathrm{op}_{\mathfrak{g},\nu}, \neg\,\mathrm{op}_{\mathfrak{g},\nu}) :\ \nu \in I_{3}^{0}\} \subseteq \mathscr{L}_{\mathfrak{g}}^{\omega}[\Omega] \times \mathscr{L}_{\mathfrak{g}}^{\kappa}[\Omega]$, where

$$(\text{A.4})\qquad \mathrm{op}_{\mathfrak{g}} \in \mathscr{L}_{\mathfrak{g}}^{\omega}[\Omega] \overset{\text{def}}{=} \{\mathrm{op}_{\mathfrak{g},0},\ \mathrm{op}_{\mathfrak{g},1},\ \mathrm{op}_{\mathfrak{g},2},\ \mathrm{op}_{\mathfrak{g},3}\} \\ = \{\mathrm{int}_{\mathfrak{g}},\ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}},\ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}},\ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}\},$$

$$(\text{A.5})\qquad \neg\,\mathrm{op}_{\mathfrak{g}} \in \mathscr{L}_{\mathfrak{g}}^{\kappa}[\Omega] \overset{\text{def}}{=} \{\neg\,\mathrm{op}_{\mathfrak{g},0},\ \neg\,\mathrm{op}_{\mathfrak{g},1},\ \neg\,\mathrm{op}_{\mathfrak{g},2},\ \neg\,\mathrm{op}_{\mathfrak{g},3}\} \\ = \{\mathrm{cl}_{\mathfrak{g}},\ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}},\ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}},\ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}\},$$

stands for the class of all possible pairs of $\mathfrak{g}$-operators and its complementary $\mathfrak{g}$-operators in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

The use of $\mathbf{op}_{\mathfrak{g}} = (\mathrm{op}_{\mathfrak{g}}, \neg\,\mathrm{op}_{\mathfrak{g}}) \in \mathscr{L}_{\mathfrak{g}}[\Omega]$ on a class of $\mathfrak{T}_{\mathfrak{g}}$-sets will construct a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, just as the use of $\mathscr{L}[\Omega] \overset{\text{def}}{=} \{\mathbf{op}_{\nu} = (\mathrm{op}_{\nu}, \neg\,\mathrm{op}_{\nu}) :\ \nu \in I_{3}^{0}\}$ on the class of $\mathfrak{T}$-sets have constructed the new class of $\mathfrak{g}$-$\mathfrak{T}$-sets. But since $\mathrm{cl}_{\mathfrak{g}} \neq \mathrm{cl}$ and $\mathrm{int}_{\mathfrak{g}} \neq \mathrm{int}$, in general, it follows that $\mathbf{op}_{\mathfrak{g},\nu} \neq \mathbf{op}_{\nu}$ for some $\nu \in I_{3}^{0}$ and therefore, the new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets that will be obtained from the first construction will, in general, differ from the new class of $\mathfrak{g}$-$\mathfrak{T}$-sets that had been obtained from the second construction. Employing the set-builder notations, the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set of category ν may then be defined as thus:

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DEFINITION A.4. Let $(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathfrak{T}_{\mathfrak{g}} \times \mathscr{T}_{\mathfrak{g}} \times \neg \mathscr{T}_{\mathfrak{g}}$ and let $\mathbf{op}_{\mathfrak{g},\nu} \in \mathscr{L}_{\mathfrak{g}}[\Omega]$ be a $\mathfrak{g}$-operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose the predicates

$$\begin{aligned}
\mathrm{P}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \subseteq, \supseteq\big) &\stackrel{\text{def}}{=} \mathrm{P}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \subseteq\big) \vee \mathrm{P}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \supseteq\big),\\
\mathrm{P}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \subseteq\big) &\stackrel{\text{def}}{=} \big(\exists\,(\mathscr{O}_{\mathfrak{g}}, \mathrm{op}_{\mathfrak{g},\nu}) \in \mathscr{T}_{\mathfrak{g}} \times \mathscr{L}^{\omega}_{\mathfrak{g}}[\Omega]\big)\\
&\qquad\qquad \big[\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}})\big],\\
\mathrm{P}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \supseteq\big) &\stackrel{\text{def}}{=} \big(\exists\,(\mathscr{K}_{\mathfrak{g}}, \neg\,\mathrm{op}_{\mathfrak{g},\nu}) \in \neg\mathscr{T}_{\mathfrak{g}} \times \mathscr{L}^{\kappa}_{\mathfrak{g}}[\Omega]\big)\\
&\qquad\qquad \big[\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g}})\big]
\end{aligned} \tag{A.6}$$

be "Boolean-valued functions" on $\mathfrak{T}_{\mathfrak{g}} \times (\mathscr{T}_{\mathfrak{g}} \cup \neg\mathscr{T}_{\mathfrak{g}}) \times \mathscr{L}_{\mathfrak{g}}[\Omega] \times \{\subseteq, \supseteq\}$, then

$$\begin{aligned}
\mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \mathrm{P}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \subseteq, \supseteq\big)\big\},\\
\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \mathrm{P}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \subseteq\big)\big\},\\
\mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \mathrm{P}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \supseteq\big)\big\},
\end{aligned} \tag{A.7}$$

respectively, are called the classes of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of category ν in $\mathfrak{T}_{\mathfrak{g}}$.

Thus, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set of category ν if and only if there exist a pair $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}}$ of $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets and a $\mathfrak{g}$-operator $\mathbf{op}_{\mathfrak{g},\nu} \in \mathscr{L}_{\mathfrak{g}}[\Omega]$ of category ν such that the following statement holds:

$$(\exists\xi)\ \big[(\xi \in \mathscr{S}_{\mathfrak{g}}) \wedge \big(\big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g}})\big)\big)\big].$$

Clearly,

$$\begin{aligned}
\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \textstyle\bigcup_{\nu \in I^0_3} \mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}_{\mathfrak{g}}] &= \bigcup_{\nu \in I^0_3} \big(\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\big)\\
&= \Big(\bigcup_{\nu \in I^0_3} \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\Big) \cup \Big(\bigcup_{\nu \in I^0_3} \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\Big)\\
&\stackrel{\text{def}}{=} \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}],
\end{aligned}$$

then, defines the class of all $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-sets as the union of the classes of all $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, defined by $\mathfrak{g}$-O $[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}$-K $[\mathfrak{T}_{\mathfrak{g}}]$ respectively.

It is interesting to view the concepts of open, semi-open, preopen, semi-preopen sets [And86, And84, CM64, Lev63, MEMED82, Nj5] as $\mathfrak{g}$-$\mathfrak{T}$-open sets of categories 0, 1, 2, and 3, respectively; likewise, to view the concepts of closed, semi-closed, preclosed, semi-preclosed sets [And96] as $\mathfrak{g}$-$\mathfrak{T}$-closed sets of categories 0, 1, 2, and 3, respectively. These can be realised by omitting the subscript "$\mathfrak{g}$" in all symbols of the above definitions. The remark follows.

REMARK A.5. Observing that, for every $\nu \in I^*_3$, the first and second components of the $\mathfrak{g}$-vector operator $\mathbf{op}_{\mathfrak{g},\nu} = \big(\mathrm{op}_{\mathfrak{g},\nu}, \neg\,\mathrm{op}_{\mathfrak{g},\nu}\big) \in \mathscr{L}_{\mathfrak{g}}[\Omega]$ are based on $\mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}}$, respectively, it follows that $\mathbf{op}_{\mathfrak{g},\nu} = \mathbf{op}_{\nu} \stackrel{\text{def}}{=} (\mathrm{op}_{\nu}, \neg\,\mathrm{op}_{\nu}) \in \mathscr{L}[\Omega]$ if based on $\mathscr{T} \times \neg\mathscr{T}$, respectively. In this way, $\mathbf{op} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is

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called a $\mathfrak{g}$-vector operator in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$. Accordingly,

$$\text{(A.8)} \qquad \mathrm{op} \in \mathscr{L}^{\omega}[\Omega] \overset{\text{def}}{=} \{\mathrm{op}_0,\ \mathrm{op}_1,\ \mathrm{op}_2,\ \mathrm{op}_3\} = \{\mathrm{int},\ \mathrm{cl}\circ\mathrm{int},\ \mathrm{int}\circ\mathrm{cl},\ \mathrm{cl}\circ\mathrm{int}\circ\mathrm{cl}\},$$

$$\text{(A.9)} \qquad \neg\,\mathrm{op} \in \mathscr{L}^{\kappa}[\Omega] \overset{\text{def}}{=} \{\neg\,\mathrm{op}_0,\ \neg\,\mathrm{op}_1,\ \neg\,\mathrm{op}_2,\ \neg\,\mathrm{op}_3\} = \{\mathrm{cl},\ \mathrm{int}\circ\mathrm{cl},\ \mathrm{cl}\circ\mathrm{int},\ \mathrm{int}\circ\mathrm{cl}\circ\mathrm{int}\},$$

and, $\mathscr{L}_{\mathfrak{g}}[\Omega] \overset{\text{def}}{=} \{\mathbf{op}_{\mathfrak{g},\nu} = (\mathrm{op}_{\mathfrak{g},\nu}, \neg\,\mathrm{op}_{\mathfrak{g},\nu}) :\ \nu \in I_3^0\} \subseteq \mathscr{L}_{\mathfrak{g}}^{\omega}[\Omega] \times \mathscr{L}_{\mathfrak{g}}^{\kappa}[\Omega]$ stands for the class of all possible pairs of $\mathfrak{g}$-operators and its complementary $\mathfrak{g}$-operators in the $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$.

By virtue of the above remark, if $(\mathscr{S}, \mathscr{O}, \mathscr{K}) \in \mathfrak{T} \times \mathscr{T} \times \neg\mathscr{T}$ and $\mathbf{op}_{\nu} \in \mathscr{L}[\Omega]$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then the predicates

$$\mathrm{P}(\mathscr{S}, \mathscr{O}, \mathscr{K}; \mathbf{op}_{\nu}; \subseteq, \supseteq) \overset{\text{def}}{=} \mathrm{P}(\mathscr{S}, \mathscr{O}; \mathbf{op}_{\nu}; \subseteq) \vee \mathrm{P}(\mathscr{S}, \mathscr{K}; \mathbf{op}_{\nu}; \supseteq),$$

$$\mathrm{P}(\mathscr{S}, \mathscr{O}; \mathbf{op}_{\nu}; \subseteq) \overset{\text{def}}{=} (\exists\,(\mathscr{O}, \mathrm{op}_{\nu}) \in \mathscr{T} \times \mathscr{L}^{\omega}[\Omega])\,[\mathscr{S} \subseteq \mathrm{op}_{\nu}(\mathscr{O})],$$

$$\text{(A.10)} \quad \mathrm{P}(\mathscr{S}, \mathscr{K}; \mathbf{op}_{\nu}; \supseteq) \overset{\text{def}}{=} (\exists\,(\mathscr{K}, \neg\,\mathrm{op}_{\nu}) \in \neg\mathscr{T} \times \mathscr{L}^{\kappa}[\Omega])\,[\mathscr{S} \supseteq \neg\,\mathrm{op}_{\nu}(\mathscr{K})]$$

are obviously "Boolean-valued functions" on $\mathfrak{T} \times (\mathscr{T} \cup \neg\mathscr{T}) \times \mathscr{L}[\Omega] \times \{\subseteq, \supseteq\}$ and,

$$\mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}] \overset{\text{def}}{=} \{\mathscr{S} \subset \mathfrak{T} :\ \mathrm{P}(\mathscr{S}, \mathscr{O}, \mathscr{K}; \mathbf{op}_{\nu}; \subseteq, \supseteq)\},$$

$$\text{(A.11)} \qquad \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}] \overset{\text{def}}{=} \{\mathscr{S} \subset \mathfrak{T} :\ \mathrm{P}(\mathscr{S}, \mathscr{O}; \mathbf{op}_{\nu}; \subseteq)\},$$

$$\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}] \overset{\text{def}}{=} \{\mathscr{S} \subset \mathfrak{T} :\ \mathrm{P}(\mathscr{S}, \mathscr{K}; \mathbf{op}_{\nu}; \supseteq)\},$$

respectively, are called the classes of all $\mathfrak{g}$-$\mathfrak{T}$-sets, $\mathfrak{g}$-$\mathfrak{T}$-open sets and $\mathfrak{g}$-$\mathfrak{T}$-closed sets of category ν in $\mathfrak{T}$. Therefore, $\mathscr{S} \subset \mathfrak{T}$ is called a $\mathfrak{g}$-$\mathfrak{T}$-set of category ν if and only if there exist a pair $(\mathscr{O}, \mathscr{K}) \in \mathscr{T} \times \neg\mathscr{T}$ of $\mathscr{T}$-open and $\mathscr{T}$-closed sets and a $\mathfrak{g}$-operator $\mathbf{op}_{\nu} \in \mathscr{L}[\Omega]$ of category ν such that the following statement holds:

$$(\exists\xi)\left[(\xi \in \mathscr{S}) \wedge \left((\mathscr{S} \subseteq \mathrm{op}_{\nu}(\mathscr{O})) \vee (\mathscr{S} \supseteq \neg\,\mathrm{op}_{\nu}(\mathscr{K}))\right)\right].$$

Evidently,

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}] \overset{\text{def}}{=} \textstyle\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}] &= \bigcup_{\nu \in I_3^0} (\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}] \cup \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}]) \\
&= \Big(\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}]\Big) \cup \Big(\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}]\Big) \\
&\overset{\text{def}}{=} \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}] \cup \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}],
\end{aligned}$$

then, defines the class of all $\mathfrak{g}$-ν-$\mathfrak{T}$-sets as the union of the classes of all $\mathfrak{g}$-ν-$\mathfrak{T}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}$-closed sets, defined by $\mathfrak{g}$-O $[\mathfrak{T}]$ and $\mathfrak{g}$-K $[\mathfrak{T}]$ respectively.

Similar to the definitions of $\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}$, those standing for $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}$ and $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}$ are defined as thus:

Definition A.6. If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space and $\mathfrak{T} = (\Omega, \mathscr{T})$ be a $\mathscr{T}$-space, then:

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- I. $\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \overset{\text{def}}{=} \{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \mathrm{P}_{\mathfrak{g}}(\mathscr{S}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =)\}$ and $\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \overset{\text{def}}{=} \{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =)\}$ denote the classes of all $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets, respectively, in $\mathfrak{T}_{\mathfrak{g}}$, with $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$;
- II. $\mathrm{O}[\mathfrak{T}] \overset{\text{def}}{=} \{\mathscr{S} \subset \mathfrak{T} : \mathrm{P}(\mathscr{S}, \mathscr{S}; \mathbf{op}_0; =)\}$ and $\mathrm{K}[\mathfrak{T}] \overset{\text{def}}{=} \{\mathscr{S} \subset \mathfrak{T} : \mathrm{P}(\mathscr{S}, \mathscr{S}; \mathbf{op}_0; =)\}$ denote the classes of all $\mathfrak{T}$-open and $\mathfrak{T}$-closed sets, respectively, in $\mathfrak{T}$, with $\mathrm{S}[\mathfrak{T}] = \mathrm{O}[\mathfrak{T}] \cup \mathrm{K}[\mathfrak{T}]$.

Remark A.7. Since

$$\mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =, =) \overset{\text{def}}{=} \mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =) \vee \mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =),$$

it is plain that $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] \overset{\text{def}}{=} \{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =, =)\}$; likewise, since

$$\mathrm{P}(\mathscr{S}, \mathscr{S}, \mathscr{S}; \mathbf{op}_{\mathfrak{g},0}; =, =) \overset{\text{def}}{=} \mathrm{P}(\mathscr{S}, \mathscr{S}; \mathbf{op}_0; =) \vee \mathrm{P}(\mathscr{S}, \mathscr{S}; \mathbf{op}_0; =),$$

it follows that $\mathrm{S}[\mathfrak{T}] \overset{\text{def}}{=} \{\mathscr{S} \subset \mathfrak{T} : \mathrm{P}(\mathscr{S}, \mathscr{S}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =, =)\}$.

Definition A.8 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Separation, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Connected). A $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separation of category ν of two nonempty $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is realised if and only if there exists either a pair $(\mathscr{O}_{\mathfrak{g},\xi}, \mathscr{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets or a pair $(\mathscr{K}_{\mathfrak{g},\xi}, \mathscr{K}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets such that:

$$\left(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{O}_{\mathfrak{g},\lambda} = \mathscr{R}_{\mathfrak{g}} \sqcup \mathscr{S}_{\mathfrak{g}}\right) \bigvee \left(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{K}_{\mathfrak{g},\lambda} = \mathscr{R}_{\mathfrak{g}} \sqcup \mathscr{S}_{\mathfrak{g}}\right). \tag{A.12}$$

Two nonempty $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ which are not $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated of category ν are said to be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected of category ν.

Thus, a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in $\mathscr{T}_{\mathfrak{g}}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected if and only if $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated if and only if $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{D}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{D}[\mathfrak{T}_{\mathfrak{g}}]$ where,

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}] \overset{\text{def}}{=} \Big\{ &\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : (\forall (\mathscr{O}_{\mathfrak{g},\lambda}, \mathscr{K}_{\mathfrak{g},\lambda})_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]) \\ &\left[\neg\left(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{O}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\right) \bigwedge \neg\left(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{O}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\right)\right]\Big\}; \end{aligned} \tag{A.13}$$

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{D}[\mathfrak{T}_{\mathfrak{g}}] \overset{\text{def}}{=} \Big\{ &\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : (\exists (\mathscr{O}_{\mathfrak{g},\lambda}, \mathscr{K}_{\mathfrak{g},\lambda})_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]) \\ &\left[\left(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{O}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\right) \bigvee \left(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{K}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\right)\right]\Big\}. \end{aligned} \tag{A.14}$$

The following remark marks the end of this pre-preliminaries section.

Remark A.9. For each, $\nu \in I_3^0$, the dependence of $\mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-}\nu\text{-}\mathrm{D}[\mathfrak{T}_{\mathfrak{g}}]$ on both $\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ is clear from their definitions. Thus, to define the pairs $(\nu\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}], \nu\text{-}\mathrm{D}[\mathfrak{T}_{\mathfrak{g}}])$, $(\mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}[\mathfrak{T}], \mathfrak{g}\text{-}\nu\text{-}\mathrm{D}[\mathfrak{T}])$, and $(\nu\text{-}\mathrm{Q}[\mathfrak{T}], \nu\text{-}\mathrm{D}[\mathfrak{T}])$, respectively, it suffices to let them be dependent on the pairs $(\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}], \nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}])$, $(\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}], \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}])$, and $(\nu\text{-}\mathrm{O}[\mathfrak{T}], \nu\text{-}\mathrm{K}[\mathfrak{T}])$. Further, in defining $\mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-}\nu\text{-}\mathrm{D}[\mathfrak{T}_{\mathfrak{g}}]$, it is clear that by the statement $(\mathscr{O}_{\mathfrak{g},\lambda}, \mathscr{K}_{\mathfrak{g},\lambda})_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times$

$\mathfrak{g}$-ν-K$\left[\mathfrak{T}_{\mathfrak{g}}\right]$ is meant a pair of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets. Furthermore, by $\Omega \in \mathfrak{g}$-ν-Q$\left[\mathfrak{T}_{\mathfrak{g}}\right]$ or $\Omega \in \mathfrak{g}$-ν-D$\left[\mathfrak{T}_{\mathfrak{g}}\right]$ is meant a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connection of category ν or a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separation of category ν of the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is realised.

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